RATIONAL POINTS ON K3 SURFACES AND DERIVED EQUIVALENCE

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The geometry of vector bundles and derived categories on complex K3 surfaces has developed rapidly since Mukai's seminal work [Muk87]. Many foundational questions have been answered,

- the existence of vector bundles and twisted sheaves with prescribed invariants;
- geometric interpretations of isogenies between K3 surfaces [Orl97, Cal00];
- the global Torelli theorem for holomorphic symplectic manifolds [Ver13, Huy12b];
- the analysis of stability conditions and its implications for birational geometry of moduli spaces of vector bundles and more general objects in the derived category [BMT14, BM13, Bri07].

Given the precision and power of these results, it is natural to seek arithmetic applications of this circle of ideas. Questions about zero cycles on K3 surfaces have attracted the attention of Beilinson, Beauville-Voisin [BV04], and Huybrechts [Huy12a].

Our focus in this note is on *rational points* over non-closed fields of arithmetic interest. We seek to relate the notion of derived equivalence to arithmetic problems over various fields. Our guiding questions are:

Question 1. Let X and Y be K3 surfaces, derived equivalent over a field F. Does the existence/density of rational points of X imply the same for Y?

Given $\alpha \in \mathrm{Br}(X)$, let (X, α) denote the twisted K3 surface associated with α: If $\mathcal{P} \to X$ is an étale projective bundle representing α, of relative dimension $r-1$, then $(X, \alpha) = [\mathcal{P}/\,\mathrm{SL}_r]$.

Question 2. Suppose that (X, α) and (Y, β) are derived equivalent over F. Does the existence of a rational point on the former imply the same for the latter?

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Note that an F-rational point of (X, α) corresponds to an $x \in X(F)$ such that $\alpha|x = 0 \in \mathrm{Br}(F)$.

We shall consider these questions for F finite, p-adic, real, and local with algebraically-closed residue field. These will serve as a foundation for studying how the geometry of K3 surfaces interacts with Diophantine questions over local and global fields. We first review general properties of derived equivalence over arbitrary base fields. We then offer examples which illuminate some of the challenges in applying derived category techniques. The case of finite and real fields is presented first—here the picture is well developed. Local fields of equicharacteristic zero are also fairly well understood, at least for K3 surfaces with semistable or other mild reduction. The analogous questions in mixed characteristic remain largely open, but comparison with the geometric case suggests a number of avenues for future investigation.

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1. Generalities on derived equivalence for K3 surfaces

1.1. **Definitions.** Let X and Y denote K3 surfaces over a field F. Let p and q be the projections from $X \times Y$ to X and Y respectively.

Definition 3. Let $\mathcal{E} \in D^b(X \times Y)$ be an element of the bounded derived category, which may be represented by a perfect complex. The *Fourier-Mukai transform* is defined

$$\begin{array}{rcl} \Phi_{\mathcal{E}} : D^b(X) & \rightarrow & D^b(Y) \\ \mathcal{F} & \mapsto & q_*(\mathcal{E} \otimes p^*\mathcal{F}), \end{array}$$

where push-forward and tensor product are the derived operations.

Consider the Mukai lattice of X

$$\tilde{H}(X) = \tilde{H}(X, \mathbb{Z}) := H^0(X, \mathbb{Z})(-1) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})(1),$$

where we apply Tate-twists to get a Hodge structure/Galois module of weight two. Let $(,)$ denote the natural nondegenerate pairing on

$\tilde{H}(X)$. There is an induced homomorphism on the level of cohomology [LO, Sect. 2]:

$$\begin{array}{rcl} \phi_{\mathcal{E}} : \tilde{H}(X) & \to & \tilde{H}(Y) \\ \eta & \mapsto & q_*(\mathrm{ch}(\mathcal{E}) \cup p^*\mathcal{F}\sqrt{\mathrm{Td}_{X\times Y}}). \end{array}$$

Observe that $\phi_{\mathcal{E}}$ is also defined on Hodge structures, de Rham cohomologies, and ℓ-adic cohomologies—and these are all compatible. Note that

$$\phi_{\mathcal{E}}\,\mathrm{ch}(\mathcal{F}) = \mathrm{ch}(\Phi_{\mathcal{E}}(\mathcal{F})).$$

Definition 4. X and Y are *derived equivalent* if there exists an object $\mathcal{E} \in D^b(X \times Y)$ such that

$$\Phi_{\mathcal{E}} : D^b(X) \to D^b(Y)$$

is an equivalence of triangulated categories.

1.2. **Characterizations over the complex numbers.**

Theorem 5. [Orl97, §3] *Let X and Y be K3 surfaces over $\mathbb{C}$, with transcendental cohomology groups*

$$T(X) := \mathrm{Pic}(X)^{\perp} \subset H^2(X,\mathbb{Z}), \quad T(Y) := \mathrm{Pic}(Y)^{\perp} \subset H^2(Y,\mathbb{Z}).$$

The following are equivalent

- *there exists an isomorphism of Hodge-structures $T(X) \simeq T(Y)$;*
- *X and Y are derived equivalent;*
- *Y is isomorphic to a moduli space of stable vector bundles over X, admitting a universal family $\mathcal{E} \to X \times Y$, i.e., $Y = M_v(X)$ for a Mukai vector v such that there exists a Mukai vector w with $(v,w) = 1$.*

See [Orl97, §3.8] for discussion of the third condition. This has been extended to arbitrary fields as follows:

Theorem 6. [LO, Th. 1.1] *Let X and Y be K3 surfaces over an algebraically closed field F of characteristic $\neq 2$. Then the second and third statements are equivalent.*

1.3. **Descending derived equivalence.**

Lemma 7. *Let X and Y be K3 surfaces projective over a field F of characteristic zero. Let v and w be primitive Mukai vectors, invariant under the Galois group $\mathrm{Gal}(\bar{F}/F)$, such that there exists an isomorphism*

$$\iota : M_v(\bar{X}) \xrightarrow{\sim} M_w(\bar{Y})$$

inducing

$$\iota^* : H^2(M_w(\bar{Y}), \mathbb{Z}_\ell) \xrightarrow{\sim} H^2(M_v(\bar{X}), \mathbb{Z}_\ell),$$

compatible with Galois actions. Then

$$M_v(X) \simeq M_w(Y)$$

over F.

