FIBRATIONS IN SEXTIC DEL PEZZO SURFACES WITH MILD SINGULARITIES

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1. Introduction

In this paper we study families of degree 6 del Pezzo surfaces over higher-dimensional bases, with a view toward existence of good models. There is an extensive literature on degenerations of del Pezzo surfaces, see, e.g., [10], [8], and specifically sextic del Pezzo surfaces, e.g., [11]. These degenerations play a role in the minimal model program and enter the study of related moduli problems, e.g., [26], [12]. They are also relevant in arithmetic applications and the study of rationality, see, e.g., [1].

Our starting point is the work of Blunk [5] and Kuznetsov [22], describing isomorphism classes of sextic del Pezzo surfaces in terms of supplementary data, taking the form of étale algebras over a given base field of degrees 2 and 3, together with Brauer group elements.

This paper continues our previous work, in which we studied families of del Pezzo surfaces of degree 9 (Brauer-Severi surfaces [17], [18]) and 8 (involution surfaces [20], [19], which include the case of quadric surfaces). (The case of del Pezzo surfaces of degree 7 is not interesting, since they are never minimal.) The passage to families entails a study of ramification patterns for the étale algebras and the Brauer group elements.

In our previous work, as well as here, the main results concern the existence of families, with specified fiber types and global singularity descriptions. This is applied, for instance, to Brauer group computations and constructions in families, e.g., in the study of rationality problems as in [15], [16].

In this paper, we require a *regular branch divisor* for the covers and allow only limited ramification of Brauer group elements. From a geometric point of view, we need to specify coverings of the base of degrees 2 and 3 with Brauer classes satisfying compatibility conditions. We obtain five degeneration types, corresponding to the possible ramification of the pair of coverings, which we call *basic*. These are parametrized by conjugacy classes of cyclic subgroups of $\mathfrak{S}_2 \times \mathfrak{S}_3$; see Section 3. While

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the structure of degree 2 covers is well understood, the branch locus of degree 3 covers is typically singular and becomes nonsingular only after birational modification of the base; see [21] and references therein. In this paper we do not address codimension 2 phenomena, e.g., when the branch loci of the covers intersect. Similar restrictions were present in the work of Auel, Parimala, and Suresh on quadric surface bundles [3].

To draw a comparison with involution surface bundles, basic degeneration corresponds to Type I in [20], while the more general notion from op. cit. of mild degeneration (essentially, degenerations that occur in families over a regular base with singular fibers over a regular divisor) encompasses three additional types that involve ramification of the Brauer group element. In the case of Brauer-Severi surface bundles [17], there is only one kind of degeneration that occurs over a regular divisor, connected with ramification of the Brauer group element.

Definition 1. Let S be a regular scheme. A *sextic del Pezzo surface bundle*, or *DP6 bundle*, over S is a flat projective morphism $\pi\colon X \to S$ such that the locus $U \subset S$ over which π is smooth is dense in S and the fibers of π over points of U are del Pezzo surfaces of degree 6.

Now we suppose that 2 and 3 are invertible in the local rings of S.

Definition 2. A sextic del Pezzo surface bundle $\pi\colon X \to S$ has *basic degeneration* if every singular fiber is geometrically isomorphic to one of the following schemes:

- (Type I) four copies of the blow-up of 3 collinear points on $\mathbb{P}^2$ with the (-2)-curve contracted, with each copy glued to the other three along the three exceptional curves;
- (Type II) the blow-up of two general points on a quadric surface with A_1-singularity;
- (Type III) "pinched" $\mathbb{F}_2$, obtained by gluing a Hirzebruch surface $\mathbb{F}_2$ to itself along a degree 2 covering $((-2)\text{-curve}) \cong \mathbb{P}^1 \to \mathbb{P}^1$;
- (Type IV) three copies L_i, $i \in \mathbb{Z}/3\mathbb{Z}$, of the blow-up of two general points on a fiber of $\mathbb{F}_2$, with the proper transform of the fiber collapsed to an A_1-singular point and images of exceptional divisors denoted by ℓ_i, ℓ_i', and of the (-2)-curve, by λ_i, together with three quadric surfaces M_i, $i \in \mathbb{Z}/3\mathbb{Z}$, with rulings m_i and m_i' meeting at a point on a smooth conic μ_i, each with multiplicity 2, with following pairs of curves glued:
$$(\lambda_i, \mu_i),\ (\ell_i, m_{i+1}),\ (\ell_i', m_{i+2}'),\qquad \forall\, i \in \mathbb{Z}/3\mathbb{Z};$$
- (Type V) Hirzebruch surface $\mathbb{F}_4$ with fiber and (-4)-curve glued.

For the gluing of a scheme to itself along a finite morphism from a closed subscheme to another scheme, see [9]. In particular, the fibers of Types I, III, IV, and V are not normal, and Type IV fibers are not even reduced.

Remark 3. Du Val degenerations of del Pezzo surfaces have been studied classically, see, e.g., [7]; above, only Type II is of this form. Degenerations of sextic del Pezzo surfaces with special fiber that is irreducible but not normal are considered in [11, Table 3]; our Types III and V are listed there. Types I and IV have not appeared in the literature; these surfaces are only embedded by a nontrivial multiple of the anticanonical class.

