CYCLE CLASS MAPS AND BIRATIONAL INVARIANTS

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Abstract. We introduce new obstructions to rationality for geometrically rational threefolds arising from the geometry of curves and their cycle maps.

1. Introduction

Let X be a smooth projective variety over a field $k \subset \mathbb{C}$ with $X_{\mathbb{C}}$ rational. When is X rational over k?

It is necessary that $X(k) \neq \emptyset$. This is also sufficient if X has dimension 1. In dimension 2, this is not sufficient, but there are effective criteria for rationality, due to Enriques, Iskovskikh, Manin, and others. For example, minimal del Pezzo surfaces of degree ≤ 4 are never rational. Indeed, the Galois action on the Néron-Severi group – the lines especially – governs the rationality of X.

The case of threefolds remains open. The Galois action on the Néron-Severi group can still be used to obtain nonrationality in some cases, but never when that group has rank one or is split over the ground field. The case of complete intersections of two quadrics was considered in depth in [HT19]; we gave a complete characterization of rationality over $k = \mathbb{R}$. Benoist and Wittenberg [BW19a] developed an approach inspired by the Clemens-Griffiths method of intermediate Jacobians. If a threefold X is rational then its cohomology reflects invariants of curves blown up in parametrizations $\mathbb{P}^3 \dashrightarrow X$. Over $\mathbb{C}$, the intermediate Jacobian of X must be isomorphic to a product of Jacobians of curves. When k is not algebraically closed, one may endow the intermediate Jacobian with the structure of a principally polarized abelian variety over k [ACMV18]. This must be isomorphic to a product of Jacobians of (not necessarily geometrically connected) curves over k, if X is to be rational over k. Benoist and Wittenberg exhibit geometrically rational conic bundles over $\mathbb{P}^2$ where the latter condition fails to hold, e.g., over $k = \mathbb{R}$.

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Now suppose that X is a smooth projective geometrically rational threefold as above, with rank-one Néron-Severi group and intermediate Jacobian $J^2(X)$ isomorphic to a product of Jacobians of curves over k. We introduce new obstructions to rationality of such X over k based on the geometry of curves on X and provide examples where they apply.

The idea is that the cycle class map on curves of given degree d naturally takes its values in a principal homogeneous space for $J^2(X)$. Moreover, if $J^2(X) \simeq J^1(C)$ for a smooth geometrically irreducible curve C of genus $g \geq 2$ over k, then this homogeneous space is isomorphic to a component of the Picard scheme of C provided X is rational over k. This is a strong constraint as the order of any such component in the Weil-Châtelet group divides $2g-2$.

As an application, we completely characterize (in Theorem 24) the rationality of smooth intersections of two quadrics $X \subset \mathbb{P}^5$: It is necessary and sufficient that X admit a line over the ground field.

Here is a roadmap of the paper: We review constructions of cycle class maps over the complex numbers in Section 2; this serves as motivation for our arithmetic approach. Cycle class maps take values in abelian varieties; we discuss Albanese morphisms from singular varieties in Section 3. We turn to nonclosed fields in Section 4, discussing how to define cycle maps over the relevant fields of definition. Our approach is a geometric implementation of the ℓ-adic Abel-Jacobi map studied by Jannsen. The key invariant is presented in Section 5, in arbitrary dimensions. An application to threefolds can be found in Section 6.

It would be interesting to find nontrivial examples of geometrically rational fourfolds where this machinery applies. Which principal homogeneous spaces for abelian varieties are realized by zero cycles on curves and surfaces?

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2. Review of the complex case

Let X be a smooth complex projective variety.

2.1. **Cycle class maps.** Regard X as a complex manifold. We consider *Deligne cohomology*, following [EV88, §1]. For each integer $p \geq 0$ we have the complex $\mathbb{Z}(p)_{\mathcal{D}}$ of complex analytic sheaves

$$0 \to \mathbb{Z}(p) \to \mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^{p-1}_X \to 0,$$

where $\mathbb{Z}(p) \to \mathcal{O}_X$ takes 1 to $(2\pi i)^p$ and the subsequent arrows are exterior differentiation. Deligne cohomology is defined as the hypercohomology of this complex

$$H^q_{\mathcal{D}}(X, \mathbb{Z}(p)) := \mathbb{H}^q(\mathbb{Z}(p)_{\mathcal{D}}).$$

For $p = 0$ we recover ordinary singular cohomology

$$H^q_{\mathcal{D}}(X, \mathbb{Z}(0)) = H^q(X, \mathbb{Z}).$$

When $p = q = 1$ we have

$$H^1_{\mathcal{D}}(X, \mathbb{Z}(0)) = H^0(X, \mathcal{O}^*_X).$$

The exponential exact sequence gives

$$H^1_{\mathcal{D}}(X, \mathbb{Z}(1)) = H^1(X, \mathcal{O}^*_X) = \mathrm{Pic}(X).$$

More generally, there is a cycle class mapping [EV88, §7]

$$\psi^p : \mathrm{CH}^p(X) \to H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)).$$

The target fits into a short exact sequence [EV88, 7.9]

$$0 \to J^p(X) \to H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)) \to \mathrm{Hg}^p(X) \to 0.$$

Here the right term is the *Hodge cycles*, the kernel of the homomorphisms

$$H^{2p}(X, \mathbb{Z}(p)) \to H^{2p}(X, \mathbb{C}) \to \oplus_{j=0,\ldots,p-1} H^{2p-j}(X, \Omega^j_X)$$

coming from Hodge theory. The left term is the *intermediate Jacobian*, a complex torus

$$J^p(X) := H^{2p-1}(X, \mathbb{C}) / \left(H^{2p-1}(X, \mathbb{Z}(p)) \oplus_{j=p}^{2p-1} H^{2p-1-j}(X, \Omega^j_X) \right).$$

The cycle class map admits an interpretation in terms of extensions of mixed Hodge structures (see [Jan90, §9.1] and [EV88, §7.12]) that is useful in drawing comparisons among cohomology theories. Suppose that $Z \subset X$ is a codimension-p compact complex submanifold with complement $U = X \setminus Z$. Consider the exact sequence for cohomology with supports

(1)
$$\cdots H^{2p-1}(X,\mathbb{Z}(p)) \to H^{2p-1}(U,\mathbb{Z}(p)) \to H^{2p}_{|Z|}(X,\mathbb{Z}(p)) \to H^{2p}(X,\mathbb{Z}(p)) \cdots$$

and the associated mixed Hodge structure on U. This is an extension of the pure weight $2p$ Tate Hodge structure associated with the cycle class of Z by the degree-$(2p-1)$ cohomology of X, yielding an element

$$\eta(Z) \in \mathrm{Ext}^1_{MHS}(\mathbb{Z}(-p), H^{2p-1}(X,\mathbb{Z})) \simeq J^p(X), \tag{2}$$

where the last identification is discussed in [Car87].

2.2. **Algebraicity and cycle maps.** Let B be a smooth connected complex variety and

$$\begin{array}{ccc} \mathcal{Z} & \hookrightarrow & X \times B \\ \downarrow & & \\ B & & \end{array}$$

a flat family of codimension-p subschemes. Then the induced cycle map

$$\Psi^p_B : B \to H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p))$$

has the following properties:

- the image lies in a coset I for $J^p(X) \subset H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p))$;
- the induced map $B \to I$ is holomorphic for the complex structure associated with an identification $I \simeq J^p(X)$;
- the smallest complex torus $P_B \subset I$ containing the image of B carries the structure of an abelian variety and the induced $B \to P_B$ is algebraic with respect to that structure.