Proof. Consider the scheme $\mathrm{Isom}(M_v(\bar{X}), M_w(\bar{Y}))$ parametrizing isomorphisms from $M_v(\bar{X})$ to $M_w(\bar{Y})$; since $M_v(X)$ and $M_w(Y)$ are defined over F, this scheme is defined over F. Since we are working in characteristic zero and K3 surfaces have no infinitesimal automorphisms, Galois-fixed points of $\mathrm{Isom}(M_v(\bar{X}), M_w(\bar{Y}))$ correspond to morphisms between the moduli spaces defined over F.

The Torelli theorem implies that the automorphism group of a K3 surface has a faithful representation in its second cohomology. This holds true for manifolds of K3 type as well—see [Mar10, Prop. 1.9] as well as previous work of Beauville and Kaledin-Verbitsky. Moduli spaces with primitive Mukai vectors are of K3 type, so our assumption on the Galois invariance of ι^* implies that ι is defined over F. □

1.4. **Cycle-theoretic invariants of derived equivalence.**

Proposition 8. *Let X and Y be derived equivalent K3 surfaces over a field F of characteristic $\neq 2$. Then* $\mathrm{Pic}(X)$ *and* $\mathrm{Pic}(Y)$ *are stably isomorphic as* $\mathrm{Gal}(\bar{F}/F)$*-modules, and* $\mathrm{Br}(X)[n] \simeq \mathrm{Br}(Y)[n]$ *provided n is not divisible by the characteristic.*

Even over $\mathbb{C}$, this result does not extend to higher dimensional varieties [Add13].

Proof. The statement on the Picard groups follows from the Chow realization of the Fourier-Mukai transform—see [LO, §2.7] for discussion. The étale realization

$$\phi : \tilde{H}_{\text{ét}}(X, \mu_n) \to \tilde{H}_{\text{ét}}(Y, \mu_n)$$

gives the equality of Brauer groups, after modding out by the images of the cycle class maps. □

Recall that the *index* $\mathrm{ind}(X)$ of a smooth projective variety X over a field F is the greatest common divisor of the degrees of field extensions F'/F over which $X(F') \neq \emptyset$.

Lemma 9. *If (S,h) is a smooth projective surface over F then*

$$\begin{aligned}\operatorname{ind}(S) &= \gcd\{c_2(E) : E \text{ vector bundle on } S\} \\ &= \gcd\{c_2(E) : E \in D^b(S)\}.\end{aligned}$$

Proof. Consider the 'decomposible index'

$$\operatorname{inddec}(S) := \gcd\{D_1 \cdot D_2 : D_1, D_2 \text{ very ample divisors on } S\}$$

which is equal to

$$\gcd\{D_1 \cdot D_2 : D_1, D_2 \text{ divisors on } S\},$$

because for any divisor D the divisor $D + Nh$ is very ample for $N \gg 0$. All three quantities above divide $\operatorname{inddec}(S)$, so we work modulo this quantity.

Given a bounded complex of vector bundles

$$E = \{E_{-n} \to E_{-n+1} \to \cdots \to E_n\}$$

we may define the Chern character

$$\operatorname{ch}(E) = \sum_j (-1)^j \operatorname{ch}(E_j)$$

in Chow groups with $\mathbb{Q}$ coefficients. This yields definitions of the rank and first Chern class of E as alternating sums of the ranks and determinants of the terms, respectively. We then may take

$$c_2(E) = \sum_j (-1)^j c_2(E_j) \pmod{\operatorname{inddec}(S)};$$

this is well defined, because for an exact complex of locally free sheaves the alternating sum of the second Chern classes is trivial modulo products of first Chern classes. This makes sense even with integer coefficients.

By this analysis, the second and third quantities agree. Given a reduced zero-dimensional subscheme $Z \subset S$ we have a resolution

$$0 \to E_{-2} \to E_{-1} \to \mathcal{O}_S \to \mathcal{O}_Z \to 0$$

with E_{-2} and E_{-1} vector bundles. This implies that

$$\gcd\{c_2(E) : E \text{ vector bundle on } S\} \,|\, \operatorname{ind}(S).$$

Conversely, given a vector bundle E there exists a twist $E \otimes \mathcal{O}_S(Nh)$ that is globally generated and

$$c_2(E \otimes \mathcal{O}_S(Nh)) \equiv c_2(E) \pmod{\operatorname{inddec}(S)}.$$

Thus there exists a zero-cycle Z with degree $c_2(E \otimes \mathcal{O}_S(Nh))$ and

$$\operatorname{ind}(S) | \gcd\{c_2(E) : E \text{ vector bundle on } S\}.$$

□

Proposition 10. *If X is a K3 surface over a field F then*

$$\operatorname{ind}(X) | \gcd\{24, D_1 \cdot D_2 \text{ where } D_1, D_2 \text{ are divisors on } X\}.$$

This follows from Lemma 9 and the fact that $c_2(T_X) = 24$. Beauville-Voisin [BV04] and Huybrechts [Huy10] have studied the corresponding subgroup of $\mathrm{CH}_0(X_{\bar{F}})$.

Proposition 11. *Let X and Y be derived equivalent K3 surfaces over a field F. Then* $\operatorname{ind}(X) = \operatorname{ind}(Y)$.

Proof. Using Lemma 7, express $Y = M_v(X)$ for $v = (r, ah, s)$ where h is a polarization on X and $a^2h^2 = 2rs$. Theorem 6 implies there exists a Mukai vector $w = (r', bg, s') \in \tilde{H}^{1,1}(X, \mathbb{Z})$ with

$$(v, w) = abg \cdot h - rs' - sr' = 1.$$

Thus we have

$$\langle r, s \rangle = \langle 1 \rangle \pmod{g \cdot h}.$$

Consider the Fourier-Mukai functor

$$\Phi : D^b(X) \to D^b(Y)$$

and the induced homomorphism ϕ on the Mukai lattice. Note that

$$\phi(v) = (0, 0, 1)$$

reflecting the fact that a point on Y corresponds to a sheaf on X with Mukai vector v.