Our approach to the construction of families is a systematic application of root stacks and descent, as in [15], [17], [20]. In essence, we exchange ramification of the covers for extra stacky structure. Other approaches have been used; for instance, the Minimal Model Program has been applied by Corti [8] to produce degenerations over a DVR, with normal irreducible special fiber, but only under the assumption of an algebraically closed residue field.

The families with basic degeneration that we will construct (Theorems 8 and 9) will have explicit global singularity descriptions. In particular, we observe new phenomena:

- degenerations with fibers embedded by multiples of the anticanonical class, according to Gorenstein index of $\mathbb{Q}$-Gorenstein singularities of the total space (Type I);
- singularities that are not $\mathbb{Q}$-Gorenstein (Type IV), requiring adjustment by components of the fibers to turn (an appropriate multiple of) the anticanonical class into a Cartier divisor class.

In Section 2, we recall the supplementary data attached to the classification of sextic del Pezzo surfaces over fields and exhibit smooth families of sextic del Pezzo surfaces, as models in the case of a general base with *unramified* covers and Brauer group elements (Theorem 5). In Section 3, we analyze the five degeneration types. Section 4 carries out the construction and establishes the main theorems (Theorem 8, with the basic construction, and Theorem 9, exhibiting the desired families).

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2. Smooth families of sextic del Pezzo surfaces

Let K be a field. A smooth del Pezzo surface X over K of degree d is a smooth projective surface with ample anticanonical class ω_X^{-1}, and

$$\mathsf{d} = \mathsf{d}(X) := \omega_X^2.$$

Here d takes the values $1, \dots, 9$. When K is algebraically closed, we have the following description: when $\mathsf{d} = 9$, $X \cong \mathbb{P}^2$; when $\mathsf{d} = 8$, X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or the blowup of $\mathbb{P}^2$ in one point; when $\mathsf{d} \le 7$, X may be obtained by blowing up $\mathbb{P}^2$ in $9 - \mathsf{d}$ points in general position. Smooth irreducible curves C on X with $C^2 = -1$ are called *exceptional curves*, and the union of the exceptional curves is the *exceptional locus*. In this paper, we focus on the case $\mathsf{d} = 6$, in which case the geometric automorphism group of X fits into a split exact sequence

$$1 \ \to T \to \mathrm{Aut}(\overline{X}) \to \mathfrak{S}_2 \times \mathfrak{S}_3 \to 1. \tag{2.1}$$

Proposition 4 (Blunk construction [5])**.** *A degree 6 del Pezzo surface over a field K is classified by*

- *étale K-algebras L and M of respective degrees* 2 *and* 3*,*
- *Brauer group elements $\eta \in \mathrm{Br}(L)[3]$ and $\zeta \in \mathrm{Br}(M)[2]$, each restricting to* 0 *in* $\mathrm{Br}(L \otimes_K M)$ *and corestricting to* 0 *in* $\mathrm{Br}(K)$*.*

Blunk's construction builds on work of Colliot-Thélène, Karpenko, and Merkurjev [6]. A table listing possibilities for the Blunk data, with corresponding arithmetic invariants, is given in [2, Table 4].

The following are the essential ingredients to the Blunk construction.

- A two-dimensional torus T is canonically associated with L and M, together with a toric variety X_0 for T that is a sextic del Pezzo surface. The parameter scheme of unordered triples of pairwise disjoint exceptional curves of X_0 is $\mathrm{Spec}(L)$ and of unordered pairs of opposite exceptional curves is $\mathrm{Spec}(M)$.
- Sextic del Pezzo surfaces for T (i.e., with T as identity component of the automorphism group scheme) correspond to torsors under T, i.e., elements of $H^1(K,T)$.
- Exact sequences lead to a description of $H^1(K,T)$ in terms of pairs of Brauer group elements satisfying the stated conditions.

The inverse map $T \to T$ extends to an automorphism of X_0. We call a pair of points, or curves, of X_0 *opposite* if they are exchanged under this automorphism. The same terminology applies to exceptional curves of X, and to the singular points of the exceptional locus of X.

We describe X as *rigidified* by L/K and M/K when the parameter scheme of pairwise disjoint exceptional curves of X, respectively of unordered pairs of opposite exceptional curves, is identified with $\mathrm{Spec}(L)$, respectively $\mathrm{Spec}(M)$. The same terminology will be used for $\pi\colon X \to S$, for any integral scheme S with rational function field K. For instance, $\pi\colon X \to S$ could be a smooth DP6 bundle, i.e., a DP6 bundle such that π is smooth. We also allow S to be a disjoint union of finitely many integral schemes, in which case the rigidification data consist of étale algebras over $K := K_1 \times \cdots \times K_n$, where $K_1, \ldots, K_n$ are the residue fields at the generic points of the components of S. Such étale algebras will be called *rigidification data*.