The first statement is clear as B is connected. The second may be found in [Gri70, Ap. A]. For the third, take the closure $\overline{B}$ of B in the Hilbert scheme and choose a projective resolution of singularities $\beta : \widetilde{B} \to \overline{B}$ that leaves B unchanged. Pulling back the universal flat family over the Hilbert scheme to $\widetilde{B}$, we obtain a flat family of cycles

$$\widetilde{\mathcal{Z}} \to \widetilde{B}$$

and an induced proper holomorphic $\Psi^p_{\widetilde{B}}$ extending Ψ^p_B. Note that $P_{\widetilde{B}} = P_B$ is dominated by the Albanese $\mathrm{Alb}(\widetilde{B})$, thus is an abelian variety. Since $\Psi^p_{\widetilde{B}}$ is a holomorphic map of projective varieties it is algebraic, thus Ψ^p_B is algebraic as well.

We fix X and p as above and consider families of codimension-p cycles $Z \subset X$. Each family yields a translate of an abelian subvariety of $J^p(X)$. Let $J^p_{cyc}(X) \subset J^p(X)$ denote the distinguished maximal (connected) abelian subvariety arising from such families of cycles. We have

$$J^p_{cyc}(X) \subset E^p(X) := \psi^p(\mathrm{CH}^p(X)) \subset H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))$$

and the quotient $G^p(X)$ of the second group by the first is countable, as there are countably-many irreducible components of the Hilbert scheme parametrizing subschemes of X.

Recall the Griffiths group [Gri69]

$$\mathrm{CH}^p(X)_{hom}/\,\mathrm{CH}^p(X)_{alg} =: \mathrm{Griff}^p(X) \subset B^p(X) := \mathrm{CH}^p(X)/\,\mathrm{CH}^p(X)_{alg}.$$

We have a surjective homomorphism

$$B^p(X) \twoheadrightarrow G^p(X)$$

with kernel consisting of cycles Abel-Jacobi equivalent to zero. Thus we obtain a diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{CH}^p(X)_{alg} & \to & \mathrm{CH}^p(X) & \to & B^p(X) & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & J^p_{cyc}(X) & \to & D^p(X) & \to & B^p(X) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & J^p_{cyc}(X) & \to & E^p(X) & \to & G^p(X) & \to & 0
\end{array}$$

where the second row is induced by the third row. We summarize this as follows:

Proposition 1. *The cycle class map induces homomorphisms*

$$\psi^p : \mathrm{CH}^p(X) \xrightarrow{\chi^p} D^p(X) \twoheadrightarrow E^p(X),$$

where $E^p(X)$ (resp. $D^p(X)$) is an extension of a countable group $G^p(X)$ (resp. $B^p(X)$) by an abelian variety $J^p_{cyc}(X)$.

Given a family of cycles over a connected base B, there is an algebraic morphism

$$B \to P$$

to a principal homogeneous space for $J^p_{cyc}(X)$.

2.3. **Vanishing results.** Let X have dimension n.

- $\mathrm{Griff}^1(X) = 0$ by the Lefschetz $(1,1)$ theorem;
- $\mathrm{Griff}^n(X) = 0$ as all zero cycles of degree zero are algebraically trivial.

We recall further results along these lines.

Definition 2. We say that X admits a *decomposition of the diagonal* if there exist a point $x \in X$, an $N \in \mathbb{N}$, and a rational equivalence on $X \times X$

$$N\Delta_X \equiv N\{x\} \times X + Z',$$

where Z' is supported on $X \times D$ for some subvariety $D \subsetneq X$.

Rationally connected varieties admit decompositions of the diagonal.

Theorem 3. [BS83, Thm. I(i)] *If X admits a decomposition of the diagonal then ψ^2 is an isomorphism. Thus we obtain an isomorphism of abelian groups*

$$\psi^2_\circ : \mathrm{CH}^2(X)_{hom} \xrightarrow{\sim} J^2(X).$$

This actually holds under weaker assumptions: It suffices that the Chow group of zero cycles on X be supported on a curve. Furthermore, $\mathrm{Griff}^2(X) = 0$ provided the Chow group of zero cycles on X is supported on a surface [BS83, Thm. I(ii)]. And the Hodge conjecture for codimension-two cycles holds provided the zero cycles are supported on a threefold [BS83, Thm. I(iv)].

Let X be rationally connected of dimension n. Voisin has asked whether $\mathrm{Griff}^{n-1}(X)$ always vanishes when X is Fano. This is known for certain complete intersections:

Theorem 4. [TZ14, Thm. 1.7] *Let $X \subset \mathbb{P}^{n+c}$ be a smooth complete intersection of hypersurfaces of degrees $d_1, \dots, d_c$ with $d_1 + \cdots + d_c \leq n-1$. Then $\mathrm{CH}^{n-1}(X)$ is generated by lines.*

Varieties of lines on complete intersections are connected when their expected dimension is positive [DM98, Thm. 2.1] – the quadric surface $X \subset \mathbb{P}^3$ being the only exception. Thus we find that $\mathrm{Griff}^{n-1}(X) = 0$.

2.4. **Chow varieties.** Let $\mathrm{Chow}^p(X)$ denote the monoid of effective codimension-p cycles on X and $\mathrm{Chow}^p_d(X)$ the cycles of degree d for each $d \in \mathrm{Hg}^p(X)$. This carries the structure of a projective semi-normal scheme [Kol96, Thm. 3.21]. There is a well-defined addition operation

$$\mathrm{Chow}^p(X) \times \mathrm{Chow}^p(X) \to \mathrm{Chow}^p(X)$$

endowing $\mathrm{Chow}^p(X)$ with the structure of a monoid.

Let $C^p(X)$ denote the free abelian group generated by the connected components $\mathcal{C} \subset \mathrm{Chow}^p(X)$. We obtain surjective homomorphisms

$$C^p(X) \twoheadrightarrow B^p(X) \twoheadrightarrow G^p(X) \twoheadrightarrow \mathrm{CH}^p(X)/\,\mathrm{CH}^p(X)_{hom},$$

where the first three groups are countably-generated and the last is finitely-generated. Furthermore, $\mathrm{Griff}^p(X)$ is a subquotient of $C^p(X)$ – cycles parametrized by a connected component of the Chow variety are algebraically equivalent to each other.

For each ample divisor h on X, we have a filtration by finitely-generated subgroups

$$F_n C^p(X) = \bigoplus_{\mathcal{C} \subset \mathrm{Chow}^p_d(X) \text{ such that } h \cdot d \leq n} \mathbb{Z}[\mathcal{C}]$$

This gives $C^p(X), B^p(X)$, and $G^p(X)$ compatible structures of inductive limits of finitely generated groups, independent of the choice of h. The same holds for $\mathrm{Griff}^p(X)$, i.e., we restrict to cycles expressible as sums of terms of bounded degree.

