EQUIVARIANT BURNSIDE GROUPS: STRUCTURE AND OPERATIONS

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Abstract. We introduce and study functorial and combinatorial constructions concerning equivariant Burnside groups.

1. Introduction

Let G be a finite group, and k a field of characteristic zero containing all roots of unity of order dividing the order of G. In this paper, we continue the study of an invariant in G-equivariant birational geometry over k, the *equivariant Burnside group*

$$\mathrm{Burn}_n(G),$$

introduced in [8], building on [6], [5], [7], and [2].

The class of a projective n-dimensional G-variety in this group is computed from an appropriate smooth G-equivariant birational model X. For instance, X may be taken to be in *standard form* [16], meaning:

- there is an invariant open $U \subset X$, on which the G-action is free,
- the complement $X \setminus U$ is a simple normal crossing divisor, and
- for every $g \in G$ and every irreducible component D of $X \setminus U$, either $g(D) = D$ or $g(D) \cap D = \emptyset$.

Standard form implies abelian stabilizers [16, Thm. 4.1]. On such a model, the class of $X \circlearrowleft G$ is defined by:

$$[X \circlearrowleft G] := \sum_{H \subseteq G} \sum_{F} \mathfrak{s}_F \in \mathrm{Burn}_n(G), \tag{1.1}$$

with summation over conjugacy classes of abelian subgroups $H \subseteq G$ and strata $F \subseteq X$ with generic stabilizer H; the symbol

$$\mathfrak{s}_F := (H, N_G(H)/H \circlearrowright k(F), \beta_F(X))$$

records the action of the normalizer $N_G(H)$ of H on $k(F)$, the product of the function fields of the components of F, as well as the generic normal bundle representation $\beta_F(X)$ of H. The class in $[X \circlearrowleft G]$ takes its value in the quotient of the free abelian group on symbols by certain

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blow-up relations, spelled out in [8, Definition 4.2]; these ensure that this expression is a well-defined G-birational invariant [8, Theorem 5.1].

In [2], we presented first geometric applications of this invariant. Here, after generalities in Section 2 we explore functorial and combinatorial properties of $\mathrm{Burn}_n(G)$, specifically:

- filtrations on $\mathrm{Burn}_n(G)$ (Section 3),
- an indexed variant (Section 4), applied to get a formula for the class of a projectivization of a sum of line bundles (Section 5),
- products (Section 6),
- the restriction homomorphism
$$\mathrm{Burn}_n(G) \to \mathrm{Burn}_n(G'),$$
where $G' \subset G$ is any subgroup (Section 7),
- a combinatorial analog $\mathcal{BC}_n(G)$ of $\mathrm{Burn}_n(G)$ (Section 8), which keeps only discrete invariants encoded in a symbol.

One of the motivating problems is to distinguish equivariant birational types of (projectivizations of) linear actions (see, e.g., [15], [3]). A sample question, raised in [1, Section 8], is: *Are there isomorphic finite subgroups of* PGL_3 *which are not conjugate in the plane Cremona group?*

Examples of equivariantly nonbirational representations in [17] require G to contain an abelian subgroup of rank equal to the dimension of the representation. Our formalism yields new examples without this condition; see Example 5.4.

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2. Generalities

We adopt notational conventions from [8]:

- G is a finite group,
- k is a field of characteristic 0, containing a primitive eth root of unity, where e is the least common multiple of the exponents of the abelian subgroups of G,
- we write $H \subseteq G$ for an abelian subgroup, with character group
$$H^\vee := \mathrm{Hom}(H, k^\times),$$
- $\mathrm{Bir}_d(k)$ is the set of birational equivalence classes of d-dimensional algebraic varieties over k , i.e., the isomorphism classes of finitely generated fields of transcendence degree d over k; we identify a field with its isomorphism class in $\mathrm{Bir}_d(k)$,

- $\mathrm{Alg}_N(K_0)$ is the set of isomorphism classes of Galois algebras K over $K_0 \in \mathrm{Bir}_d(k)$ for the group
$$N := N_G(H)/H,$$
satisfying

  Assumption 1: the composite homomorphism
$$\mathrm{H}^1(N_G(H), K^\times) \to \mathrm{H}^1(H, K^\times)^N \to H^\vee \tag{2.1}$$
is surjective (see [8, Section 2] for more details).
- More generally, for a subgroup $M \subseteq N$ we denote by $\mathrm{Alg}_M(K_0)$ the set of isomorphism classes of M-Galois algebras K/K_0 (i.e., Galois algebras K over K_0 for the group M), such that $\mathrm{Ind}_M^N(K)$ satisfies Assumption 1. Of particular interest is
$$Z := Z_G(H)/H \subseteq N.$$

Lemma 2.1. *Let $K' \in \mathrm{Alg}_N(K_0)$. Then*
$$K' \cong \mathrm{Ind}_Z^N(K)$$
for some $K \in \mathrm{Alg}_Z(K_0)$.

Proof. The $N_G(H)$-action on $K'^\times$ restricts to a trivial action of H, so
$$\mathrm{H}^1(H, K'^\times) = \mathrm{Hom}(H, K'^\times).$$
We write $K' \cong K'^1 \times \cdots \times K'^\ell$ and choose the projection to the first factor K'^1 to define the rightmost map in (2.1) (as we may do, by [8, Rmk. 2.1]):
$$\mathrm{Hom}(H, K'^\times) \to \mathrm{Hom}(H, (K'^1)^\times) \cong H^\vee.$$
This is Y-equivariant, where $Y \subseteq N$ denotes the subgroup, defined by the condition of mapping K'^1 to itself. Here the Y-action on $H^\vee$ is induced by the conjugation action on H. Of course, Y-invariant elements map to Y-invariant elements. So Assumption 1 implies that the conjugation action of Y on H is trivial, i.e., $Y \subseteq Z$. With
$$K = \mathrm{Ind}_Y^Z(K'^1) \tag{2.2}$$
we have the result. □

Remark 2.2. Assumption 1, for an N-Galois algebra K'/K_0, of the form $K' = \mathrm{Ind}_Z^N(K)$, where K/K_0 is a Z-Galois algebra, may be expressed as the surjectivity of
$$\mathrm{H}^1(Z_G(H), K^\times) \to H^\vee. \tag{2.3}$$
This is equivalent to the existence of an equivalence of categories between

- $Z_G(H)$-Galois algebras L/L_0 with equivariant $K \to L$ and
- H-Galois algebras $\widetilde{L}_0/L_0$

over étale K_0-algebras L_0, as we see by the proof of [8, Prop. 2.2]. A further equivalent condition is the existence of an object of the first category for $L_0 = K_0$. (The existence of $Z_G(H)$-Galois algebra over K_0 with equivariant homomorphism from K leads to an equivalence of the two categories by the construction of [12, Lem. 3.21].) Consequently we may view Assumption 1 as a lifting problem of Galois cohomology

$$\mathrm{H}^1(\mathrm{Gal}_{K_0}, Z_G(H)) \to \mathrm{H}^1(\mathrm{Gal}_{K_0}, Z).$$

We remark that the machinery of nonabelian cohomology (cf. [14, §1.3.2]) supplies an obstruction to lifting in $\mathrm{H}^2(\mathrm{Gal}_{K_0}, H)$.