Blunk's description of sextic del Pezzo surfaces over a field is best expressed as a description of isomorphism classes of sextic del Pezzo surfaces with rigidification. A degree 6 del Pezzo surface rigidified by L and M is determined uniquely up to isomorphism by $\eta \in \mathrm{Br}(L)[3]$ and $\zeta \in \mathrm{Br}(M)[2]$ satisfying the indicated conditions.

We seek a generalization to an arbitrary regular base. For this, we need to consider smooth families of sextic del Pezzo surfaces with rigidification as *equivalent* when there exists a birational equivalence between them that restricts over the generic point (of every component of the base) to an isomorphism of rigidified sextic del Pezzo surfaces.

Theorem 5. *Let S be a quasi-compact separated regular scheme with rigidification data L/K, M/K, and let S_L and S_M denote the respective normalizations of S in L/K and M/K. We suppose that S_L and S_M are étale over S. Then the Blunk construction gives rise to a bijection between:*

- *Equivalence classes of smooth DP6 bundles*
$$\pi\colon X \to S$$
rigidified by L/K and M/K, and
- *pairs of Brauer group elements $\eta \in \mathrm{Br}(S_L)[3]$ and $\zeta \in \mathrm{Br}(S_M)[2]$ which restrict to zero in $\mathrm{Br}(L \otimes_K M)$ and corestrict to zero in $\mathrm{Br}(K)$.*

Proof. A smooth DP6 bundle ridigified by L/K and M/K determines Brauer-Severi schemes of relative dimension 2 over S_L (by birationally contracting a triple of pairwise disjoint exceptional curves) and of relative dimension 1 over S_M (by a corresponding fibration in conics). These have classes in $\mathrm{Br}(S_L)[3]$ and $\mathrm{Br}(S_M)[2]$, respectively; since their restrictions to the generic points of all the components are the ones that correspond to X_K under Blunk's correspondence, they satisfy the indicated conditions on restriction and corestriction. Restriction to the generic point induces an injective map on the Brauer group [14, II.1.10], hence the Brauer group elements are uniquely determined by the equivalence class of a given rigidified smooth DP6 bundle.

It remains to show that any pair of Brauer group elements arises from some smooth DP6 bundle. We first treat the case $S = \mathrm{Spec}(R)$, where R is a semi-local Dedekind domain. Then, we claim that η and ζ determine an element of $H^1(\mathrm{Spec}(R), T)$, where T denotes the two-dimensional torus associated with S_L/S and S_M/S. The argument in [5] carries over: in addition to basic functoriality of the Brauer group one needs only the concrete description of H^1 and H^2 of a norm one torus given at the top of page 47 of op. cit. Following the diagram in Figure 1 and using the fact that a semi-local Dedekind domain has trivial Picard group, we see that the description remains valid.

Next we treat the general case. The field case of Blunk's construction yields a sextic del Pezzo surface over K, which spreads out to a smooth DP6 bundle

$$X' \to U$$

over a Zariski open dense subscheme $U \subset S$. The complement $S \setminus U$ has finitely many irreducible components of codimension 1, whose generic points we denote by $x_1, \dots, x_m$. By [25, Prop. VIII.1], affine open subsets are cofinal among all open subsets V of Σ containing $x_1, \dots, x_m$. Hence

$$\varprojlim_{\substack{V \subset S \\ x_1, \dots, x_m \in V}} V$$

is an affine scheme of the form $\mathrm{Spec}(R_1 \times \dots \times R_n)$ where each R_i is a semi-local Dedekind domain. By the case already treated, there is, for some V as above, a smooth DP6 bundle

$$X'' \to V$$

where, shrinking V if necessary while maintaining $x_1, \dots, x_m \in V$, we may suppose

$$X' \times_U (U \cap V) \cong X'' \times_V (U \cap V).$$

Then it is possible to glue X' and X'' to obtain a smooth DP6 bundle over $S^\circ := S \setminus Z$ for some closed Z that, at each of its points, has codimension at least 2.

We conclude by showing that restriction to S° induces an equivalence of categories between smooth DP6 bundles over S and over S°. A smooth DP6 bundle $\pi^\circ \colon X^\circ \to S^\circ$ extends canonically to a projective scheme over S as follows: X° embeds in $\mathbb{P}((\pi^\circ_*\omega^\vee_{X^\circ/S^\circ})^\vee)$; we apply direct image by $\iota \colon S^\circ \to S$ and closure to obtain X in $\mathbb{P}(\iota_*(\pi^\circ_*\omega^\vee_{X^\circ/S^\circ})^\vee)$. These operations are compatible with étale base change. So, it suffices to show, for $z \in Z$, with the strictly henselian local ring of S at z denoted by $\mathcal{O}_{S,\bar{z}}$, that $X^\circ \times_S \mathrm{Spec}(\mathcal{O}_{S,\bar{z}})$ extends to a smooth DP6 bundle over $\mathcal{O}_{S,\bar{z}}$. This is clear, since T becomes a split torus after base change to $\mathcal{O}_{S,\bar{z}}$. □