Proposition 5. *There is a unique cycle class morphism of complex analytic spaces*

$$\Psi^p_d : \mathrm{Chow}^p_d(X) \to I,$$

where I is a coset for $J^p(X) \subset H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))$, compatible with the cycle class mapping ψ^p. Its image generates a finite union of translates of abelian subvarieties in the intermediate Jacobian, each contained in $J^p_{cyc}(X)$.

Proof. As the Chow variety is seminormal by definition and the cycle class is well-defined set-theoretically, it suffices to construct Ψ^p_d on each connected component W of the normalization.

In Section 2.2 we discussed how to define the desired projective morphism on a resolution $\beta : \widetilde{W} \to W$. As it is constant on the fibers of β, Stein factorization gives the desired descent to W. □

3. Albanese varieties

Here we work over a field $k \subset \bar{k} \subset \mathbb{C}$ with absolute Galois group $\Gamma = \mathrm{Gal}(\bar{k}/k)$. Our goal is to recast classical work of Lang, Serre [Ser60], and others, with a view toward analyzing cycle maps. A good recent survey of Albanese constructions over general fields is the appendix of [ACMV19a].

Given a principal homogeneous space P for an abelian variety J, we use $[P]$ to denote the associated class in the Weil-Châtelet group of J, which may be interpreted as the Galois cohomology group $H^1(\Gamma, J_{\bar{k}})$. If P and P' and principal homogenous spaces over an abelian variety J then multiplication induces a morphism

$$P \times_{\mathrm{Spec}(k)} P' \to P''$$

to a principal homogeneous space P'' for J satisfying

$$[P] + [P'] = [P''].$$

The original construction of the Albanese goes back to [Ser60, Exp. 10, §4]. Brian Conrad [Con17] establishes the result we require using the duality between Albanese and Picard varieties:

Proposition 6. *Let T be projective, geometrically reduced, and geometrically connected, over k. Then there exists an abelian variety* $\mathrm{Alb}(T)$, *a principal homogeneous space P for* $\mathrm{Alb}(T)$*, and a morphism*

$$i_T : T \to P,$$

all defined over k, with the following properties:

- *given a morphism $T \to T'$ over k, where T and T' satisfy our hypotheses, there is an induced morphism*
$$\mathrm{Alb}(T) \to \mathrm{Alb}(T');$$
- *each morphism $T \to P'$ to a principal homogeneous space for an abelian variety defined over k admits a factorization through i_T.*

The first property is dual to the functorial pull-back homomorphism $\mathrm{Pic}(T') \to \mathrm{Pic}(T)$. Conrad actually gives a stronger universal property: For any scheme S/k and morphism

$$T \times S \to P'$$

to a principal homogeneous space for an abelian variety over S, there is a factorization through $(i_T)_S : T \times S \to P \times S$. However, the behavior of the Albanese under basechange can be quite subtle for non-proper varieties and inseparable field extensions [ACMV19a, App.].

Corollary 7. *Retain the notation of Proposition 6. For each $d \in \mathbb{N}$ there is a natural morphism*

$$i_T^d : \mathrm{Sym}^d(T) \to P_d,$$

where P_d is the principal homogeneous space over $\mathrm{Alb}(T)$*, satisfying*

$$[P_d] = d[P]$$

in the Weil-Châtelet group of $\mathrm{Alb}(T)$*. When $d \gg 0$ the morphism i_T^d is dominant.*

Hence for d sufficiently large and divisible – e.g., when T admits a point over a degree d extension – $\mathrm{Alb}(T)$ is dominated by $\mathrm{Sym}^d(T)$.

Proof. Indeed, i_T gives

$$\mathrm{Sym}^d(T) \to \mathrm{Sym}^d(P)$$

and addition induces

$$\underbrace{P \times \cdots \times P}_{d \text{ times}} \to P_d,$$

compatible with permutations of the factors.

For the last statement: If W is smooth, projective, and geometrically integral then $i_W^e : \mathrm{Sym}^e(W) \to P_{e,W}$ – the morphism onto the degree-e torsor over $\mathrm{Alb}(W)$ – is dominant for large e. Let d be the sum of the

e's taken over all (geometrically) irreducible components of a resolution of T. As $\mathrm{Alb}(T)$ is a quotient of the product of the Albanese varieties of these components, we find that i_T^d is dominant as well. $\square$

4. Passage to nonclosed fields

We continue to work over a field $k \subset \mathbb{C}$ with absolute Galois group Γ. Let X be a smooth projective variety over k and write $\bar{X} = X_{\bar{k}}$.

4.1. **ℓ-adic cycle maps.** One formulation goes back to Bloch [Blo79]: Let ℓ be a prime and $\mathrm{CH}^p(\bar{X})(\ell)$ the ℓ-primary part of the torsion, i.e.,

$$\mathrm{CH}^p(\bar{X})(\ell) = \varinjlim_{\nu \to \infty} \mathrm{CH}^p(\bar{X})[\ell^\nu].$$

Then there is a functorial cycle class homomorphism

$$\lambda_\ell^p : \mathrm{CH}^p(\bar{X})(\ell) \to H^{2p-1}(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(p)).$$

It is an isomorphism when $p = n := \dim(X)$, in which case the target may be interpreted as the ℓ-primary part of the torsion of the Albanese $\mathrm{Alb}(\bar{X})$ [Blo79, 3.9].

Jannsen [Jan88, §3] has defined cycle class maps to continuous étale cohomology

$$\psi_\ell^p : \mathrm{CH}^p(X) \to H^{2p}_{cont}(X, \mathbb{Z}_\ell(p)).$$

The main advantage of continuous cohomology is the existence of a Hochschild-Serre type spectral sequence under field extensions. Fix the cohomology class of an algebraic cycle

$$[Z_0] \in H^0_{cont}(\Gamma, H^{2p}(\bar{X}, \mathbb{Z}_\ell(p)))$$

and consider the induced

$$\psi_\ell^p : \{Z \in \mathrm{CH}^p(X) : [Z] = [Z_0]\} \to H^1_{cont}(\Gamma, H^{2p-1}(\bar{X}, \mathbb{Z}_\ell(p))), \tag{3}$$

the ℓ-adic analog of the Abel-Jacobi map [Jan88, §6]. The target group is equal to

$$\mathrm{Ext}^1_\Gamma(\mathbb{Z}_\ell, H^{2p-1}(\bar{X}, \mathbb{Z}_\ell(p)))$$

so we may compare with the extension (2). When $p = n = \dim(X)$ we obtain

$$\psi_\ell^n : \mathrm{CH}^n(X)_{hom} \to H^1_{cont}(\Gamma, H^{2n-1}(\bar{X}, \mathbb{Z}_\ell(n))).$$

When k is finitely generated over $\mathbb{Q}$, the Mordell-Weil theorem yields an injection [Jan90, 9.14]

$$\mathrm{Alb}(X)(k) \otimes \mathbb{Z}_\ell \hookrightarrow H^1_{cont}(\Gamma, H^{2n-1}(\bar{X}, \mathbb{Z}_\ell(n))). \tag{4}$$

Suppose we are given a smooth, projective, and geometrically connected B of dimension b. Given a flat family of codimension-p subschemes

$$\begin{array}{ccc} \mathcal{Z} & \hookrightarrow & X \times B \\ \downarrow & & \\ B & & \end{array}$$

flat pullback followed by pushforward induces

$$H^{2b-1}_{cont}(B, \mathbb{Z}_\ell(b)) \to H^{2p-1}_{cont}(X, \mathbb{Z}_\ell(p)),$$
$$\mathrm{Alb}(\bar{B})(\ell) \simeq H^{2b-1}(\bar{B}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(b)) \to H^{2p-1}(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(p)),$$

the latter compatible with Galois actions.