Suppose that Y has a rational point over a field of degree n over F; let $Z \subset Y$ denote the corresponding subscheme of length n. Applying Φ^{-1} to $\mathcal{O}_Z$ gives an element of the derived category with Mukai vector (nr, nah, ns) and

$$c_2(\Phi^{-1}(\mathcal{O}_Z)) = \frac{c_1(\Phi^{-1}(\mathcal{O}_Z))^2}{2} - \chi(\Phi^{-1}(\mathcal{O}_Z)) + 2\operatorname{rank}(\Phi^{-1}(\mathcal{O}_Z))$$

which equals $n(nrs + r - s)$. Following the proof of Lemma 9, we compute

$$c_2(\Phi^{-1}(\mathcal{O}_Z)) \pmod{\operatorname{inddec}(X)}.$$

First suppose that r and s have different parity, so that

$$\gcd(nrs + r - s, 2rs) = \gcd(nrs + r - s, rs).$$

Then we have

$$\langle nrs + r - s, rs\rangle = \langle r - s, rs\rangle = \langle r, s\rangle^2 = \langle 1\rangle \pmod{g \cdot h}.$$

If they have the same parity then $g \cdot h$ must be odd and

$$\langle nrs + r - s, 2rs\rangle = \langle nrs + r - s, rs\rangle \pmod{g \cdot h}$$

and repeating the argument above gives the desired conclusion. Thus we find

$$c_2(\Phi^{-1}(\mathcal{O}_Z)) \equiv n \pmod{\operatorname{inddec}(X)},$$

whence $\operatorname{ind}(X)|n$. Varying over all degrees n, we find

$$\operatorname{ind}(X)|\operatorname{ind}(Y)$$

and the Proposition follows. □

A *spherical object* on a K3 surface X/F is an element $\mathcal{S} \in D^b(X)$ with

$$\operatorname{Ext}^0(\mathcal{S},\mathcal{S}) = \operatorname{Ext}^2(\mathcal{S},\mathcal{S}) = F, \quad \operatorname{Ext}^i(\mathcal{S},\mathcal{S}) = 0, \quad i \neq 0, 2.$$

These satisfy the following

- $(v(\mathcal{S}), v(\mathcal{S})) = -2$;
- rigid simple vector bundles are spherical;
- each spherical object $\bar{\mathcal{S}}$ on $X_{\bar{F}}$ is defined over a finite extension F'/F [Huy10, 5.4];
- over $\mathbb{C}$, each $v = (r, D, s) \in \tilde{H}(X,\mathbb{Z}) \cap H^{1,1}$ with $(v, v) = -2$ arises from a spherical object, which may be taken to be a rigid vector bundle E if $r > 0$ [Kul89];
- under the same assumptions, for each polarization h on X there is a *unique* h-slope stable vector bundle E with $v(E) = v$ [Huy12a, 5.1.iii].

The last result raises the question of whether spherical objects are defined over the ground field:

Question 12. Let X be a K3 surface over a field F. Suppose that $\bar{\mathcal{S}}$ is a spherical object on $X_{\bar{F}}$ such that $c_1(\bar{\mathcal{S}}) \in \operatorname{Pic}(X_{\bar{F}})$ is a divisor defined over X. When does $\bar{\mathcal{S}}$ come from an object $\mathcal{S}$ on X?

Kuleshov [Kul89, Kul90] gives a partial description of how to generate all exceptional bundles on K3 surfaces of Picard rank one through ‘restructuring’ operations and ‘dragons’. It would be worthwhile to analyze which of these operations could be defined over the ground field.

Example 13. We give an example of a K3 surface X over a field F with

$$\mathrm{Pic}(X) = \mathrm{Pic}(X_{\bar{F}}) = \mathbb{Z}h$$

and a rigid sheaf E over $X_{\bar{F}}$ that fails to descend to F.

Choose (X, h) to be a degree fourteen K3 surface defined over $\mathbb{R}$ with $X(\mathbb{R}) = \emptyset$. This may be constructed as follows: Fix a smooth conic C and quadric threefold Q with

$$C \subset Q \subset \mathbb{P}^4, \quad Q(\mathbb{R}) = \emptyset.$$

Let X' denote a complete intersection of Q with a cubic containing C; we have $X'(\mathbb{R}) = \emptyset$ and X' admits a lattice polarization

$$\begin{array}{c|cc} & g & C \\ \hline g & 6 & 2 \\ C & 2 & -2 \end{array}.$$

Write $h = 2g - C$ so that (X', h) is a degree 14 K3 surface containing a conic. Let X be a small deformation of X' with $\mathrm{Pic}(X_{\mathbb{C}}) = \mathbb{Z}h$.

The K3 surface X is Pfaffian if and only if it admits a vector bundle E with $v(E) = (2, h, 4)$ corresponding to the classifying morphism $X \to \mathrm{Gr}(2, 6)$. However, note that

$$c_2(E) = 5$$

which would mean that $\mathrm{ind}(X) = 1$. On the other hand, if $X(\mathbb{R}) = \emptyset$ then $\mathrm{ind}(X) = 2$.

2. Examples of derived equivalence

2.1. **Elliptic fibrations.**

Proposition 14. *Let F be algebraically closed of characteristic zero. Let $\phi : X \to \mathbb{P}^1$ be an elliptic K3 surface with Jacobian fibration $J(X) \to \mathbb{P}^1$. Let $\alpha \in \mathrm{Br}(J(X))$ denote the Brauer class associated with $[X]$ in the Tate-Shafarevich group of $J(X) \to \mathbb{P}^1$. Then X is derived equivalent to the pair $(J(X), \alpha)$.*

This follows from the proof of Căldăraru's conjecture; see [HS06, 1.vi] as well as [Cal00, 4.4.1] for the fundamental identification between the twisting data and the Tate-Shafarevich group.

Proposition 15. *Let F be of characteristic zero. Let X and Y be K3 surfaces which are derived equivalent over F. Then X is elliptic over*

F if and only if Y is elliptic over F. Moreover, they admit a common Jacobian elliptic K3 surface $J \to \mathbb{P}^1$ such that

$$\mathrm{Pic}^d(X/\mathbb{P}^1) = Y, \quad \mathrm{Pic}^e(Y/\mathbb{P}^1) = X,$$

as principal homogeneous spaces over $J \to \mathbb{P}^1$, for suitable $d, e \in \mathbb{Z}$.