$$
\begin{array}{ccccc}
\frac{\mathbb{G}_m(S)}{N(\mathbb{G}_m(S_L))} \times \frac{\mathbb{G}_m(S)}{N(\mathbb{G}_m(S_M))} & \to & H^1(S,R^1_L) \times H^1(S,R^1_M) & \to & \ker(\mathrm{Nm}_L) \times \ker(\mathrm{Nm}_M) \\
\downarrow & & \downarrow & & \downarrow \\
\frac{\mathbb{G}_m(S)}{N(\mathbb{G}_m(S_{LM}))} & \longrightarrow & H^1(S,R^1_{LM}) & \longrightarrow & \ker(\mathrm{Nm}_{LM}) \\
 & & \downarrow & & \\
 & & H^1(S,T) & & \\
 & & \downarrow & & \\
\mathrm{coker}(\mathrm{Nm}_L) \times \mathrm{coker}(\mathrm{Nm}_M) & \to & H^2(S,R^1_L) \times H^2(S,R^1_M) & \to & \ker(\mathrm{cores}_L) \times \ker(\mathrm{cores}_M) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{coker}(\mathrm{Nm}_{LM}) & \longrightarrow & H^2(S,R^1_{LM}) & \longrightarrow & \ker(\mathrm{cores}_{LM})
\end{array}
$$

FIGURE 1. Exact sequence involving $H^1(S,T)$, from short exact $0 \to R^1_L \times R^1_M \to R^1_{LM} \to T \to 0$ of [5]. Here, S_{LM} denotes $S_L \times_S S_M$ and R^1 stands for the kernel of the norm N from the restriction of scalars $R(\mathbb{G}_m)$ under the indicated extension (e.g., S_L) of S to $\mathbb{G}_m$. Rows are short exact sequences (with leading and trailing 0 omitted to save space), Nm denotes the norm map on Picard groups (for the indicated extension, e.g., $\mathrm{Nm}_L\colon \mathrm{Pic}(S_L) \to \mathrm{Pic}(S)$), and cores denotes corestriction of Brauer groups.

Remark 6. It is impossible to strengthen Theorem 5 to a uniqueness statement for the isomorphism class of a rigidified smooth DP6 bundle, even if we replace the Brauer classes by Azumaya algebra representatives (or, what amounts to the same, Brauer-Severi schemes). This is in constrast to the main theorem of [3], which proves such a result for quadric surface bundles. Indeed, let k be an algebraically closed field and let K, L, and M be rational function fields over k of transcendence degree 1, such that $L \otimes_K M$ is a function field of positive genus. Take S to be the complement of a suitable finite set of points in $\mathbb{P}^1_k$. The hypotheses of Theorem 5 are satisfied, and any smooth DP6 bundle over S rigidified by L/K and M/K has associated Brauer-Severi schemes $\mathbb{P}^2_{S_L}$ and $\mathbb{P}^1_{S_M}$ (by Tsen's theorem and the fact that the coordinate rings of S_L and S_M are PIDs). However, we have nontrivial $H^1(S,T) \cong \mathrm{Pic}(S_L \times_S S_M)$ (cf. Figure 1). Thus, there is more than one isomorphism class of smooth DP6 bundles over S rigidified by L/K and M/K.

3. Basic degenerations

In this section we analyze the five degeneration types in DP6 bundles with basic degeneration. At the generic point of a divisor on the base

	(L_u)	(L_r)
(M_u)	smooth	I
(M_r)	II	III
(M_t)	IV	V

Table 1. Basic degenerations

they correspond to the possible degeneration types of the rigidification data.

Let $\mathfrak{o}_K$ be a DVR with fraction field K and residue field κ of characteristic different from 2 and 3. Let L/K and M/K be rigidification data. Let $\mathfrak{o}_L$, respectively $\mathfrak{o}_M$ denote the integral closure of $\mathfrak{o}_K$ in L, respectively in M.

There are the following possibilities for L:

(L_u) L/K is unramified.
(L_r) L/K is ramified.

The following are the possibilities for M:

(M_u) M/K is unramified.
(M_r) M/K is simply ramified.
(M_t) M/K is totally ramified.

The possible combinations are listed in Table 1.

We describe, for each type, the root constructions that replace ramified covers of schemes by unramified covers of stacks. We work over $S = \mathrm{Spec}(\mathfrak{o}_K)$, and introduce $S_L = \mathrm{Spec}(\mathfrak{o}_L)$ and $S_M = \mathrm{Spec}(\mathfrak{o}_M)$.

- *Type I:* Replace S by $\sqrt{(S, \mathrm{Spec}(\kappa))}$, make a corresponding replacement of S_M.
- *Type II:* Replace S by $\sqrt{(S, \mathrm{Spec}(\kappa))}$, make a corresponding replacement of S_L, replace S_M by its root stack along the unramified closed point.
- *Type III:* Replace S by $\sqrt{(S, \mathrm{Spec}(\kappa))}$, replace S_M by its root stack along the unramified closed point.
- *Type IV:* Replace S by $\sqrt[3]{(S, \mathrm{Spec}(\kappa))}$, make a corresponding replacement of S_L.
- *Type V:* Replace S by $\sqrt[6]{(S, \mathrm{Spec}(\kappa))}$, perform cube- respectively square-root replacements of S_L, respectively S_M.