4.2. **Descent of intermediate Jacobians.** We recall a result on descending Abel-Jacobi maps:

Theorem 8. [ACMV18, Thm. A], [ACMV19b, Thm. 1] *There exists an abelian variety J over k with the following properties:*

- *there exists an isomorphism*
$$\iota : J^p_{cyc}(X_{\mathbb{C}}) \xrightarrow{\sim} J_{\mathbb{C}};$$
- *given a pointed scheme $(B, 0)$ over k that is smooth and geometrically connected and a family of codimension-p cycles*
$$\begin{array}{ccc} \mathcal{Z} & \hookrightarrow & X \times B \\ \downarrow & & \\ B & & \end{array}$$
defined over k, there exists a morphism
$$\Phi : B \to J$$
over k such that
$$\Phi_{\mathbb{C}}(b) = \iota \circ \Psi^p_B([\mathcal{Z}_b] - [\mathcal{Z}_0]).$$

Moreover, J is unique up to isomorphism over k and compatible with field extensions; Φ is unique and compatible with field extensions.

Proof. We sketch the construction of J: Fix a family of codimension-p cycles over a smooth geometrically-connected base, all defined over k:

$$\begin{array}{ccc} \mathcal{Z} & \hookrightarrow & X \times B \\ \downarrow & & \\ B & & \end{array}$$

Consider the induced morphisms of complex abelian varieties

$$\mathrm{Alb}(B_{\mathbb{C}}) \to J^p(X_{\mathbb{C}})$$

and homomorphisms of ℓ-adic representations

$$\mathrm{Alb}(\bar{B})(\ell) \to H^{2p-1}(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(p)).$$

Recall that $J^p_{cyc}(X_\mathbb{C}) \subset J^p(X_\mathbb{C})$ is the maximal abelian subvariety spanned by the images of these morphisms.

We claim that this maximal subvariety can be realized over the ground field: There exists an abelian variety J' over k – the Albanese of a geometrically-connected base of some family of cycles defined over k – such that the cycle class map induces a surjection

$$J'_\mathbb{C} \twoheadrightarrow J^p_{cyc}.$$

Suppose the maximal subvariety is associated with a family of cycles over a smooth geometrically-connected base B over a finite extension L/k. Recall that the restriction of scalars $\mathbf{R}_{L/k}(B)$ parametrizes k-morphisms $\mathrm{Spec}(L) \to B$, giving a diagram of k-schemes:

$$\begin{array}{ccc} \mathbf{R}_{L/k}(B) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L) & \to & B \\ \downarrow & & \\ \mathbf{R}_{L/k}(B) & & \end{array}$$

A family of cycles $\mathcal{Z} \to B$ pulls back to a family over $\mathbf{R}_{L/k} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L)$ [Kol96, I.3.18]; pushing forward by the vertical finite morphism yields a family of cycles over the restriction of scalars

$$\mathcal{Z}' \to \mathbf{R}_{L/k}(B).$$

Informally, we are summing cycles over points conjugate over k. The image of the associated

$$\overline{\mathbf{R}_{L/k}(\mathrm{Alb}(B))}(\ell) \to H^{2p-1}(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(p))$$

contains the image of the original homomorphism. Thus maximal images are achievable over the ground field.

It remains to show that our surjection

$$J'_\mathbb{C} \twoheadrightarrow J^p_{cyc}$$

descends naturally to k. Consider the induced homomorphisms of torsion subgroups

$$J'_\mathbb{C}[\ell^\nu] \to J^p_{cyc}[\ell^\nu] \subset J^p(X)[\ell^\nu].$$

The associated Galois data is encoded by

$$\rho_\ell : \bar{J}'(\ell) \to H^{2p-1}(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(p)),$$

which is compatible with Γ-actions. Descent for homorphisms of abelian varieties – quotients of a given abelian variety over k can be read off from its ℓ-adic representations – yields a unique $J' \twoheadrightarrow J^p_{cyc}$ factoring

$$\rho_\ell : \bar{J}'(\ell) \to J(\ell) \hookrightarrow H^{2p-1}(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(p)),$$

for every ℓ. Indeed, first find an isogeny $J'' \to J^p_{cyc}$ and then quotient out by torsion subgroups in the kernel. The resulting J is defined over k because each ρ_ℓ is Galois invariant. $\square$

Proposition 9. *Retain the set-up of Theorem 8, with B smooth and geometrically connected over k and $\mathcal{Z} \subset X \times B$ a family of codimension-p cycles. Then for each d there is a morphism over k*

$$\phi : \mathrm{Sym}^d(B) \times \mathrm{Sym}^d(B) \to J$$

such that

$$\phi_{\mathbb{C}}(\sum_i b_i, \sum_i b'_i) = \iota \circ \Psi^p_{\mathrm{Sym}^d(B)\times\mathrm{Sym}^d(B)}(\sum_i [\mathcal{Z}_{b_i}] - \sum_i [\mathcal{Z}_{b'_i}]).$$

This morphism is compatible with field extensions.

Proof. First, pass to a resolution $B^{[d]} \to \mathrm{Sym}^d(B)$; our family of cycles $\mathcal{Z}$ induces a family of cycles over $B^{[d]}$ by summing over d-tuples of points. Taking differences yields a family of cycles over $B^{[d]} \times B^{[d]}$. Choose a field extension L/k so that $B^{[d]}$ admits an L-rational point. Theorem 8 gives a morphism

$$\Phi_L : (B^{[d]} \times B^{[d]})_L \to J_L;$$

translate in J_L so it takes the diagonal to zero. Then Galois descent yields a morphism over k

$$\phi^\nu : B^{[d]} \times B^{[d]} \to J$$

which induces a morphism on symmetric powers

$$\phi : \mathrm{Sym}^d(B) \times \mathrm{Sym}^d(B) \to J,$$

by the standard Stein factorization argument. $\square$

Corollary 10. *Retain the assumptions of Proposition 9 and assume there is a k-rational point $\sum_i b_i^\circ \in \mathrm{Sym}^d(B)$. Then there is a morphism over k*

$$\Phi : \mathrm{Sym}^d(B) \to J$$

such that

$$\Phi_{\mathbb{C}}(\sum_i b_i) = \iota \circ \Psi^p_{\mathrm{Sym}^d(B)}(\sum_i [\mathcal{Z}_{b_i}] - \sum_i [\mathcal{Z}_{b_i^\circ}]).$$

Proposition 11. *Let B be seminormal and geometrically connected over k and fix a family of codimension-p cycles*

$$\begin{array}{ccc} \mathcal{Z} & \hookrightarrow & X \times B \\ \downarrow & & \\ B & & \end{array}$$

as before. Then there is a morphism over k

$$\phi : B \times B \to J$$

such that

$$\phi_{\mathbb{C}}(b, b') = \iota \circ \Psi^p_{B\times B}([\mathcal{Z}_b] - [\mathcal{Z}_{b'}]).$$

This is compatible with field extensions.