Proof. Let $\bar{J} \to \mathbb{P}^1$ denote the Jacobian of $X \to \mathbb{P}^1$ over $\bar{F}$. Over an algebraically closed field this follows from Proposition 14: $\bar{Y}$ gives rise to a twisted structure over $\bar{J} \to \mathbb{P}^1$ which may be interpreted as an element of the Tate-Shafarevich group. The fact that the images of $\mathrm{Br}(\bar{X}), \mathrm{Br}(\bar{Y})$ in $\mathrm{Br}(\bar{J})$ coincide means that $[\bar{X}]$ and $[\bar{Y}]$ generate the same subgroup in this group. This gives the identifications

$$\mathrm{Pic}^d(\bar{X}/\mathbb{P}^1) = \bar{Y}, \quad \mathrm{Pic}^e(\bar{Y}/\mathbb{P}^1) = \bar{X}.$$

We know that $\bar{J} \to \mathbb{P}^1$ descends to $J \to \mathbb{P}^1$ over F, e.g., as the relative Jacobian of $X \to \mathbb{P}^1$. There is a corresponding fibration $I \to \mathbb{P}^1$ for Y which is derived equivalent to J. However, [HLOY04, Cor. 2.7.3] implies that an elliptic K3 surface with section is unique up to isomorphism in its derived equivalence class, over $\bar{F}$. Lemma 7 implies that J and I are in fact isomorphic over F; the same reasoning shows that the identifications descend to F. □

Corollary 16. *Let X and Y be elliptic K3 surfaces derived equivalent over a field F of characteristic zero. If $X(F) \neq \emptyset$ then $Y(F) \neq \emptyset$. The same holds for Zariski density of rational points.*

The identifications given in Proposition 15 imply that X dominates Y over F, and *vice versa*.

2.2. **Rank one K3 surfaces.** We recall the general picture:

Proposition 17. [Ogu02, Prop. 1.10] *Let $X/\mathbb{C}$ be a K3 surface with $\mathrm{Pic}(X) = \mathbb{Z}h$, where $h^2 = 2n$. Then the number m of isomorphism classes of K3 surfaces Y derived equivalent to X is given by*

$$m = 2^{\tau(n)-1}, \quad \text{where} \quad \tau(n) = \text{number of prime factors of } n.$$

Example 18. The first case where there are multiple isomorphism classes is degree twelve. Let (X, h) be such a K3 surface and $Y = M_{(2,h,3)}(X)$ the moduli space of stable vector bundles $E \to X$ with

$$\mathrm{rk}(E) = 2, \quad c_1(E) = h, \quad \chi(E) = 2 + 3 = 5,$$

whence $c_2(E) = 5$. Note that if $Y(F) \neq \emptyset$ then X admits an effective zero-cycle of degree five and therefore a zero-cycle of degree one. Indeed, if $E \to X$ is a vector bundle corresponding to $[E] \in Y(F)$ then a generic $\sigma \in \Gamma(X, E)$ vanishes at five points on X. As we vary σ, we get a four-parameter family of such cycles. Moreover, the cycle h^2 has degree twelve, relatively prime to five.

Is $X(F) \neq \emptyset$ when $Y(F) \neq \emptyset$?

2.3. **Rank two K3 surfaces.** Exhibiting pairs of non-isomorphic derived equivalent complex K3 surfaces of rank two is a problem on quadratic forms [HLOY04, §3]. Suppose that $\mathrm{Pic}(X_{\mathbb{C}}) = \Pi_X$ and $\mathrm{Pic}(Y_{\mathbb{C}}) = \Pi_Y$ and X and Y are derived equivalent. Orlov's Theorem implies $T(X) \simeq T(Y)$ which means that Π_X and Π_Y have isomorphic discriminant groups/p-adic invariants. Thus we have to exhibit p-adically equivalent rank-two even indefinite lattices that are not equivalent over $\mathbb{Z}$.

Example 19. We are grateful to Sho Tanimoto and Letao Zhang for assistance with this example. Consider the lattices

$$\Pi_X = \begin{array}{c|cc} & C & f \\ \hline C & 2 & 13 \\ f & 13 & 12 \end{array} \qquad \Pi_Y = \begin{array}{c|cc} & D & g \\ \hline D & 8 & 15 \\ g & 15 & 10 \end{array}$$

which both have discriminant 145. Note that Π_X represents -2

$$(2f - C)^2 = (25C - 2f)^2 = -2$$

but that Π_Y fails to represent -2.

Let X be a K3 surface over F with split Picard group Π_X over a field F. We assume that C and f are ample. The moduli space

$$Y = M_{(2,C+f,10)}(X)$$

has Picard group

$$\begin{array}{c|cc} & 2C & (C+f)/2 \\ \hline 2C & 8 & 15 \\ (C+f)/2 & 15 & 10 \end{array} \simeq \Pi_Y$$

while $M_{(2,D,2)}(Y)$ has Picard group

$$\begin{array}{c|cc} & D/2 & 2g \\ \hline D/2 & 2 & 15 \\ 2g & 15 & 40 \end{array} \simeq \Pi_X$$

and is isomorphic to X.

These surfaces have the following properties:

- X and Y admit decomposable zero cycles of degree one over F;

- $X(F) \neq \emptyset$: the rational points arise from the smooth rational curves with classes $2f - C$ and $25C - 2f$, both of which admit zero-cycles of odd degree and thus are $\simeq \mathbb{P}^1$ over F;
- $Y(F')$ is dense for some finite extension F'/F, due to the fact that $|\operatorname{Aut}(Y_{\mathbb{C}})| = \infty$.

We do not know whether

- $X(F')$ is dense for any finite extension F'/F;
- $Y(F) \neq \emptyset$.

3. Finite and real fields

The ℓ-adic interpretation of the Fourier-Mukai transform yields

Theorem 20. [LO] *Let X and Y be K3 surfaces derived equivalent over a finite field F. Then for each finite extension F'/F we have*

$$|X(F')| = |(Y(F')|.$$

We have a similarly complete picture over the real numbers. We review results of Nikulin [Nik79, §3] [Nik08, §2] on real K3 surfaces.