In summary, we replace S by $\sqrt[n]{(S, \mathrm{Spec}(\kappa))}$, where $n = 2$ for Types I, II, and III; $n = 3$ for Type IV; and $n = 6$ for Type V. For the corresponding replacements S'_L of S_L and S'_M of S_M we have achieved the following:

Proposition 7. *In every case, the morphisms*

$$S'_L \to \sqrt[n]{(S, \mathrm{Spec}(\kappa))} \qquad \text{and} \qquad S'_M \to \sqrt[n]{(S, \mathrm{Spec}(\kappa))}$$

are finite and étale.

The classification of basic degeneration types reflects the classification of conjugacy classes of cyclic subgroups of $\mathfrak{S}_2 \times \mathfrak{S}_3$, acting on toric X_0 with $T \subset X_0$, and in particular, on the set of exceptional curves; see (2.1).

- *Type I:* The factor $\mathfrak{S}_2$, which acts by the inverse map on T and by exchanging opposite pairs of exceptional curves.
- *Type II:* Cyclic subgroup of order 2 contained in the factor $\mathfrak{S}_3$, which acts, fixing a smooth conic in X_0.
- *Type III:* The remaining order 2 case, where the fixed locus is a smooth quartic curve in X_0 passing through a pair of opposite singular points of the exceptional locus.
- *Type IV:* Cyclic subgroup of order 3, acting with three fixed points on T and with two orbits on the set of exceptional curves.
- *Type V:* Cyclic subgroup of order 6, acting with one fixed point in T, one orbit of three points with stabilizer μ_2, and one orbit of two points with stabilizer μ_3.

4. Construction

In this section, we describe the construction of basic degenerations. The construction proceeds in three steps:

(1) Root stack construction on the base to permit a DP6 bundle to extend smoothly across a given divisor. (See Section 3.)
(2) Birational modification of the smooth DP6 bundle on the root stack. This step combines the operations of blowing up, contracting [17, Prop. A.9], and pinching [9].
(3) Descent from the root stack to the original base [17, Prop. 2.5].

Step (1), when the base is $S = \mathrm{Spec}(\mathfrak{o}_K)$, replaces S by

$$\sqrt[n]{(S, \mathrm{Spec}(\kappa))},$$

where $n = 2$ for Types I, II, and III; $n = 3$ for Type IV; and $n = 6$ for Type V. In the case of a semi-local Dedekind domain, we perform the corresponding root stack replacement at each closed point. The passage from semi-local Dedekind domain case to general case will proceed just as in the proof of Theorem 5.

Step (2) breaks up according to the Type. We let s denote the gerbe of the root stack; X_s is a smooth DP6 with μ_n-action.

- *Type I:* The fixed-point locus consists, geometrically, of four points in X_s, which we blow up. The special fiber becomes a union of four copies of $\mathbb{P}^2$, attached along lines to four (-1)-curves on a resolved DP2, double cover of the plane branched along the union of four lines. The resolved DP2 has six (-2)-curves, which may be flopped to obtain the singular DP2 with four attached copies of the blow-up of a plane at three collinear points. The DP2 contracts, leaving four copies of the plane blown up at three collinear points with (-2)-curve contracted, attached to each other along exceptional divisors. This is $\mathbb{Q}$-Gorenstein of index 2. Twice the anticanonical class gives rise to an embedding in $\mathbb{P}^{18}$, where the image of each component has degree 6.
- *Type II:* Let $C \subset X_s$ be the fixed-point locus, a smooth conic incident to two opposite exceptional curves. The blow-up $B\ell_C X$ has, as fiber over s, the union of X_s and $C \times \mathbb{P}^1$, and the two incident exceptional curves on X_s may be flopped to obtain a DP8 joined along C with the blow-up of $C \times \mathbb{P}^1$ at two points on C. (Note that the exceptional curves, and the points, may be Galois conjugated.) The DP8 may be contracted to an ordinary double point singularity on a new flat family of sextic del Pezzo surfaces over S with Type II fiber over s.
- *Type III:* Let $C \subset X_s$ be the fixed-point locus, a quartic rational curve C through two opposite singular points of the exceptional locus. The blow-up $B\ell_C X$ has, as fiber over s, the union of X_s and a Hirzebruch surface $\mathbb{F}_2$, where C is identified with the (-2)-curve of $\mathbb{F}_2$. There is a conic bundle $X_s \to \mathbb{P}^1$, which restricts to a degree 2 morphism $C \to \mathbb{P}^1$. A corresponding contraction of $B\ell_C X$ has the effect, on the special fiber, of "pinching" $\mathbb{F}_2$ onto its image under the incomplete linear system of four times a ruling plus the (-2)-curve, consisting of sections whose restriction to C are pullbacks of sections of $\mathcal{O}_{\mathbb{P}^1}(1)$ under $C \to \mathbb{P}^1$.
- *Type IV:* The fixed-point locus consists, geometrically, of three points in X_s, which we blow up. The special fiber becomes three copies of $\mathbb{P}^2$, attached along lines ℓ, ℓ', ℓ'' to a smooth cubic surface with 18 Eckhardt points (and thus, geometrically, the Fermat cubic surface). In each copy of $\mathbb{P}^2$ the action of μ_3 fixes a point and a line; we blow these up, which replaces the cubic surface by its blow-up at six of the Eckhardt points, upon which the self-intersection number changes from -1 to -3 for nine exceptional curves: ℓ, ℓ', ℓ'', and six others. The six others each have normal bundle $\mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$, and we flop these to obtain curves, along which the total space has singularities of type $\mathsf{A}_{3,1}$, cf. [4,

