BIRATIONAL TYPES OF ALGEBRAIC ORBIFOLDS

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ABSTRACT. We introduce a variant of the birational symbols group of Kontsevich, Pestun, and the second author, and use this to define birational invariants of algebraic orbifolds.

1. INTRODUCTION

Let k be a field of characteristic zero and X a smooth projective variety over k, of dimension n; we require our varieties to be irreducible, but not necessarily geometrically irreducible. The paper [14] introduced the *Burnside group* of varieties

$$\mathrm{Burn}_n = \mathrm{Burn}_{n,k},$$

the free abelian group on isomorphism classes of finitely generated fields of transcendence degree n over k; for such a field K we denote the corresponding generator by $[K]$. To X one associates its class

$$[X] := [k(X)] \in \mathrm{Burn}_n,$$

extended by additivity for smooth projective schemes that are not necessarily irreducible. To

$$U = X \setminus D,$$

the complement to a simple normal crossing divisor

$$D = D_1 \cup \cdots \cup D_\ell,$$

one may also associate a class in Burn_n:

$$[U] := [X] - \sum_{1 \le i \le \ell} [D_i \times \mathbb{P}^1] + \sum_{1 \le i < j \le \ell} [(D_i \cap D_j) \times \mathbb{P}^2] - \dots . \qquad (1.1)$$

This is not only an invariant of the isomorphism type of U, but is a birational invariant in the following sense: $[U] = [U']$ in Burn_n if there exist a quasiprojective variety V and birational projective morphisms

$$V \to U \qquad \text{and} \qquad V \to U'.$$

This formalism was used to establish specialization of rationality.

Now we suppose that X is equipped with a faithful action of a finite abelian group G. Then there is a G-equivariant birational invariant of X introduced in [13], taking its value in a group

$$\mathcal{B}_n(G)$$

which records the normal bundle representation generically along components of the fixed locus X^G.

This paper concerns birational invariants of *orbifolds*. An (algebraic) orbifold is a smooth separated irreducible finite-type Deligne-Mumford stack over k with trivial generic stabilizer. Such a stack has a coarse moduli space [11], separated and of finite type over k. We call the orbifold *quasiprojective* (respectively *projective*) when the coarse moduli space is a quasiprojective (respectively projective) variety (see [15]). For instance, the G-action on X determines a projective orbifold $[X/G]$. The orbifolds in this article are always quasiprojective.

We will introduce a group

$$\overline{\mathrm{Burn}}_n$$

that combines features of the groups Burn_n and $\mathcal{B}_n(G)$. We only carry in $\overline{\mathrm{Burn}}_n$ the information of representations of finite abelian groups, up to automorphisms of those groups. Working with $\overline{\mathrm{Burn}}_n$, we will exhibit a birational invariant of a quasiprojective n-dimensional orbifold.

It suffices to consider finite *abelian* groups thanks to divisorialification [6], a sequence of blow-ups in smooth centers which, when applied to a general orbifold, yields an orbifold with only abelian groups as geometric stabilizer groups. Weak factorization [2], in a functorial form proved in [3], is used to exhibit the desired birational invariance.

In Section 2 we establish a presentation of the Burnside group of varieties with relations that are analogous to the scissors relations, used to define the Grothendieck group of varieties. Section 3 introduces the group $\overline{\mathrm{Burn}}_n$. In Section 4 the class of an algebraic orbifold in $\overline{\mathrm{Burn}}_n$ is defined. Section 5 confirms a connection between invariants of orbifold surfaces and modular curves.

2. Burnside group via scissors relations

Let k be a field of characteristic zero. The Grothendieck group

$$\mathrm{K}_0(\mathrm{Var}_k)$$

may be approached in two ways, as an abelian group generated by the classes of algebraic varieties over k with the classical scissors relations (where it makes no difference if we restrict to just smooth quasiprojective varieties), or via the Bittner presentation [8], which only involves smooth projective varieties. We do not concern ourselves in this article with the further structure of $\mathrm{K}_0(\mathrm{Var}_k)$ as a ring.

In this section we record the observation that the Burnside group Burn_n also admits a description in terms of scissors relations. As mentioned in the Introduction, we only require our varieties to be irreducible (but not necessarily geometrically irreducible).