Let X be a K3 surface over $\mathbb{R}$, $X_{\mathbb{C}}$ the corresponding complex K3 surface, and ϕ the action of the anti-holomorphic involution (complex conjugation) of $X_{\mathbb{C}}$ on $H^2(X_{\mathbb{C}}, \mathbb{Z})$. Let $\Lambda_{\pm} \subset H^2(X_{\mathbb{C}}, \mathbb{Z})$ denote the eigenlattices where ϕ acts via ± 1. If D is a divisor on X defined over $\mathbb{R}$ then

$$\phi([D]) = -D;$$

the sign reflects the fact that complex conjugation reverses the sign of $(1,1)$ forms. In Galois-theoretic terms, the cycle class of a divisor lives naturally $H^2(X_{\mathbb{C}}, \mathbb{Z}(1))$ and twisting by -1 accounts for the sign change. Let $\tilde{\Lambda}_{\pm}$ denote the eigenlattices of the Mukai lattice; note that $\tilde{\Lambda}_{-}$ contains the degree zero and four summands. Again, the sign change reflects the fact that these are twisted in the Mukai lattice.

We introduce the key invariants: Let $\mathbf{r}$ denote the rank of Λ_{-}. The discriminant groups of $\Lambda_{\pm}$ are two-elementary groups of order 2^a where a is a non-negative integer. Note that $\tilde{\Lambda}_{\pm}$ have discriminant groups of the same order. Finally, we set

$$\delta_\phi = \begin{cases} 0 \text{ if } (\lambda, \phi(\lambda)) \equiv 0 \pmod 2 \text{ for each } \lambda \in \Lambda \\ 1 \text{ otherwise.} \end{cases}$$

Note that δ_ϕ can be computed via the Mukai lattice

$$\delta_\phi = 0 \text{ iff } (\lambda, \phi(\lambda)) \equiv 0 \pmod 2 \text{ for each } \lambda \in \tilde{\Lambda},$$

as the degree zero and four summands always give even intersections.

We observe the following:

Proposition 21. *Let X and Y be K3 surfaces over $\mathbb{R}$, derived equivalent over $\mathbb{R}$. Then*

$$(\mathbf{r}(X), a(X), \delta_{\phi,X}) = (\mathbf{r}(Y), a(Y), \delta_{\phi,Y}).$$

Proof. The derived equivalence induces an isomorphism

$$\tilde{H}(X_{\mathbb{C}}, \mathbb{Z}) \simeq \tilde{H}(Y_{\mathbb{C}}, \mathbb{Z})$$

compatible with the conjugation actions. Since $(\mathbf{r}, a, \delta_\phi)$ can be read off from the Mukai lattice, the equality follows. ☐

The topological type of a real K3 surface is governed by these invariants. Let Σ_g denote a compact orientable surface of genus g.

Proposition 22. [Nik79, Th. 3.10.6] [Nik08, 2.2] *Let X be a real K3 surface with invariants $(\mathbf{r}, a, \delta_\phi)$. Then the manifold $X(\mathbb{R})$ is orientable and*

$$X(\mathbb{R}) = \begin{cases} \emptyset & \text{if } (\mathbf{r}, a, \delta_\phi) = (10, 10, 0) \\ T_1 \sqcup T_1 & \text{if } (\mathbf{r}, a, \delta_\phi) = (10, 8, 0) \\ T_g \sqcup (T_0)^k & \text{otherwise, where} \\ & \quad g = (22 - \mathbf{r} - a)/2, k = (\mathbf{r} - a)/2 \end{cases}.$$

Corollary 23. *Let X and Y be K3 surfaces defined and derived equivalent over $\mathbb{R}$. Then $X(\mathbb{R})$ and $Y(\mathbb{R})$ are diffeomorphic. In particular, $X(\mathbb{R}) \neq \emptyset$ if and only if $Y(\mathbb{R}) \neq \emptyset$.*

The last statement also follows from Proposition 11: A variety over $\mathbb{R}$ has a real point if and only if its index is one. (This was pointed out to us by Colliot-Thélène.)

Example 24. Let X and Y be derived equivalent K3 surfaces, defined over $\mathbb{R}$; assume they have Picard rank one. Then $Y = M_v(X)$ for some isotropic Mukai vector $v = (r, f, s) \in \tilde{H}(X(\mathbb{C}), \mathbb{Z})$ with $(r, s) = 1$. For a vector bundle E of this type note that

$$c_2(E) = c_1(E)^2/2 + r\chi(\mathcal{O}_X) - \chi(E) = rs + r - s,$$

which is odd as r and s are not both even. Then a global section of E gives an odd-degree cycle on X over $\mathbb{R}$, hence an $\mathbb{R}$-point.

4. Geometric case: local fields with complex residue field

4.1. **Monodromy and models of K3 surfaces.** We assume that $F = \mathbb{C}((t))$ with valuation ring $R = \mathbb{C}[[t]]$; let $\Delta = \operatorname{Spec}(R)$ and $\Delta^\circ = \operatorname{Spec}(F)$.

Let X be a K3 surface over F. Let $T \in \operatorname{Aut}(H^2(X_{\mathbb{C}}, \mathbb{Z}))$ denote its monodromy, which satisfies

$$(T^e - I)^f = 0$$

for some $e, f \in \mathbb{N}$. We take e and f minimal with this property.

The semistable reduction theorem [KKMSD73] implies there exists an integer $n \geq 1$ such that after basechange to

$$R_2 = \mathbb{C}[[t_2]], F_2 = \mathbb{C}((t_2)), \quad t_2^n = t,$$

there exists a flat proper

$$\pi_2 : \mathcal{X}_2 \to \Delta_2 = \operatorname{Spec}(R_2)$$

such that

- the generic fiber is the basechange of X to F_2;
- the central fiber $\pi_2^{-1}(0)$ is a reduced normal crossings divisor.

We call this a *semistable model* for X. It is well-known that semistable reductions have unipotent monodromy so $e|n$.

By work of Kulikov and Persson-Pinkham [Kul77, PP81], there exists a semistable modification of $\mathcal{X}_2$

$$\varpi : \widetilde{\mathcal{X}} \to \Delta_2$$

with trivial canonical class, i.e., there exists a birational map $\varphi : \mathcal{X}_2 \dashrightarrow \widetilde{\mathcal{X}}$ that is an isomorphism away from the central fibers. We call this a *Kulikov model* for X. Furthermore, the structure of the central fiber $\widetilde{\mathcal{X}}_0$ can be described in more detail:

Type I $\widetilde{\mathcal{X}}_0$ is a K3 surface and $f = 1$.