Proof. Pick a resolution $\beta : \widetilde{B} \to B$ whose exceptional locus is a strict normal crossings divisor. Consider the stratification $\{S_\xi\}$ of $\widetilde{B}$ associated with its connected components, the components of the strict normal crossings divisor, or intersections of these components. Choose a finite extension L/k and a finite set $\Xi \subset \widetilde{B}(L)$ of base points for the strata such that

- each stratum S_ξ admits a distinguished $\xi \in \Xi$ and thus is defined over L;
- if $\beta(S_{\xi_1})$ and $\beta(S_{\xi_2})$ have the same closure then $\beta(\xi_1) = \beta(\xi_2)$.

To produce Ξ and L, enumerate all the closed subvarieties of B arising as images of strata, fix open subsets of these subvarieties contained in the images of all the corresponding strata, specify a $\bar{k}$-point in each of these open subsets, and then select a point in each stratum lying over the specified point.

We construct

$$(\widetilde{B} \times \widetilde{B})_L \to J_L$$

as in the proof of Proposition 9 with the diagonal of each component mapped to zero. *A priori*, the morphism on the 'off-diagonal' components would only be determined up to translation. Thus we insist that if $b_{\xi_1}, b_{\xi_2} \in \Xi$ are identified in $B(L)$ then (b_{ξ_1}, b') and (b_{ξ_2}, b') go to the same point in J. We impose the same condition on (b, b_{ξ_1}) and (b, b_{ξ_2}). As B is geometrically connected, the resulting

$$\widetilde{\phi}_L : (\widetilde{B} \times \widetilde{B})_L \to J_L$$

is well-defined, because the rational points in Ξ control how the connected components of $\widetilde{B}$ fit together. Note further that the resulting morphism is independent of the choice of base points. This gluing data satisfies the requisite compatibilities – we may verify this over $\mathbb{C}$ where Proposition 5 applies. Since $B \times B$ is also seminormal [GT80, 5.9], these gluing data induce

$$\phi_L : (B \times B)_L \to J_L.$$

Our construction is compatible with Galois actions over k so ϕ_L descends to the desired morphism over k. □

4.3. **Application of the Albanese to cycle maps.** Fix

$$\mathcal{C} \subset \mathrm{Chow}^p_d(X),$$

a connected component of the Chow variety that is geometrically connected. Let $P_{\mathcal{C}}$ denote the principal homogeneous space for $\mathrm{Alb}(\mathcal{C})$ and $i_{\mathcal{C}} : \mathcal{C} \to P_{\mathcal{C}}$ the morphism constructed in Section 3.

Our next result extends [ACMV18, Thm. B]:

Theorem 12. *There is a homomorphism of abelian varieties over k*

$$\varphi : \mathrm{Alb}(\mathcal{C}) \to J,$$

where J is the model defined in Theorem 8, with the following property: Consider the principal homogeneous space

$$\begin{array}{ccc} J \times P & \to & P \\ (j, p) & \mapsto & j \cdot p, \end{array}$$

where $[P] = \varphi([P_{\mathcal{C}}])$, and the morphism

$$\Phi : \mathcal{C} \to P$$

induced from $i_{\mathcal{C}}$. For each $c_1, c_2 \in \mathcal{C}$ and corresponding cycles $\mathcal{Z}_{c_1}$ and $\mathcal{Z}_{c_2}$, we have

$$\Phi_{\mathbb{C}}(c_2) = \iota(\Psi^p_{\mathcal{C}}([\mathcal{Z}_{c_2}] - [\mathcal{Z}_{c_1}])) \cdot \Phi_{\mathbb{C}}(c_1).$$

Proof. Corollary 7 – the Albanese is dominated by large symmetric powers – shows that it suffices to construct compatible morphisms over symmetric powers of $\mathcal{C}$. Propositions 9 and 11 explain the passage to symmetric powers and to singular parameter spaces. Thus we obtain the homomorphism of abelian varieties over k

$$\varphi : \mathrm{Alb}(\mathcal{C}) \to J$$

such that each zero cycle $\sum_i n_i c_i$ of degree zero goes to the corresponding cycle class $\iota(\Psi^p_{\mathcal{C}}(\sum_i n_i \mathcal{Z}_{c_i}))$ in J. It follows immediately that Φ admits the desired interpretation as a cycle class map to a principal homogeneous space for J. ∎

4.4. **Compatibility under addition.** Let $\mathcal{C}$ and $\mathcal{C}'$ denote geometrically connected components of Chow^p defined over k, so that

$$\mathcal{C} \times_{\mathrm{Spec}(k)} \mathcal{C}'$$

is geometrically connected as well. It is also seminormal [GT80, 5.9]. Let $\mathcal{C}''$ denote the geometrically connected component of Chow^p obtained via addition

$$\begin{array}{rcl} \alpha : \mathcal{C} \times_{\mathrm{Spec}(k)} \mathcal{C}' & \to & \mathcal{C}'' \\ (Z, Z') & \mapsto & Z + Z'. \end{array}$$

We refer the reader to [Kol96, I.3.21] for a discussion of the representability properties of Chow^p underlying these morphisms.

Proposition 13. *Retain the notation of Theorem 12. Suppose that P, P', and P'' are the principal homogeneous spaces over J associated with $\mathcal{C}$, $\mathcal{C}'$, and $\mathcal{C}''$. Then we have*

$$[P] + [P'] = [P''].$$

Proof. The additivity is clear for the fiber product of $\mathcal{C}$ and $\mathcal{C}'$. The morphism α induces a morphism of the corresponding principal homogeneous spaces for J. It is evidently an isomorphism over extensions over which $\mathcal{C}$ and $\mathcal{C}'$ admit rational points. Thus it must also be an isomorphism over its field of definition. □

5. Construction of invariants

We continue to use the notation of Section 4.

5.1. Galois actions on cycle groups.

Proposition 14. *Fix a finite Galois extension L/k, $\sigma \in \mathrm{Gal}(L/k)$, and an embedding $L \subset \mathbb{C}$.*

1. *If Z_1 and Z_2 are defined and algebraically equivalent over L. Then ${}^\sigma Z_1$ and ${}^\sigma Z_2$ are as well.*
2. *If Z_1 and Z_2 are defined over k and are algebraically equivalent over L then there exists an N dividing $[L : k]$ such that NZ_1 and NZ_2 are algebraically equivalent over k.*
3. *If Z_1 and Z_2 are defined over L, are algebraically equivalent over some extension of L, and Abel-Jacobi equivalent to zero then ${}^\sigma Z_1$ and ${}^\sigma Z_2$ are as well.*

Proof. The first assertion is trivial. The second is standard: Suppose C is a smooth connected curve over L with rational points $c_1, c_2 \in C(L)$ admitting a family of cycles $\mathcal{Z} \to C$ with $\mathcal{Z}_{c_1} = Z_1$ and $\mathcal{Z}_{c_2} = Z_2$. Then the restriction of scalars $\mathbf{R}_{L/k}(C)$ is defined over k and admits a family of cycles

$$\mathcal{Z}' \to \mathbf{R}_{L/k}(C)$$

obtained by summing over the conjugates. The fibers over c_1 and c_2 are $[L : k]Z_1$ and $[L : k]Z_2$ as Z_1 and Z_2 are Galois invariant. This gives the desired algebraic equivalence.