Lemma 2.1. *Let k be a field of characteristic zero, and let W be a smooth quasiprojective variety over k. For any nonempty open $U \subset W$ there exist divisors $D_1, \ldots, D_\ell$ such that $W \smallsetminus D_1$ is contained in U, and $D_1 \smallsetminus D_2, \ldots, D_{\ell-1} \smallsetminus D_\ell, D_\ell$ are all smooth.*

Proof. Let $Z = W \smallsetminus U$. By [12, Thm. 7], given an embedding of W in projective space, a general hypersurface of sufficiently large degree containing Z defines a divisor D_1 on W whose singular locus D_1^{sing} is contained in Z and does not contain any irreducible component of Z. If D_1 is smooth, then we are done with $\ell = 1$. Otherwise, we have $\dim(D_1^{\mathrm{sing}}) < \dim(Z)$, and we conclude by induction on $\dim(Z)$. $\square$

Proposition 2.2. *Let k be a field of characteristic zero and n a natural number. Then the assignment to $[k(X)]$ of $[X]$ for smooth projective varieties X of dimension n over k defines an isomorphism*

$$\mathrm{Burn}_n \xrightarrow{\sim} \Big(\bigoplus_{[U],\, \dim(U)=n} \mathbb{Z} \cdot [U] \Big) \Big/ \textit{modified-scissors,}$$

where, on the right, we have the quotient of the free abelian group on isomorphism classes of smooth quasiprojective varieties of dimension n over k by the modified scissors relations

$$[U] = [V \times \mathbb{P}^{n-d}] + [U \smallsetminus V]$$

for smooth closed subvarieties $V \subset U$ of dimension $d < n$. The inverse isomorphism is given by the formula (1.1).

Proof. We check that the map from the statement of the proposition is well-defined, i.e., the classes of any pair of birationally equivalent smooth projective n-dimensional varieties are equal modulo the modified scissors relations.

By weak factorization, it suffices to consider the case of X and $B\ell_Y X$, where X is smooth and projective of dimension n and Y is a smooth

subvariety of X of dimension $d < n$. By the modified scissors relations we have

$$[X] = [Y \times \mathbb{P}^{n-d}] + [X \smallsetminus Y],$$
$$[B\ell_Y X] = [\mathbb{P}(N_{Y/X}) \times \mathbb{P}^1] + [X \smallsetminus Y],$$

where $N_{Y/X}$ denotes the normal bundle. We are done if we can show that $[\mathbb{P}(N_{Y/X}) \times \mathbb{P}^1] = [Y \times \mathbb{P}^{n-d}]$. We will show, more generally, that for any smooth quasiprojective variety W of dimension $e < n$ and vector bundle F on W of rank $r \leq n + 1 - e$, we have

$$[\mathbb{P}(F) \times \mathbb{P}^{n+1-e-r}] = [W \times \mathbb{P}^{n-e}]. \tag{2.1}$$

For any smooth quasiprojective variety Z of dimension $n - 1$ we have $[Z \times \mathbb{A}^1] = 0$ (by considering $Z \times \{\infty\} \subset Z \times \mathbb{P}^1$), and hence

$$[W \times \mathbb{P}^{n-e}] = [W \times (\mathbb{P}^1)^{n-e}]$$

(by considering $W \times \mathbb{P}^{n-e-1} \subset W \times \mathbb{P}^{n-e}$). We prove (2.1) by induction on e; the case $e = 0$ is now clear. Let $U \subset W$ be a nonempty open subset on which F is trivial, and D_1, ..., D_ℓ, divisors as in Lemma 2.1. The modified scissors relation and the induction hypothesis lead to

$$\begin{aligned}
[\mathbb{P}(F) \times \mathbb{P}^{n+1-e-r}] = {} & [D_\ell \times \mathbb{P}^{n+1-e}] + [(D_{\ell-1} \smallsetminus D_\ell) \times \mathbb{P}^{n+1-e}] \\
& + \cdots + [(D_1 \smallsetminus (D_2 \cup \cdots \cup D_\ell)) \times \mathbb{P}^{n+1-e}] \\
& + [(W \smallsetminus (D_1 \cup \cdots \cup D_\ell)) \times \mathbb{P}^{n-e}].
\end{aligned}$$

We conclude with the relations, for $1 \leq i \leq \ell$:

$$\begin{aligned}
[(W \smallsetminus (D_{i+1} \cup \cdots \cup D_\ell)) \times \mathbb{P}^{n-e}] = {} & [(D_i \smallsetminus (D_{i+1} \cup \cdots \cup D_\ell)) \times \mathbb{P}^{n+1-e}] \\
& + [(W \smallsetminus (D_i \cup \cdots \cup D_\ell)) \times \mathbb{P}^{n-e}].
\end{aligned}$$

