# ON THE MODULI OF DEGREE 4 DEL PEZZO SURFACES

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*To Professor Mukai, with admiration*

ABSTRACT. We study irreducibility of families of degree 4 Del Pezzo surface fibrations over curves.

## 1. INTRODUCTION

Let $X \subset \mathbb{P}^4$ be a smooth surface defined by the intersection of two quadrics over an algebraically closed field $k$ of characteristic different from 2. It is known that $X$ is characterized up to isomorphism by the degeneracy locus of the pencil of quadrics containing $X$, i.e., by the form

$$(1.1) \qquad f(u, v) = \det(uP + vQ),$$

where $P$ and $Q$ are symmetric $5{\times}5$ matrices whose associated quadratic forms define $X$. The two-dimensional space of binary quintic forms with nonvanishing discriminant up to linear change of variable serves as a moduli space of smooth Del Pezzo surfaces of degree 4. This space compactifies to weighted projective space $\mathbb{P}(1, 2, 3)$. Mabuchi and Mukai [29] studied this compactification from the perspective of Geometric Invariant Theory.

Any (nontrivial) family $\mathcal{X} \to \mathbb{P}^1$ of degree 4 Del Pezzo surfaces necessarily contains singular fibers. Generically, these are Del Pezzo surfaces with a single $\mathsf{A}_1$-singularity. So a study of families of degree 4 Del Pezzo surfaces necessarily entails a moduli problem which admits Del Pezzo surfaces with one $\mathsf{A}_1$-singularity.

A general smooth Del Pezzo surface of degree 4 has automorphism group $(\mathbb{Z}/2\mathbb{Z})^4$. A notable incongruity with standard moduli problems such as stable curves is that automorphism groups can *decrease*, rather than increase, upon specialization. Indeed, the general degree 4 Del

Pezzo surface with an $A_1$-singularity has only 8 automorphisms [22]. This reflects the failure of the underlying moduli space to be *separated*.

In this paper we compare several approaches to the moduli problem of degree 4 Del Pezzo surfaces. One involves the moduli problem of log general type surfaces, as worked out by Hacking, Keel, and Tevelev [20], where the nonseparatedness disappears. A second involves *spectral covers*: Given a family of Del Pezzo surfaces of degree 4 over a base $T$, the fiberwise vanishing locus of the form (1.1) determines a degree 5 cover over $T$. Assuming the family is sufficiently general, the singular members of the pencils of quadrics are of nodal type, hence contain two families of planes. This defines a double cover of the spectral cover. In the case of a family over $\mathbb{P}^1$, we get

$$(1.2) \qquad\qquad \tilde{D} \to D \to \mathbb{P}^1.$$

Our approach is to describe families of Del Pezzo surfaces of degree 4 over $\mathbb{P}^1$ in terms of the spectral curve $D$ together with tower (1.2), and to relate the moduli problem for families to the moduli problem for such towers. The machinery of moduli of log general type surfaces is used to show that every such tower does indeed come from a family of Del Pezzo surfaces of degree 4. Our main result (Theorem 10.2) is an explicit description of families of degree 4 Del Pezzo surfaces over $\mathbb{P}^1$, general in the sense of having smooth total space, fibers with at most one $A_1$-singularity, and maximal monodromy of the lines in smooth fibers. A single discrete invariant, the *height*, is proportional to the number of singular fibers. We establish the irreducibility of the space of general families of given height, with exceptional behavior for a few small heights.

In Section 2, we recall basic properties of Del Pezzo surfaces of degree 4 and their spectral covers. In particular, the spectral curve in (1.2) comes embedded in a Hirzebruch surface $\mathbb{F} \to \mathbb{P}^1$. It is then relevant to understand the monodromy of 2-torsion of the Jacobian of curves in Hirzebruch surfaces, which is described in Section 3. Section 4 contains a classical treatment of degree 4 Del Pezzo surfaces containing a line disjoint from the singular locus, an ingredient in the comparison of moduli problems. We discuss properties of binary quintic forms in Section 5; the connection between binary quintics and degree 4 Del Pezzo surfaces allows us to compute invariants of their moduli. In Section 6 we begin the study of moduli of degree 4 Del Pezzo surfaces, their invariants, and relations with binary quintics. Section 7 recalls the moduli problem of log general type surfaces, which is related to

our moduli problem in Section 8. We introduce genericity conditions on families of degree 4 Del Pezzo surfaces in Section 9. In Section 10 we state and prove our main theorems, describing and enumerating the components of general families of degree 4 Del Pezzo surfaces over $\mathbb{P}^1$. In an Appendix, we show that the discrete invariant of families introduced here agrees with the height defined by the first and third authors in [21].

## 2. Degree 4 Del Pezzo surfaces and spectral covers

We work over a perfect base field $k$.

A Del Pezzo surface is a smooth projective surface (always assumed geometrically integral) $X$ over $k$ with ample anticanonical line bundle $\omega_X$. The degree of $X$ is the self-intersection of the anticanonical class. A Del Pezzo surface of degree 4 is embedded by the anticanonical linear system as a complete intersection of two quadrics in $\mathbb{P}^4$, and is geometrically isomorphic to the blow-up of $\mathbb{P}^2$ at 5 points in general position (i.e., no three on a line). Geometrically, the curves with self-intersection $-1$ are the 16 lines on such a surface, the Picard group has rank 6, and the primitive Picard group (i.e., the subgroup orthogonal to the anticanonical class) is a root lattice of type $\mathsf{D}_5$. In particular, all Galois symmetries factor through the Weyl group $W(\mathsf{D}_5)$.

Singular Del Pezzo surfaces, i.e., normal projective surfaces with rational double points (ADE-singularities) and ample anticanonical class, are extensively studied, e.g., [9, 12, 25]. Such a surface $X$ has a minimal resolution $\widetilde{X}$. The anticanonical linear system (or a suitable multiple) induces the morphism $\widetilde{X} \to X$.

Now assume the characteristic of $k$ is different from 2. When $X$ is geometrically a Del Pezzo surface of degree 4 with one $\mathsf{A}_1$-singularity, $\widetilde{X}$ has one curve with self-intersection $-2$, contracted under the morphism $\widetilde{X} \to X \subset \mathbb{P}^4$, and has, geometrically, 12 curves with self-intersection $-1$. Unlike the smooth case, the pencil of quadrics containing $X$ has a distinguished member, appearing with multiplicity 2 in the degeneracy locus.

Let $T$ be a $k$-scheme of finite type. Any flat family of possibly singular degree 4 Del Pezzo surfaces $\pi\colon \mathcal{X} \to T$ gives rise to a degree 5 cover $D \to T$ which encapsulates the degeneracy loci of the pencils of quadrics associated with the fibers of $\pi$, as follows. The relative anticanonical line bundle $\omega_\pi^{-1}$ is ample and induces a closed immersion

$$\mathcal{X} \to \mathbb{P}((\pi_*\omega_\pi^{-1})^\vee)$$

over $T$; we let $\pi$ also denote projection $\mathbb{P}((\pi_*\omega_\pi^{-1})^\vee) \to T$. The composition

$$(\wedge^5(\pi_*\omega_\pi^{-1})^\vee)^{\otimes 2} \overset{\det}{\to} \mathrm{Sym}^5(\mathrm{Sym}^2((\pi_*\omega_\pi^{-1})^\vee))$$

$$(2.1) \qquad\qquad \cong \mathrm{Sym}^5((\mathrm{Sym}^2\pi_*\omega_\pi^{-1})^\vee) \to \mathrm{Sym}^5((\pi_*\mathcal{I}_\mathcal{X}(2))^\vee)$$

gives rise to an ideal sheaf on $\mathbb{P}(\pi_*(\mathcal{I}_\mathcal{X}(2)))$ and hence the *spectral cover*

$$D \subset \mathbb{P}(\pi_*(\mathcal{I}_\mathcal{X}(2))) \to T.$$

When $T = \mathbb{P}^1$ (the main case of interest in this paper), $\mathbb{P}(\pi_*(\mathcal{I}_\mathcal{X}(2)))$ is a Hirzebruch surface. In fact, for general families over $\mathbb{P}^1$ in a sense made precise below (see Theorem 10.2), the Hirzebruch surface is always $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$.

## 3. Curves and their monodromy

In this section, we study monodromy groups of some families of curves in Hirzebruch surfaces; these results are a key ingredient of Theorem 10.2. We assume the base field $k$ is algebraically closed of characteristic different from 2.

Our definition of monodromy depends on the context. Given a smooth projective variety $D$, we write $H^1(D, \mathbb{Z}/2\mathbb{Z})$ for either singular cohomology of the associated complex variety (in characteristic zero, where we may embed the field of definition into $\mathbb{C}$) or étale cohomology (in characteristic $p > 0$).

**Lemma 3.1.** *Let $D$ be a smooth curve on $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(a, b)$, with $a \geq b \geq 3$, respectively a smooth curve on the Hirzebruch surface $\mathbb{F}_1$ in the class $af + b\xi$, with $a > b \geq 3$, where $f$ denotes the class of a fiber of $\mathbb{F}_1 \to \mathbb{P}^1$ and $\xi$ denotes the class of the $(-1)$-curve. The monodromy action on $H^1(D, \mathbb{Z}/2\mathbb{Z})$ of the space of smooth curves in the same curve class as $D$ is the full symplectic group $\mathrm{Sp}(H^1(D, \mathbb{Z}/2\mathbb{Z}))$, in each of the following cases:*

   (i) *$D \subset \mathbb{P}^1 \times \mathbb{P}^1$ with $a$ or $b$ odd;*
   (ii) *$D \subset \mathbb{F}_1$ with $a$ even or $b$ odd.*

**Remark 3.2.** In characteristic zero, the argument also yields that the monodromy on integral cohomology is the full symplectic group.

*Proof.* We recall that Hirzebruch surfaces, being smooth complete toric varieties, have the property [10, Thm. 6.1.15] that every ample line bundle is very ample. In each case the class of $D$ is very ample. Hence the discriminant hypersurface in $|D|$ is irreducible. The general point of the discriminant hypersurface in $|D|$ corresponds to a curve with a single node, and the general pencil in $|D|$ is a Lefschetz pencil; see, e.g., [27] for a treatment in characteristic 0 and [11, Exp. XVII 3.3,3.5.0] for the general case.

Lefschetz theorems [15, p. 151],[11, Exp. XVIII 6.1.6] imply the full monodromy group of the linear series is generated by this pencil. The monodromy of the pencil contains symplectic reflections by vanishing cycles associated with each nodal fiber. These are conjugate under the full monodromy [11, Exp. XVIII 6.1], as the curves in $|D|$ with precisely one node form a connected set.

The method outlined by Beauville in [7] is applicable, provided that we verify:

- $D$ may be degenerated to acquire an $\mathsf{E}_6$-singularity;
- $D$ degenerates to a union $D' \cup D''$ of two smooth curves meeting transversally in an odd number of points.

In characteristics $\neq 2, 3$ an $\mathsf{E}_6$ singularity is characterized by the normal form $x^3 + y^4$; in characteristic 3 there is a second normal form $x^3 + y^4 + x^2y^2$ [5]. In any event, the associated surface singularity $z^2 = x^3 + y^4$ (or $z^2 = x^3 + y^4 + x^2y^2$) has vanishing cycles corresponding to the negative definite $\mathsf{E}_6$ lattice. This governs the $\mathbb{Z}/2\mathbb{Z}$ intersection properties of the vanishing cycles of our curve singularity, which is the key to applying the method of [7, Thm. 3].

We analyze the monodromy representation mod 2. (In characteristic zero, squares of symplectic reflections generate the kernel of

$$\mathrm{Sp}(H^1(D, \mathbb{Z})) \to \mathrm{Sp}(H^1(D, \mathbb{Z}/2\mathbb{Z})),$$

so the remark follows from the mod 2 result.) The first assumption and the conjugacy of the vanishing cycles imply that the monodromy contains $W(\mathsf{E}_6)$, as it contains all the reflections associated with simple roots. It is thus either $\mathrm{Sp}(H^1(D, \mathbb{Z}/2\mathbb{Z}))$ or a subgroup $O(q) \subset \mathrm{Sp}(H^1(D, \mathbb{Z}/2\mathbb{Z}))$ preserving a quadratic form $q$ with $q(\delta) = 1$ for each vanishing cycle $\delta$ [7, Th. 3]. The monodromy can factor through such a subgroup, e.g., when $a$ and $b$ are both even, in case (i). The second assumption precludes this: Given a smoothing of $D' \cup D''$

in $|D|$, fix vanishing cycles $\delta_1, \ldots, \delta_{2p+1}$ indexed by the nodes; the sum $\sum_{j=1}^{2p+1} \delta_j$ is homologous to zero. Since $q(\sum_j \delta_j) = 0$ we must have $q(\delta_j) = 0$ for some $j$, a contradiction.