For the third statement: All our fields are embedded in $\mathbb{C}$ and Abel-Jacobi equivalence is defined in terms of complex cycles. Thus we may pass to a Galois extension L'/L with $L' \subset \mathbb{C}$ over which our cycles are algebraically equivalent via C as above. We have a morphism $C \to J_{L'}$

such that c_1 and c_2 map to the same L-rational point of J. Then the same holds after conjugating the points and the morphism. $\square$

The groups $C^p(\bar{X})$, $B^p(\bar{X})$, and $\mathrm{Griff}^p(\bar{X})$ all admit actions of Γ compatible with the homomorphisms and inductive structures we introduced previously. The situation for $G^p(\bar{X})$ is less straightforward:

Question 15. Is Abel-Jacobi triviality an algebraic notion? Let X be a smooth projective variety and Z a codimension-p cycle on X homologous to zero, both defined over a field k.

- Given embeddings $i_1, i_2 : k \hookrightarrow \mathbb{C}$, if $i_1(Z)$ is Abel-Jacobi equivalent to zero does it follow that $i_2(Z)$ is as well? cf.[ACMV19b, Conj. 2]
- Suppose that k is finitely generated over $\mathbb{Q}$ and assume that $i(Z)$ is Abel-Jacobi equivalent to zero for some embedding i. Does it follow that
$$\psi^p_\ell(Z) = 0 \in H^1_{cont}(\Gamma, H^{2p-1}(\bar{X}, \mathbb{Z}_\ell(p))),$$
for each ℓ?

The first question is of interest for cycles not algebraically equivalent to zero. The latter statement over number fields k should be compared to the Bloch-Beilinson conjectures – see [Jan90, Conj. 9.12] and the discussion there for context.

5.2. **The key homomorphism.** Consider codimension-p cycles on X over geometrically connected projective schemes over k. We discussed in Section 4.4 how to add two such families. Two families are *equivalent* if they admit fibers over $\bar{k}$ that are algebraically equivalent. The resulting group $B^p(X)$ is generated by geometrically-connected connected components $\mathcal{C}$ of Chow^p. Indeed, given a family $\mathcal{Z} \to B$ over a geometrically connected base as indicated, the classifying map from the seminormalization of B
$$B^\nu \to \mathrm{Chow}^p$$
maps to a distinguished such component. The fiber map yields an injection
$$B^p(X) \hookrightarrow B^p(\bar{X})^\Gamma$$
so this notation is compatible with what was introduced in Section 2.2.

Example 16. Observe that $B^1(X) = \mathrm{NS}(\bar{X})^\Gamma$. Indeed, for sufficiently ample divisor classes, the corresponding divisors are parametrized by a Brauer-Severi scheme over a principal homogeneous space for the identity component of the Picard scheme. This parameter space is geometrically integral.

An element $\zeta \in B^p(X)$ need not be represented by a cycle over k – the base of the family representing ζ might not admit rational points.

Theorem 17. *Let J be the abelian variety produced in Theorem 8. Then there is a homomorphism*

$$\tau : B^p(X) \to H^1_\Gamma(\bar{J})$$

with the following properties:

- *For each $\zeta \in B^p(X)$, there is an isomorphism*

$$\iota_\zeta : (P_{\tau(\zeta)})_{\mathbb{C}} \xrightarrow{\sim} J^p_{cyc}(X_{\mathbb{C}}) + \zeta$$

  of $J^p_{cyc}(X_{\mathbb{C}})$ principal homogeneous spaces.
- *Given a flat family of codimension-p cycles over a geometrically connected base*

$$\begin{array}{ccc} \mathcal{Z} & \hookrightarrow & X \times B \\ \downarrow & & \\ B & & \end{array}$$

  there is a morphism

$$\Phi : B \to P_{\tau([\mathcal{Z}_b])}$$

  over k such that

$$\iota_{[\mathcal{Z}_b]} \circ \Phi_{\mathbb{C}} = \Psi^p_{B_{\mathbb{C}}}.$$

- *These structures are compatible with addition of cycles and field extensions.*

This is a reformulation of Theorem 12. The compatibility under addition is Proposition 13.

Remark 18. The ℓ-primary parts of this homomorphism are natural from the perspecive of the Jannsen's ℓ-adic Abel-Jacobi map (3)

$$\psi^p_\ell : \mathrm{CH}^p(X)_{hom} \to H^1_{cont}(\Gamma, H^{2p-1}(\bar{X}, \mathbb{Z}_\ell(p))).$$

Let $r = \dim J$ and consider the homomorphism of Galois representations

$$H^{2r-1}(\bar{J}, \mathbb{Z}_\ell(r)) \to H^{2p-1}(\bar{X}, \mathbb{Z}_\ell(p))$$

associated with the construction of J. This is the ℓ-adic analog of the homomorphism arising from the inclusion

$$J_{\mathbb{C}} = J^p_{cyc}(X_{\mathbb{C}}) \hookrightarrow J^p(X_{\mathbb{C}})$$

of complex tori. Fixing a class in $B^p(X)$ allows us to restrict the 1-cocyle from $H^{2p-1}(\bar{X}, \mathbb{Z}_\ell(p))$ to

$$H^{2r-1}(\bar{J}, \mathbb{Z}_\ell(r)).$$

Our construction shows that the cocycle lies in the image of the homomorphism cf. (4):

$$T_\ell \bar{J} \to H^1_{cont}(\Gamma, H^{2r-1}(\bar{J}, \mathbb{Z}_\ell(r))),$$

where T_ℓ is the Tate module. The inclusion

$$T_\ell \bar{J} \hookrightarrow \bar{J}$$

yields principal homogeneous spaces for J over k with ℓ-primary order.

Remark 19. One expects that these invariants should vanish on cycles Abel-Jacobi equivalent to zero, i.e., τ factors through $G^p(X)$. This should follow from a positive resolution of both parts of Question 15.

5.3. **Sample cases.** Suppose that $p = 1$ so that

$$\tau : \mathrm{NS}(\bar{X})^\Gamma \to H^1_\Gamma(\mathrm{Pic}^0(\bar{X}))$$

is the tautological map assigning a Galois-invariant connected component of the Picard scheme to the corresponding principal homogeneous space for the identity component. The image is finite because classes of divisors over the ground field form a finite-index subgroup of the source group.

Suppose that $p = n = \dim(X)$ so that

$$\tau : H^{2n}(X_\mathbb{C}, \mathbb{Z}(n)) \to H^1_\Gamma(\mathrm{Alb}(\bar{X}))$$

assigns to each degree the corresponding principal homogeneous space for the Albanese (cf. Corollary 7).