We now recall the definition of the *equivariant Burnside group*

$$\mathrm{Burn}_n(G) = \mathrm{Burn}_{n,k}(G)$$

following [8, Section 4]: it is a $\mathbb{Z}$-module, generated by symbols

$$\mathfrak{s} := (H, N \circlearrowright K, \beta),$$

where

- $H \subseteq G$ is an abelian subgroup,
- $K \in \mathrm{Alg}_N(K_0)$, with $K_0 \in \mathrm{Bir}_d(k)$, and $d \le n$,
- $\beta = (b_1, \dots, b_{n-d})$, a sequence of nonzero elements of the character group $H^\vee$, that generate $H^\vee$.

The sequence of characters β determines a faithful $(n-d)$-dimensional representation of H over k, with trivial space of invariants. As every $(n-d)$-dimensional representation of H over k splits as a sum of one-dimensional representations, any faithful $(n-d)$-dimensional representation of H over k determines a sequence of characters, generating $H^\vee$, up to order. The ambiguity of order gives us the first of several relations that we impose on symbols:

(O): $(H, N \circlearrowright K, \beta) = (H, N \circlearrowright K, \beta')$ if β' is a reordering of β.

The further relations are **conjugation** and **blowup** relations:

(C): $(H, N \circlearrowright K, \beta) = (H', N' \circlearrowright K, \beta')$, when $H' = gHg^{-1}$ and $N' = N_G(H')/H'$, with $g \in G$, and β and β' are related by conjugation by g.

(B1): $(H, N \circlearrowright K, \beta) = 0$ when $b_1 + b_2 = 0$.

(B2): $(H, N \circlearrowright K, \beta) = \Theta_1 + \Theta_2$, where

$$\Theta_1 = \begin{cases} 0, & \text{if } b_1 = b_2, \\ (H, N \circlearrowright K, \beta_1) + (H, N \circlearrowright K, \beta_2), & \text{otherwise,} \end{cases}$$

with

$$\beta_1 := (b_1, b_2 - b_1, b_3, \dots, b_{n-d}), \quad \beta_2 := (b_2, b_1 - b_2, b_3, \dots, b_{n-d}), \quad (2.4)$$

and

$$\Theta_2 = \begin{cases} 0, & \text{if } b_i \in \langle b_1 - b_2 \rangle \text{ for some } i, \\ (\overline{H}, \overline{N} \circlearrowright \overline{K}, \bar{\beta}), & \text{otherwise,} \end{cases}$$

with

$$\overline{H}^\vee := H^\vee / \langle b_1 - b_2 \rangle, \quad \bar{\beta} := (\bar{b}_2, \bar{b}_3, \dots, \bar{b}_{n-d}), \quad \bar{b}_i \in \overline{H}^\vee, \tag{2.5}$$

and $\overline{K}$ carries the action described in Construction **(A)** in [8, Section 2], applied to the character $b_1 - b_2$.

We remark that, following [10, §10], the presentation of $\mathrm{Burn}_n(G)$ may be simplified by allowing an arbitrary sequence of elements $(b_1, \dots, b_{n-d})$ generating $H^\vee$ as β in a symbol and imposing the additional relation, that a symbol $(H, N \circlearrowright K, \beta)$ vanishes whenever $b_i = 0$ for some i. Then relations **(B1)** and **(B2)** may be combined into the single relation

(B):
$(H, N \circlearrowright K, \beta) = (H, N \circlearrowright K, \beta_1) + (H, N \circlearrowright K, \beta_2) + (\overline{H}, \overline{N} \circlearrowright \overline{K}, \bar{\beta})$.

We permit ourselves to write a symbol in the form

$$(H, M \circlearrowright K, \beta) \tag{2.6}$$

with a subgroup $M \subset N$ and $K \in \mathrm{Alg}_M(K_0)$, with

$$(H, M \circlearrowright K, \beta) := (H, N \circlearrowright \mathrm{Ind}_M^N(K), \beta).$$

We further allow K_0 to be a *product* of fields; then (2.6) will denote the corresponding sum of symbols, one for each factor.

By Lemma 2.1, any symbol in $\mathrm{Burn}_n(G)$ is of the form

$$(H, Z \circlearrowright K, \beta),$$

with $K \in \mathrm{Alg}_Z(K_0)$.

In this notation, Construction **(A)** has a compact formulation. Applied to a single character $b \in H^\vee$, this yields the subgroup

$$\overline{H} := \ker(b) \subset H$$

and the symbol

$$(\overline{H}, Z_G(H)/\overline{H} \circlearrowright K(t), \bar{\beta}),$$

where a $Z_G(H)$-action on $K(t)$ arises by lifting b via (2.3) and is trivial on $\overline{H}$, and $\bar{\beta}$ is obtained from β as above.

Remark 2.3. Construction **(A)** may be applied to a collection of characters, yielding the same outcome as when applied iteratively, one character at a time.

We recall the convention on G-varieties from [8, §3], which allows X to have several components but demands that the generic point of any component has dense orbit in X.

A G-variety in standard form always satisfies

Assumption 2: The stabilizers for the G-action on X are abelian, and for every H and F in (1.1) the composite homomorphism

$$\mathrm{Pic}^G(X) \to \mathrm{H}^1(N_G(H), k(F)^\times) \to H^\vee$$

is surjective, where the first map is given by restriction and the second is the map from Assumption 1, with $K = k(F)$.

Note that Assumption 2 implies Assumption 1, for every H and every $N_G(H)/H \circlearrowright k(F)$ (see [8, Rmk. 3.2(i)]). Thus, (1.1) defines an element of $\mathrm{Burn}_n(G)$. Alternative expressions may be given, with normalizers replaced by centralizers as in Lemma 2.1 or further by the subgroup from which the Galois algebra is induced as in (2.2):

$$[X \circlearrowleft G] = \sum_{(H, \mathrm{Pic}^G(X) \to H^\vee)} \sum_F (H, Z \circlearrowright k(F), \beta_F(X)) \tag{2.7}$$

$$= \sum_{x \in X} (H_x, G_x/H_x \circlearrowright k(x), \beta_{\overline{\{x\}}}(X)) \tag{2.8}$$

The first sum is over conjugacy class representatives of pairs, consisting of an abelian subgroup of G and a surjective homomorphism to the group of characters, while the second is over orbit representatives for the G-action on the points (in the scheme-theoretic sense, i.e.., generic points of subvarieties) of X, where $x \in X$ has stabilizer G_x and $\overline{\{x\}}$ has generic stabilizer H_x; cf. [9, Defn. 2.3] for (2.7) and [11, p. 3027] for (2.8).