To verify the $\mathsf{E}_6$ condition, we use the plane quartic $C = \{y^3 = x^4\}$, which has a singularity of this type. The image of $C$ under the linear system of quadrics through two general points of $C$ gives a nodal curve in $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(3,3)$ with the same singularity. Adding appropriate fibers gives curves with all desired bidegrees. Similarly, blowing up a generic point of $C$ gives a curve in $\mathbb{F}_1$ with the desired singularity and class $4f + 3\xi$. Adding lines (with class $\xi + f$) and fibers gives the classes we seek. The final condition can be checked case by case, e.g., for $\mathbb{F}_1$ with $a$ even consider $[D'] = f + \xi$ and $[D''] = (a-1)f + (b-1)\xi$, so $[D'] \cdot [D''] = a - 1$. $\qquad\square$

**Example 3.3.** Consider hyperelliptic curves of genus $g > 1$. Note that the general such curve—with the datum of a line bundle of degree $g + 1$—arises as a curve of bidegree $(g+1, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. The monodromy of such curves has been studied by A'Campo [2]: It is an explicit subgroup $\Gamma \subset \mathrm{Sp}(H^1(D, \mathbb{Z})) = \mathrm{Sp}_{2g}(\mathbb{Z})$, where

$$\Gamma = \{\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z}) : \gamma(\mathrm{mod}\ 2) \in \mathfrak{S}_{2g+2} \subset \mathrm{Sp}_{2g}(\mathbb{Z}/2\mathbb{Z})\}.$$

The symmetric group comes from the monodromy action on the branch points $r_1, \ldots, r_{2g+2}$ of the degree-two map $D \to \mathbb{P}^1$. Indeed, any two-torsion point of the Jacobian $\eta \in \mathrm{J}(D)[2]$ admits a unique expression

$$\eta = \sum_{j \in S} r_j - n g_2^1, \quad S \subset \{1, \ldots, 2g + 2\}, |S| = 2n, 0 \le n \le g/2.$$

The monodromy representation on $\mathrm{J}(D)[2]$ for $D$ hyperelliptic therefore factors through the permutation representation on even subsets of the branch points. The orbits of $\mathrm{J}(D)[2]$ correspond to integers $n = 0, 1, \ldots, \lfloor g/2 \rfloor$.

## 4. Nonsingular lines on degree 4 Del Pezzo surfaces

We summarize the results of this section: Assume the base field $k$ is perfect with characteristic different from 2. Let $X \subset \mathbb{P}^4$ be a complete intersection of two quadrics which is normal and contains a line $L$ disjoint from the singular locus of $X$. We will call such $L$ a *nonsingular line*. Projection from $L$ identifies $X$ with the blow-up of the projective plane along a degree 5 subscheme $\Xi$ of a smooth conic $B$, making $X$ a degree 4 Del Pezzo surface with restricted singularities. The conic, which is the image of $L$ under the projection, is canonically

identified with the pencil of quadric hypersurfaces containing $X$, so that the locus of singular members of the pencil (which carries a natural scheme structure) corresponds to $\Xi$.

Let us write $\mathbb{P}^4 = \mathbb{P}(k^5)$ and $L = \mathbb{P}(V)$ with $V \subset k^5$ a subspace of dimension 2.

**Proposition 4.1.** *With the above notation, the morphism $L \to \mathbb{P}(k^5/V)$, sending $p \in L$ to the tangent plane $T_pX$, is an isomorphism onto a conic $B \subset \mathbb{P}(k^5/V)$.*

*Proof.* Under the morphism $L \to \mathbb{P}(k^5/V) \cong \mathbb{P}^2$, the tautological rank two quotient bundle $\mathcal{O}_{\mathbb{P}^2}^3 \to \mathcal{Q}$ pulls back to $\mathcal{O}_L^3 \to \mathcal{O}_L(1)^2$ given by a $2 \times 3$ matrix of linear forms. The morphism is therefore a closed immersion of degree 2. $\square$

Projection from $L$ is a morphism

$$\psi \colon X \to \mathbb{P}(k^5/V) \cong \mathbb{P}^2$$

sending $p \in X \setminus L$ to the linear span of $L$ and $p$, and sending $p \in L$ to $T_pX$. A general hyperplane in $\mathbb{P}^4$ containing $L$ intersects $X$ in the union of $L$ and a residual cubic curve having intersection number 2 with $L$. The residual cubic curve belongs to the linear system $\mathbb{P}(H^0(X, \psi^*\mathcal{O}_{\mathbb{P}^2}(1)))$. Its self-intersection number is 1, hence the morphism $\psi$ is *birational*.

Under $\psi$, a general member $D$ of the linear system $\mathbb{P}(H^0(X, \mathcal{O}_X(1)))$ maps to an irreducible cubic curve $C \subset \mathbb{P}^2$, and the image linear system is spanned by:

- $B \cup \ell$, $B \cup \ell'$, $B \cup \ell''$ where $\ell$, $\ell'$, $\ell''$ span $\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)))$;
- an irreducible cubic curve $C$ as above;
- another such irreducible cubic curve $C'$, with $B \cap C' \neq B \cap C$.

Let $\Xi = B \cap C \cap C'$ be the base locus of the linear system; comparing self-intersection numbers of $D$ and $C$ we see that $\deg(\Xi) = 5$, and the linear system determines a morphism

$$(4.1) \qquad\qquad \rho \colon \mathrm{Bl}_\Xi(\mathbb{P}^2) \to X.$$

It is known that $\mathrm{Bl}_\Xi(\mathbb{P}^2)$ is normal, and the fiber over a geometric point of $\Xi$ is a copy of $\mathbb{P}^1$ mapping to a line in $X$. It follows that $\rho$ is birational and finite, hence by Zariski's main theorem is an isomorphism.

Let $Q$ be a quadric hypersurface containing $X$. To $Q$ there is an associated symmetric bilinear form on $k^5$ (defined up to scalar multiplication). Since $L$ avoids the singular locus of $Q$, this is a bilinear form

of rank $\geq 4$, and $\mathbb{P}(V^\perp)$ is a plane containing $L$; we therefore have a morphism
$$\varphi \colon \mathbb{P}(H^0(\mathbb{P}^4, \mathcal{I}_X(2))) \to \mathbb{P}(k^5/V).$$

**Lemma 4.2.** *For any plane $\Pi \subset \mathbb{P}^4$ containing $L$ exactly one of the following statements is true:*

(i) *$\Pi \notin B$, and as schemes we have $\Pi \cap X = L \cup \{p\}$ for a point $p \in X$ not in $L$ nor in any line $L' \subset X$ satisfying $L \cap L' \neq \emptyset$;*

(ii) *We have $\Pi \in B$, i.e., $\Pi = T_p X$ for a unique $p \in L$, and the scheme $\Pi \cap X$ is irreducible, has $L$ as reduced subscheme, and has a unique embedded point, located at $p$;*

(iii) *$\Pi \in B$, and $\Pi \cap X = L \cup L'$ where $L' \subset X$ is a line satisfying $L \cap L' = \{p\}$, with $\Pi = T_p X$.*

*Proof.* If the pencil of quadric hypersurfaces containing $X$ restricts to a pencil of conics in $\Pi$, then it decomposes as $L$ plus a residual pencil of lines with a base point $p$. If $p \notin L$ then we are in case (i), with $\Pi$ the linear span of $L$ and $p$ and no line on $X$ through $p$ meeting $L'$. If $p \in L$ then we are in case (ii). Otherwise some member of the pencil of quadric hypersurfaces contains $\Pi$, and for any other member $Q$ of the pencil we have $\Pi \cap X = \Pi \cap Q$ of degree 2 in $\Pi$ and containing $L$. We exclude $L$ having multiplicity 2 by Proposition 4.1, and obtain therefore $L \cup L'$ as in case (iii). $\qquad\square$

By Lemma 4.2, the morphism $\varphi$ factors through $B$. Since by a Chern class computation the morphism $\varphi$ has degree 2, the morphism $\varphi$ determines an *isomorphism*

(4.2) $$\mathbb{P}(H^0(\mathbb{P}^4, \mathcal{I}_X(2))) \cong L.$$

The morphism $\varphi$ sends singular members of the pencil of quadric hypersurfaces to $\Xi$ and nonsingular members to $B \smallsetminus \Xi$.

**Theorem 4.3.** *Let $k$ be a perfect field of characteristic different from 2, and let $X \subset \mathbb{P}^4 = \mathbb{P}(k^5)$ be a Del Pezzo surface of degree 4 containing a nonsingular line $L = \mathbb{P}(V)$; projection from $L$ identifies $X \cong \mathrm{Bl}_\Xi(\mathbb{P}(k^5/V))$ with $\Xi \subset B = \{T_p X | p \in L\}$ uniquely determined of degree 5. Then the isomorphism $\mathbb{P}(H^0(\mathbb{P}^4, \mathcal{I}_X(2))) \cong B$, sending a quadric hypersurface $Q \supset X$ to the plane $\mathbb{P}(V^\perp)$ with $V^\perp$ the orthogonal space to $V$ under a symmetric bilinear form corresponding to $Q$, identifies the scheme of singular members of the pencil of quadric hypersurfaces containing $X$ (defined by the vanishing of a determinant in an evident fashion) with $\Xi$.*

When $X$ is smooth the isomorphism (4.2) and statement of Theorem 4.3 are classical; see [32]. For singular $X$, projection from nonsingular lines appears as a key ingredient already in Segre's classical treatment [31].

*Proof of Theorem 4.3.* Let $U$ be an irreducible smooth affine variety with $k$-point $u \in U$, $\pi\colon \mathcal{X} \to U$ a generically smooth family of Del Pezzo surfaces of degree 4, and $\mathcal{L} \subset \mathcal{X}$ a family of lines; fix an identification $\mathcal{X}_u = \pi^{-1}(u) \cong X$ sending $\mathcal{L}_u$ to $L$. There are versions for families of the identifications $\mathrm{Bl}_\Xi(\mathbb{P}(k^5/V)) \cong X$ and $\mathbb{P}(H^0(\mathbb{P}^4, \mathcal{I}_X(2))) \cong L$ of (4.1) and (4.2) respectively. So we have subschemes of $\mathcal{L}$ finite and flat over $U$, corresponding to the spectral cover and the center of the blow-up, respectively. Since they agree over the generic point of $U$, they must be equal. $\qquad\square$

## 5. INVARIANTS OF BINARY QUINTICS

In this section, we review the classical theory of binary quintics from a stack-theoretic perspective, with a view toward computing numerical invariants of the underlying quotients.

We consider the space $\mathbb{P}^5$ of binary quintic forms $\sum_{i=0}^5 A_i x^{5-i} y^i$ with standard $\mathrm{PGL}_2$-action. Let $U \subset \mathbb{P}^5$ denote the binary quintic forms with at most double roots. Over any field, this is the semistable and stable locus.

The algebra of invariant homogeneous polynomials is generated by classically known invariants $I_d$ of degrees $d = 4$, 8, 12, and 18, given explicitly by Schur in [30, pp. 87-89] with coefficients in $\mathbb{Q}$. Moreover, $I_{18}^2$ may be expressed as a weighted-homogeneous form $F \in \mathbb{Q}[I_4, I_8, I_{12}]$ of degree 36 (see, e.g., [18, § 7.2]). So the invariant-theoretic quotient is

$$\mathrm{Proj}(\mathbb{Q}[I_4, I_8, I_{12}]) = \mathbb{P}(1, 2, 3)_{\mathbb{Q}}.$$

The $\mathrm{PGL}_2$-action on $U$ has reduced finite stabilizer group schemes. By [28, (8.1)] the stack quotient $[U/\mathrm{PGL}_2]$ is a Deligne-Mumford stack, and is in fact a separated Deligne-Mumford stack (the standard valuative criterion for separation may be checked easily). It follows that the evident morphism $[\overline{M}_{0,5}/\mathfrak{S}_5] \to [U/\mathrm{PGL}_2]$ contracting any component of a stable 5-pointed genus 0 curve with exactly two marked points is proper; it is as well surjective and is an isomorphism on the locus of five distinct points on $\mathbb{P}^1$. In particular, $[U/\mathrm{PGL}_2]$ is proper over $\mathrm{Spec}(\mathbb{Z})$.

Choose $J_4, J_8, J_{12} \in \mathbb{Z}[A_0, \dots, A_5]$ such that

$$\mathbb{Z}[J_4, J_8, J_{12}]_d = \mathbb{Z}[A_0, \dots, A_5]_d \cap \mathbb{Q}[I_4, I_8, I_{12}], \quad \text{for } d = 4, 8, 12,$$

i.e., $J_4$, $J_8$, and $J_{12}$ generate the invariants in degrees $4, 8$, and $12$ over $\mathbb{Z}$. Let $J_{18}$ be a multiple of $I_{18}$ with relatively prime integer coefficients. A direct computation via Gröbner bases shows the invariants $J_4$, $J_8$, and $J_{12}$ define a morphism $[U/\mathrm{PGL}_2] \to \mathbb{P}(1, 2, 3)$ *over the integers.* It follows from $\mathrm{Pic}(\overline{M}_{0,5})^{\mathfrak{S}_5} \cong \mathbb{Z}$ that the composite

$$[\overline{M}_{0,5}/\mathfrak{S}_5] \to [U/\mathrm{PGL}_2] \to \mathbb{P}(1, 2, 3)$$

is quasi-finite, hence each individual morphism is proper and quasi-finite.