We verify that the map in the reverse direction, given by the formula (1.1), is well-defined, i.e., respects the modified scissors relations. Let V be a smooth closed subvariety of U of dimension d. Then U may be presented as the complement in a smooth projective variety X of a simple normal crossing divisor $D_1 \cup \cdots \cup D_\ell$, with which a smooth subvariety $Y \subset X$ has normal crossing, such that $Y \cap U = V$. We have $[U]$, given by the formula (1.1). For $[V \times \mathbb{P}^{n-d}]$ we have the embedding in $Y \times \mathbb{P}^{n-d}$, complement to the simple normal crossing divisor

$$(D_1 \cap Y) \times \mathbb{P}^{n-d} \cup \cdots \cup (D_\ell \cap Y) \times \mathbb{P}^{n-d},$$

and thus an analogous formula in Burn_n. The blow-up $B\ell_Y X$ has the simple normal crossing divisor $\widetilde{D}_1 \cup \cdots \cup \widetilde{D}_\ell \cup E$, where $\widetilde{D}_i$ denotes the proper transform of D_i, and E, the exceptional divisor, leading to a formula for $[U \smallsetminus V]$ in Burn_n. Comparing formulas and using that any intersection not involving E is birational to the corresponding intersection

in X, while any intersection involving E is birational to the product of an intersection in Y with projective space of the appropriate dimension, we obtain the desired relation.

It is clear that the composite $\mathrm{Burn}_n \to \mathrm{Burn}_n$ of the two maps is the identity. The composite in the other order is seen to be the identity using the modified scissors relations. $\qquad\square$

3. Burnside group for stacks

In this section we introduce the group $\overline{\mathrm{Burn}}_n$.

Definition 3.1. We define the $\mathbb{Z}[t]$-module $\overline{\mathcal{B}}$ by starting with the free $\mathbb{Z}$-module on pairs (A, S) consisting of a finite abelian group A and finite generating system S of A, where the action of t is to append the element 0 to S, and passing to the quotient by the following relations:

- (A, S) and (A, S') are equivalent if S' is a permutation of S.
- (A, S) and (A', S') are equivalent if some isomorphism $A \cong A'$ transforms S to S'.
- (A, S), $S = (a_1, \ldots, a_m)$, is equivalent, for any $2 \le j \le m$, to

$$\sum_{\emptyset \neq I \subset \{1,\ldots,j\}} (-t)^{|I|-1} \Big(A/\langle a_i - a_{i_0} \rangle_{i \in I},$$

$$(\bar{a}_{i_0}, \bar{a}_1 - \bar{a}_{i_0}, \ldots (\text{omitting all } i \in I) \ldots, \bar{a}_j - \bar{a}_{i_0}, \bar{a}_{j+1}, \ldots, \bar{a}_m) \Big),$$

where inside the sum i_0 denotes an element of I, with sequence of elements of $A/\langle a_i - a_{i_0} \rangle_{i \in I}$ of length $1 + (j - |I|) + (m - j)$ that is independent of the choice of i_0.

Example 3.2. When $m = j = 2$, we obtain $(A, (a_1, a_2))$ equivalent to

$$(A, (a_1, a_2 - a_1)) + (A, (a_2, a_1 - a_2)) - t(A/\langle a_1 - a_2 \rangle, (\bar{a}_1)).$$

We let $[A, S]$ denote the class in $\overline{\mathcal{B}}$ of a pair (A, S). We define a grading on $\overline{\mathcal{B}}$ by assigning degree $|S|$ to $[A, S]$:

$$\overline{\mathcal{B}} = \bigoplus_{n=0}^{\infty} \overline{\mathcal{B}}_n.$$

With this grading, $\overline{\mathcal{B}}$ is a graded $\mathbb{Z}[t]$-module, for the natural grading on $\mathbb{Z}[t]$.

Representations determine, via their weights, elements of $\overline{\mathcal{B}}$. If G is a finite diagonalizable group scheme with faithful representation

$$\rho \colon G \to GL_n$$

(over an arbitrary field), then there is a pair (A, S), where A is the Cartier dual group to G and S is the sequence of weights supplied by a

decomposition of ρ as a sum of n one-dimensional linear representations. The element

$$[\rho] := [A, S] \in \overline{\mathcal{B}}_n$$

is canonically determined by ρ.

Restricting to e-torsion groups A for a positive integer e, respectively, to p-primary A for a prime number p, leads to a $\mathbb{Z}[t]$-module $\overline{\mathcal{B}}^{[e]}$, respectively $\overline{\mathcal{B}}^{(p)}$. The evident homomorphisms from these modules to $\overline{\mathcal{B}}$ are split monomorphisms, with splittings given by

$$[A, S] \to [A/eA, S], \qquad \text{respectively,} \qquad [A, S] \to [A(p), S],$$

where $A(p)$ denotes the p-primary subgroup of A. We have

$$\overline{\mathcal{B}} = \bigoplus_p \overline{\mathcal{B}}^{(p)}, \qquad \overline{\mathcal{B}}^{(p)} = \varinjlim_j \overline{\mathcal{B}}^{[p^j]}.$$

Definition 3.3. Let k be a field of characteristic zero and n a natural number. The group

$$\overline{\mathrm{Burn}}_n$$

is the abelian group generated by pairs (K, α), where

- K is a finitely generated field of transcendence degree $d \leq n$ over k and
- $\alpha \in \overline{\mathcal{B}}_{n-d}$,

modulo the identification of $(K(t), \beta)$ and $(K, t\beta)$ for $\beta \in \overline{\mathcal{B}}_{n-d-1}$.