Type II $\widetilde{\mathcal{X}}_0$ is a chain of surfaces glued along elliptic curves, with rational surfaces at the end points and elliptic ruled surfaces in between; here $f = 2$.

Type III $\widetilde{\mathcal{X}}_0$ is a union of rational surfaces and $f = 3$.

We will say more about the Type III case: It determines a combinatorial triangulation of the sphere with vertices indexed by irreducible components, edges indexed by double curves, and 'triangles' indexed by triple points [Mor84]. We analyze this combinatorial structure of $\widetilde{\mathcal{X}}_0$ in terms of the integer m.

Let $\widetilde{\mathcal{X}}_0 = \cup_{i=1}^n V_i$ denote the irreducible components, $\tilde{V}_i$ their normalizations, and $D_{ij} \subset \tilde{V}_i$ the double curves over $V_i \cap V_j$.

Definition 25. $\widetilde{\mathcal{X}}_0$ is in *minus-one* form if for each double curve D_{ij} we have $({D'_{ij}}^2)_{V_i} = -1$ if D'_{ij} is a smooth component of D_{ij} and $(D_{ij}^2)_{V_i} = 1$ if D_{ij} is nodal.

Miranda-Morrison [MM83] have shown that after elementary transformation of $\widetilde{\mathcal{X}}$, we may assume that $\widetilde{\mathcal{X}}$ is in minus-one form.

The following are equivalent [Fri83, §3],[FS85, 0.5,7.1]:

- the logarithm of the monodromy is m times a primitive matrix;
- $\widetilde{\mathcal{X}}_0$ admits a 'special μ_m action', i.e., acting trivially on the sets of components, double/triple points, and Picard groups of the irreducible components;
- $\widetilde{\mathcal{X}}_0$ admits 'special m-bands of hexagons', i.e., the triangulation coming from the components of $\widetilde{\mathcal{X}}_0$ arises as a degree m refinement of another triangulation.

In other words, $\widetilde{\mathcal{X}}_0$ 'looks like' it is obtained from applying semistable reduction to the degree m basechange of a Kulikov model. Its central fiber $\widetilde{\mathcal{X}}'_0$ can readily be described [Fri83, 4.1]—its triangulation is the one with refinement equal to the triangulation of $\widetilde{\mathcal{X}}_0$, and its components are contractions of the corresponding components of $\widetilde{\mathcal{X}}_0$.

For Type II we can do something similar [FS85, 0.3]. After elementary modifications, we may assume the elliptic surfaces are minimal. Then following are equivalent:

- the logarithm of the monodromy is m times a primitive matrix;
- $\widetilde{\mathcal{X}}_0 = V_0 \cup_E \ldots \cup_E V_m$ is a chain of $m+1$ surfaces glued along copies of an elliptic curve E, where V_0 and V_m are rational and $V_1, \ldots, V_{m-1}$ are minimal surfaces ruled over E.

Again $\widetilde{\mathcal{X}}_0$ 'looks like' it is obtained from applying semistable reduction to another Kulikov model with central fiber $\widetilde{\mathcal{X}}'_0 = V_0 \cup_E V_m$.

There are refined Kulikov models taking into account polarizations: Let (X, g) be a polarized K3 surface over F of degree $2d$. Shepherd-Barron [SB83] has shown there exists a Kulikov model $\varpi : \widetilde{\mathcal{X}} \to \Delta_2$ with the following properties:

- there exists a specialization of g to a nef Cartier divisor on the central fiber $\widetilde{\mathcal{X}}_0$;

- g is semi-ample relative to Δ_2, inducing

$$\begin{array}{ccc} \widetilde{\mathcal{X}} & \stackrel{|g|}{\to} & \mathcal{Z} \\ & \searrow \quad \swarrow & \\ & \Delta_2 & \end{array}$$

  where $\widetilde{\mathcal{X}}_0 \to \mathcal{Z}_0$ is birational and $\mathcal{Z}_0$ has rational double points, normal crossings, or singularities with local equations

$$xy = zt = 0.$$

These will be called *quasi-polarized Kulikov models* and their central fibers *admissible degenerations of degree* $2d$.

Recall the construction in sections five and six of [FS85]: Let $\mathcal{D}$ denote the period domain for degree $2d$ K3 surfaces and Γ the corresponding arithmetic group—the orientation-preserving automorphisms of the cohomology lattice $H^2(X,\mathbb{Z})$ fixing g. Fix an admissible degeneration $(\mathcal{Y}_0, g)$ of degree $2d$ and its image $(\mathcal{Z}_0, h)$, with deformation spaces $\mathrm{Def}(\mathcal{Y}_0, g) \to \mathrm{Def}(\mathcal{Z}_0, h)$; the morphism arises because g is semiample over the deformation space. Let

$$\overline{\Gamma\backslash\mathcal{D}}_{N_{\mathcal{Y}_0}} \supset \Gamma\backslash\mathcal{D}$$

denote the partial toroidal compactification parametrizing limiting mixed Hodge structures with monodromy weight filtration given by a nilpotent $N_{\mathcal{Y}_0}$ associated with $\mathcal{Y}_0$ (see [FS85, p.27]). We do keep track of the stack structure. Given a holomorphic mapping

$$f : \{t : 0 < |t| < 1\} \to \Gamma\backslash\mathcal{D},$$

that is locally liftable (lifting locally to $\mathcal{D}$), with unipotent monodromy Γ-conjugate to $N_{\mathcal{Y}_0}$, then f extends to

$$f : \{t : |t| < 1\} \to \overline{\Gamma\backslash\mathcal{D}}.$$

The period map extends to an étale morphism [FS85, 5.3.5,6.2]

$$\mathrm{Def}(\mathcal{Y}_0, g) \to \overline{\Gamma\backslash\mathcal{D}}.$$

Thus the partial compactification admits a (local) universal family.

Proposition 26. *The smallest positive integer n for which we have a Kulikov model equals the smallest positive integer e such that T^e is unipotent.*

Proof. It suffices to show that a Kulikov model exists provided the monodromy is unipotent. Suppose we have unipotent monodromy over $R_1 = \mathrm{Spec}(\mathbb{C}[[t_1]]), t_1^e = t$, and semistable reduction

$$\mathcal{X}_2 \to \mathrm{Spec}(R_2), \quad R_2 = \mathrm{Spec}(\mathbb{C}[[t_2]]), t_2^{me} = t.$$

Let $\widetilde{\mathcal{X}} \to \mathrm{Spec}(R_2)$ denote a Kulikov model, obtained after applying elementary transofrmations as specified above. Write

$$mN = \log(T^e) = (T^e - I) - \frac{1}{2}(T^e - I)^2$$

where $m \in \mathbb{N}$ and N is primitive (cf.[FS85, 1.2] for the Type III case).