Since, over a field of characteristic zero, $\mathbb{P}(1, 2, 3)$ is the coarse moduli space of $[U/\mathrm{PGL}_2]$, and also of $[\overline{M}_{0,5}/\mathfrak{S}_5]$, the same is then true over $\mathrm{Spec}(\mathbb{Z})$ by an application of Zariski's Main Theorem to the normal scheme $\mathbb{P}(1, 2, 3)$, and over a field of positive characteristic by the same argument, noting that the scheme loci of $[\overline{M}_{0,5}/\mathfrak{S}_5]$ and of $[U/\mathrm{PGL}_2]$ are dense over any field.

**Lemma 5.1.** *The stacks $[\mathbb{P}^5/\mathrm{PGL}_2]$ and $[U/\mathrm{PGL}_2]$ have Picard group isomorphic to $\mathbb{Z}$, generated by $H := [Z(J_{18})] - 4[Z(J_4)]$.*

*Proof.* We may identify $[\mathbb{P}^5/\mathrm{PGL}_2]$ with $[(\mathbb{A}^6 \smallsetminus \{0\})/\mathrm{GL}_2]$ where the standard $\mathrm{GL}_2$-action is twisted by the $(-2)$-power of the determinant representation. Since the open immersions of regular stacks

$$[U/\mathrm{PGL}_2] \subset [\mathbb{P}^5/\mathrm{PGL}_2] \cong [(\mathbb{A}^6 \smallsetminus \{0\})/\mathrm{GL}_2] \subset [\mathbb{A}^6/\mathrm{GL}_2]$$

each have complement of codimension $\geq 2$, they induce isomorphisms of Picard groups, hence the Picard groups are identified with

$$\mathrm{Pic}([\mathbb{A}^6/\mathrm{GL}_2]) \cong \mathrm{Pic}(B\mathrm{GL}_2) \cong \mathbb{Z}.$$

(For these identifications of Picard groups, see Edidin-Graham [13, Lem. 2].) By comparing characters of $\mathrm{GL}_2$ we see that $H$ generates the Picard group. $\square$

For the rest of this section we work over a field $k$ of characteristic different from $2$. On the space of binary quintic forms the discriminant is an invariant of degree $8$, defining a divisor $\Delta \subset \mathbb{P}^5$ which on $U$ has singularities along quintic forms with two double roots.

**Lemma 5.2.** *The canonical class $K_{[U/\mathrm{PGL}_2]}$ is $-3H$.*

*Proof.* The coarse moduli space $\mathbb{P}(1, 2, 3)$ has a standard affine chart isomorphic to $\mathbb{A}^2$ and on it a standard generator of the canonical bundle on this chart. This pulls back to a rational section of the canonical

bundle of $[U/\mathrm{PGL}_2]$ vanishing to order 1 along $Z(J_{18})$ and having a pole of order 6 along $Z(J_4)$. So we have

$$K_{[U/\mathrm{PGL}_2]} = [Z(J_{18})] - 6[Z(J_4)] = 9H - 12H = -3H.$$

$\square$

**Lemma 5.3.** *We have*

$$\mathrm{Pic}([\overline{M}_{0,5}/\mathfrak{S}_5]) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z},$$

*where the first summand is the Picard group of $[U/\mathrm{PGL}_2]$ and the second summand is generated by $[\partial] - 2H$, where $\partial = [\partial\overline{M}_{0,5}/\mathfrak{S}_5]$ with $\partial\overline{M}_{0,5}$ denoting the boundary (complement of $M_{0,5}$) of $\overline{M}_{0,5}$.*

*Proof.* As in the proof of Lemma 5.1 we may remove any codimension 2 substack without changing the Picard group. Removing the locus of curves with three irreducible components, respectively quintic forms with two double roots, the morphism $[\overline{M}_{0,5}/\mathfrak{S}_5] \to [U/\mathrm{PGL}_2]$ restricts to a morphism

$$(5.1) \qquad\qquad [\overline{M}'_{0,5}/\mathfrak{S}_5] \to [U'/\mathrm{PGL}_2],$$

such that $\Delta \cap U'$ is a smooth divisor on $U'$. A construction called the *root stack* adds stabilizer along a divisor: with standard $\mathbb{G}_m$-action on $\mathbb{A}^1$ the stack $[\mathbb{A}^1/\mathbb{G}_m]$ is identified with pairs consisting of a line bundle and a global section, which is determined by an effective Cartier divisor so we have a morphism $[U'/\mathrm{PGL}_2] \to [\mathbb{A}^1/\mathbb{G}_m]$; then the root stack of interest is

$$[U'/\mathrm{PGL}_2] \times_{[\mathbb{A}^1/\mathbb{G}_m],\theta_2} [\mathbb{A}^1/\mathbb{G}_m]$$

where

$$\theta_2 \colon [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$$

is the morphism induced by squaring on both $\mathbb{A}^1$ and $\mathbb{G}_m$; cf. [8, §2] and [1, App. B]. Since $\Delta$ acquires multiplicity 2 upon pullback to $\overline{M}_{0,5}$, the morphism (5.1) factors through the root stack; the morphism to the root stack is quasi-finite, proper, and representable, hence by Zariski's Main Theorem is an isomorphism. The result now follows by the discussion of the Picard group of a root stack in [8, §3.1] and the fact that the $\mathrm{PGL}_2$-equivariant class of $\Delta$ is $4H \in \mathrm{Pic}([U/\mathrm{PGL}_2])$. $\square$

## 6. Moduli stacks of degree 4 Del Pezzo surfaces

In this section, we use the classical constructions presented earlier to relate moduli stacks of degree 4 Del Pezzo surfaces to stacks associated with binary quintics. We analyze the *height* of a fibration of degree 4 Del Pezzo surfaces, which is important because it is the unique numerical invariant of such a fibration; this is Proposition 6.5, which relies on the connection with binary quintic forms. Finally, we present geometric constraints on the height.

The moduli stack $\mathcal{M}^{\mathrm{DP4}}$, of degree 4 Del Pezzo surfaces, where we allow arbitrary rational double points (ADE-singularities), is an Artin stack with good geometric properties. We may work in an absolute setting, over $\mathrm{Spec}(\mathbb{Z})$, then this is a smooth finite-type Artin stack of relative dimension 2 over $\mathrm{Spec}(\mathbb{Z})$. The diagonal is separated and quasicompact.

Given a flat family of degree 4 Del Pezzo surfaces $\mathcal{X} \to S$ over an arbitrary scheme $S$, standard Hilbert scheme machinery gives rise to a scheme of nonsingular lines $\mathcal{L} \to S$. By the associated deformation theory, specifically the result [17, Cor. 5.4] and the fact that nonsingular line $L \subset X$ satisfies $H^i(L, N_{L/X}) = 0$ for $i = 0$ and 1, the morphism $\mathcal{L} \to S$ is étale. In particular, containing a nonsingular line is an open condition in moduli, and there is a corresponding moduli stack $\mathcal{M}^\circ$ of degree 4 Del Pezzo surfaces containing a nonsingular line. This comes with a representable étale covering

$$(6.1) \qquad\qquad\qquad \mathcal{M}'^\circ \to \mathcal{M}^\circ,$$

where $\mathcal{M}'^\circ$ is the moduli stack of degree 4 Del Pezzo surface with choice of nonsingular line. The stacks $\mathcal{M}^\circ$ and $\mathcal{M}'^\circ$ are also smooth, of finite type, and of relative dimension 2 over the base ring.

We now work over a perfect field $k$ with char $k \neq 2$. Let $X \subset \mathbb{P}^4$ be a smooth Del Pezzo surface of degree 4 over $k$. The classical characterization of $X$ up to isomorphism mentioned in the Introduction for $k$ algebraically closed, in terms of the singular locus $D \subset \mathbb{P}(H^0(X, \mathcal{I}_X(2)))$ up to projective equivalence, admits the following arithmetic refinement, which makes use of the restriction of scalars $R_{k[D]/k}$ from the $k$-algebra of regular functions on $D$. According to Skorobogatov [32, §2.2], $X$ is specified uniquely up to isomorphism by the singular locus up to projective equivalence together with an isomorphism class of $k$-torsors under the group scheme $R_{k[D]/k}(\mu_2)/\mu_2$. This description extends naturally to singular degree 4 Del Pezzo surfaces containing a nonsingular line (Theorem 6.1 and Corollary 6.2).

**Theorem 6.1.** *The blow-up of the projective plane along the zero locus of a binary quintic form on the Veronese-embedded projective line yields an isomorphism of stacks*

$$(6.2) \qquad\qquad [\mathbb{P}^5/\mathrm{PGL}_2] \xrightarrow{\sim} \mathcal{M}'^\circ$$

*with inverse isomorphism given by the spectral cover construction (Section 2). The spectral cover morphism*

$$(6.3) \qquad\qquad \mathcal{M}^\circ \to [\mathbb{P}^5/\mathrm{PGL}_2]$$

*is an étale gerbe (an étale surjective morphism with étale surjective relative diagonal) which is neutral, i.e., admits a section; a section is the composite of* (6.1) *and* (6.2).

*Proof.* That we have the morphism (6.2) is clear. As remarked in the proof of Theorem 4.3, the treatment given in Section 4 can be carried out in a relative setting over a smooth $k$-scheme, so by Theorem 4.3 in a relative setting we have the isomorphism as claimed. The claim about the section to the morphism (6.3) is clear, and implies that the morphism (6.3) is étale surjective. It remains, therefore, only to show that the relative diagonal is étale surjective, i.e., that two families in $\mathcal{M}^\circ$ having a common spectral cover are locally isomorphic. We can étale locally make choices of nonsingular lines in the fibers, and then we apply the isomorphism (6.2) to establish the assertion. $\qquad\square$

**Corollary 6.2.** *Given a Del Pezzo surface $X \subset \mathbb{P}^4$ of degree 4 over $k$ the set of nonsingular lines defined over $k$ is either empty or is acted upon simply transitively by the kernel of $\mathrm{Aut}(X) \to \mathrm{PGL}(H^0(X, \mathcal{I}_X(2)))$.*

In the stack of binary quintic forms there is the open substack $[U/\mathrm{PGL}_2]$ treated in Section 5, consisting of binary quintic forms with at most double roots. We let $\mathcal{M}$ denote the corresponding open substack of $\mathcal{M}^\circ$, under the morphism (6.3). Concretely, $\mathcal{M}$ is the moduli stack of degree 4 Del Pezzo surfaces with at most one $\mathsf{A}_1$-singularity, or with two $\mathsf{A}_1$-singularities connected by a line. The morphism (6.3) restricts to

$$(6.4) \qquad\qquad \mathcal{M} \to [U/\mathrm{PGL}_2],$$

also a neutral étale gerbe. However, even though $[U/\mathrm{PGL}_2]$ is separated (and, in fact, is proper), the stack $\mathcal{M}$ is nonseparated, since as remarked in the Introduction (see also [21, Rem. 2]), the order of the geometric stabilizer group in a family may decrease under specialization.

If we let $U^\circ$ denote the open subset of $U$ with nonvanishing discriminant and $\mathcal{M}^{\mathrm{sm}}$ denote the moduli stack of smooth degree 4 Del Pezzo surfaces, then (6.4) restricts to a neutral étale gerbe

$$(6.5) \qquad \mathcal{M}^{\mathrm{sm}} \to [U^\circ/\mathrm{PGL}_2].$$

**Remark 6.3.** The morphism $\mathcal{M} \to \mathbb{P}(1, 2, 3)$ is one-to-one on geometric points by [21, Prop. 1]. By combining a standard property of gerbes [28, Lem. 3.8] with the fact that $[U/\mathrm{PGL}_2] \to \mathbb{P}(1, 2, 3)$ is a coarse moduli space, we deduce that the morphism $\mathcal{M} \to \mathbb{P}(1, 2, 3)$ is also universal for morphisms to algebraic spaces. However, as $\mathcal{M}$ is nonseparated, the property of being étale locally on a coarse moduli space a quotient of a scheme by a finite group (a standard property for separated Deligne-Mumford stacks and more generally for Deligne-Mumford stacks with finite stabilizer [23]) fails to hold for the stack $\mathcal{M}$.

**Corollary 6.4.** *The singular locus of the total space of the universal family over*

$$[\overline{M}_{0,5}/\mathfrak{S}_5] \times_{[U/\mathrm{PGL}_2]} \mathcal{M}$$

*consists of ordinary double points in the fibers over the locus of singular degree 4 Del Pezzo surfaces.*

**Proposition 6.5.** *The morphisms* (6.3) *and* (6.4) *induce isomorphisms on Picard groups. In particular, we have* $\mathrm{Pic}(\mathcal{M}^{\mathrm{DP4}}) \cong \mathrm{Pic}(\mathcal{M}) \cong \mathbb{Z}$.