§III.5], also known as $\frac{1}{3}(1,1)$. The remaining three are fibers of Hirzebruch surfaces $\mathbb{F}_1$ (blow-up of a point on $\mathbb{P}^2$), which we contract to lines. After the flops and contractions the cubic surface has become a surface with singularity type $9\mathsf{A}_{3,1}$ and $K^2=0$; in fact, it is geometrically the quotient of $E\times E$ by the diagonal $\mu_3\subset\mu_3\times\mu_3$, where E is an elliptic curve of j-invariant 0 and μ_3 acts by elliptic curve automorphisms (cf., e.g., [28]). The components of the special fiber of multiplicity 2 have pointwise trivial but scheme-theoretically nontrivial action of μ_3, making the quotient map a finite flat morphism of degree 2 to the respective reduced subscheme. The operation of pinching by this morphism [9] introduces singularities along these components but transforms the non-Gorenstein $\mathsf{A}_{3,1}$-singularities into hypersurface singularities. Then we contract the non-normal component of the special fiber to a point.

- *Type V:* On the special fiber DP6 there is a unique μ_6-fixed point, which we blow up to obtain a DP5 joined along exceptional curve ℓ to a copy of $\mathbb{P}^2$ on which the action of $\mu_2\subset\mu_6$ is trivial and the action of $\mu_3\subset\mu_6$ fixes a point and a line; the line m that is fixed meets ℓ at a point q. On the DP5 there are three (possibly Galois conjugated) exceptional curves ℓ', ℓ'', ℓ''' incident to ℓ. As well, the proper transforms of cubic curves on DP6 through the μ_6-fixed point comprise two pencils of conics on DP5, out of which we are interested in the unique members passing through q. We blow up m, which has normal bundle isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)$, so the exceptional divisor is a copy of the Hirzebruch surface $\mathbb{F}_2$, and the action of μ_2 on $\mathbb{F}_2$ fixes a pair of sections, while the action of μ_3 is trivial. The proper transforms of the two conics through q are disjoint, each with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)$; they may be flopped, so that the Hirzebruch surface is blown up at two points on a fiber. We contract ℓ', ℓ'', and ℓ''' to points, so that the special fiber becomes the union of the blown up Hirzebruch surface and two copies of $\mathbb{P}^2$ meeting along a line on which the total space has, geometrically, three ordinary double point singularities. The union of the two copies of $\mathbb{P}^2$ is a Cartier divisor, which may be contracted to a threefold singularity of type D_4. Since, then, the action of μ_3 on the special fiber is trivial, we may descend so that on the base there is only μ_2-stabilizer, and the fiber is the A_2-singular surface that is obtained by contracting the pair of (-2)-curves of the blown up Hirzebruch surface; the μ_2-fixed locus is a rational curve n. Blowing up n, which has normal bundle isomorphic to $\mathcal{O}_{\mathbb{P}^1}(3)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)$, produces exceptional divisor $\mathbb{F}_4$ and replaces the A_2-singularity with an A_1-singularity. The A_1-singular surface may be contracted to a curve, leaving $\mathbb{F}_4$ with (-4)-curve glued to a fiber.

Step (3) is straightforward in Types I through IV. In Type V there is an intermediate descent step which replaces the original μ_6-action by a μ_2-action; a similar two-stage descent construction has been employed in the proof of [20, Thm. 6].

Theorem 8. *Let S be a quasi-compact separated regular scheme, in whose local rings 2 and 3 are invertible, and let D_1, …, D_5 be disjoint regular divisors. Then the construction described above identifies, up to unique isomorphism:*

- *smooth* DP6 *bundles $\mathcal{X}\to\mathcal{S}$, where $\mathcal{S}$ denotes the iterated root stack*
$$\sqrt{(S,D_1\cup D_2\cup D_3)}\times_S\sqrt[3]{(S,D_4)}\times_S\sqrt[6]{(S,D_5)},\qquad(4.1)$$
such that on geometric fibers over the gerbe of the root stack over D_i the action of μ_2 ($i\le 3$), μ_3 ($i=4$), μ_6 ($i=5$) is a toric action of Type i.
- *basic* DP6 *bundles $X\to S$ with Type i fibers over D_i for $i=1$, …, 5, where X is regular, except for isolated singularities of type cone over Veronese surface over D_1 and of type cone over rational normal scroll $S_{2,2}$ over D_4.*

Proof. The construction described above is checked, in each Type, to transform a smooth DP6 bundle over the root stack to a basic DP6 bundle. The construction may be reversed. The pattern of argument follows [17, §5–§6], except that for the reverse construction there is a new ingredient: the pinching operation in Type IV is reversed by normalization.