Let $\widetilde{\mathcal{X}}_0'$ be the candidate for the 'replacement' Kulikov model, i.e., the central fiber of the Kulikov model we expect to find

$$\widetilde{\mathcal{X}}' \to \mathrm{Spec}(R_1).$$

In the Type I case $\widetilde{\mathcal{X}}_0' = \widetilde{\mathcal{X}}_0$ by Torelli, so we focus on the Type II and III cases.

Lemma 27. *Suppose that $\widetilde{\mathcal{X}}_0$ admits a degree $2d$ semiample divisor g. Then $\widetilde{\mathcal{X}}_0'$ admits one as well, denoted by g'.*

Proof. For Type II, let L_j denote the class of a ruling in V_j, for $j = 1, \ldots, m-1$. Consider the collection of nonnegative numbers

$$(g_1, \ldots, g_{m-1}), \quad g_j = g \cdot L_j.$$

We claim there exist integers $a_0, \ldots, a_{m-1}$ such that

$$\mathcal{O}_{\widetilde{\mathcal{X}}}(g + a_0 V_0 + \ldots + a_{m-1} V_{m-1})$$

remains nef but intersects each L_j trivially. Note that

$$V_i \cdot L_{i+1} = 1, i = 0, \ldots, m-2, \quad V_i \cdot L_{i-1} = 1, i = 2, \ldots, m$$

and also

$$L_i \cdot V_i = -2, i = 1, \ldots, m-1.$$

Thus to intersect L_j trivially we need

$$g_j - 2a_j + a_{j-1} + a_{j+1} = 0, \quad j = 1, \ldots, m-2,$$

and $g_{m-1} - 2a_{m-1} + a_{m-2} = 0$ if $j = m-1$. We first choose $a_2, \ldots, a_{m-1}$ so that

$$(a_2 V_2 + \cdots + a_{m-1} V_{m-1}) \cdot L_j = -g_j, \quad j = 2, \ldots m-1;$$

necessarily we have $a_2, \ldots, a_{m-1} \geq 0$, whence

$$(a_2 V_2 + \cdots + a_{m-1} V_{m-1}) \cdot L_1 \geq 0.$$

Now we choose $a_0 \leq 0$ so that $a_0 + g_1 + a_2 = 0$. It follows that $g + a_0V_0 + \ldots + a_{m-1}V_{m-1}$ is nef: By construction, it meets $L_1, \ldots, L_m$ trivially, and it is nef on V_0 and V_m because the conductor curves on those surfaces are themselves nef. Indeed, they are irreducible curves with self-intersection zero.

For Type III, we rely on Proposition 4.2 of [Fri83], which gives an analogous process for modifying the coefficients of h so that it is trivial or a sum of fibers on the special bands of hexagons. However, Friedman's result does not indicate whether the resulting line bundle is nef. This can be achieved after birational modifications of the total space [SB83, Th. 1]. □

We can apply the Friedman-Scattone compactification construction to both $(\widetilde{\mathcal{X}}_0, g)$ and $(\widetilde{\mathcal{X}}'_0, g')$, with $N = N_{\widetilde{\mathcal{X}}'_0}$ and $mN = N_{\widetilde{\mathcal{X}}_0}$. Thus we obtain *two* compactifications

$$\overline{\Gamma\backslash\mathcal{D}}_{mN} \to \overline{\Gamma\backslash\mathcal{D}}_N \supset \Gamma\backslash\mathcal{D},$$

both with universal families of degree $2d$ K3 surfaces and admissible degenerations.

To construct $\widetilde{\mathcal{X}}' \to \operatorname{Spec}(R_1)$ we use the diagram

$$\begin{array}{ccc} \operatorname{Spec}(R_2) & \to & \overline{\Gamma\backslash\mathcal{D}}_{mN} \\ \downarrow & & \downarrow \\ \operatorname{Spec}(R_1) & & \overline{\Gamma\backslash\mathcal{D}}_N. \end{array}$$

The liftability criterion for mappings to the toriodal compactifications gives an arrow

$$\operatorname{Spec}(R_1) \to \overline{\Gamma\backslash\mathcal{D}}_N$$

making the diagram commute. The induced universal family on this space induces a family

$$\widetilde{\mathcal{X}}' \to \operatorname{Spec}(R_1),$$

agreeing with our original family for $t_1 \neq 0$ by the Torelli Theorem. This is the desired model. □

4.2. **Applications.**

Corollary 28. *Suppose that X and Y are derived equivalent K3 surfaces over $F = \mathbb{C}((t))$. If X admits a Kulikov model then Y admits a Kulikov model with central fiber diffeomorphic to that of X. In particular, both $X(F)$ and $Y(F)$ are nonempty.*

Orlov's Theorem implies they have the same (unipotent!) monodromy so Proposition 26 applies. The last assertion follows from Hensel's Lemma.

Proposition 29. *Suppose that X and Y are derived equivalent K3 surfaces over $F = \mathbb{C}((t))$. Then the following conditions are equivalent:*

- *X (equivalently, Y) has monodromy acting via an element of a product of Weyl groups;*
- *X and Y admit models with central fiber consisting of a K3 surface with ADE singularities.*

Thus both $X(F)$ and $Y(F)$ are nonempty.

Proof. We elaborate on the first condition: Let T denote the monodromy of X. Then there exist vanishing cycles $\gamma_1, \ldots, \gamma_s$ for X such that each $\gamma_i^2 = -2$, $\langle \gamma_1, \ldots, \gamma_s \rangle$ is negative definite, and T is a product of reflections associated with the γ_i. If g is any polarization on X then the γ_i are orthogonal to g. Thus the Fourier-Mukai transform restricts to an isomorphism on the sublattice generated by the γ_i. In particular, the monodromy of Y admits the same interpretation as a product of reflections.