*Proof.* Let

$$\beta \colon [\mathbb{P}^5/\mathrm{PGL}_2] \to \mathcal{M}^\circ$$

denote the composite of (6.1) and (6.2), mentioned in the statement of Theorem 6.1. The Leray spectral sequence gives an exact sequence

$$0 \to \mathrm{Pic}([\mathbb{P}^5/\mathrm{PGL}_2]) \to \mathrm{Pic}(\mathcal{M}^\circ) \to \mathrm{Hom}(\mathrm{Aut}(\beta), \mathbb{G}_m) \to 0.$$

Already the restriction of the sheaf $\mathcal{H}om(\mathrm{Aut}(s), \mathbb{G}_m)$ to $[U^\circ/\mathrm{PGL}_2]$ has no nontrivial sections. Since the stacks in question all have separated diagonal, this is enough to deduce the vanishing of $\mathrm{Hom}(\mathrm{Aut}(\beta), \mathbb{G}_m)$. We conclude by appealing to Lemma 5.1. $\qquad\square$

We call attention to the restriction of $\beta$ to $[U/\mathrm{PGL}_2]$, a section

$$(6.6) \qquad [U/\mathrm{PGL}_2] \to \mathcal{M}$$

of the gerbe (6.4), and to $[U^\circ/PGL_2]$, a section

$$(6.7) \qquad [U^\circ/\mathrm{PGL}_2] \to \mathcal{M}^{\mathrm{sm}}$$

of the gerbe (6.5).

**Corollary 6.6.** *For a flat family* $\pi\colon \mathcal{X} \to \mathbb{P}^1$ *of degree 4 Del Pezzo surfaces with* ADE-*singularities, we have*

$$(6.8) \qquad\qquad \deg(\pi_*\omega_\pi^{-1}) = \deg(\pi_*(\mathcal{I}_{\mathcal{X}}(2))),$$

*with the notation of Section 2.*

*Proof.* By Proposition 6.5, the degrees in (6.8) must be related by a constant proportionality. We deduce their equality from any of the worked out examples, e.g., Case 1 on page 11 of [21] with $\pi_*\omega_\pi^{-1} \cong \mathcal{O}_{\mathbb{P}^1}(-2n)^5$ and $\pi_*(\mathcal{I}_{\mathcal{X}}(2)) \cong \mathcal{O}_{\mathbb{P}^1}(-5n)^2$. $\qquad\square$

**Definition 6.7.** The height of a flat family $\pi\colon \mathcal{X} \to \mathbb{P}^1$ of degree 4 Del Pezzo surfaces with ADE-singularities is the quantity

$$h(\mathcal{X}) = -2\deg(\pi_*\omega_\pi^{-1}) = -2\deg(\pi_*(\mathcal{I}_{\mathcal{X}}(2))).$$

The constant 2 in the definition of height is a convention. This height agrees with the height defined in [21, §3]; see the Appendix.

**Lemma 6.8.** *Let* $\mathbb{F}_n$ *be the Hirzebruch surface, with Picard group generated by* $(-n)$-*curve* $\xi$ *and fiber* $f$. *If* $D \subset \mathbb{F}_n$ *is a reduced divisor in class* $[D] = af + b\xi$ *then* $(b-1)n \leq a$. *If* $D \subset \mathbb{F}_n$ *is irreducible then either* $a \geq bn$ *or* $a = 0$.

*Proof.* The first assertion is a consequence of Proposition 2.2 of [26]. The second encodes the fact that $D\cdot\xi \geq 0$ unless $D$ contains a multiple of $\xi$. $\qquad\square$

For $d \leq e$, on the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(d) \oplus \mathcal{O}_{\mathbb{P}^1}(e)) \cong \mathbb{F}_n$ with $n = e - d$ we have $c_1(\mathcal{O}_{\mathbb{P}(\mathcal{O}(d)\oplus\mathcal{O}(e))}(1)) = -df + \xi$.

**Lemma 6.9.** *Let* $E$ *be a vector bundle on a scheme* $B$, *let* $\pi : \mathbb{P}(E^\vee) \to B$ *be the projectivization of the dual of* $E$, *let* $F$ *be a vector bundle of rank* $f$ *on* $B$, *and let* $F \to \mathrm{Sym}^d(E)$ *for some* $d \geq 1$ *be given, defining* $\mathcal{X} \subset \mathbb{P}(E^\vee)$ *(locally by* $f$ *homogeneous equations of degree* $d$). *Assume that the fibers of* $\mathcal{X} \to B$ *are of codimension* $f$. *Then*

$$[\mathcal{X}] = c_f(\mathcal{O}_{\mathbb{P}(E^\vee)}(d) \otimes \pi^*F^\vee)$$

*in the Chow group of* $\mathbb{P}(E^\vee)$.

*Proof.* From $F \to \mathrm{Sym}^d(E) = \pi_*\mathcal{O}_{\mathbb{P}(E^\vee)}(d)$ we get $\pi^*F \to \mathcal{O}_{\mathbb{P}(E^\vee)}(d)$ and hence a global section of $\mathcal{O}_{\mathbb{P}(E^\vee)}(d)\otimes\pi^*F^\vee$, whose vanishing defines $\mathcal{X} \subset \mathbb{P}(E^\vee)$. $\qquad\square$

We will consider flat families of degree 4 Del Pezzo surfaces over $\mathbb{P}^1$ with smooth general fiber. The geometric fibers over closed points of $\mathbb{P}^1$ are allowed to have arbitrary ADE-singularities. Let $h = h(\mathcal{X})$ be the height of such a family $\pi \colon \mathcal{X} \to \mathbb{P}^1$.

**Proposition 6.10.** *A generically smooth family $\pi \colon \mathcal{X} \to \mathbb{P}^1$ of degree 4 Del Pezzo surfaces of height $h = h(\mathcal{X})$ has discriminant divisor $\Delta(\pi) \subset \mathbb{P}^1$ of degree $2h$.*

*Proof.* The discriminant divisor is defined by the vanishing of

$$\mathcal{O}_{\mathbb{P}^1}(-10h) \cong ((\wedge^2 \pi_*(\mathcal{I}_\mathcal{X}(2))^\vee)^\vee)^{\otimes 20}$$
$$\to \mathrm{Sym}^8(\mathrm{Sym}^5(\pi_*(\mathcal{I}_\mathcal{X}(2))^\vee)^\vee) \to (\wedge^5 \pi_* \omega_\pi^{-1})^{\otimes 16} \cong \mathcal{O}_{\mathbb{P}^1}(-8h).$$

We are using the eighth symmetric power of the dual of the linear transformation coming from (2.1). $\qquad \square$

**Corollary 6.11.** *For a family $\pi \colon \mathcal{X} \to \mathbb{P}^1$ of height $h$ the morphism $\mathbb{P}^1 \to \mathbb{P}(1,2,3)$ has degree $6h$, i.e., the image of the class $[\mathbb{P}^1]$ is $6h$ times the positive generator of the divisor class group of $\mathbb{P}(1,2,3)$.*

**Proposition 6.12.** *Let $\pi \colon \mathcal{X} \to \mathbb{P}^1$ be a nonconstant flat generically smooth family of degree 4 Del Pezzo surfaces with ADE-singularities. Then:*

- *We have $h(\mathcal{X}) \geq 4$.*
- *If the spectral cover $D$ is irreducible then $h(\mathcal{X}) \neq 6$.*
- *If in addition the monodromy action on the lines does not factor through $\mathfrak{S}_5 \subset W(\mathsf{D}_5)$ then $h(\mathcal{X}) \neq 4$.*

*Proof.* Letting $h = h(\mathcal{X})$, it follows from the definition of height that $(\wedge^5 \pi_* \omega_\pi^{-1})^{\otimes 2} \cong \mathcal{O}_{\mathbb{P}^1}(-h)$. Lemma 6.9 yields

$$[D] = c_1(\mathcal{O}_{\mathbb{P}(\pi_*(\mathcal{I}_\mathcal{X}(2)))}(5)) - hf.$$

Let us write

$$\pi_*(\mathcal{I}_\mathcal{X}(2)) \cong \mathcal{O}(a) \oplus \mathcal{O}(-\frac{h}{2} - a)$$

with $a \leq -h/4$. We set $n = -2a - h/2$. Then,

(6.9) $$[D] = (-5a - h)f + 5\xi$$

on $\mathbb{P}(\pi_*(\mathcal{I}_\mathcal{X}(2))) \cong \mathbb{F}_n$.

Now assume that the generic fiber of $\pi$ is smooth. Then $D$ is reduced, so by Lemma 6.8 we have $-3a - h \leq 0$. Combining the facts, we have

$$-\frac{h}{3} \leq a \leq -\frac{h}{4}.$$

So, $h = 2$ is impossible.

Suppose further that $D$ is irreducible. Then the second part of Lemma 6.8 implies $-5a - h \geq 5n = 5(-2a - h/2)$, hence

$$-\frac{3h}{10} \leq a \leq -\frac{h}{4}.$$

This excludes $h = 6$.

Finally, suppose $h = 4$. The analysis above implies $a = -1, n = 0$ and $[D] = f + 5\xi$, i.e., $D$ has bidegree $(1, 5)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Since $D$ is irreducible, it is necessarily isomorphic to $\mathbb{P}^1$. Since $\mathbb{P}^1$ is simply connected, the monodromy action on the families of planes in the singular quadric hypersurfaces consists of two $\mathfrak{S}_5$-orbits. $\square$

## 7. Log general type surfaces

In this section we sketch a compactifcation of the moduli space of degree 4 Del Pezzo surfaces via stable log surfaces. We work over an algebraically closed field of characteristic different from 2.

**Proposition 7.1.** *Let $X$ be a smooth quartic Del Pezzo surface with lines $D_1, \ldots, D_{16} \subset X$. Then the pair $(X, (D_1, \ldots, D_{16}))$ has log canonical singularities and ample log canonical class.*

*Proof.* Computing in the Picard group, we find that

$$D_1 + \cdots + D_{16} \equiv -4K_X$$

thus the log canonical class $K_X + D_1 + \cdots + D_{16}$ is ample. Recall that at most two $D_i$ can be incident at any point $x \in X$—this is straightforward from the classical realization of $X$ as the blow-up of $\mathbb{P}^2$ at five distict points with no three collinear. Since each $D_i$ is smooth, the union $\cup_{i=1}^{16}$ is strict normal crossings; thus the pair is log canonical. $\square$

We would like to compactify the moduli space of quartic Del Pezzo surfaces, considered as a moduli space of 'stable log surfaces', in the sense of Kollár, Shepherd-Barron, and Alexeev [3, 19]. In positive characteristic the general construction of moduli spaces of stable log surfaces is not fully worked out, but the specific space we require can be obtained via other techniques [20].

The definition of a family of stable log surfaces is still evolving; we refer the reader to Kollár [24] for more detailed discussion. For our purposes, we may use the following restricted definition:

**Definition 7.2.** Let $B$ be a scheme of finite type over the base field. A *family of mildly singular stable log varieties* consists of

- a scheme $\pi : \mathcal{X} \to B$ with $\pi$ flat, proper, and Gorenstein;
- effective Cartier divisors $\mathcal{D}_1, \ldots, \mathcal{D}_r \subset \mathcal{X}$ flat over $B$;

satisfying the following: For each closed point $b \in B$,

- the pair $(\mathcal{X}_b, (\mathcal{D}_{1b}, \ldots, \mathcal{D}_{rb}))$ is semilog canonical;
- $\omega_\pi(\mathcal{D}_1 + \cdots + \mathcal{D}_r)|\mathcal{X}_b$ is ample, where $\omega_\pi$ is the relative dualizing sheaf.

Note that $\omega_\pi(\mathcal{D}_1 + \cdots + \mathcal{D}_r)|\mathcal{X}_b = \omega_{\mathcal{X}_b}(\mathcal{D}_{1b} + \cdots + \mathcal{D}_{rb})$ for each $b \in B$, by standard properties of the dualizing sheaf.

In general, stable varieties need not be Gorenstein and the boundary components need not be Cartier, which is why we describe these as 'mild' singularities. The mildness conditions behave extremely well in families—both are open conditions: If $\mathcal{X}_b$ is Gorenstein then $\pi$ is Gorenstein over some neighborhood of $b$; if $\mathcal{D}_{j,b}$ is Cartier (and $\mathcal{D}_j$ is flat over $B$) then $\mathcal{D}_j$ is Cartier near $b$. In practice, this means that the mildly singular varieties are open in moduli spaces of stable varieties.

Let $\widetilde{\mathcal{M}}^+$ denote the connected component of the moduli stack of stable log surfaces $(X, (D_1, \ldots, D_{16}))$ containing the pairs introduced in Proposition 7.1. We recall some key properties:

- $\widetilde{\mathcal{M}}^+ \simeq \overline{M}_{0,5}$ [20, Rem. 1.3, Thm. 10.19], with the smooth quartic Del Pezzos identified with $M_{0,5}$;
- the singularities of the fibers are 'stably toric', obtained by gluing together toroidal varieties along their boundaries [20, Thm. 1.1].