Example 3.4. For $\overline{\mathcal{B}}_2^{[5]}$ we have generators $t^2[0, ()]$, $t[C_5, (1)]$, $[C_5, (1, 1)]$, $[C_5, (1, 2)]$, $[C_5, (1, 4)]$, $[C_5 \oplus C_5, ((1, 0), (0, 1))]$, and relations:

$$t[C_5, (1)] = [C_5, (1, 4)] + t[C_5, (1)] - t^2[0, ()],$$
$$[C_5, (1, 1)] = 2t[C_5, (1)] - t[C_5, (1)],$$
$$[C_5, (1, 2)] = [C_5, (1, 1)] + [C_5, (1, 2)] - t^2[0, ()],$$
$$[C_5, (1, 4)] = 2[C_5, (1, 2)] - t^2[0, ()],$$
$$[C_5 \oplus C_5, ((1, 0), (0, 1))] = 2[C_5 \oplus C_5, ((1, 0), (0, 1))] - t[C_5, (1)],$$

where $C_5 = \mathbb{Z}/5\mathbb{Z}$. We deduce

$$[C_5 \oplus C_5, ((1, 0), (0, 1))] = [C_5, (1, 1)] = [C_5, (1, 4)] = t[C_5, (1)] = t^2[0, ()],$$

with

$$2\big([C_5, (1, 2)] - t^2[0, ()]\big) = 0.$$

Hence $\overline{\mathcal{B}}_2^{[5]} \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

4. BIRATIONAL INVARIANTS OF ORBIFOLDS

In this section we introduce birational invariants of n-dimensional orbifolds over a field k of characteristic zero, taking values in $\overline{\mathrm{Burn}}_n$.

Let $\mathcal{X}$ be an orbifold. We recall from [5] (see also [6]): if $D_1 \cup \cdots \cup D_\ell$ is a simple normal crossing divisor on $\mathcal{X}$, then $\mathcal{X}$ is called *divisorial* with respect to $D_1, \ldots, D_\ell$ if the morphism

$$\mathcal{X} \to B\mathbb{G}_m^\ell,$$

determined by $\mathcal{O}_{\mathcal{X}}(D_i)$, for $i = 1, \ldots, \ell$, is representable; this implies that the stabilizer group schemes of $\mathcal{X}$ are diagonalizable. We will apply this terminology more generally to any finite collection of line bundles.

Divisorialification is a procedure that, when applied to an orbifold $\mathcal{X}$, yields a succession of blow-ups along smooth centers

$$\mathcal{Y} \to \cdots \to \mathcal{X},$$

such that $\mathcal{Y}$ is divisorial with respect to a suitable simple normal crossing divisor. This is given as Algorithm C in [5], initially with a requirement to have abelian geometric stabilizer groups, later with this requirement removed [6].

As explained in the introduction, invariance under birational projective morphisms is the statement of invariance under the equivalence relation of existence of a third object (variety or Deligne-Mumford stack) with birational projective morphisms to two given objects. In this section we are interested in quasiprojective orbifolds $\mathcal{X}$ and $\mathcal{X}'$, and the equivalence takes the form of existence of a Deligne-Mumford stack $\mathcal{Y}$ with birational projective morphisms

$$\mathcal{Y} \to \mathcal{X} \qquad \text{and} \qquad \mathcal{Y} \to \mathcal{X}'. \tag{4.1}$$

We recall that a morphism of stacks is projective if it factors up to 2-isomorphism as a closed immersion followed by projection from a projective bundle $\mathbb{P}(\mathcal{E})$ for some quasi-coherent sheaf $\mathcal{E}$ of finite type; in particular, projective morphisms are always representable.

In the situation (4.1) there is no loss of generality in supposing $\mathcal{Y}$ as well to be an orbifold, since resolution of singularities in a functorial form as in [20] and [7] is applicable to algebraic stacks. When $\mathcal{X}$ and $\mathcal{Y}$ are quasiprojective orbifolds, a morphism $\mathcal{Y} \to \mathcal{X}$ is projective if and only if it is representable and proper. (Every projective morphism is representable and proper. The reverse implication uses that $\mathcal{Y} \to \mathcal{X}$ factors up to 2-isomorphism through $\mathcal{X} \times_X Y$, where X and Y denote the respective coarse moduli spaces, that $\mathcal{X} \to X$ and $\mathcal{Y} \to Y$ induce bijections on geometric points, and that a representable proper morphism inducing a bijection on geometric points is finite, hence projective.)