Let L denote the smallest saturated sublattice of $H^2(X, \mathbb{Z})$ containing $\gamma_1, \ldots, \gamma_s$—the classification of Dynkin diagrams implies it is a direct sum of lattices of ADE type. Let M denote the corresponding lattice in $H^2(Y, \mathbb{Z})$, which is isomorphic to L.

After a basechange

$$\mathrm{Spec}(R_1) \to \mathrm{Spec}(R), \quad t_1^e = t$$

where e is the order of T, the Torelli Theorem gives smooth (Type I Kulikov) models

$$\mathcal{X}_1, \mathcal{Y}_1 \to \mathrm{Spec}(R_1)$$

with central fibers having ADE configurations of type M, consisting of smooth rational curves. Blowing these down yield models

$$\mathcal{X}_1', \mathcal{Y}_1' \to \mathrm{Spec}(R_1)$$

which descend to

$$\mathcal{X}, \mathcal{Y} \to \mathrm{Spec}(R),$$

i.e., ADE models of X and Y.

An application of Hensel's Lemma gives that $X(F), Y(F) \neq \emptyset$. □

5. Semistable models over p-adic fields

Let F be a p-adic field with ring of integers R. A K3 surface X over F has *good reduction* if there exists a smooth proper algebraic space $\mathcal{X} \to \operatorname{Spec}(R)$ with generic fiber X. It has *ADE reduction* if the central fiber has just rational double points.

We start with the case of good reduction, which follows from Theorem 6 and Hensel's Lemma:

Corollary 30. *Let X and Y be K3 surfaces over F, with good reduction and derived equivalent over F. Then $X(F) \neq \emptyset$ if and only if $Y(F) \neq \emptyset$.*

We can extend this as follows:

Proposition 31. *Assume that the residue characteristic $p \geq 7$. Let X and Y be K3 surfaces over F, with ADE reduction and derived equivalent over F. Then $X(F) \neq \emptyset$ if and only if $Y(F) \neq \emptyset$.*

Proof. Let k be the finite residue field, $\mathcal{X}, \mathcal{Y} \to \operatorname{Spec}(R)$ proper models of X and Y, $\mathcal{X}_0$ and $\mathcal{Y}_0$ denote the resulting reductions, and $\tilde{\mathcal{X}}_0$ and $\tilde{\mathcal{Y}}_0$ their minimal resolutions over $\bar{k}$. Applying Artin's version of Brieskorn simultaneous resolution [Art74, Th. 2], there exists a finite extension

$$\operatorname{Spec}(R_1) \to \operatorname{Spec}(R)$$

and proper models

$$\tilde{\mathcal{X}} \to \mathcal{X} \times_{\operatorname{Spec}(R)} \operatorname{Spec}(R') \to \operatorname{Spec}(R'),$$

$$\tilde{\mathcal{Y}} \to \mathcal{Y} \times_{\operatorname{Spec}(R)} \operatorname{Spec}(R') \to \operatorname{Spec}(R'),$$

in the category of algebraic spaces, with central fibers $\tilde{\mathcal{X}}_0$ and $\tilde{\mathcal{Y}}_0$.

The Fourier-Mukai transform specializes to give an isomorphism

$$\psi : H^2_{\text{ét}}(\tilde{\mathcal{X}}_0, \mathbb{Q}_\ell) \to H^2_{\text{ét}}(\tilde{\mathcal{Y}}_0, \mathbb{Q}_\ell).$$

Note that since $\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}$ are not projective over $\operatorname{Spec}(R')$, there is not an evident interpretation of this as a derived equivalence over $\operatorname{Spec}(R')$. (See [BM02] for such interpretations for K3 fibrations over complex curves.) Furthermore ψ is far from unique, as we may compose with reflections arising from exceptional curves in either $\tilde{\mathcal{X}}_0 \to \mathcal{X}_0$ or $\tilde{\mathcal{Y}}_0 \to \mathcal{Y}_0$ associated with vanishing cycles of $\mathcal{X}$ or $\mathcal{Y}$.

Let L (resp. M) denote the lattice of vanishing cycles in $H^2(X, \mathbb{Q}_\ell)$ (resp. $H^2(Y, \mathbb{Q}_\ell)$), with orthogonal complement $L^\perp$ (resp.$M^\perp$). The isomorphism ψ does induce a *canonical* isomorphism

$$L^\perp \simeq M^\perp$$

compatible with Galois actions. As in the proof of Proposition 29, the lattices L and M are isomorphic once we fix an interpretation via vanishing cycles of our models.

Our assumption on p guarantee that the classification and deformations of rational double points over k coincides with the classification in characteristic 0 [Art77]. Choose new *regular* models for X and Y

$$\mathcal{X}'', \mathcal{Y}'' \to \operatorname{Spec}(R)$$

whose central fibers $\mathcal{X}_0''$ and $\mathcal{Y}_0''$ are obtained from $\tilde{\mathcal{X}}_0$ and $\tilde{\mathcal{Y}}_0$ by blowing down the (-2)-curves classes associated with L and M respectively. Let $\mathcal{X}_\circ \subset \mathcal{X}_0''$ and $\mathcal{Y}_\circ \subset \mathcal{Y}_0''$ denote the smooth loci, i.e., the complements of the rational curves associated with L and M respectively.

We claim that ψ induces an isomorphism on compactly supported cohomology

$$H^2_{c,\text{ét}}(\bar{\mathcal{X}}_\circ, \mathbb{Q}_\ell) \simeq H^2_{c,\text{ét}}(\bar{\mathcal{Y}}_\circ, \mathbb{Q}_\ell),$$

compatible with Galois actions. Indeed, these may be identified with $L^\perp$ and $M^\perp$, respectively. The Weil conjectures yield then that

$$|\mathcal{X}_\circ(k)| = |\mathcal{Y}_\circ(k)|$$

and Hensel's Lemma implies our claim. □

Question 32. Is admitting a model with good or ADE reduction a derived invariant?

Y. Matsumoto [Mat] has recently shown that having *potentially* good reduction is governed by whether $H^2_{\text{ét}}(\bar{X}, \mathbb{Q}_\ell)$ is unramified, under some technical hypotheses. This condition depends only on the ℓ-adic cohomology and thus depends only on the derived equivalence class. Proposition 29 suggests a monodromy characterization of ADE reduction in the mixed characteristic case.

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