For our purposes, we enumerate the singular fibers over the zero-dimensional and one-dimensional boundary strata of $\overline{M}_{0,5}$. This description is implicit in [20, Rem. 1.3(5)]:

**one-dimensional**: The surface $X$ consists of six components:

- $X_1$, the minimal resolution of a quartic Del Pezzo surface with a single node with conductor divisors
    - $B_{12}$ the exceptional divisor over the node;
    - $B_{1k}, k = 3, 4, 5, 6$ the proper transforms of the lines meeting the node;
- $X_2$, the blow up of a quadric surface at four coplanar points $p_3, p_4, p_5, p_6$ with conductor divisors
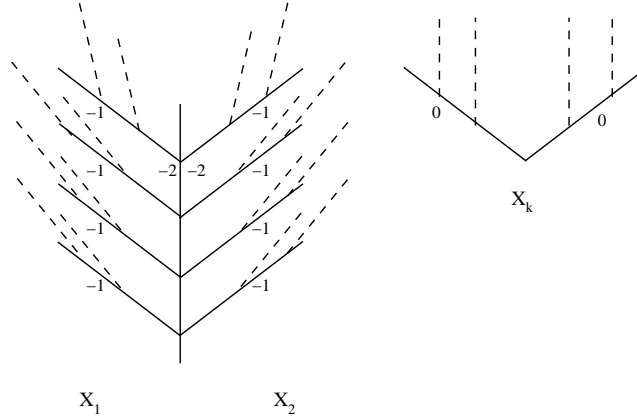    - $B_{12}$ the proper transform of the hyperplane sections;

FIGURE 1. the components, with conductors solid and
lines dashed

- $B_{2k}, k = 3, 4, 5, 6$ the exceptional divisors over the $p_k$;
- $X_k, k = 3, 4, 5, 6$ copies of the Hirzebruch surface $\mathbb{F}_0$ with distinguished rulings
  - $B_{k1} \in |f_k|$ ;
  - $B_{k2} \in |f'_k|$;

Note that $X_1$ and $X_2$ are in fact isomorphic.

The limits of the 16 lines are

- $D_{1k}$: union of the eight lines of $X_1$ *not* incident to the node and the ruling $\in |f'_k|$ meeting it;
- $D_{2k}$: union of the eight lines of $X_2$ *not* incident to the node and the ruling $\in |f_k|$ meeting it.

In particular, $X$ is D-semistable in the sense of Friedman [14]; each line $D_{ij} \subset X$ is cut out transversally. Thus the singularities are mild.

**zero-dimensional**: The surface $X$ consists of 12 components of two types:

- four components $X_1, X_2, X_3, X_4$, each isomorphic to $\mathbb{P}^2$ blown up at four non-collinear points $\{x'_{i,i-1}, x''_{i,i-1}, x'_{i,i+i}, x''_{i,i+1}\}, i \in \mathbb{Z}/4\mathbb{Z}$, with conductor divisor
  - the four exceptional divisors $B'_{i,i-1}, B''_{i,i-1}, B'_{i,i+1}, B''_{i,i+1}$ in $X_i$;
  - the proper transforms $B_{i,i-1}$ and $B_{i,i+1}$ of the lines joining $\{x'_{i,i-1}, x''_{i,i-1}\}$ and $\{x'_{i,i+1}, x''_{i,i+1}\}$ respectively;
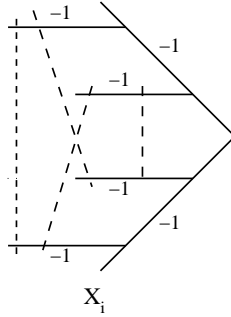
FIGURE 2. the four new components $X_1, \ldots, X_4$

- eight components $X'_{12}, X''_{12}, X'_{23}, X''_{23}, X'_{34}, X''_{34}, X'_{41}, X''_{41}$ isomorphic to $\mathbb{F}_0$, with conductor divisor consisting of representatives from each ruling.

Note that the components of the second type appear over the codimension 1 boundary points; however, we now have eight such components rather than four.

We describe the limit of one of the 16 lines, the others being defined symmetrically. It has three irreducible components:

- In $X_1$, take the proper transform $D_{1,4',2'}$ of the line joining $\{x'_{1,4}, x'_{1,2}\}$.
- In $X'_{41}$ take the ruling incident to $D_{1,4',2'}$ in one point.
- In $X'_{12}$ take the ruling incident to $D_{1,4',2'}$ in one point.

In particular, $X$ is D-semistable except at one point, where $X_1, X_2, X_3, X_4$ all intersect. Here $X$ is locally the cone over a cycle of four lines in $\mathbb{P}^3$, e.g.,

$$\{x = z = 0\} \cup \{y = z = 0\} \cup \{y = x + z - 1 = 0\} \cup \{x = y + z - 1\}$$

which is a complete intersection

$$\{xy = z(x + y + z - 1) = 0\}$$

thus Gorenstein. Each line avoids this point and is cut out transversally in $X$. These singularities are mild as well.

From this analysis, we deduce

**Proposition 7.3.** *The action of $W(\mathsf{D}_5)$ on the moduli space of marked quartic Del Pezzo surfaces extends naturally to a regular action on $\widetilde{\mathcal{M}}^+$. The distinguished subgroup*

$$(\mathbb{Z}/2\mathbb{Z})^4 = \ker(W(\mathsf{D}_5) \to \mathfrak{S}_5)$$

*acts via automorphisms on the universal family. The induced action of* $\mathfrak{S}_5$ *on* $\widetilde{\mathcal{M}}^+$ *coincides with the standard relabelling action on* $\overline{M}_{0,5}$.

**Definition 7.4.** Let $\widetilde{\mathcal{M}} = [\widetilde{\mathcal{M}}^+/W(\mathsf{D}_5)]$ denote the moduli stack of stable log surfaces as above, with *unordered* boundary divisors.

This comes with a morphism

$$(7.1) \qquad \widetilde{\mathcal{M}} \cong [\overline{M}_{0,5}/W(\mathsf{D}_5)] \to [\overline{M}_{0,5}/\mathfrak{S}_5]$$

fitting into a fiber diagram with the morphism of classifying stacks $BW(\mathsf{D}_5) \to B\mathfrak{S}_5$. In particular, $\widetilde{\mathcal{M}}$ is proper with coarse moduli space $\mathbb{P}(1,2,3)$. The morphism (7.1) restricts to a morphism

$$(7.2) \qquad \mathcal{M}^{\mathrm{sm}} \cong [M_{0,5}/W(\mathsf{D}_5)] \to [M_{0,5}/\mathfrak{S}_5] \cong [U^\circ/\mathrm{PGL}_2].$$

## 8. Comparison of moduli spaces

The compactification of Section 7 yields natural modifications of Del Pezzo fibrations satisfying our genericity condition. This will allow us to compare the underlying moduli stacks.

We start by recalling a well-known geometric description of torsors under symmetric and hyperoctahedral groups. Let $n$ be a positive integer, and let $T$ be a scheme. There is an equivalence

$$\mathcal{C}ov^n(T) \cong B\mathfrak{S}_n(T)$$

between the category of degree-$n$ étale covers of $T$, and the category of $\mathfrak{S}_n$-torsors over $T$, where $\mathfrak{S}_n$ denotes the symmetric group on $n$ letters, sending an étale cover $U \to T$ to the open subscheme of the $n$-fold fiber product $U^{\times n}$ over $T$ of pairwise distinct points, and in the other direction associating to an $\mathfrak{S}_n$-torsor $E \to T$ the cover $E \times^{\mathfrak{S}_n} \{1, \ldots, n\}$.

There is a similar equivalence

$$\mathcal{C}ov^{2,n}(T) \cong BW(\mathsf{B}_n)(T)$$

between the category of towers of étale coverings $V \to U \to T$ with $V \to U$ of degree 2 and $U \to T$ of degree $n$ and the category of torsors under the hyperoctahedral group $W(\mathsf{B}_n) = (\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n$. For torsors under the index 2 subgroup $W(\mathsf{D}_n) \subset W(\mathsf{B}_n)$ (the type $\mathsf{D}$ Weyl group) there is a description via towers $V \to U \to T$ together with a section of $E/W(\mathsf{D}_n) \to T$, where $E$ denotes the associated $W(\mathsf{B}_n)$-torsor. If $n$ is *odd*, then the involution of $V \to U$ induces the involution of the degree 2 cover $E/W(\mathsf{D}_n) \to T$.

The next result uses the above language to make explicit the morphism (7.2).

**Proposition 8.1.** (i) *Let $T$ be a scheme and $T \to [U^\circ/\mathrm{PGL}_2]$ a morphism, corresponding to the cover $D \subset \mathbb{P}(E) \to T$ for a rank 2 vector bundle $E$ on $T$. Then $D \to T$ is canonically identified with the degree 5 cover associated with the composite*

$$T \to [U^\circ/\mathrm{PGL}_2] \cong [M_{0,5}/\mathfrak{S}_5] \to B\mathfrak{S}_5.$$

(ii) *Let $T$ be a scheme and $\pi\colon \mathcal{X} \to T$ a smooth family of Del Pezzo surfaces of degree 4, corresponding to a morphism $T \to \mathcal{M}^{\mathrm{sm}}$. Let*

$$D \subset \mathbb{P}(\pi_*(\mathcal{I}_{\mathcal{X}}(2))) \to T$$

*be the spectral cover, parametrizing singular members of the pencils of quadric hypersurfaces, and $\tilde{D} \to D$ the degree 2 cover of singular quadric hypersurface with family of rulings, cf. [21, §3]. Then*

$$\tilde{D} \to D \to T$$

*is canonically identified with the tower of coverings associated with the composite morphism*

$$T \to \mathcal{M}^{\mathrm{sm}} \cong [M_{0,5}/W(\mathsf{D}_5)] \to BW(\mathsf{D}_5).$$

(iii) *Let $T$ be a scheme, $\pi\colon \mathcal{X} \to T$ a smooth family of Del Pezzo surfaces of degree 4 with family of lines $\mathcal{L} \subset \mathcal{X}$, corresponding to a morphism $T \to [U^\circ/\mathrm{PGL}_2] \cong \mathcal{M}'^{\circ} \times_{\mathcal{M}^\circ} \mathcal{M}^{\mathrm{sm}}$. Then with the notation of (ii) we obtain a canonical identification $\tilde{D} \cong D \times \mathbb{Z}/2\mathbb{Z}$ by labeling with $0$ the family of planes containing the linear span of the vertex of a singular quadric and the chosen line of $\mathcal{L}$ and with $1$ the opposite family.*

*Proof.* The first assertion is clear. For the second assertion, given a smooth Del Pezzo surface of degree 4, each singular member of the pencil of quadric hypersurfaces determines a partition of the set of 16 lines into two disjoint sets each consisting of 4 pairs of intersecting lines. Then $W(\mathsf{D}_5)$ acts on the 5 such pairs of sets via the standard inclusion in the hyperoctahedral group $(\mathbb{Z}/2\mathbb{Z})^5 \rtimes \mathfrak{S}_5$, cf. [25], and the assertion is clear. For the third assertion, we merely recall that the choice of family of lines $\mathcal{L}$ dictates a distinguished plane in every singular member of the pencil of quadrics, namely the linear span with the vertex.     $\square$

**Corollary 8.2.** *The morphism* (6.7) *fits into a fiber diagram*

$$
\begin{array}{ccc}
[U^\circ/\mathrm{PGL}_2] & \longrightarrow & B\mathfrak{S}_5 \\
\downarrow & & \downarrow \\
\mathcal{M}^{\mathrm{sm}} & \longrightarrow & BW(\mathsf{D}_5)
\end{array}
$$

*with the right-hand morphism induced by* $\mathfrak{S}_5 \subset W(\mathsf{D}_5)$.

**Proposition 8.3.** (i) *There is a canonical étale representable morphism*

$$[\overline{M}_{0,5}/\mathfrak{S}_5] \times_{[U/\mathrm{PGL}_2]} \mathcal{M} \to \widetilde{\mathcal{M}},$$

*extending the identity morphism on smooth families of degree 4 Del Pezzo surfaces.*

(ii) *The morphism* $[\overline{M}_{0,5}/\mathfrak{S}_5] \to [\overline{M}_{0,5}/\mathfrak{S}_5] \times_{[U/\mathrm{PGL}_2]} \mathcal{M}$ *determined by the section* (6.6), *composed with the morphism in* (i), *is canonically 2-isomorphic to the morphism*

$$[\overline{M}_{0,5}/\mathfrak{S}_5] \to [\overline{M}_{0,5}/W(\mathsf{D}_5)] \cong \widetilde{\mathcal{M}}$$

*coming from* $\mathfrak{S}_5 \subset W(\mathsf{D}_5)$.