Theorem 4.1. *Let k be a field of characteristic zero, n a natural number, and $\mathcal{X}$ an n-dimensional quasiprojective orbifold over k. The following recipe, assigning to $\mathcal{X}$ a class $[\mathcal{X}] \in \overline{\mathrm{Burn}}_n$ gives an invariant under birational projective morphisms:*

- *Use divisorialification to replace $\mathcal{X}$ by a quasiprojective orbifold $\mathcal{Y}$ that is divisorial with respect to some finite collection of line bundles.*
- *Stratify $\mathcal{Y}$ by the isomorphism type of the geometric stabilizer group and attach to each component the normal bundle:*

$$\mathcal{Y} = \coprod_G \mathcal{Y}_G, \qquad N_{Y,G} = N_{\mathcal{Y}_G/\mathcal{Y}}.$$

- *Writing the coarse moduli space of $\mathcal{Y}_G$, for each G, as Y_G, we assign the element*

$$[\mathcal{X}] := \sum_G ([Y_G], [N_{Y,G}]) \in \overline{\mathrm{Burn}}_n.$$

In the last step, if Y_G is irreducible of dimension d, then we understand $[Y_G]$ to be the associated element of Burn_d, with $[N_{Y,G}] \in \overline{\mathcal{B}}_{n-d}$ associated to the representation of G at the geometric generic point of $\mathcal{Y}_G$. In general, we understand $([Y_G], [N_{Y,G}])$ to be the sum of the elements of $\overline{\mathrm{Burn}}_n$ attached to the irreducible components.

Proof. Let $\mathcal{X}'$ be a quasiprojective orbifold with birational projective morphism to $\mathcal{X}$. We divisorialize $\mathcal{X}'$ to obtain $\mathcal{Y}'$. The diagram

$$
\begin{array}{ccc}
 & & \mathcal{Y}' \\
 & & \downarrow \\
\mathcal{Y} & \longrightarrow & \mathcal{X}
\end{array}
$$

may be completed to a 2-commutative square of birational projective morphisms of quasiprojective orbifolds by desingularizing the closure in the fiber product of a nonempty open substack where the morphisms are isomorphisms. This way, we are reduced to showing that for a birational projective morphism $\mathcal{Z} \to \mathcal{Y}$ of quasiprojective orbifolds we have

$$\sum_G ([Y_G], [N_{Y,G}]) = \sum_G ([Z_G], [N_{Z,G}]) \in \overline{\mathrm{Burn}}_n. \qquad (4.2)$$

Let $\mathcal{L}_1, \ldots, \mathcal{L}_\ell$ be line bundles, relative to which $\mathcal{Y}$ is divisorial. The functorial form of weak factorization in [3] is applicable to stacks and yields a factorization of $\mathcal{Z} \to \mathcal{Y}$ as a composite of maps *of divisorial projective orbifolds* (with respect to pullbacks of $\mathcal{L}_1, \ldots, \mathcal{L}_\ell$), each equal to or inverse to a blow-up along a smooth center.

Let $\mathcal{V}$ be a smooth closed substack of $\mathcal{Y}$ of dimension $< n$, with coarse moduli space V, and let $\mathcal{Z} = Bl_{\mathcal{V}}\mathcal{Y}$. We verify (4.2) in this case. On the left, we break up $\mathcal{Y}_G$ into the unions of components $\mathcal{Y}'_G$ disjoint from $\mathcal{V}$ and $\mathcal{Y}''$, meeting $\mathcal{V}$ nontrivially, and apply the modified scissors relation to $\mathcal{Y}''_G$:

$$\sum_G ([Y_G], [N_{Y,G}]) = \sum_G ([Y'_G], [N_{Y,G}])$$
$$+ \sum_G ([Y''_G \cap V], t^{\dim(\mathcal{Y}''_G) - \dim(\mathcal{Y}''_G \cap \mathcal{V})}[N_{Y,G}]) + \sum_G ([Y''_G \smallsetminus V], [N_{Y,G}]),$$

where in the second sum on the right, the dimensions are understood to be taken componentwise. Breaking up the sum on the right of (4.2) in a similar fashion, we obtain an expression with identical first and third sums and a second sum that differs from the second sum in the expression above by relations in $\overline{\mathcal{B}}$. $\qquad\square$

Remark 4.2. Over an algebraically closed field of characteristic zero, if we consider orbifold surfaces whose only nontrivial stabilizer groups are of order 5, then the parity of the number of isolated points with C_5-stabilizer and unequal weights not summing to zero remains unchanged under blow-up of points. This observation is reflected in the 2-torsion in $\overline{\mathcal{B}}_2^{[5]}$ obtained in Example 3.4 and the birational invariance in Theorem 4.1. In this context we mention [5, Exa. 4.3], the observation that a single such point with C_5-stabilizer persists under blow-up.