The vanishing results in Section 2.3 allow some sharpened statements:

- Suppose that $p = 2$ and write
$$N^2(\bar{X}) = \mathrm{CH}^2(\bar{X}) / \mathrm{CH}^2(\bar{X})_{hom} \subset \mathrm{Hg}^2(X_\mathbb{C}).$$
If the Chow group of zero cycles on X is supported on a surface then $\mathrm{Griff}^2(X_\mathbb{C}) = 0$ and
$$B^2(X) \subset N^2(\bar{X})^\Gamma,$$
with finite index as the Hodge conjecture holds in this case.
- If X is a uniruled threefold then the integral Hodge conjecture holds [Voi06] and
$$N^2(\bar{X}) = N^2(X_\mathbb{C}) = \mathrm{Hg}^2(X_\mathbb{C}).$$
- If X is a rationally connected threefold then
$$H^4(X_\mathbb{C}, \mathbb{Z}(2)) = \mathrm{Hg}^2(X_\mathbb{C}),$$
and
$$B^2(X) \subset H^4(X_\mathbb{C}, \mathbb{Z}(2))^\Gamma$$

with finite index. The Galois action on H^4 reflects the fact that the cohomology is generated by algebraic cycles. The homomorphism τ has finite image because the classes of intersections of divisors over the ground field span a finite index subgroup of $H^4(X_{\mathbb{C}}, \mathbb{Z}(2))^\Gamma$.

- If X is a prime Fano threefold then the variety of lines on X is geometrically connected by the classification [IP99]. In this case
$$B^2(X) = H^4(X_{\mathbb{C}}, \mathbb{Z}(2)),$$
hence
$$\tau : H^4(X_{\mathbb{C}}, \mathbb{Z}(2)) \to H^1(\Gamma, \bar{J}).$$

6. Threefolds

Our main interest in these invariants is in their application to rationality questions for threefolds that are geometrically rational. We spell out their birational implications and explore these in representative examples.

6.1. **A preliminary result.**

Proposition 20. *If X is a smooth projective threefold, rational over k, then $B^2(X) = H^4(X_{\mathbb{C}}, \mathbb{Z}(2))^\Gamma$.*

Proof. This boils down to two observations, valid for arbitrary fields k:

- given a Galois-invariant collection of points $S = \{s_1, \dots, s_r\} \in \mathbb{P}^3$, there exists a smooth rational curve in $\mathbb{P}^3$ containing S and defined over k;
- given a smooth projective curve $A \subset \mathbb{P}^3$ and $e \in \mathbb{N}$, there exists a geometrically integral family of rational curves intersecting A in a reduced subscheme supported in a generic configuration of e points.

The first assertion is a standard interpolation result; the second follows from the first by working over the function field L of $\mathrm{Sym}^e(A)$, yielding a rational curve defined over L with the desired incidences.

The general argument is inspired by Example 1.4 and Proposition 4.7 of [Kol08]; the latter statement establishes the birational nature of this interpolation property.

Consider the birational map $\mathbb{P}^3 \dashrightarrow X$ and a factorization

$$\begin{tikzcd} & Y \arrow[dl, "\beta"'] \arrow[dr, "\beta'"] & \\ \mathbb{P}^3 & & X \end{tikzcd}$$

where β is a sequence of blow-ups along smooth centers, defined over k. Pushing forward by β' gives a split surjection

$$H^4(Y_{\mathbb{C}}, \mathbb{Z}(2)) \twoheadrightarrow H^4(X_{\mathbb{C}}, \mathbb{Z}(2)),$$

compatible with Galois actions. Thus

$$H^4(Y_{\mathbb{C}}, \mathbb{Z}(2))^{\Gamma} \twoheadrightarrow H^4(X_{\mathbb{C}}, \mathbb{Z}(2))^{\Gamma}$$

and families of cycles in Y over a geometrically connected base project down to such cycles in X.

It suffices to establish the following interpolation result: Fix a finite Galois extension L/k and let $S \subset Y$ denote a Galois-invariant collection of smooth points in the exceptional locus of β defined over L; then there exists a smooth rational curve in Y over L meeting the exceptional locus precisely along S with multiplicity one. Choose formal arcs of smooth curves in Y over L transverse to the exceptional locus at S. Post-composing by β yields formal maps of smooth curves to $\mathbb{P}^3$. Morphisms $\mathbb{P}^1 \to \mathbb{P}^3$ approximating these to sufficiently high order – but otherwise disjoint from the center of β – have proper transforms in Y with the desired intersection property. We are free to take the image curves in $\mathbb{P}^3$ to arbitrarily large degree, so Lagrange interpolation allows us to produce the desired curves over L. □

Remark 21. Thus the invariant τ – introduced in Theorem 17 – is defined on $H^4(X_{\mathbb{C}}, \mathbb{Z}(2))^{\Gamma}$ when X is rational over k.

6.2. **Rationality criterion.** Let X be a smooth and projective threefold over $k \subset \mathbb{C}$ such that $X_{\mathbb{C}}$ is rational. It follows then that $H^3(X_{\mathbb{C}}, \mathbb{Z}(2))$ is torsion-free and $J^2(X_{\mathbb{C}})$ is a principally polarized abelian variety isomorphic to a (nonempty) product of Jacobians of curves. Let J denote its model over k from section 4.2.

Benoist and Wittenberg have established [BW19a, Cor. 2.8] that J is isomorphic to the Jacobian of a smooth projective (not necessarily geometrically connected) curve over k whenever X is rational over k.

We state a refinement of the results of [HT19, §11.5]:

Theorem 22. *Retain the notation introduced above and assume that X is rational over k. Let*

$$\tau : H^4(X_{\mathbb{C}}, \mathbb{Z}(2))^{\Gamma} \to H^1_{\Gamma}(\bar{J})$$

denote the invariant constructed in Theorem 17. Then there exist a smooth projective curve C with positive genus components and an isomorphism

$$i : J \to J^1(C),$$

over k, along with a Galois equivariant homomorphism

$$b : H^2(C_{\mathbb{C}}, \mathbb{Z}(1)) \to H^4(X_{\mathbb{C}}, \mathbb{Z}(2)),$$

such that the composition

$$H^2(C_{\mathbb{C}}, \mathbb{Z}(1))^{\Gamma} \xrightarrow{b} H^4(X_{\mathbb{C}}, \mathbb{Z}(2))^{\Gamma} \xrightarrow{\tau} H^1_{\Gamma}(\bar{J}) \xrightarrow{i} H^1_{\Gamma}(\overline{J^1(C)})$$

is the canonical homomorphism assigning each component of C over k to the corresponding principal homogeneous space for its Jacobian $J^1(C) = \mathrm{Alb}(C)$.

On the notation: C is defined over k so that Γ acts via permutation on its geometric components and thus on $H^2(C_{\mathbb{C}}, \mathbb{Z}(1))$.

After this manuscript was written, Olivier Wittenberg informed us that he obtained a version of Theorem 22, with Theorem 24 below as a corollary. This was developed in correspondence with Colliot-Thélène [Wit19a, Wit19b] and Kuznetsov [Kuz19].

Proof. Our approach follows [Man68] in spirit.

Consider a birational map $\mathbb{P}^3 \dashrightarrow X$ and a factorization

$$\begin{tikzcd} & Z \arrow[dl, "\beta"'] \arrow[dr, "\beta'"] & \\ \mathbb{P}^3 & & X \end{tikzcd}$$

where β is a sequence of blow-ups along smooth centers, defined over k. The blow-up formula [Ful98, §6.7] tells us that

$$\mathrm{CH}^2(\bar{Z}) = \mathbb{Z} \oplus P \oplus (\oplus_{i=1}^{N} \mathrm{CH}^1(\bar{A}_i))$$

where P is a permutation module associated with the points blown up and the A_i are the smooth irreducible curves blown up.