If X is not assumed to be projective, but only quasi-projective, then there are two ways to attach a class in $\mathrm{Burn}_n(G)$, as described in [8, §5]. The *naive class* $[X \circlearrowleft G]^{\mathrm{naive}}$ is defined by the same formula as in the projective case, i.e., equivalently (1.1), (2.7), or (2.8). With a suitable equivariant compactification, the (non-naive) class $[X \circlearrowleft G]$ is the sum of the class of the compactification and an adjustment which takes the form of an alternating sum of naive classes of normal bundles of strata.

3. Filtrations

In this section, we explore additional combinatorial constructions on equivariant Burnside groups $\mathrm{Burn}_n(G)$, reflecting the geometry of the G-action on strata with given generic stabilizers.

Definition 3.1. A G-prefilter is a collection $\mathbf{H}$ of pairs (H, Y) consisting of an abelian subgroup $H \subseteq G$ and a subgroup

$$Y \subseteq Z = Z_G(H)/H,$$

such that $\mathbf{H}$ is closed under conjugation, i.e., for $(H, Y) \in \mathbf{H}$ we have

$$(gHg^{-1}, gYg^{-1}) \in \mathbf{H}, \quad \text{for all } g \in G.$$

A G-filter is a G-prefilter $\mathbf{H}$, such that $(H, Y) \in \mathbf{H}$ with H nontrivial and $g \in Z_G(H)$ with $\bar{g} \in Y$ and $Y \subseteq Z_G(g)/H$ implies $(\langle H, g\rangle, Y/\langle \bar{g}\rangle) \in \mathbf{H}$.

Definition 3.2. Given a G-prefilter $\mathbf{H}$, we let

$$\mathrm{Burn}_n^{\mathbf{H}}(G)$$

be the quotient of $\mathrm{Burn}_n(G)$ by the subgroup generated by classes of the form

$$(H, Y \circlearrowright K, \beta),$$

where $K \in \mathrm{Alg}_Y(K_0)$ is a *field*, and

$$(H, Y) \notin \mathbf{H}.$$

Proposition 3.3. *Let* $\mathbf{H}$ *be a* G*-filter Then* $\mathrm{Burn}_n^{\mathbf{H}}(G)$ *is generated by triples*

$$(H, Y \circlearrowright K, \beta),$$

where $K \in \mathrm{Alg}_Y(K_0)$ *is a field and* $(H, Y) \in \mathbf{H}$*, subject to relations* **(O)**, **(C)**, *and* **(B)** *applied to these triples.*

Proof. We consider

$$(H, Y \circlearrowright K, \beta)$$

with $K \in \mathrm{Alg}_Y(K_0)$ a field, $Y = Y_0/H$ for some $Y_0 \subseteq Z_G(H)$. In the term $(\overline{H}, \overline{Y} \circlearrowright \overline{K}, \bar{\beta})$ from **(B)** (where $\overline{Y} = Y_0/\overline{H}$ and $\overline{K} = K(t)$), if $\overline{H} = \ker(b_1 - b_2)$ is nontrivial, then $(\overline{H}, \overline{Y}) \in \mathbf{H}$ implies $(H, Y) \in \mathbf{H}$. This observation establishes the proposition. □

Example 3.4.

- For G abelian, the G-filter $\{(G, \mathrm{triv})\}$ leads to

  $$\mathrm{Burn}_n^G(G) = \mathrm{Burn}_n^{\{(G,\mathrm{triv})\}}(G),$$

  introduced in [8, §8].
- For the G-prefilter $\mathbf{H}$ consisting of all (H, Y) with H nontrivial cyclic and Y noncyclic, $\mathrm{Burn}_2^{\mathbf{H}}(G)$ appeared in [2, §7.4].

Remark 3.5. Let $(H, Y \circlearrowright K^1, \beta) \in \mathrm{Burn}_n(G)$, where, as in Lemma 2.1, the symbol has been written with $Y \subseteq Z$ acting on a field K^1 and $Y = Y^1/H$ for some subgroup $Y^1 \subseteq Z_G(H)$ containing H. Let F be a smooth projective model, $k(F) \cong K^1$. According to [4, Thm. IV.5.5], formation of the MRC quotient F' of F is functorial, so we get a rational action of Y^1 on F' with some generic stabilizer U^1 and Galois algebra over $k(F')^{Y^1}$ for the group Y^1/U^1. The association, to the given symbol, of U^1 and $Y^1/U^1 \circlearrowright k(F')$, up to conjugation, survives the defining relations of $\mathrm{Burn}_n(G)$, thus giving a direct sum decomposition of $\mathrm{Burn}_n(G)$ according to the conjugacy class of the pair $(U^1, Y^1/U^1 \circlearrowright L^1)$, where $L^1 = k(F')$.

Remark 3.6. One can additionally suppress the field information, which will lead to *combinatorial* analogues of Burnside groups. We will explore this in Section 8.

4. Nontrivial generic stabilizers

In this section, we introduce a version of the equivariant Burnside group, relevant for considerations of actions with *nontrivial* generic stabilizer.

Let G be a finite group. A variant of the equivariant Burnside group takes the additional data of a finite index set

$$I \subset \mathbb{N}.$$

The *equivariant indexed Burnside group*

$$\mathrm{Burn}_{n,I}(G),$$

is defined as a quotient of the $\mathbb{Z}$-module generated by symbols

$$(H \subseteq H', N' \circlearrowright K, \beta, \gamma),$$

where

- $H \subseteq H'$ are abelian subgroups of G,
- $N' := N_{N_G(H)}(H')/H'$,
- $K \in \mathrm{Alg}_{N'}(K_0)$, with $K_0 \in \mathrm{Bir}_d(k)$, and $d \le n - |I|$,
- $\beta = (b_1, \dots, b_{n-d-|I|})$, a sequence of nonzero characters of H', trivial upon restriction to H, that generate $(H'/H)^\vee$,
- $\gamma = (c_i)_{i \in I}$ is a sequence of elements of $H'^\vee$, such that the images of c_i in $H^\vee$ generate $H^\vee$.

As in Section 2, we permit ourselves to write a symbol in the form

$$(H \subseteq H', M' \circlearrowright K, \beta, \gamma),$$

where $M' \subset N'$ is a subgroup. Every symbol may be expressed as

$$(H \subseteq H', Z' \circlearrowright K, \beta, \gamma), \qquad Z' := Z_G(H')/H'.$$

(Notice that $Z_G(H') = Z_{N_G(H)}(H')$.)

These symbols are subject to relations:

(O): $(H \subseteq H', N' \circlearrowright K, \beta, \gamma) = (H \subseteq H', N' \circlearrowright K, \beta', \gamma)$ if β' is a reordering of β.