*Proof.* For both statements, we use Nagata-Zariski purity, (cf. [16, Cor. X.3.3]) which tells us that the restriction functor, from finite étale covers of a regular locally noetherian scheme $X$ to covers of a dense open subscheme $U$ is fully faithful, and is an equivalence of categories if $X \smallsetminus U$ is of codimension at least 2. The statement is equally valid if $X$ is an algebraic stack. For (i), by the fiber diagram mentioned just after Definition 7.4 it suffices to show that the tautological $W(\mathsf{D}_5)$-torsor over $\mathcal{M}^{\mathrm{sm}} \cong [M_{0,5}/W(\mathsf{D}_5)]$ extends to a $W(\mathsf{D}_5)$-torsor over the complement of a codimension 2 substack of $[\overline{M}_{0,5}/\mathfrak{S}_5] \times_{[U/\mathrm{PGL}_2]} \mathcal{M}$. We consider the substack of Del Pezzo surfaces with at most one $\mathsf{A}_1$-singularity. Then the recipe to produce a log canonical model is to blow up the singular locus described in Corollary 6.4, then to blow up the $(-1)$-curves in the fibers meeting the relative singular locus. The proof of (ii) is similar, using Corollary 8.2.                $\square$

By Proposition 8.3(i) we have a 2-commutative diagram with cartesian square.

(8.1)

$$[\overline{M}_{0,5}/\mathfrak{S}_5] \times_{[U/\mathrm{PGL}_2]} \mathcal{M} \longrightarrow \mathcal{M}$$

$$\widetilde{\mathcal{M}} \cong [\overline{M}_{0,5}/W(\mathsf{D}_5)] \longrightarrow [\overline{M}_{0,5}/\mathfrak{S}_5] \longrightarrow [U/\mathrm{PGL}_2]$$

Given a family of Del Pezzo surfaces of degree 4 over a regular base whose discriminant divisor is a multiple of 2, Proposition 8.3(i) supplies a family of log general type surfaces, which we call the *associated family* of log general type surfaces.

**Proposition 8.4.** *Let $R$ be a discrete valuation ring over $k$ with algebraically closed residue field, and let $\mathcal{X} \to \mathrm{Spec}(R)$ and $\mathcal{X}' \to \mathrm{Spec}(R)$ be generically smooth families of Del Pezzo surfaces of degree 4, each with central fiber having a single $\mathsf{A}_1$-singularity and with discriminant of valuation 2. Let*

$$\widetilde{\mathcal{X}} = \mathrm{Bl}_{4\,\mathrm{lines}}(\mathrm{Bl}_{\mathcal{X}^{\mathrm{sing}}}(\mathcal{X}))$$

*where we recognize by the hypotheses that $\mathcal{X}$ has one singular point, an ordinary double point, which is the center of the first blow-up, and where the second blow-up is along the proper transform of the four lines in the central fiber meeting $\mathcal{X}^{\mathrm{sing}}$. Define $\widetilde{\mathcal{X}}'$ similarly, and let*

$$\varphi \colon \widetilde{\mathcal{X}} \to \widetilde{\mathcal{X}}'$$

*be an isomorphism over $\mathrm{Spec}(R)$. Then the rational map*

$$\mathcal{X} \dashrightarrow \mathcal{X}'$$

*induced by $\varphi$ is a morphism when $\varphi$ sends exceptional divisors to exceptional divisors and is only a rational map otherwise.*

*Proof.* Under the composite blow-down morphism $\widetilde{\mathcal{X}} \to \mathcal{X}$ the preimage of the complement of the four lines meeting $\mathcal{X}^{\mathrm{sing}}$ is the complement of the exceptional divisors in $\widetilde{\mathcal{X}}$. If $\varphi$ sends exceptional divisors to exceptional divisors then $\varphi$ induces an isomorphism

$$\mathcal{X} \smallsetminus \{4\ \mathrm{lines}\} \to \mathcal{X}' \smallsetminus \{4\ \mathrm{lines}\}.$$

Since $\mathcal{X}$ is normal, this extends to an isomorphism $\mathcal{X} \to \mathcal{X}'$. Conversely, an isomorphism $\mathcal{X} \to \mathcal{X}'$ induces an isomorphism of blow-ups sending exceptional components to exceptional components. $\square$

**Proposition 8.5.** *Let $R$ be a strictly henselian discrete valuation ring over $k$ with $\varpi \in R$ a uniformizer, and let*

$$D \subset \mathbb{P}^1_R$$

*be a divisor, with $D$ isomorphic over $\mathrm{Spec}(R)$ to*

$$\mathrm{Spec}(R(\sqrt{\varpi})) \amalg \coprod_{i=1}^{3} \mathrm{Spec}(R).$$

*Let*

$$\iota \colon \mathbb{P}^1_R \to \mathbb{P}^2_R,$$

*denote the Veronese embedding, and define*

$$\mathcal{X}_0 = \mathrm{Bl}_{\iota(D)}(\mathbb{P}^2_R), \qquad \mathcal{X} = \mathrm{Spec}(R(\sqrt{\varpi})) \times_{\mathrm{Spec}(R)} \mathcal{X}_0.$$

*Define $\widetilde{\mathcal{X}}$ as in Proposition 8.4, i.e., as the blow-up along four lines of the blow-up of the singular point of $\mathcal{X}$. Notice that $\widetilde{\mathcal{X}}$ has a unique exceptional divisors isomorphic to the minimal resolution of an $\mathsf{A}_1$-singular Del Pezzo surface of degree 4; we call this the Del Pezzo exceptional divisor. Let $\tau\colon R(\sqrt{\varpi}) \to R(\sqrt{\varpi})$ be the nontrivial Galois automorphism. Then the automorphism $\mathrm{Spec}(\tau) \times \mathrm{id}_{\mathcal{X}_0}$ of $\mathcal{X}$ induces an automorphism of $\widetilde{\mathcal{X}}$ which maps the Del Pezzo exceptional divisor nontrivially to itself.*

*Proof.* Since the blow-ups have Galois-invariant centers, the automorphism of $\mathcal{X}$ induces an automorphism of $\widetilde{\mathcal{X}}$ mapping exceptional divisors to exceptional divisors, hence the Del Pezzo exceptional divisor to itself. The Del Pezzo exceptional divisor is birational to the projectivized normal cone to $\mathcal{X}^{\mathrm{sing}}$. A computation in formal local coordinates establishes the result. $\qquad\square$

## 9. Genericity conditions

Here we develop genericity conditions that will make it possible to classify fibrations in degree 4 Del Pezzo surfaces. These are also quite convenient in analyzing local-global and Brauer-Manin obstructions for such surfaces over function fields.

We continue to work over a perfect field $k$ of characteristic different from 2. The primary case of interest is a generically smooth family $\pi\colon \mathcal{X} \to \mathbb{P}^1$ of degree 4 Del Pezzo surfaces with square-free discriminant. We introduce two conditions for $\pi$ to be general.

**Definition 9.1.**

(G1) $\pi$ has reduced discriminant divisor, or equivalently, $\mathcal{X}$ is smooth and each fiber of $\pi$ has at worst a single $\mathsf{A}_1$-singularity,

(G2) $\pi$ has full $W(\mathsf{D}_5)$-monodromy of lines in smooth fibers.

For $D \to \mathbb{F} \to \mathbb{P}^1$ with $\mathbb{F} \to \mathbb{P}^1$ a Hirzebruch surface with divisor $D$ such that $D \to \mathbb{P}^1$ is finite and flat of degree 5, there are two related conditions:

**Definition 9.2.**

(G1)′ $D$ is simply branched over $\mathbb{P}^1$.

(G2)′ The normal closure of $k(D)$ over $k(\mathbb{P}^1)$ has Galois group $\mathfrak{S}_5$.

Given a generically smooth family $\pi\colon \mathcal{X} \to \mathbb{P}^1$ of Del Pezzo surfaces of degree 4, we will say that $\pi$ satisfies (G2)′ if the spectral cover $D \subset \mathbb{P}(\pi_*(\mathcal{I}_{\mathcal{X}}(2))) \to \mathbb{P}^1$ satisfies (G2)′. We remark that in $W(\mathsf{D}_5)$

there are no proper subgroups strictly containing $\mathfrak{S}_5$. So, (G2)$'$ implies either $\mathfrak{S}_5$-monodromy or full $W(\mathsf{D}_5)$-monodromy of lines in smooth fibers.

**Proposition 9.3.** *A family $\pi\colon \mathcal{X} \to \mathbb{P}^1$ of height $h = h(\mathcal{X})$ satisfying* (G1) *has $2h$ singular fibers. If, furthermore, $\pi$ satisfies* (G2)$'$, *then the spectral curve is an irreducible nonsingular curve of genus $h - 4$.*

*Proof.* This is immediate from Proposition 6.10. $\qquad\qquad\square$

Condition (G1) is open in moduli. In the locus of moduli where (G1) is satisfied, condition (G2) is an open and closed condition. This means, if we let $\mathrm{Hom}(\mathbb{P}^1, \mathcal{M}; h)$ denote the moduli stack of height $h$ families of Del Pezzo surfaces of degree 4, then the families satisfying (G1) are the points of a well-defined open substack $\mathrm{Hom}_{(G1)}(\mathbb{P}^1, \mathcal{M}; h)$. There is an open and closed substack

$$\mathrm{Hom}_{(G1),(G2)}(\mathbb{P}^1, \mathcal{M}; h) \subset \mathrm{Hom}_{(G1)}(\mathbb{P}^1, \mathcal{M}; h)$$

where both (G1) and (G2) are satisfied.

If $D \subset \mathbb{P}(\pi_*(\mathcal{I}_{\mathcal{X}}(2))) \to \mathbb{P}^1$ is the spectral cover of $\pi$, then it satisfies (G1)$'$ if and only if $\pi$ satisfies (G1). Furthermore, property (G2) for $\pi$ implies that $D \subset \mathbb{P}(\pi_*(\mathcal{I}_{\mathcal{X}}(2))) \to \mathbb{P}^1$ satisfies (G2)$'$, although the reverse implication does not hold. Because any subgroup of $\mathfrak{S}_5$ containing a transposition and acting transitively on $\{1, \ldots, 5\}$ is the full group $\mathfrak{S}_5$, if in Definition 9.2 (G1)$'$ holds and $D$ is irreducible, then (G2)$'$ holds as well.

There are open and closed substacks

$$\mathrm{Hom}_{(G1),(G2)'}(\mathbb{P}^1, \mathcal{M}; h) \subset \mathrm{Hom}_{(G1)}(\mathbb{P}^1, \mathcal{M}; h)$$

and

$$\mathrm{Hom}_{(G1)',(G2)'}(\mathbb{P}^1, [U/\mathrm{PGL}_2]; h) \subset \mathrm{Hom}_{(G1)'}(\mathbb{P}^1, [U/\mathrm{PGL}_2]; h),$$

where the notation is self-explanatory. We will remove $h$ from the notation when we do not want to constrain the height.

**Lemma 9.4.** *The stack $\mathrm{Hom}_{(G1)}(\mathbb{P}^1, \mathcal{M}; h)$ is smooth of dimension $\frac{3}{2}h + 2$.*

*Proof.* For $h = 0$ the assertion is trivial, so we assume $h > 0$. As remarked above, (G1) implies that the spectral curve $D$ is smooth. Since $\mathcal{M}$ is étale over $[U/\mathrm{PGL}_2]$, it suffices to show that the space of maps $\mathbb{P}^1 \to [U/\mathrm{PGL}_2]$ of degree $6h$ with smooth spectral curve is smooth.