We give extensive details in Type III and sketch the arguments in Types IV and V.

Type III: The construction evidently leads to a DP6 bundle with Type III fibers over D_3. It remains to verify that the total space is regular. For this we may work locally, and assume we are at a point of D_3 with separably closed residue field, D_3 defined by $f=0$, and root stack with gerbe of the root stack defined by $t=0$ where $t^2=f$. Write DP6 has the hypersurface $xyz=\tilde{x}\tilde{y}\tilde{z}$ in $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$, with homogeneous coordinates x, $\tilde{x}$, etc., on the respective $\mathbb{P}^1$ factors, where μ_2 acts by swapping the first two factors. The conic bundle is given by projection to the third factor, and C is defined by $x\tilde{y}=\tilde{x}y$ in the DP6, passing through $(0,0,\infty)$ and $(\infty,\infty,0)$. We denote the exceptional curves on the DP6 by $E_{0\infty\mathbb{P}}$, etc., reflecting the images under projection to the three $\mathbb{P}^1$ factors. Then we compute that multiplication by $t/(x\tilde{x}^{-1} - y\tilde{y}^{-1})$ identifies

$$\mathcal{O}(E_{0\infty\mathbb{P}} + E_{\mathbb{P}\infty 0} + E_{\infty\mathbb{P}0} + E_{\infty 0\mathbb{P}})$$

with $\mathcal{I}/\mathcal{I}^2$, where $\mathcal{I}$ denotes the ideal sheaf of the family of DP6 over the gerbe of the root stack, which is to be contracted. By [17, Prop. A.9], contraction yields $\mathbb{P}^1$ over the gerbe of the root stack whose normal cone is identified with $\mathrm{Spec}(\bigoplus_{n\geq 0}\psi_*(\mathcal{I}/\mathcal{I}^2)^n)$, where ψ denotes the contraction to $\mathbb{P}^1$. We compute:

$$\psi_*(\mathcal{O}(E_{0\infty\mathbb{P}} + E_{\mathbb{P}\infty 0} + E_{\infty\mathbb{P}0} + E_{\infty 0\mathbb{P}})) \cong \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1),$$

where global sections $x\tilde{x}^{-1}$, $y\tilde{y}^{-1}$, 1, $x\tilde{x}^{-1}y\tilde{y}^{-1}$, on the left correspond to $(1,0,0)$, $(0,1,0)$, $(0,0,z)$, $(0,0,\tilde{z})$, respectively, on the right; the action of μ_2 swaps the first two factors. The anti-invariant section t corresponds to $(1,-1,0)$. There is a quadratic relation with term t^2, defining a hypersurface singularity of type D_∞. (Such singularities have been encountered in [20, §3].) After descent this yields a hypersurface whose defining equation includes a nontrivial linear term f, hence the total space is regular.

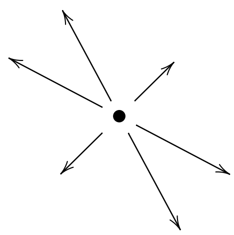
Type IV: We indicate, in coordinates, the curves in the blown-up DP6 that are flopped. Write the DP6 in the standard way as compactification of $\mathbb{G}_m^2$ with coordinates x and y, with action of primitive third root of unity ω by

$$(x,y) \mapsto (x^{-1}y, x^{-1}).$$

Then, ℓ, ℓ', ℓ'' are the exceptional curves obtained by blowing up

$$p := (1,1), \qquad p' := (\omega,\omega^2), \qquad p'' := (\omega^2,\omega).$$

The exceptional curves on the cubic surface are listed in Table 2. The cubic surface is blown up at two points on each of ℓ, ℓ', ℓ'', namely the Eckhardt points, incident to the proper transforms of the cubic curves in Table 2. Those become the six (-3)-curves that are flopped. Then, the blown-up DP6 is contracted to a surface with singularity type $6\mathsf{A}_{3,1}$, with three floating (-1) curves (i.e., contained in the smooth locus) corresponding to the three quartic curves in Table 2. If the floating (-1) curves would be contracted, the surface would have anticanonical degree 2 and would be the surface listed in [23] as No. 9 in [23, Table 1]. In fact, this is a toric surface, with fan



and three points, incidence points to fibers of each of the three visible conic bundle structures, whose blow-ups recover the three floating (-1)-curves. The three remaining (-3)-curves correspond to ℓ, ℓ', ℓ''. When contracted, the surface acquires anticanonical degree 0, and is transformed by the pinching operation to the degree 3 cover of $\mathbb{P}^1\times\mathbb{P}^1$ of the form $\mathrm{Spec}(\mathcal{A})$, where $\mathcal{A}$ is a coherent sheaf of algebras of the form

$$\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-2)\oplus\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,-4),$$

with algebra structure determined by a morphism

$$\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-3,-6)\to\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}$$