(C): $(H \subseteq H', N' \circlearrowright K, \beta, \gamma) = (gHg^{-1} \subseteq gH'g^{-1}, gN'g^{-1} \circlearrowright K, \beta', \gamma')$ for $g \in G$, with β and β', respectively γ and γ', related by conjugation by g.

(B1): $(H \subseteq H', N' \circlearrowright K, \beta, \gamma) = 0$ when $b_1 + b_2 = 0$.

(B2): $(H \subseteq H', N' \circlearrowright K, \beta, \gamma) = \Theta_1 + \Theta_2$, where Θ_1 and Θ_2 are as in Section 2, with H prepended and γ, respectively $\bar{\gamma}$, appended to the corresponding symbols.

As in Section 2, we may allow symbols where β contains the zero character, impose a relation of the vanishing of such symbols, and combine **(B1)** and **(B2)** into a single relation **(B)**.

Remark 4.1. By analogy with Remark 2.2, we may express Assumption 1, for the Galois algebra K, as the surjectivity of the middle vertical map

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{H}^1(Z_G(H')/H, K^\times) & \to & \mathrm{H}^1(Z_G(H'), K^\times) & \to & \mathrm{H}^1(H, K^\times)^{Z_G(H')/H} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (H'/H)^\vee & \longrightarrow & H'^\vee & \longrightarrow & H^\vee & \longrightarrow & 0
\end{array}$$

Here, the top row comes from the Hochschild-Serre spectral sequence. In a symbol, we have β generating the left-hand group in the bottom row, while γ is a sequence of characters of H', whose images generate $H^\vee$. Consequently, β and γ together generate $H'^\vee$. Thus we have a homomorphism

$$\psi_I \colon \mathrm{Burn}_{n,I}(G) \to \mathrm{Burn}_n(G),$$

sending $(H \subseteq H', Z' \circlearrowright K, \beta, \gamma)$ to

$$(H', Z' \circlearrowright K, \beta \cup \gamma).$$

(More generally there is $\psi_{I,J}$ for $J \subseteq I$, mapping to $\mathrm{Burn}_{n,J}(G)$, cf. [9, Defn. 4.2].)

In order to explain the relevance of this definition, we introduce a map which converts some of the characters in γ to a transcendental extension of the Galois algebra. Let

$$J \subseteq I$$

be a subset. Given a symbol $(H\subseteq H', Z'\circlearrowright K,\beta,\gamma)$, we define subgroups

$$\overline{H}' := \bigcap_{i\in I\setminus J}\ker(c_i)\subseteq H',$$
$$\overline{H} := H\cap \overline{H}'\subseteq H.$$

Then we define

$$\omega_{I,J}\colon \mathrm{Burn}_{n,I}(G)\to \mathrm{Burn}_{n,J}(G),$$

by applying Construction **(A)** to the characters of γ indexed by $I\setminus J$:

$$(H\subseteq H', Z'\circlearrowright K,\beta,\gamma)\mapsto (\overline{H}\subseteq \overline{H}', Z_G(H')/\overline{H}'\circlearrowright K((t_i)_{i\in I\setminus J}),\bar\beta,\bar\gamma),$$

where $\bar\gamma=(\bar c_j)_{j\in J}$. This is compatible with relations, as we see using Remark 2.3.

We recall the setting of [8, Defn. 5.4]: Let X be a smooth projective variety of dimension n, with a generically free action of G, satisfying Assumption 2. Let $D_1,\dots,D_\ell$ be G-stable divisors, with

$$D_I := \bigcap_{i\in I} D_i,\qquad \text{for } I\subseteq \mathcal{I}:=\{1,\dots,\ell\},\quad D_\emptyset = X.$$

We suppose, for notational simplicity, that for every I the generic stabilizers of the components of D_I belong to a single conjugacy class of subgroups, and take H_I to be a representative. Then to $I\subseteq M\subseteq\mathcal{I}$ we attach the following class in $\mathrm{Burn}_{n,I}(G)$:

$$\chi_{M,I}(X\circlearrowleft G,(D_i)_{i\in\mathcal{I}}) := \sum_{H'\supseteq H_I}\ \sum_{\substack{W\subset D_I\\ \{i\in\mathcal{I}\,|\,W\subset D_i\}=M}} (H_I\subseteq H', N'\circlearrowright k(W),\beta,\gamma),$$

where

- the first sum is over conjugacy class representatives H' of abelian subgroups of $N_G(H_I)$, containing H_I,
- the second sum is over $N_{N_G(H_I)}(H')$-orbits of components W with generic stabilizer H', contained in components of D_I with generic stabilizer H_I and satisfying $\{i\in\mathcal{I}\,|\,W\subset D_i\}=M$,
- $\beta=\beta_W(D_I)$ encodes the normal bundle to W in D_I, and
- $\gamma=(c_i)_{i\in I}$, the characters coming from D_i with $i\in I$.

Then

$$[\mathcal{N}_{D_I/X}\circlearrowleft G]^{\mathrm{naive}} = \sum_{I\subseteq M\subseteq\mathcal{I}}\ \sum_{M\setminus I\subseteq J\subseteq M}\psi_{I\cap J}(\omega_{I,I\cap J}(\chi_{M,I}(X\circlearrowleft G,(D_i)_{i\in\mathcal{I}}))).$$

The summand may be rewritten as $\psi_J(\omega_{M,J}(\chi_{M,M}(X\circlearrowleft G,(D_i)_{i\in\mathcal{I}})))$ using [9, Exa. 4.3]. Now we obtain some insight into [8, Lemma 5.7] by recognizing the summand with $J=\emptyset$ as $[\mathcal{N}^\circ_{D_I/X}\circlearrowleft G]^{\mathrm{naive}}$.

5. Fibrations

In this section, we define a projectivized version of the equivariant indexed Burnside group and use it to give a formula for the class in $\mathrm{Burn}_n(G)$ of the projectivization of a sum of line bundles.

Let G be a finite group and $I \subset \mathbb{N}$ a *nonempty* finite index set. The *equivariant projectively indexed Burnside group*

$$\mathrm{Burn}_{n,\mathbb{P}(I)}(G)$$

is defined with generators and relations as in Section 4, where

- β consists of $n-d-|I|+1$ characters (so $d \le n-|I|+1$),
- the *differences* of pairs of characters of γ should generate $H^\vee$,
- and there is an additional relation:

(P): If $\gamma'-\gamma$ is a constant sequence then

$$(H \subseteq H', N' \circlearrowright K, \beta, \gamma) = (H \subseteq H', N' \circlearrowright K, \beta, \gamma').$$

We define

$$\omega_{\mathbb{P}(I),J}\colon\ \mathrm{Burn}_{n,\mathbb{P}(I)}(G) \to \mathrm{Burn}_{n,J}(G),$$

for a *proper* subset

$$J \subsetneq I,$$

by

- choosing $i_0 \in I \setminus J$,
- applying **(P)** to get a representative symbol
$$(H \subseteq H', N' \circlearrowright K, \beta, \gamma)$$
with $\gamma_{i_0}=0$, and
- applying $\omega_{I\setminus\{i_0\},J}$ to the class of $(H \subseteq H', N' \circlearrowright K, \beta, (c_i)_{i\in I\setminus\{i_0\}})$ in $\mathrm{Burn}_{n,I\setminus\{i_0\}}(G)$.