Consider the A_i that 'survive' in X, i.e., positive genus curves whose Jacobians appear as principally polarized factors in the intermediate Jacobian of X. A positive genus curve A_i that does not survive is explained by β' blowing down a curve A' with $J^1(A') \simeq J^1(A)$ as principally polarized abelian varieties.

Let C denote the disjoint union of the surviving A_i, which is Γ-invariant, as the Galois action respects the decomposition of the intermediate Jacobians into simple factors. It follows that $J \simeq J^1(C)$. Let b be obtained by assigning to each surviving A_i the total transforms in X of the exceptional fibers over A_i at the point where it is blown up in β. Thus we obtain that

$$\mathrm{CH}^2(\bar{X}) = P' \oplus \mathrm{CH}^1(\bar{C})$$

where P' is a permutation module over Γ reflecting its action on punctual and genus zero centers of β.

Suppose we are given a family of curves

$$\begin{array}{ccc} \mathcal{Z} & \hookrightarrow & X \times B \\ \downarrow & & \\ B & & \end{array}$$

over a geometrically connected base B. Passing to residual curves in complete intersections on X if necessary, we may assume that the generic fibers intersect the total transforms of the exceptional divisors associated with the components $C_1, \dots, C_r \subset C$ properly. Intersecting, we obtain a morphism

$$\sigma : B \to \operatorname{Pic}^{e_1}(C_1) \times \cdots \times \operatorname{Pic}^{e_r}(C_r),$$

where $e_1, \dots, e_r$ depend only on the homology class of $[\mathcal{Z}_b]$. Over $\mathbb{C}$, σ coincides with the cycle map ψ^2 from B to the appropriate component of $E^2(X_{\mathbb{C}})$, a principal homogeneous space for $J^2(X_{\mathbb{C}})$.

This geometric construction shows that τ fits into the stipulated factorization, i.e., the principal homogeneous space for J receiving the cycles parametrized by B is necessarily of the form

$$\operatorname{Pic}^{e_1}(C_1) \times \cdots \times \operatorname{Pic}^{e_r}(C_r),$$

where the e_i depend on the homology classes of the corresponding curves. $\square$

Remark 23.

(1) This result is most useful when C is determined uniquely by X. This follows from the Torelli Theorem [Lau01] provided the geometric components of C all have genus at least two. Indeed, over nonclosed fields there may be numerous genus one curves with a given elliptic curve as their Jacobian.

(2) It would be interesting to have explicit examples of rational threefolds X admitting a diagram

$$\begin{array}{ccccc} & & X & & \\ & \overset{\beta}{\swarrow} & & \overset{\beta'}{\searrow} & \\ \mathbb{P}^3 & & & & \mathbb{P}^3 \end{array}$$

where β and β' are blowups along smooth centers over k, satisfying the following:

- the only positive genus centers of β and β' are irreducible genus one curves E and E';
- E and E' are not isomorphic over k;
- $J^1(E) \simeq J^1(E')$ in such a way that the subgroups $\mathbb{Z}[E]$ and $\mathbb{Z}[E']$ in the Weil-Châtelet group coincide cf. [AKW17].

The order of $[E]$ in the Weil-Châtelet group must be at least five. Examples in this vein, with centers K3 surfaces instead of elliptic curves, exist for complex fourfolds [HL18].

(3) If C is a geometrically irreducible curve of genus $g > 1$ then $\mathrm{Pic}^e(C)$ has order dividing $2g-2$, as the canonical divisor gives $\mathrm{Pic}^{2g-2}(C) \simeq J^1(C)$.

6.3. **Complete intersections of two quadrics.** The invariant gives the following extension of Theorem 36 of [HT19].

Theorem 24. *Let $X \subset \mathbb{P}^5$ be a smooth complete intersection of two quadrics over a field $k \subset \mathbb{C}$. Then X is rational over k if and only if X admits a line defined over k.*

We refer the reader to [HT19, §11] for specific situations where there are no lines, e.g., the isotopy classes over $\mathbb{R}$ not admitting lines.

Wittenberg pointed out at the Schiermonnikoog conference that his improvements to the argument of [HT19] also yield this result over arbitrary fields. This is presented in [BW19b], including the case of positive characteristic.

Proof. The reverse implication is classical so we focus on proving that every rational X admits a line.

The behavior of our invariant τ was analyzed by X. Wang in [Wan18, BGW17]:

- $J \simeq J^1(C)$, where C is the genus two curve associated with the pencil of quadrics cutting out X;
- the variety of lines $F_1(X)$ is a principal homogeneous space for $J^1(C)$ satisfying
$$2[F_1(X)] = [\mathrm{Pic}^1(C)].$$

Assuming X is rational, there exists a genus two curve C' blown up in $\mathbb{P}^3 \dashrightarrow X$ such that $F_1(X) \simeq \mathrm{Pic}^e(C')$ for some degree e. However, the Torelli Theorem [Lau01] implies that $C \simeq C'$, whence
$$2[\mathrm{Pic}^e(C)] = [\mathrm{Pic}^1(C)].$$
It follows that $\mathrm{Pic}^1(C)$ and $F_1(X)$ are trivial as principal homogeneous spaces for $J^1(C)$, i.e.,
$$F_1(X)(k) \neq \emptyset.$$
Thus we obtain a line over k. □

The geometry of rational curves on X is a good testing ground for the constructions underlying the formulation of τ:

- $\mathrm{Chow}^2_1(X)$ coincides with $F_1(X)$.

- $\mathrm{Chow}^2_2(X)$ admits two components, $\mathrm{Sym}^2(F_1(X))$ and the variety of conics which is an étale $\mathbb{P}^3$-bundle over C [HT19, §2] – these meet along a Kummer surface bundle over C with fibers realized as 16-nodal quartic surfaces. Both map naturally to the same principal homogeneous space for $J^1(C)$, which may be interpreted as both $2[F_1(X)]$ and $\mathrm{Pic}^1(C)$.
- $\mathrm{Chow}^2_3(X)$ admits three components
  1. $\mathrm{Sym}^3(F_1(X))$;
  2. the product of $F_1(X)$ and the variety of conics;
  3. the variety of rational cubic curves, which carries the structure of a $\mathrm{Gr}(2,4)$-bundle over $F_1(X)$ [HT19, §4.2].
- $\mathrm{Chow}^2_4(X)$ admits a number of components – in addition to those parametrizing reducible curves we have the rational normal quartic curves in X and the codimension-two linear sections of X, both of dimension eight. These together map naturally to the trivial principal homogeneous space for $J^1(C)$ although only the latter obviously admits a rational point.

Remark 25. Kuznetsov proposes [Kuz16, §2.4] [Kuz19] invariants of Fano threefolds X with $J^2(X) \simeq J^1(C)$, relating the derived category $\mathsf{D}^b(X)$ to derived categories of twisted sheaves on C. The Brauer group of C is related to principal homogeneous spaces for $J^1(C)$. It would be interesting to compare this approach with our invariant.

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