Example 4.3. Functorial destackification [5] of an orbifold provides a sequence of blow-ups along smooth centers and root stack operations along smooth divisors that simplify the stack structure. The root stack operation adds stabilizer μ_n (for some positive integer n) along a divisor [10, §2], [1, App. B], and the outcome of destackification is an orbifold that is obtained from a smooth variety by iterating root stack operations along components of a simple normal crossing divisor. Blow-ups alone are, as noted in Remark 4.2, insufficient to bring a general orbifold into this form. Correspondingly, we may view the quotient $\overline{\mathcal{B}}/\mathcal{C}$, where $\mathcal{C}$ denotes the submodule generated by the classes of pairs

$$(C_{a_1} \oplus \cdots \oplus C_{a_r}, (g_1, \ldots, g_r))$$

of direct sums of finite cyclic groups ($r \geq 0$ arbitrary) and tuples of generators, as an obstruction to destackification with blow-ups alone.

We have

$$\overline{\mathcal{B}}^{[p]} \subset \mathcal{C} \qquad \text{for} \qquad p \in \{2, 3\},$$

since blow-ups suffice for the destackification in these cases [16], [19].

p	$\overline{\mathcal{B}}_2^{[p]}/(\mathcal{C} \cap \overline{\mathcal{B}}_2^{[p]})$	p	$\overline{\mathcal{B}}_2^{[p]}/(\mathcal{C} \cap \overline{\mathcal{B}}_2^{[p]})$	p	$\overline{\mathcal{B}}_2^{[p]}/(\mathcal{C} \cap \overline{\mathcal{B}}_2^{[p]})$
5	$\mathbb{Z}/2\mathbb{Z}$	17	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$	31	$\mathbb{Z}^2$
7	0	19	$\mathbb{Z}$	37	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^2$
11	$\mathbb{Z}$	23	$\mathbb{Z}^2$	41	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^3$
13	$\mathbb{Z}/2\mathbb{Z}$	29	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^2$	43	$\mathbb{Z}^3$

TABLE 1. Isomorphism type of $\overline{\mathcal{B}}_2^{[p]}/(\mathcal{C} \cap \overline{\mathcal{B}}_2^{[p]})$

Table 1 records the isomorphism type of $\overline{\mathcal{B}}_2^{[p]}/(\mathcal{C} \cap \overline{\mathcal{B}}_2^{[p]})$ for some primes $p \geq 5$. The next result confirms the evident pattern.

Proposition 4.4. *For a prime $p \geq 5$ let*

$$g = g(X_0(p))$$

denote the genus of the modular curve, i.e.,

$$g = \begin{cases} \left[\frac{p}{12}\right] \mp 1, & \text{when } p \equiv \pm 1 \bmod 12, \\ \left[\frac{p}{12}\right], & \text{otherwise.} \end{cases}$$

Then

$$\overline{\mathcal{B}}_2^{[p]}/(\mathcal{C} \cap \overline{\mathcal{B}}_2^{[p]}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^g, & \text{if } p \equiv 1 \bmod 4, \\ \mathbb{Z}^g, & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

The proof of Proposition 4.4, based on computations with Manin's modular symbols [17], is given in the next section.

The entry 0 in Table 1 for $p = 7$ indicates that $\overline{\mathcal{B}}_2^{[7]} \subset \mathcal{C}$. In fact, we have $\overline{\mathcal{B}}_3^{[7]} \subset \mathcal{C}$ as well. But we find

$$\overline{\mathcal{B}}_4^{[7]}/(\mathcal{C} \cap \overline{\mathcal{B}}_4^{[7]}) \cong \mathbb{Z}/2\mathbb{Z}.$$

5. Modular symbols and the proof of Proposition 4.4

The equivariant Burnside group introduced in [13] is shown to exhibit a connection with the modular curves $X_1(N)$ for various N. In this section the modular curves

$$X_0(p) = \Gamma_0(p) \backslash \mathbb{H} \cup \{0, \infty\}$$

and the corresponding modular symbols [17] play a role; the connection between 2-dimensional birational geometry and modular curves remains mysterious to us.

Fix a prime $p \geq 5$; we are interested in the abelian group

$$\overline{\mathcal{B}}_2^{[p]}/(\mathcal{C} \cap \overline{\mathcal{B}}_2^{[p]})$$

with generators
$$[C_p, (1, a)], \qquad 2 \le a \le p - 2,$$
and relations

$$[C_p, (1, a)] = [C_p, (1, a^{-1})] \qquad \text{for all } a, \tag{5.1}$$
$$2[C_p, (1, 2)] = 0, \tag{5.2}$$
$$[C_p, (1, 2)] = -[C_p, (1, p - 2)], \tag{5.3}$$
$$[C_p, (1, a)] = [C_p, (1, a - 1)] + [C_p, (1, a^{-1} - 1)] \tag{5.4}$$
$$\text{for } a \in \{3, \dots, \tfrac{p-1}{2}\} \cup \{\tfrac{p+3}{2}, \dots, p - 2\},$$

where a^{-1} denotes the positive integer less than p, inverse to $a \bmod p$. (We have $[C_p, (1, 1)] = t[C_p, (1)] \in \mathcal{C}$ and $[C_p, (1, p - 1)] = t^2[0, ()] \in \mathcal{C}$.)