Let X be a smooth projective variety over k. Assume that X carries a G-action, and let $L_0, \ldots, L_r$ be G-linearized line bundles on X. The next statement examines the condition, for G to act generically freely on $\mathbb{P}(L_0 \oplus \cdots \oplus L_r)$, so that Assumption 2 satisfied. The statement uses the variant of Assumption 2, where $\mathrm{Pic}^G(X)$ is replaced by the subgroup of $\mathrm{Pic}^G(X)$ generated by a given collection of G-linearized line bundles. Given this, we will say that Assumption 2 holds for the given collection of G-linearized line bundles.

Lemma 5.1. *Let X be a smooth projective variety over k with a G-action and G-linearized line bundles $L_0, \ldots, L_r$. Let H be the stabilizer at the generic point of a component of X, and let us denote the $N_G(H)$-orbit of the component by X'. The following are equivalent.*

(i) *The N-action on X' satisfies Assumption 2, and H is abelian with $H^\vee$ spanned by the differences of characters determined by $L_0, \dots, L_r$.*
(ii) *The G-action on $\mathbb{P}(L_0 \oplus \cdots \oplus L_r)$ is generically free and satisfies Assumption 2.*
(iii) *The G-action on $\mathbb{P}(L_0 \oplus \cdots \oplus L_r)$ is generically free and satisfies Assumption 2 for $L_0, \dots, L_r$, together with the G-linearized line bundles on X associated with N-linearized line bundles on X'.*

The statement is inspired by [8, Lemma 7.3]. The association of line bundles in (iii) refers to the equivalence of categories (by restriction) between G-linearized line bundles on X and $N_G(H)$-linearized line bundles on X'. Since $N = N_G(H)/H$ and H acts trivially on X', an $N_G(H)$-linearization is determined by inflation from an N-linearization of a line bundle on X'. An $N_G(H)$-linearization arises in this manner if and only if it induces the trivial H-representation at points of X'.

Proof. The action of G on $\mathbb{P}(L_0 \oplus \cdots \oplus L_r)$ is generically free if and only if the action of $N_G(H)$ on $\mathbb{P}(L_0|_{X'} \oplus \cdots \oplus L_r|_{X'})$ is generically free. The latter has generic stabilizer $\bigcap_{i=1}^r \ker(b_i - b_0)$. Thus the condition on H in (i) is equivalent to the condition of generically free action in (ii) and in (iii). We assume this from now on.

We start by showing (i) implies (iii), using the interpretation of Assumption 2 in terms of the representability of a morphism from the quotient stack to a product of copies of $B\mathbb{G}_m$ as in [8, Rmk. 3.2]. Given (i), we have such a representable morphism

$$[X'/N] \to B\mathbb{G}_m \times \cdots \times B\mathbb{G}_m.$$

Correspondingly, the fibers of the composite morphism

$$[X/G] \cong [X'/N_G(H)] \to [X'/N] \to B\mathbb{G}_m \times \cdots \times B\mathbb{G}_m$$

all have constant stabilizer group H. The condition in (i) implies that the H-representation given by $b_0, \dots, b_r$ is faithful. With $r+1$ additional factors $B\mathbb{G}_m$ we get a representable morphism from $[X/G]$, hence also from $\mathbb{P}(L_0 \oplus \cdots \oplus L_r)$.

Since trivially (iii) implies (ii), it remains only to show (ii) implies (i). Since

$$\mathbb{P}(L_0 \oplus \cdots \oplus L_r) \to X$$

admits equivariant sections, (ii) implies the existence of a representable morphism

$$[X/G] \to B\mathbb{G}_m \times \cdots \times B\mathbb{G}_m, \tag{5.1}$$

determined by some finite collection of G-linearized line bundles on X. If we tensor each of these by a suitable tensor combination of $L_0, \dots, L_r$, then we may suppose that each of these comes from an N-linearized line bundle on X'; now the morphism (5.1) has fibers with constant stabilizer group H. These N-linearized line bundles determine a representable morphism

$$[X'/N] \to B\mathbb{G}_m \times \cdots \times B\mathbb{G}_m.$$

Thus we have (i). □

Example 5.2. We elaborate on the spanning of $\mathrm{Burn}_n(G)$ by the classes of quasiprojective G-varieties, claimed in [8, Rmk. 5.16]. Any symbol $(H, N \circlearrowright K, \beta)$ can be made to arise in formula (1.1), we just have to use Assumption 1 to convert the characters in β to G-linearized line bundles on a suitable model Y' of K, which we may take to be smooth projective (the ability to extend a G-linearized line bundle from an invariant dense open is guaranteed by [13, Lem. 4.1]) and satisfy Assumption 2 (for the action of N). Then, taking Y to be a disjoint union of $[G : N_G(H)]$ copies of Y' and $X = \mathbb{P}(\mathcal{O}_Y \oplus L_1 \oplus \cdots \oplus L_{n-d})$, Lemma 5.1 guarantees Assumption 2 for a naturally induced G-action on X, with

$$[X \circlearrowleft G] - [(X \setminus s(Y)) \circlearrowleft G]^{\mathrm{naive}} = (H, N \circlearrowright K, \beta) + \dots, \tag{5.2}$$

where s is the zero-section of $L_1 \oplus \cdots \oplus L_{n-d} \subset X$. Any additional terms on the right-hand side of (5.2) involve fields of smaller transcendence degree, so the classes (5.2) span $\mathrm{Burn}_n(G)$. But $X \setminus s(Y)$ is equivariantly isomorphic to the normal bundle $\mathcal{O}_{\mathbb{P}(L_1 \oplus \cdots \oplus L_{n-d})}(1)$ of the divisor of X, complement of $L_1 \oplus \cdots \oplus L_{n-d}$, thus (5.2) is the class $[L_1 \oplus \cdots \oplus L_{n-d} \circlearrowleft G]$. So $\mathrm{Burn}_n(G)$ is spanned by classes $[L_1 \oplus \cdots \oplus L_r \circlearrowleft G]$ of sums of G-linearized line bundles on smooth projective varieties, $0 \le r \le n$, satisfying the conditions of Lemma 5.1 (with a trivial bundle L_0).