We treat two cases, according to whether or not the spectral curve is irreducible. First suppose that the spectral curve is irreducible. The stack of morphisms $\mathbb{P}^1 \to B(\mathrm{PGL}_2)$ is the stack $\mathrm{Bun}_{\mathrm{PGL}_2}$ whose fiber over any scheme $T$ is the category of principal $\mathrm{PGL}_2$-bundles over $T \times \mathbb{P}^1$. It is a smooth algebraic stack with two irreducible components, each of dimension $-3$. Identifying $\mathrm{PGL}_2$-bundles with $\mathbb{P}^1$-bundles, these correspond to the Hirzebruch surfaces $\mathbb{F}_n$ with $n$ even, respectively, $n$ odd. The stack of morphisms from $\mathbb{P}^1$ to $[U/\mathrm{PGL}_2]$ is an algebraic stack $\mathrm{Hom}(\mathbb{P}^1, [U/\mathrm{PGL}_2])$, and its points correspond to covers $D \subset \mathbb{F} \to \mathbb{P}^1$ as above. Standard deformation theory in a relative setting (see [17, Cor. 5.4]) implies that the morphism

$$(9.1) \qquad \mathrm{Hom}_{(\mathrm{G1})'}(\mathbb{P}^1, [U/\mathrm{PGL}_2]; h) \to \mathrm{Bun}_{\mathrm{PGL}_2}$$

is smooth of relative dimension $\frac{3}{2}h + 5$. Indeed, in this situation $\mathcal{O}_{\mathbb{F}}(D)|_D$ has degree $5h/2$, so $H^1(D, \mathcal{O}_{\mathbb{F}}(D)|_D) = 0$ and

$$\dim H^0(D, \mathcal{O}_{\mathbb{F}}(D)|_D) = \frac{3}{2}h + 5.$$

If the spectral curve is reducible, then it must be a disjoint union of the exceptional curve $E \subset \mathbb{F}_n$ and an irreducible curve different from $E$. Then $h$ must be divisible by 6, with $n = h/6$ and spectral curve $D \cup E$ with $[D] = (2/3)hf + 4\xi$, by (6.9) (with the same notation). By deformation theory for the curve in the surface $E \subset \mathbb{F}_n$, the vanishing

$$(9.2) \qquad \mathrm{Ext}^1(\Omega^1_{\mathbb{F}_n}(\log E), \mathcal{O}_{\mathbb{F}_n}) = 0$$

implies that near a point with reducible spectral curve the morphism (9.1) factors through the locally closed substack of $\mathrm{Bun}_{\mathrm{PGL}_2}$ corresponding to the isomorphism type of $\mathbb{F}_n$. We conclude as above by computing the dimension of $|D|$ to be $\frac{5}{3}h + 4$ and noting that $\frac{5}{3}h + 4 - (n+2) = \frac{3}{2}h + 2$. The vanishing (9.2) may be seen by comparing the sequences of Ext groups associated with the standard exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^1_{\mathbb{F}_n} & \longrightarrow & \Omega^1_{\mathbb{F}_n}(\log E) & \longrightarrow & \mathcal{O}_E & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1_{\mathbb{F}_n} & \longrightarrow & \Omega^1_{\mathbb{F}_n}(\log \sum D_i) & \longrightarrow & \bigoplus \mathcal{O}_{D_i} & \longrightarrow & 0
\end{array}
$$

where $D_i$ are the toric divisors on $\mathbb{F}_n$, and applying the triviality of the middle term of the bottom sequence (see, e.g., [10, §8.1]) and the isomorphism $\mathrm{Ext}^j(\mathcal{O}_{D_i}, \mathcal{O}_{\mathbb{F}_n}) \cong H^{j-1}(D_i, \mathcal{O}_{\mathbb{F}_n}(D_i)|_{D_i})$. $\qquad \square$

**Remark 9.5.** Lemma 9.4 is consistent with the expected dimension of the Kontsevich space of maps $f : \mathbb{P}^1 \to \mathcal{M}$. Indeed, if $g : \mathbb{P}^1 \to [U/\mathrm{PGL}_2]$ denotes the composite map to $[U/\mathrm{PGL}_2]$ then the expected dimension is

$$\deg(g^* K_{[U/\mathrm{PGL}_2]}) - 1 = \frac{3}{2}h - 1.$$

## 10. Maps to $\mathcal{M}$ from covers of spectral curves

Now we may state the main results of this paper:

**Theorem 10.1.** *Let $k$ be a field of characteristic different from 2. There is an equivalence of fibered categories over $k$-schemes between*

- *families $\mathcal{X} \to T \times \mathbb{P}^1$ of degree 4 Del Pezzo surfaces (flat, proper, finitely presented, with fibers having* ADE*-singularities) satisfying* (G1) *and* (G2)′ *over all geometric points of $T$, and*
- *the fibered category whose fiber over $T$ consists of $D \subset \mathbb{F} \to T \times \mathbb{P}^1$ with $\mathbb{F} \to T \times \mathbb{P}^1$ a $\mathbb{P}^1$-bundle, $D \to T$ smooth with $D \to T \times \mathbb{P}^1$ finite flat of degree 5 and satisfying* (G1)′ *and* (G2)′ *over all geometric points of $T$, together with a section of the 2-torsion of the relative Jacobian $J(D/T)[2] \to T$,*

*given by the spectral cover construction together with the fiberwise double cover of families of planes in the singular quadric hypersurfaces.*

*Proof.* The functor in the forward direction is given as follows. To a family $\pi \colon \mathcal{X} \to T \times \mathbb{P}^1$ we associate the spectral cover

$$D \subset \mathbb{P}(\pi_*(\mathcal{I}_{\mathcal{X}}(2))) \to T \times \mathbb{P}^1$$

which parametrizes singular members of the pencils of quadric hypersurfaces of the fibers of $\pi$, together with the section of $J(D/T)[2] \to T$ corresponding to the double cover $\tilde{D} \to D$ of families of planes in the singular quadrics. Compatiblity of the construction with base change is obvious, so we have a functor between the fibered categories.

This functor is a morphism between algebraic stacks that are étale over $\mathrm{Hom}_{(\mathrm{G1})'}(\mathbb{P}^1, [U/\mathrm{PGL}_2])$. So the morphism is étale, and to verify that it is an isomorphism it suffices to show that each geometric fiber consists of a single point, with trivial stabilizer. For this we may assume that $k$ is algebraically closed and that we are given $D \subset \mathbb{F} \to \mathbb{P}^1$ with $\mathbb{F}$ a Hirzebruch surface, $D$ a nonsingular irreducible curve and $D \to \mathbb{P}^1$ finite of degree 5 and simply ramified over a divisor $\Delta \subset \mathbb{P}^1$, together with a 2-torsion element of the Jacobian $J(D)$. Let $\mathbb{P}^1 \to [U/\mathrm{PGL}_2]$

and $\tilde{D} \to D$ be the corresponding morphism, respectively, cover. Let

$$\mathcal{C} = [\overline{M}_{0,5}/\mathfrak{S}_5] \times_{[U/\mathrm{PGL}_2]} \mathbb{P}^1,$$

so we have the solid arrows of a diagram which extends the diagram (8.1):

(10.1)

In the diagram, the lower square is a fiber square, as mentioned just after Definition 7.4.

By the discussion in the proof of Lemma 5.3, $\mathcal{C}$ is a root stack over $\mathbb{P}^1$ along $\Delta$. So if we let $\mathcal{D}$ be the normalization of $\mathcal{C} \times_{\mathbb{P}^1} D$, then $\mathcal{D}$ is an étale cover of $\mathcal{C}$. Setting $\tilde{\mathcal{D}} = \tilde{D} \times_D \mathcal{D}$, we have a tower

$$(10.2) \qquad \tilde{\mathcal{D}} \to \mathcal{D} \to \mathcal{C}$$

The dashed arrows in the diagram are obtained by examining the tower (10.2) near an orbifold point of $\mathcal{C}$. If we pass to the henselization of the local ring of $\mathbb{P}^1$ at a point of $\Delta$ then we obtain, by base change, $[\mathrm{Spec}(k[t]^h_{(t)})/\mu_2] \to \mathcal{C}$. The base change of $\mathcal{D}$ is

$$(10.3) \qquad \mathrm{Spec}(k[t]^h_{(t)}) \amalg \coprod_{i=1}^{3} [\mathrm{Spec}(k[t]^h_{(t)})/\mu_2],$$

corresponding to the morphism $[\mathrm{Spec}(k[t]^h_{(t)})/\mu_2] \to B\mathfrak{S}_5$ given by a transposition in $\mathfrak{S}_5$. Since $\tilde{\mathcal{D}} \to \mathcal{D}$ is obtained by base change from $\tilde{D} \to D$, the only possibility is that $\tilde{\mathcal{D}} \to \mathcal{D}$ base-changes to a trivial cover of (10.3). In particular, if we let $\mathcal{E}$ be the open substack of $\tilde{\mathcal{D}}^{\times n}$ as in the description in Section 8, then the base change of $\mathcal{E}/W(\mathsf{D}_5) \to \mathcal{C}$ is also a trivial cover, so since $\mathcal{C}$ is an orbifold $\mathbb{P}^1$, the cover $\mathcal{E}/W(\mathsf{D}_5) \to \mathcal{C}$ must be globally trivial. So there is a canonical $W(\mathsf{D}_5)$-torsor over $\mathcal{C}$ corresponding to the tower (10.2), and hence a bottom dashed arrow in the diagram, determined up to unique 2-isomorphism. (Recall, the

involution of $\tilde{\mathcal{D}}$ over $\mathcal{D}$ switches the two sections of $\mathcal{E}/W(\mathsf{D}_5) \to \mathcal{C}$, so the $W(\mathsf{D}_5)$-torsor structure is canonical.) The next dashed arrow is obtained using the universal property of a fiber diagram, i.e., we have a canonically defined family of log general type surfaces $\epsilon\colon \mathcal{Y} \to \mathcal{C}$. To give the final dashed arrow is equivalent to giving a section of

$$(10.4) \qquad \mathcal{C} \times_{\widetilde{\mathcal{M}}} ([\overline{M}_{0,5}/\mathfrak{S}_5] \times_{[U/\mathrm{PGL}_2]} \mathcal{M}) \to \mathcal{C}.$$

This is an étale morphism, an isomorphism away from the orbifold points of $\mathcal{C}$, and of degree 2 over the orbifold points of $\mathcal{C}$. By Proposition 8.4, the fiber of (10.4) over an orbifold point of $\mathcal{C}$ is identified with the 2-element set of components of the fiber of $\epsilon$, isomorphic to a resolution of a singular Del Pezzo surface of degree 4; we call these the Del Pezzo components. Combining Propositions 8.3(ii) and 8.5, we see that $\mu_2$ acts trivially on the set of Del Pezzo components, and the $\mu_2$-action on one of the Del Pezzo components is trivial and on the other is nontrivial. So the morphism (10.4) admits sections, and a section is specified uniquely by dictating the choice of Del Pezzo component that is acted upon trivially by $\mu_2$ at each orbifold point. With this uniquely specified section we have, canonically determined, a family $\pi'\colon \mathcal{X}' \to \mathcal{C}$ of Del Pezzo surfaces and an isomorphism of the associated family of log general type surfaces with $\mathcal{Y}$. Since the $\mu_2$-action on the fiber of $\pi'$ at any orbifold point of $\mathcal{C}$ is trivial, $\pi'$ is obtained by base change from a family $\pi\colon \mathcal{X} \to \mathbb{P}^1$ of Del Pezzo surfaces, defined up to a unique isomorphism. The spectral curve is $D \subset \mathbb{F} \to \mathbb{P}^1$, and the double cover associated to the families of planes in singular quadric hypersurfaces is $\tilde{D} \to D$. So we have shown that morphism of algebraic stacks, described in the statement of the theorem, is surjective. Noting the uniqueness (up to canonical 2-isomorphisms) of the dashed arrows in (10.1) and the uniqueness of the family $\pi$ in the last step, we obtain that the morphism is an isomorphism. $\qquad\square$

We apply this description to enumerate the components of general families of given height.

**Theorem 10.2.** *Fix a height $h$, even and positive. The space of families of height $h$ satisfying Conditions* (G1) *and* (G2) *is empty when $h \leq 6$ and for $h \geq 8$ consists of:*

(i) *two components when $h = 8$ or $h = 10$;*
(ii) *one component when $h \geq 12$.*

*When $h = 8$, the spectral curve is hyperelliptic, hence the 2-torsion of the Jacobian is partitioned according to the minimal number of Weierstrass points (minus the appropriate multiple of the $g_2^1$) in the canonical represenation. The two components with monodromy $W(\mathsf{D}_5)$ correspond to sums of two, respectively four Weierstrass points. When $h = 10$, the spectral curve is a plane quintic curve, hence comes with a natural theta characteristic and an associated quadratic form on the 2-torsion of the Jacobian. In this case the two components with monodromy $W(\mathsf{D}_5)$ correspond to the two values of the quadratic form on the nonzero points of the 2-torsion of the Jacobian of $D$.*

*Proof.* By Theorem 10.1, we may consider spectral curves in Hirzebruch surfaces with choice of 2-torsion in the Jacobian. By the relative smoothness assertion of Lemma 9.4, we may restrict to case of the Hirzebruch surfaces $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1$.

Assume $h \geq 12$. By the formula (6.9), for $h$ divisible by 4 we have $D \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(h/4, 5)$, and for $h$ congruent to 2 modulo 4 we have $D \subset \mathbb{F}_1$ with $[D] = ((h+10)/4)f + 5\xi$. In each case Lemma 3.1 implies that the monodromy action on the 2-torsion in the Jacobian of the spectral curve is the full symplectic group. So the space of pairs $(D, \tilde{D} \to D)$ with $D$ as above and $\tilde{D} \to D$ a nontrivial unramified degree 2 cover consists of a single component.

When $h = 10$ we have $D \subset \mathbb{F}_1$ with $[D] = 5f + 5\xi$, i.e., if we identify $\mathbb{F}_1$ with the blow-up of a point in the projective plane then $D$ is the preimage of a smooth quintic curve not passing through the point that is blown up. In this case the restriction of $\mathcal{O}_{\mathbb{P}^2}(1)$ is a theta characteristic, so the monodromy group is cut down to $\mathrm{O}(H^2(D, \mathbb{Z}/2\mathbb{Z}), q)$ where $q$ is the corresponding quadratic form. When $h = 8$, the spectral curve is of bidegree $(2, 5)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, and Example 3.3 furnishes the complete description. $\qquad\square$

## 11. Examples of families of low height

In this section, we provide examples, in height 8 and 10, of distinct families of quartic Del Pezzo surface fibrations $\pi : \mathcal{X} \to \mathbb{P}^1$, of expected dimension and maximal monodromy.