The modular group
$$\Gamma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \bmod p \right\}$$
has index $p + 1$ in $\mathrm{SL}_2(\mathbb{Z})$, with right coset representatives
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & 0 \\ p - 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$
We let $\Gamma_0(p)$ act in the standard way on the upper half-plane $\mathbb{H}$ and as well on $\mathbb{Q} \cup \{i\infty\}$, the latter with two orbits corresponding to the cusps $0, \infty \in X_0(p)$. Here, 0 corresponds to the set of all $b/d \in \mathbb{Q}$ with $p \nmid d$ and ∞, to the set of $a/c \in \mathbb{Q}$ with $p \mid c$. The real structure on $X_0(p)$ is determined by the standard complex conjugation $\mathbb{H} \to \mathbb{H}$, $z \mapsto -\bar{z}$. It is well known that the real locus of $X_0(p)$ is connected.

With Manin's modular symbols [17], applied to $\Gamma_0(p)$, we get a presentation of $H_1(X_0(p), \mathbb{Z})$ by generators and relations. Proposition 4.4 is established by showing that these relations, together with the additional relations that the sum of any cycle and its complex conjugate is zero, match the presentation (5.1)–(5.4). In fact, we use a simpler set of relations, which yield the homology not of the Riemann surface $X_0(p)$, but rather of the corresponding *orbifold* with stabilizers at elliptic points. The quotient of $\mathbb{H}$ by $\Gamma_0(p)/\{\pm 1\}$ is an orbifold, which we compactify by adding the cusps to obtain
$$X_0(p)_{\mathrm{orb}}.$$

Orbifolds and their topological invariants are explained, for instance, in [18], while a convenient reference for orbifold curves is [4]. However, $H_1(X_0(p)_{\mathrm{orb}}, \mathbb{Z})$ may also be presented directly as the homology of the complement of the elliptic points, modulo the relation that an appropriate multiple of a small loop around an elliptic point is zero.

When $p \equiv 1 \bmod 4$ there is a complex conjugate pair of elliptic points of $X_0(p)_{\mathrm{orb}}$ where the stabilizer (of a representative point of $\mathbb{H}$) has order 2 in $\Gamma_0(p)/\{\pm 1\}$; for each of these, twice a small loop is declared to be zero in homology. When $p \equiv 1 \bmod 3$ there is a complex conjugate pair of elliptic points where the stabilizer has order 3 in $\Gamma_0(p)/\{\pm 1\}$, for which we declare 3 times a small loop to be zero in homology.

We summarize the needed results from [17], modified appropriately to the orbifold setting. We maintain the convention from (5.1)–(5.4) about a and a^{-1} and, when $a \notin \{p-2, (p-1)/2\}$ define positive integers a' and a'' less than p by the requirements

$$a' \equiv -a^{-1} - 1 \bmod p, \qquad a'' \equiv -(a+1)^{-1} \bmod p.$$

Lemma 5.1 ([17, (1.4)]). *A surjective homomorphipsm*

$$\Gamma_0(p) \to H_1(X_0(p)_{\mathrm{orb}}, \mathbb{Z})$$

is defined by sending $\gamma \in \Gamma_0(p)$ to the image

$$\{0, \gamma \cdot 0\}$$

in $X_0(p)$ of a geodesic path in $\mathbb{H} \cup \mathbb{Q}$ from 0 to $\gamma \cdot 0$. The kernel is generated by the commutator subgroup of $\Gamma_0(p)$ and the parabolic elements of $\Gamma_0(p)$.

Lemma 5.2 ([17, (1.5)–(1.9)]). *The abelian group $H_1(X_0(p)_{\mathrm{orb}}, \mathbb{Z})$ is presented by generators*

$$\left\{0, \frac{1}{a}\right\}, \qquad 2 \le a \le p-2,$$

and relations

$$\left\{0, \frac{1}{a}\right\} + \left\{0, \frac{1}{p - a^{-1}}\right\} = 0, \tag{5.5}$$

$$\left\{0, \frac{1}{a}\right\} + \left\{0, \frac{1}{a'}\right\} + \left\{0, \frac{1}{a''}\right\} = 0, \tag{5.6}$$

$$\left\{0, \frac{1}{(p-1)/2}\right\} + \left\{0, \frac{1}{p-2}\right\} = 0. \tag{5.7}$$

Now the proof of Proposition 4.4 combines an algebraic result with topological reasoning.