Proposition 5.3. *Let X be a smooth projective variety of dimension $n - r$ over k with a G-action and G-linearized line bundles $L_0, \dots, L_r$. We assume the conditions and adopt the notation of Lemma 5.1. We define $I := \{0, \dots, r\}$ and the following class in* $\mathrm{Burn}_{n,\mathbb{P}(I)}(G)$*:*

$$\xi(X \circlearrowleft G, (L_i)_{i \in I}) := \sum_{H' \supseteq H} \sum_{\substack{W \subset X' \\ \text{generic stabilizer } H'}} (H \subseteq H', N' \circlearrowright k(W), \beta, \gamma),$$

where

- *the first sum is over abelian subgroups H' of G that contain H, up to conjugacy in $N_G(H)$,*
- *the second sum is over $N_{N_G(H)}(H')$-orbits of components $W \subset X'$ where the generic stabilizer is H',*
- *$\beta = \beta_W(X')$ encodes the normal bundle to W in X', and*
- *$\gamma = (c_i)_{i \in I}$, the characters coming from L_i with $i \in I$.*

Then

$$[\mathbb{P}(L_0\oplus\cdots\oplus L_r)\circlearrowleft G]=\sum_{J\subsetneq I}\psi_J(\omega_{\mathbb{P}(I),J}(\xi(X\circlearrowleft G,(L_i)_{i\in I})))$$

in $\mathrm{Burn}_n(G)$.

Proof. We identify each contribution to $[\mathbb{P}(L_0\oplus\cdots\oplus L_r)\circlearrowleft G]$ as

$$V=\varphi_J^{-1}(W),$$

for some W in the definition of $\xi(X\circlearrowleft G,(L_i)_{i\in I})$, where φ_J denotes the projection to X from the projectivization of $\bigoplus_{i\in I\setminus J}L_i$. Now

$$(H\subseteq H',N'\circlearrowright k(W),\beta,\gamma)\in\mathrm{Burn}_{n,\mathbb{P}(I)}(G)$$

maps under $\psi_J\circ\omega_{\mathbb{P}(I),J}$ to

$$(\overline{H}',N_{N_G(H)}(\overline{H}')\circlearrowright k(V),\beta_V(X)),$$

and we have the result. □

Example 5.4. Let $G:=C_5\times\mathfrak{S}_3$, acting on $X:=\mathbb{P}^1$ via an irreducible 2-dimensional representation of $\mathfrak{S}_3$. We take L_0 to be trivial and L_1 to be the twist of $\mathcal{O}_{\mathbb{P}^1}(1)$ by a nontrivial character χ of C_5. Then we have the situation of Lemma 5.1 with $H=C_5$ and $N=\mathfrak{S}_3$, and the conditions of the lemma are satisfied. We have

$$\begin{aligned}\xi(X\circlearrowleft G,(L_0,L_1))=&\,(C_5\subseteq C_5,\mathfrak{S}_3\circlearrowright k(\mathbb{P}^1),\emptyset,(0,\chi))\\&+(C_5\subseteq C_5\times\langle(1,2)\rangle,\mathrm{triv}\circlearrowright k,(0,1),(0,(\chi,0)))\\&+(C_5\subseteq C_5\times\langle(1,2)\rangle,\mathrm{triv}\circlearrowright k,(0,1),(0,(\chi,1)))\\&+(C_5\subseteq C_5\times\mathfrak{A}_3,\mathfrak{S}_3/\mathfrak{A}_3\circlearrowright k\times k,(0,1),(0,(\chi,1))).\end{aligned}$$

The outcome of Proposition 5.3 is

$$\begin{aligned}&[\mathbb{P}(L_0\oplus L_1)\circlearrowleft G]=(\mathrm{triv},G\circlearrowright k(\mathbb{P}^1)(t),\emptyset)+(\langle(1,2)\rangle,C_5\overset{\chi}{\circlearrowright}k(t),1)\\&\quad+(C_5,\mathfrak{S}_3\circlearrowright k(\mathbb{P}^1),\chi)+(C_5\times\langle(1,2)\rangle,\mathrm{triv}\circlearrowright k,((0,1),(\chi,0)))\\&\quad+(C_5\times\langle(1,2)\rangle,\mathrm{triv}\circlearrowright k,((0,1),(\chi,1)))\\&\quad+(C_5\times\mathfrak{A}_3,\mathfrak{S}_3/\mathfrak{A}_3\circlearrowright k\times k,((0,1),(\chi,1)))\\&\quad+(C_5,\mathfrak{S}_3\circlearrowright k(\mathbb{P}^1),-\chi)+(C_5\times\langle(1,2)\rangle,\mathrm{triv}\circlearrowright k,((0,1),(-\chi,0)))\\&\quad+(C_5\times\langle(1,2)\rangle,\mathrm{triv}\circlearrowright k,((0,1),(-\chi,1)))\\&\quad+(C_5\times\mathfrak{A}_3,\mathfrak{S}_3/\mathfrak{A}_3\circlearrowright k\times k,((0,1),(-\chi,2))).\end{aligned}$$

For the G-filter (cf. Section 3)

$$\mathbf{H}:=\{(C_5,\mathfrak{S}_3)\},$$

the projection $\mathrm{Burn}_2(G) \to \mathrm{Burn}_2^{\mathbf{H}}(G)$ yields the class

$$(C_5, \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^1), \chi) + (C_5, \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^1), -\chi) \in \mathrm{Burn}_2^{\mathbf{H}}(G).$$

This class is nonzero and is different for $\chi \in \{\pm 1\}$, as compared with $\chi \in \{\pm 2\}$.

Geometrically, the situation above arises as follows: Consider the 3-dimensional representation $W_\chi = 1 \oplus (V \otimes \chi)$ of G, sum of a trivial 1-dimensional representation and twist by χ of the standard 2-dimensional representation V of $\mathfrak{S}_3$. This gives a generically free action of G on $\mathbb{P}^2 = \mathbb{P}(W_\chi)$, with a G-fixed point $\mathfrak{p}$. When we blow up $\mathfrak{p}$, we obtain $\mathbb{P}(L_0 \oplus L_1)$. So,

$$[\mathbb{P}(W_\chi) \circlearrowleft G] = [\mathbb{P}(L_0 \oplus L_1) \circlearrowleft G] \in \mathrm{Burn}_2(G).$$

6. Products

Let G' and G'' be finite groups. Define a product map

$$\mathrm{Burn}_{n'}(G') \times \mathrm{Burn}_{n''}(G'') \to \mathrm{Burn}_{n'+n''}(G' \times G'').$$

On symbols, it is given by

$$((H', Z' \circlearrowright K', \beta'), (H'', Z'' \circlearrowright K'', \beta'')) \mapsto (H, Z \circlearrowright K, \beta), \tag{6.1}$$

where

- $H = H' \times H''$,
- $Z = Z' \times Z''$,
- $K = K' \otimes_k K''$, with the natural action of Z,
- $\beta = \beta' \cup \beta''$.