vanishing along the union of three rulings with multiplicity 1 and three rulings (from the other family of rulings) with multiplicity 2. This surface is singular along the multiplicity 2 rulings. Although the structure sheaf of the surface has nonvanishing H^2, the conclusion of [17, Prop. A.9] still holds in the strong form needed here (flatness, compatibility with base change). Indeed, if $\psi\colon X\to Y$ denotes the contraction, $X\cong B\ell_FY$ (where $F\subset Y$ is a section of the restriction of Y to the gerbe of the root stack), with line bundle $L\cong\psi^*\mathcal{O}_Y(1)$ on X, then after replacing L by a suitable power we find that $R^1\psi_*(L^n)=0$ and $R^2\psi_*(L^n)$ is (the direct image under $F\to Y$ of) a locally free sheaf on F, for all $n\ge 1$. The latter has projective dimension 1 at points of F. By the machinery of cohomology and base change, the formation of $\psi_*(L^n)$ commutes with arbitrary base change, i.e., the conclusion of [17, Prop. A.8(iv)] holds.

Type V: The step which contracts a conic bundle over a rational curve with A_1-singular total space yields a singularity of type $\mathsf{J}_{2,\infty}$ [13], [27]. □

Theorem 9. *Let S be a quasi-compact separated regular scheme, in whose local rings 2 and 3 are invertible, with rigidification data L/K, M/K, and let D_1, …, D_5 be disjoint regular divisors. We let S_L and S_M denote the respective normalizations of S in L/K and M/K. We suppose that $S_L\to S$ is finite and flat, ramified over $D_1\cup D_3\cup D_5$, and*

number	description	defining equations
3	exc. of blow-up	
6	proper transf. of exc. curves	
9	proper transf. of conics	$x \in \mu_3,\ y \in \mu_3,\ y/x \in \mu_3$
6	proper transf. of cubics	$x+y+1=0$, $x+y+xy=0$, $\omega x+\omega^2 y+1=0$, $\omega x+\omega^2 y+xy=0$, $\omega^2 x+\omega y+1=0$, $\omega^2 x+\omega y+xy=0$
3	proper transf. of quartics	$xy = 1,\ y = x^2,\ x = y^2$

TABLE 2. The 27 exceptional curves on the blow-up of DP6 at three points

$S_M \to S$ is finite and flat, simply ramified over $D_2 \cup D_3$ and totally ramified over $D_4 \cup D_5$. We introduce notation $D_{L,i}$ and $D_{M,i}$ for the reduced divisor of S_L, respectively S_M, over D_i, for $i = 1, \dots, 5$, and write

$$D_{M,2} = D_{M_u,2} \cup D_{M_r,2}, \qquad D_{M,3} = D_{M_u,3} \cup D_{M_r,3},$$

to distinguish the components where the covering map is unramified and simply ramified. Then the Blunk construction gives rise to a bijection between

- *Equivalence classes of basic* DP6 *bundles with Type i fibers over D_i for $i = 1, \dots, 5$, where the total space is regular, exception for isolated singularities of type cone over Veronese surface over D_1 and of type cone over rational normal scroll $S_{2,2}$ over D_4, rigidified by L/K and M/K, and*
- *pairs of Brauer group elements*

  $$\eta \in \mathrm{Br}(S_L)[3] \setminus (D_{L,4} \cup D_{L,5})$$

  and

  $$\zeta \in \mathrm{Br}(S_M)[2] \setminus (D_{M,1} \cup D_{M_u,2} \cup D_{M_u,3} \cup D_{M,5})$$

  which restrict to zero in $\mathrm{Br}(L \otimes_K M)$ *and corestrict to zero in* $\mathrm{Br}(K)$.

Based on Proposition 7, we see that the iterated root stack (4.1) has finite étale covers

$$\sqrt{(S_L, D_{L,2})} \times_{S_L} \sqrt[3]{(S_L, D_{L,4} \cup D_{L,5})} \tag{4.2}$$

of degree 2 and

$$\sqrt{(S_M, D_{M,1} \cup D_{M_u,2} \cup D_{M_u,3} \cup D_{M,5})} \tag{4.3}$$

of degree 3.

Proof. The passage from basic DP6 bundles to Brauer group elements is just as in the first paragraph of the proof of Theorem 5. The rest of the proof is structured just as in the remainder of the proof of Theorem 5, where we note that by Theorem 8, it suffices to exhibit a smooth DP6-fibration over the iterated root stack (4.1). The argument in the case of a semi-local Dedekind domain relies on the fact that the given Brauer group elements extend to the Brauer groups of (4.2), respectively (4.3), since the orbifold structure kills any ramification, as observed, e.g., in [24, Lemma 2], and on the fact that in Figure 1, the top-right vertical map is surjective, while the bottom-left vertical map is an isomorphism. Indeed, the target of the top-right vertical map is zero in Types I, III, IV, and V, and in Type II this map is an isomorphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$. Case-by-case verification takes care of the claim concerning the bottom-left vertical map, e.g., in Type II this is $0 \to 0$ when L/K is split, and otherwise Nm_L and Nm_{LM} are both zero, while Nm_M is surjective. The remainder of the argument is exactly as in the proof of Theorem 5: restriction to the complement of a closed substack of codimension at least 2 induces an equivalence of categories between smooth DP6 bundles over the iterated root stack (4.1) and smooth DP6 bundles over the complement. □

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