Lemma 5.3. *An isomorphism*

$$\overline{\mathcal{B}}_2^{[p]}/(\mathcal{C} \cap \overline{\mathcal{B}}_2^{[p]}) \to$$

$$H_1(X_0(p)_{\mathrm{orb}}, \mathbb{Z}) \Big/ \Big\langle \left\{0, \frac{1}{a}\right\} + \left\{0, \frac{1}{p-a}\right\}, \, a \in \{2, \ldots, p-2\} \Big\rangle$$

is given by $[C_p, (1, a)] \mapsto \{0, 1/a\}$ for all a.

Proof. Suppose $2 \leq b \leq (p-3)/2$. We subtract the relations (5.4) corresponding to $a = b + 1$ and $a = p - b$, noticing that the rightmost terms cancel thanks to (5.1), to obtain

$$[C_p, (1, b+1)] - [C_p, (1, p-b)] = [C_p, (1, b)] - [C_p, (1, p-b-1)].$$

Starting from (5.3) we obtain, inductively,

$$[C_p, (1, a)] = -[C_p, (1, p-a)] \tag{5.8}$$

for all a. Using (5.8) and (5.1), we rewrite (5.4) as

$$[C_p, (1, a)] + [C_p, (1, a')] + [C_p, (1, a'')] = 0 \tag{5.9}$$

for $a \notin \{(p-1)/2, p-2\}$. We conclude by matching relations (5.1)–(5.2), (5.8)–(5.9) with (5.5)–(5.7) and the additional relations from the quotient group in the statement of the lemma. $\square$

While $H_1(X_0(p), \mathbb{Z})$ is free of rank $2g$ (where g is the genus of $X_0(p)$), there may be torsion in $H_1(X_0(p)_{\mathrm{orb}}, \mathbb{Z})$:

$$H_1(X_0(p)_{\mathrm{orb}}, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}^{2g}, & \text{if } p \equiv 1 \bmod 12, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{2g}, & \text{if } p \equiv 5 \bmod 12, \\ \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}^{2g}, & \text{if } p \equiv 7 \bmod 12, \\ \mathbb{Z}^{2g}, & \text{if } p \equiv 11 \bmod 12. \end{cases}$$

Complex conjugation acts on $H_1(X_0(p)_{\mathrm{orb}}, \mathbb{Z})$ by

$$\left\{0, \frac{1}{a}\right\} \mapsto \left\{0, \frac{1}{p-a}\right\}.$$

Lemma 5.3 identifies $\overline{\mathcal{B}}_2^{[p]}/(\mathcal{C} \cap \overline{\mathcal{B}}_2^{[p]})$ with the quotient of $H_1(X_0(p)_{\mathrm{orb}}, \mathbb{Z})$ by the elements of the form sum of a cycle and its conjugate.

Complex conjugation acts trivially on $H_1(X_0(p)_{\mathrm{orb}}, \mathbb{Z})_{\mathrm{tors}}$. When $p \equiv 1$ mod 4, intersection number mod 2 with a conjugation-invariant curve joining the order 2 elliptic points splits off $H_1(X_0(p)_{\mathrm{orb}}, \mathbb{Z})[2]$ equivariantly as a direct summand of $H_1(X_0(p)_{\mathrm{orb}}, \mathbb{Z})$. Now $\overline{\mathcal{B}}_2^{[p]}/(\mathcal{C} \cap \overline{\mathcal{B}}_2^{[p]})$ is a direct sum of $\mathbb{Z}/2\mathbb{Z}$ when $p \equiv 1$ mod 4, zero when $p \equiv 3$ mod 4, and the quotient of $H_1(X_0(p), \mathbb{Z})$ by the elements of the form sum of a cycle and its conjugate. The latter is accessed by choosing a conjugation-invariant triangulation of $X_0(p)$ and using spectral sequences relating the equivariant homology of $X_0(p)$ with the group homology of $H_j(X_0(p), \mathbb{Z})$, on the one hand, and the group homology of the groups of j-chains on the other, for $j = 0, 1, 2$; cf. [9, §VII.7]. (All group homology is for the group $\mathbb{Z}/2\mathbb{Z}$, corresponding to complex conjugation.) We omit the details and report

only the outcome:

$$H_i(\mathbb{Z}/2\mathbb{Z}, H_1(X_0(p), \mathbb{Z})) = 0 \qquad \text{for all } i \geq 1,$$

$$H_j^{\mathbb{Z}/2\mathbb{Z}}(X_0(p), \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } j = 0, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^g, & \text{if } j = 1, \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } j \geq 2. \end{cases}$$

The vanishing of $H_1(\mathbb{Z}/2\mathbb{Z}, H_1(X_0(p), \mathbb{Z}))$ has the consequence that the subgroup of $H_1(X_0(p), \mathbb{Z})$ of elements of the form sum of a cycle and its conjugate has torsion-free quotient. Hence the quotient is isomorphic to $\mathbb{Z}^g$.

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