Proposition 6.1. *The product map* (6.1) *is well-defined, and satisfies*

$$([X' \circlearrowleft G'], [X'' \circlearrowleft G'']) \mapsto [X' \times X'' \circlearrowleft G' \times G''].$$

Proof. The map clearly respects relations. The only point to remark is that in **(B2)**, the condition for nontriviality of Θ_2 holds for β' if and only if it holds for $\beta = \beta' \cup \beta''$. □

Example 6.2. If we take G'' to be trivial, then the product with the class of a rational function field of transcendence degree e is the homomorphism $\mathrm{Burn}_n(G) \to \mathrm{Burn}_{n+e}(G)$ given by

$$(H, Z \circlearrowright K, \beta) \mapsto (H, Z \circlearrowright K(t_1, \dots, t_e), \beta).$$

The kernel contains all $(H, Z \circlearrowright K, \beta)$, $\beta = (b_1, \dots, b_{n-d})$, such that some b_i has order $\leq e + 1$, by [11, Prop. 4.1].

7. Restrictions

Let G be a finite group and $G' \subset G$ a subgroup. A G-action on a quasiprojective variety X induces an action of G', and thus it is natural to propose the existence of a restriction homomorphism from $\mathrm{Burn}_n(G)$ to $\mathrm{Burn}_n(G')$, acting by

$$[X \circlearrowleft G] \mapsto [X \circlearrowleft G']. \tag{7.1}$$

In this section we establish the existence and uniqueness of this homomorphism.

Example 7.1. Suppose H is an abelian subgroup of G, contained in G'. Symbols, identified in $\mathrm{Burn}_n(G)$ by relation **(C)**, might no longer be identified in $\mathrm{Burn}_n(G')$. E.g., with $G = \mathfrak{D}_4$ and $G' = C_4$ the restriction of $(G', G/G' \circlearrowright k \times k, 1) \in \mathrm{Burn}_1(G)$ to $\mathrm{Burn}_1(G')$ has to be a sum of two symbols with distinct characters:

$$(G', G/G' \circlearrowright k \times k, 1) \mapsto (G', \mathrm{triv} \circlearrowright k, 1) + (G', \mathrm{triv} \circlearrowright k, 3).$$

Theorem 7.2. *For all $n \geq 0$, there exists a unique homomorphism of abelian groups*

$$\mathrm{res}^G_{G'} \colon \mathrm{Burn}_n(G) \to \mathrm{Burn}_n(G'),$$

compatible with (7.1).

Proof. By Lemma 2.1, it suffices to consider symbols of the form

$$\mathfrak{s} = (H, Z \circlearrowright K, \beta).$$

When we act by conjugation by some element of G, we obtain an equivalent symbol, where H is replaced by a conjugate, the corresponding centralizer quotient replaces Z, and conjugation is used to form from β a sequence of characters of the conjugate of H. By conjugation we have a transitive action of G on a set $\mathfrak{S}$ of symbols, where $\mathfrak{s} \in \mathfrak{S}$ has stabilizer $Z_G(H)$. The restriction of the action to G' consists of finitely many orbits; in the formula below the sum is over orbit representatives

$$\mathfrak{s}' = (H', Z_G(H')/H' \circlearrowright K, \beta'),$$

where one of the orbit representatives may be taken to be $\mathfrak{s}$ itself. We define the restriction to G' by

$$\mathfrak{s} \mapsto \sum_{\mathfrak{s}'} (H' \cap G', (Z_G(H') \cap G')/(H' \cap G') \circlearrowright K, \beta'|_{H' \cap G'}). \tag{7.2}$$

It is clear that the map respects relations **(O)** and **(C)**. To see that it respects **(B)**, we first give a formula, valid for a symbol of the form

$$(H, Y \circlearrowright K, \beta),$$

with $Y \subseteq Z$. Let us write $Y = \widetilde{Y}/H$ with $H \subseteq \widetilde{Y} \subseteq Z_G(H)$. We restrict the transitive action of G on $G/\widetilde{Y}$ to G'. Then $(H, Y \circlearrowright K, \beta)$ is mapped to the sum over orbit representatives $g'\widetilde{Y}$ for the G'-action

$$\sum_{g'\widetilde{Y}} (H' \cap G', (g'\widetilde{Y}g'^{-1} \cap G')/(H' \cap G') \circlearrowright K, \beta'|_{H' \cap G'}), \tag{7.3}$$

where H' denotes $g'Hg'^{-1}$, with corresponding characters β'. (Choose $Z_G(H) = g_1\widetilde{Y} \cup \cdots \cup g_d\widetilde{Y}$, $d = [Z_G(H) : \widetilde{Y}]$, $g_1 = 1$, write $\mathrm{Ind}_Y^Z K = K^d$ in the standard way, then a $(Z_G(H) \cap G')$-orbit $\{g_{i_1}\widetilde{Y}, \dots, g_{i_r}\widetilde{Y}\}$ in $Z_G(H)/\widetilde{Y}$ determines $(Z_G(H) \cap G')$-invariant $Ke_{i_1} \oplus \cdots \oplus Ke_{i_r}$, which with $i = i_1$ we explicitly identify with $\mathrm{Ind}_{(g_i\widetilde{Y}g_i^{-1} \cap G')/(H \cap G')}^{(Z_G(H) \cap G')/(H \cap G')} K$ to see that the summand $\mathfrak{s}' = \mathfrak{s}$ of (7.2) is the contribution to (7.3) from the $g'\widetilde{Y}$ with $g' \in Z_G(H)$. A similar consideration with H replaced by H' identifies an arbitrary summand of (7.2) with a corresponding contribution to (7.3).)

The formula (7.3), applied to the term $(\overline{H}, \overline{N} \circlearrowright \overline{K}, \bar{\beta})$ from relation **(B)**, may be expressed as a sum

$$\sum_{\mathfrak{s}'} (\overline{H}' \cap G', (Z_G(H') \cap G')/(\overline{H}' \cap G') \circlearrowright K(t), \bar{\beta}'|_{\overline{H}' \cap G'})$$

as in (7.2), where $\overline{H}' = \ker(b_1' - b_2')$, $\beta' = (b_1', \dots, b_{n-d}')$. This is the contribution from the final term of relation **(B)** for the sum (7.2).

Since $\mathrm{Burn}_n(G)$ is spanned by the classes of quasiprojective G-varieties (as discussed in Example 5.2), we have the uniqueness. ☐

Remark 7.3. Clearly, we also have

$$\mathrm{res}_{G'}^G([X \circlearrowleft G]^{\mathrm{naive}}) = [X \circlearrowleft G']^{\mathrm{naive}}.$$

As an application of the restriction construction, we obtain a map

$$\mathrm{Burn}_{n'}(G) \times \mathrm{Burn}_{n''}(G) \to \mathrm{Burn}_{n'+n''}(G),$$

using the product construction in Section 6 with $G' = G'' = G$, followed by restriction to the diagonal

$$G \subseteq G \times G.$$

This map on Burnside groups satisfies

$$([X' \circlearrowleft G], [X'' \circlearrowleft G]) \mapsto [X' \times X'' \circlearrowleft G].$$

8. Combinatorial analogs

Here we define and study a *combinatorial* version $\mathcal{BC}_n(G)$ of the equivariant Burnside group $\mathrm{Burn}_n(G)$, and a homomorphism

$$\mathrm{Burn}_n(G) \to \mathcal{BC}_n(G)$$

which forgets the information about the Galois algebra.