**Height 8**: We recall the construction from [21, Remark 15]. Let

$$\mathcal{X} \subset \mathbb{P}^1 \times \mathbb{P}^5$$

be a complete intersection of a form of bidegree $(1, 1)$ and two forms of bidegree $(0, 2)$. The projection to the second factor $\mathcal{X} \to \mathcal{Y} \subset \mathbb{P}^5$ gives a

complete intersection of two quadrics, and the quartic Del Pezzo surface fibration $\pi : \mathcal{X} \to \mathbb{P}^1$ corresponds to a pencil of hyperplane sections, with base locus a smooth curve $E$ of genus 1. In turn, projection from a line $\ell$ in $\mathcal{Y}$ is the blowup of $\mathbb{P}^3$ in a quintic curve of genus 2, so that

$$\mathrm{IJ}(\mathcal{X}) \simeq \mathrm{J}(C) \times E.$$

**Height 8**: Consider the vector bundle

$$E = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}^2 \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$$

and its projectivization $\phi : \mathbb{P}(E) \to \mathbb{P}^1 \times \mathbb{P}^1$. Let $\phi_i : \mathbb{P}(E) \to \mathbb{P}^1$ denote the resulting projections. Let $W \subset \mathbb{P}(E)$ be a conic fibration corresponding to $\mathcal{O}_{\mathbb{P}(E)}(2) \otimes \phi_2^* \mathcal{O}_{\mathbb{P}^1}(1)$, given by a section of

$$\mathrm{Sym}^2(E^\vee) \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, 1).$$

Such a section corresponds to a symmetric matrix of forms

$$A := \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

where $A_{11}, A_{12}, A_{22}$ have bidegree $(0, 1)$, $A_{13}$ and $A_{23}$ have bidegree $(1, 2)$, and $A_{33}$ has bidegree $(2, 3)$. Note that

$$h^0(\mathrm{Sym}^2(E^\vee) \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, 1)) = 30$$

and $h^0(\mathrm{End}(E)) = 13$, so $\dim \mathrm{Aut}(\mathbb{P}(E)) = 18$ and hence $W$ depends on 11 parameters.

The discriminant curve $D \subset \mathbb{P}^1 \times \mathbb{P}^1$ has bidegree $(2, 5)$, thus is hyperelliptic of genus four. Thus the projection $\phi_1 : W \to \mathbb{P}^1$ has fibers isomorphic to conic bundles with five degenerate fibers, whence cubic surfaces. Consider the subvariety

$$W' = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}^2) \cap W \subset \mathbb{P}_1^1 \times \mathbb{P}_2^1 \times \mathbb{P}_3^1;$$

we order so that $\phi_1$ and $\phi_2$ map to $\mathbb{P}_1^1$ and $\mathbb{P}_2^1$ and $\mathbb{P}_3^1$ is the fiber of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}^2) \to \mathbb{P}_1^1 \times \mathbb{P}_2^1$. We regard $W'$ as a bisection of $W \to \mathbb{P}_1^1 \times \mathbb{P}_2^1$. A degree computation shows that $W'$ is the preimage of a curve $B \subset \mathbb{P}_2^1 \times \mathbb{P}_3^1$ of bidegree $(1, 2)$.

Now $f_D : D \to \mathbb{P}_2^1$ and $f_B : B \to \mathbb{P}_2^1$ both have degree two with ten and two branch points respectively.

**Lemma 11.1.** *The branch locus of $f_B$ is contained in the branch locus of $f_D$.*

*Proof.* Recall that $D = \{\det(A) = 0\}$. The branch locus of $f_B$ is given by $\{A_{11}A_{22} - A_{12}^2 = 0\}$. Note that

$$\det(A) \equiv -A_{11}A_{23}^2 - A_{22}A_{13}^2 + 2A_{12}A_{23}A_{13} \pmod{A_{11}A_{22} - A_{12}^2}.$$

Let $u$ and $v$ be homogeneous coordinates of $\mathbb{P}_1^1$; write

$$A_{13} = A_{13}'u + A_{13}''v, \quad A_{23} = A_{23}'u + A_{23}''$$

and expand

$$-A_{11}A_{23}^2 - A_{22}A_{13}^2 + 2A_{12}A_{23}A_{13} = au^2 + 2buv + cv^2, \quad a, b, c \in \Gamma(\mathcal{O}_{\mathbb{P}_2^1}(5)).$$

The branch locus of $f_D$ equals $\{ac - b^2 = 0\}$ modulo $A_{11}A_{22} - A_{12}^2$. A direct computation shows

$$\begin{aligned}
ac - b^2 &= (A_{11}A_{22} - A_{12}^2)(A_{13}'A_{23}'' - A_{13}''A_{23}')^2 \\
&= \det\begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix} \det\begin{pmatrix} A_{13}' & A_{13}'' \\ A_{23}' & A_{23}'' \end{pmatrix}^2.
\end{aligned}$$

$\square$

This covering data determines the intermediate Jacobian $\mathrm{IJ}(W)$: The general theory of conic bundles over rational surfaces [6] implies $\mathrm{IJ}(W) = \mathrm{Prym}(\tilde{D} \to D)$ for the étale double cover arising from the irreducible components of the singular conics over the discriminant $D$. In our situation, $\tilde{D}$ is the normalization of the fiber product $D \times_{\mathbb{P}_2^1} B$. It follows that

$$\mathrm{Prym}(\tilde{D} \to D) = \mathrm{J}(C),$$

where $C$ is a double curve of $\mathbb{P}_2^1$ branched over the complement of the branch locus of $f_B$ in the branch locus of $f_D$ [4, p. 303]. In particular, $C$ is hyperelliptic of genus three.

Fixing $p \in \mathbb{P}_1^1$, we find that

$$\mathbb{P}(E)|_{\{p\} \times \mathbb{P}_2^1} \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^2 \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \subset \mathbb{P}^3 \times \mathbb{P}_2^1,$$

i.e., the planes containing a fixed line $\ell \subset \mathbb{P}^3$. Morever $W|_{\{p\} \times \mathbb{P}_2^1}$ may be interpreted as one of the cubic surfaces containing this line, and $W'|_{\{p\} \times \mathbb{P}_2^1}$ as the bisection induced by $\ell$. Blowing down $\ell$ in the generic fiber of $\phi_1$, we obtain a fibration in quartic Del Pezzo surfaces

$$\pi : \mathcal{X} \to \mathbb{P}_1^1.$$

**Height 10**: Consider

$$\mathcal{X} \subset \mathbb{P}^1 \times \mathbb{P}^4,$$

given as a complete intersection of forms of bidegree $(0,2)$ and $(1,2)$. Then $\mathcal{X} = \mathrm{Bl}_C(Q)$, the blowup of a smooth quadric $Q \subset \mathbb{P}^4$ in a smooth canonical curve $C$ of genus 5, the base locus of a pencil of quadrics on $Q$ defining $\pi : \mathcal{X} \to \mathbb{P}^1$. Thus

$$\mathrm{IJ}(\mathcal{X}) \simeq \mathrm{J}(C).$$

**Height 10**: Fix a smooth cubic threefold $\mathcal{W} \subset \mathbb{P}^4$ and a conic curve $Q \subset \mathcal{W}$; this data depends on 14 parameters. Let $\ell \subset \mathcal{W}$ denote the line residual to $Q$ in $P = \mathrm{span}(Q)$ and set $\ell \cap Q = \{w_1, w_2\}$. Consider the pencil of hyperplane sections of $\mathcal{W}$ associated with $P$, which induces a cubic surface fibration

$$\mathrm{Bl}_{\ell \cup Q}(\mathcal{W}) \dashrightarrow \mathbb{P}^1.$$

The total space has two ordinary threefold singularities over $w_1$ and $w_2$; these are in the fibers associated with the tangent hyperplanes to $\mathcal{W}$ at $w_1$ and $w_2$. A small resolution $\widetilde{\mathcal{W}} \to \mathrm{Bl}_{\ell \cup Q}$ may be obtained by blowing up $Q$ and then $\ell$. Each fiber contains of $\varpi : \widetilde{\mathcal{W}} \to \mathbb{P}^1$ contains $\ell$ as well as $Q$. Blowing down the exceptional divisor $\ell \times \mathbb{P}^1 \subset \widetilde{\mathcal{W}}$ yields a fibration

$$\pi : \mathcal{X} \to \mathbb{P}^1$$

in quartic Del Pezzo surfaces.

The intermediate Jacobian $\mathrm{IJ}(\mathcal{X}) \simeq \mathrm{IJ}(\mathcal{W})$ has numerous Prym interpretations: For each line $\ell \subset \mathcal{W}$ projecting from $\ell$ induces a conic bundle structure

$$\mathrm{Bl}_\ell(\mathcal{W}) \to \mathbb{P}^2$$

with discriminant curve a plane quintic $D$;

$$\mathrm{IJ}(\mathcal{W}) \simeq \mathrm{Prym}(\tilde{D} \to D).$$

The conic $Q$ corresponds to fixing a point $q \in \mathbb{P}^2$, and the spectral cover $D \to \mathbb{P}^1$ arises from projection from $q$.

To get explicit equations for $\mathcal{X}$, consider the vector bundle

$$V = \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^3 + \mathcal{O}_{\mathbb{P}^1}$$

and the associated projective bundle $\mathbb{P}(V)$ with relative hyperplane class $\eta$. Let $h$ be the pull back of the hyperplane class from $\mathbb{P}^1$. Let $\mathcal{X}$ be a complete intersection of divisors in $\mathbb{P}(V)$ of degree $2\eta - 2h$ and $2\eta - h$. The canonical class of $\mathbb{P}(V)$ is $-5\eta + 3h$ so the canonical class of $\mathcal{X}$ is $-\eta$.

We have a natural inclusion of $V \subset \mathcal{O}_{\mathbb{P}^1}^{10}$ inducing a morphism

$$\mathcal{X} \to \mathbb{P}^9$$

with image $\mathcal{Y}$ singular at the image of the summand $\mathcal{O}_{\mathbb{P}^1} \subset V$. This is a singular Fano threefold of genus eight; the smooth varieties in this class arise as codimension 5 linear sections of $\mathrm{Gr}(2,6)$.

## Appendix A. Alternative characterization of height

Working over $\mathbb{C}$, we consider a flat family $\pi \colon \mathcal{X} \to \mathbb{P}^1$ of degree 4 Del Pezzo surfaces with ADE-singularities. In Definition 6.7 we defined the height $h(\mathcal{X})$ as the degree of a vector bundle on $\mathbb{P}^1$. If we assume that family is generically smooth with square-free discriminant, then $\mathcal{X}$ is a smooth projective threefold. In this case in [21] the height is defined as a triple intersection number on $\mathcal{X}$. Here we show that these two definitions agree.

**Proposition A.1.** *Let* $\pi \colon \mathcal{X} \to \mathbb{P}^1_{\mathbb{C}}$ *be a generically smooth family of degree 4 Del Pezzo surfaces with square-free discriminant. Then*

$$\int_{\mathcal{X}} c_1(\omega_\pi)^3 = -2 \deg(\pi_* \omega_\pi^{-1}).$$

*Proof.* Applying Lemma 6.9, we have

$$[\mathcal{X}] = c_2(\mathcal{O}_{\mathbb{P}((\pi_* \omega_\pi^{-1})^\vee)}(2) \otimes \pi^* \pi_*(\mathcal{I}_{\mathcal{X}}(2)))$$
$$= 4c_1(\mathcal{O}_{\mathbb{P}((\pi_* \omega_\pi^{-1})^\vee)}(1))^2 - 2\pi^* c_1(\pi_*(\mathcal{I}_{\mathcal{X}}(2))) c_1(\mathcal{O}_{\mathbb{P}((\pi_* \omega_\pi^{-1})^\vee)}(1)).$$

Therefore,

$$\int_{\mathcal{X}} c_1(\omega_\pi^{-1})^3 = \int_{\mathbb{P}((\pi_* \omega_\pi^{-1})^\vee)} [\mathcal{X}] \cdot c_1(\mathcal{O}_{\mathbb{P}((\pi_* \omega_\pi^{-1})^\vee)}(1))^3$$

$$= 4 \int_{\mathbb{P}((\pi_* \omega_\pi^{-1})^\vee)} c_1(\mathcal{O}_{\mathbb{P}((\pi_* \omega_\pi^{-1})^\vee)}(1))^5$$

$$- 2 \int_{\mathbb{P}((\pi_* \omega_\pi^{-1})^\vee)} \pi^* c_1(\pi_*(\mathcal{I}_{\mathcal{X}}(2))) c_1(\mathcal{O}_{\mathbb{P}((\pi_* \omega_\pi^{-1})^\vee)}(1))^4$$

$$= 4 \int_{\mathbb{P}^1} c_1(\pi_* \omega_\pi^{-1}) - 2 \int_{\mathbb{P}^1} c_1(\pi_*(\mathcal{I}_{\mathcal{X}}(2)))$$

$$= 2 \int_{\mathbb{P}^1} c_1(\pi_* \omega_\pi^{-1}).$$

where at the last step we have used (6.8).                     $\square$

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