Definition 8.1. The *combinatorial symbols group*

$$\mathcal{BC}_n(G)$$

is the $\mathbb{Z}$-module, generated by symbols

$$(H, Y, \beta)$$

with H abelian, $Y \subseteq Z_G(H)/H$, and β a sequence of nonzero elements generating $H^\vee$, of length at most n, modulo relations:

(O): $(H, Y, \beta) = (H, Y, \beta')$ if β' is a reordering of β.

(C): $(H, Y, \beta) = (gHg^{-1}, gYg^{-1}, \beta')$ for $g \in G$, with β and β' related by conjugation by g.

(B1): $(H, Y, \beta) = 0$ when $b_1 + b_2 = 0$.

(B2): $(H, Y, \beta) = \Theta_1 + \Theta_2$, where Θ_1 and Θ_2 are as in Section 2, i.e.,

$$\Theta_1 = \begin{cases} 0, & \text{if } b_1 = b_2, \\ (H, Y, \beta_1) + (H, Y, \beta_2), & \text{otherwise,} \end{cases}$$

with β_1 and β_2 as in (2.4), and

$$\Theta_2 = \begin{cases} 0, & \text{if } b_i \in \langle b_1 - b_2 \rangle \text{ for some } i, \\ (\overline{H}, \overline{Y}, \bar{\beta}), & \text{otherwise,} \end{cases}$$

where $\overline{H}$ and $\bar{\beta}$ are as in (2.5).

We remark that, as in Section 2, we may obtain a simplified presentation of $\mathcal{BC}_n(G)$ by allowing symbols where β contains the trivial character, imposing the vanishing of all such symbols, and combining relations **(B1)** and **(B2)** into a single relation **(B)**.

Proposition 8.2. *The map sending the class of a triple*

$$(H, Y \circlearrowright K, \beta) \in \mathrm{Burn}_n(G),$$

for fields $K \in \mathrm{Alg}_Y(K_0)$, $K_0 \in \mathrm{Bir}_d(k)$, with $d \le n$, to

$$[k' : k](H, Y, \beta) \in \mathcal{BC}_n(G),$$

where k' is the algebraic closure of k in K_0, gives a homomorphism

$$\mathrm{Burn}_n(G) \to \mathcal{BC}_n(G).$$

Proof. This is clear from the description of the relations in $\mathrm{Burn}_n(G)$ from Section 2. □

Remark 8.3. Although the map of Proposition 8.2 kills geometric information, an advantage of working in $\mathcal{BC}_n(G)$ is a direct sum structure, established in [18]:

$$\mathcal{BC}_n(G) \cong \bigoplus_{(H,Y)} \mathcal{B}_n(H)/(\mathbf{C}_{(H,Y)}).$$

Here, the direct sum is over conjugacy class representatives of pairs (H,Y) as in Definition 8.1, $\mathcal{B}_n(H)$ is the birational symbols group defined in [5], and the quotient by $(\mathbf{C}_{(H,Y)})$ signifies the imposition of conjugacy relations, under the stabilizer of (H,Y) for the conjugation action of G.

Example 8.4. As mentioned in [18], there is in general no direct sum decomposition of $\mathrm{Burn}_n(G)$ that is compatible with that Remark 8.3 under the map in Proposition 8.2. For simplicity, let us suppose that k is algebraically closed. Then a relation such as $(H,Y,(b)) = (H,Y,(b,b))$ in $\mathcal{BC}_2(G)$, for H nontrivial cyclic with primitive character b and Y nontrivial, has no geometric counterpart. We analyze in detail the case $G = \mathfrak{K}_4$, the Klein 4-group. Then $\mathrm{Burn}_2(G)$ is the direct sum of

- *trivial* symbols $(\mathrm{triv}, G \circlearrowright K, ())$,
- *incompressible* symbols [9, Defn. 3.3] $(H, G/H \circlearrowright K, (1))$, with $|H| = 2$ and K of transcendence degree 1 over k, $K \ncong k(t)$, and
- a free abelian group of rank 3, generated by $(H, G/H \circlearrowright k(t), (1))$ for $|H| = 2$ and $(G, \mathrm{triv} \circlearrowright k, (b,b'))$ for distinct nontrivial characters b and b' up to order, modulo relation **(B2)**.

The third summand receives a contribution, isomorphic to $\mathbb{Z}$, from every subgroup of G of order 2. These span an index 4 subgroup of the free abelian group of rank 3, with quotient $(\mathbb{Z}/2\mathbb{Z})^2 \cong \mathcal{B}_2(G)$. Whereas $\mathcal{BC}_2(G) = \mathcal{BC}_2(G)^{\mathrm{triv}} \oplus \mathcal{BC}_2(G)^{\mathrm{nontriv}}$, with $\mathcal{BC}_2(G)^{\mathrm{nontriv}} = \mathcal{B}_2(G)$.

Definition 8.5. Given a G-prefilter $\mathbf{H}$, we let

$$\mathcal{BC}_n^{\mathbf{H}}(G)$$

be the quotient of $\mathcal{BC}_n(G)$ by the subgroup generated by classes (H,Y,β) with $(H,Y) \notin \mathbf{H}$.

Exactly as in Section 3 we have

Proposition 8.6. *Let* $\mathbf{H}$ *be a* G*-filter. Then* $\mathcal{BC}_n^{\mathbf{H}}(G)$ *is generated by symbols* (H,Y,β) *for* $(H,Y) \in \mathbf{H}$*, subject to relations* **(O)***,* **(C)***,* **(B1)***, and* **(B2)** *applied to these symbols.*

Additionally, upon passage to the combinatorial analogue we also have the other structures developed in this paper:

- equivariant (projectively) indexed combinatorial Burnside group;
- product map;
- restriction homomorphisms.

Example 8.7. Suppose that G is abelian.

- We have (cf. [8, §8])
$$\mathcal{B}_n(G) = \mathcal{BC}_n^{(G,\mathrm{triv})}(G),$$
where $\mathcal{B}_n(G)$ is the symbols group from [5].
- There is a commutative diagram
$$\begin{array}{ccc} \mathrm{Burn}_n(G) & \longrightarrow & \mathcal{BC}_n(G) \\ \downarrow & & \downarrow \\ \mathrm{Burn}_n^G(G) & \longrightarrow & \mathcal{B}_n(G) \end{array}$$
(The factor factor $[k' : k]$ in Proposition 8.2 matches the similar factor in [8, Prop. 8.1].)

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