SPACES OF SECTIONS OF QUADRIC SURFACE FIBRATIONS OVER CURVES

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1. Introduction

Let k be a field of characteristic not equal to two, B a smooth projective curve of genus $\mathsf{g}(B)$ over k, and F its function field. A *quadric hypersurface fibration* is a flat projective morphism $\pi : \mathcal{X} \to B$ such that each geometric fiber is a quadric hypersurface with at worst an isolated singularity and the generic fiber is smooth. Sections $\sigma : B \to \mathcal{X}$ of π are in bijection with rational points $X(F)$.

Our study is motivated by arithmetic applications and analogies between function fields of curves and number fields. When k is a finite field, the following problems have been studied by various research groups:

(1) existence of rational points, see, e.g., [CTK11], [CTSD10];
(2) bounding the smallest height of a rational point;
(3) weak approximation [Har68] and its effective versions;
(4) asymptotic distribution of rational points with respect to heights, e.g., [LY02], [Pey05], [Bou03].

All of these questions ultimately rely on algebro-geometric properties of spaces of sections. In this paper we study in detail these spaces. We relate computable invariants of quadric surfaces over function fields of curves, like the discriminant, to geometric invariants of spaces of sections such as the maximally rationally connected quotients of the section spaces.

In general, spaces of rational curves on rationally connected three-folds have intricate geometry, even for cubic threefolds or complete intersections of two quadrics in $\mathbb{P}^5$ (see e.g., [Cas04], [HRS05, HRS02]). Rational surface fibrations over $\mathbb{P}^1$ appear to be much easier. In our case, the spaces of sections turn out to be projective bundles over the Jacobian of the discriminant curve. This allows us to answer the questions above.

The geometry of the degenerations of spaces of sections serves as a prototype for investigations of sections of more complicated rational surface fibrations over curves. However, even in the case of quadric surfaces, our inductive approach has implications for enumerative geometry, e.g., the Gromov-Witten invariants associated with sections of height passing through prescribed points and curves in the fibers. We expect an inductive formula for this, expressed in terms of the numerical invariants of the fibration.

We summarize the contents of this paper. Section 2 develops general notions of height and discriminant for quadric hypersurface fibrations. Section 3 presents the key construction of reduction to the discriminant. This is fundamental and well-known in the algebraic study of quadratic forms, but here we recast it in geometric terms. We also include numerical estimates on the dimension of spaces of sections, from various points of view. In Section 4, we show how our classification techniques apply over $\mathbb{P}^1$ and offer explicit equations for the quadric surface fibrations in each case. Quadric surface fibrations admit numerous birational modifications; the structure of these is indicated in Section 5. We review some general facts about stability of bundles in Section 6. Sections 7 and 8 demonstrate how spaces of sections of quadric fibrations specialize as the discriminant curve acquires nodes. We expect there exists a compactification for the space of quadric surface fibrations with square-free discriminant over the moduli space of admissible discriminant covers (cf. [HM82]), sharing many properties with Pandharipande's compactification of the moduli space of vector bundles over moduli space of stable curves [Pan96]. The theory of Néron models in the context of limiting mixed Hodge structures offers a useful framework for the analysis of components of the space of sections as the discriminant breaks. Section 9 and 10 are devoted to arithmetic applications, e.g., effective weak approximation, which entail effective estimates of vanishing of cohomology.

2. Quadratic forms, discriminants, and heights

Let $\pi : \mathcal{X} \to B$ be a quadric hypersurface fibration of relative dimension n, as defined in the introduction. Let ω_π denote the relative dualizing sheaf, an invertible sheaf that commutes with basechange. The *height* of $\mathcal{X}$ is defined as

$$h(\mathcal{X}) = -\deg(c_1(\omega_\pi^{-1})^{n+1}).$$

Note that

- If $\mathcal{X} \to B$ is trivial, i.e., $\mathcal{X} \simeq \mathcal{X}_b \times B$ for some smooth quadric $\mathcal{X}_b$, then $h(\mathcal{X}) = 0$.
- If $B' \to B$ is a finite morphism of smooth projective curves then

$$h(\mathcal{X} \times_B B') = \deg(B'/B)h(\mathcal{X}).$$

- Every smooth quadric fibration $\mathcal{X} \to B$ also has $h(\mathcal{X}) = 0$.

To deduce the last statement, it suffices to observe that a smooth quadric fibration may be trivialized after a finite flat base change $B' \to B$.

We define the *height* of a section $\sigma : B \to \mathcal{X}$ of π to be

$$h_{\omega_\pi^{-1}}(\sigma) = \deg(\sigma^* \omega_\pi^{-1}).$$

If $\mathcal{X}$ is smooth then this equals the degree of the normal bundle N_σ. We are interested in spaces of sections

$$\mathrm{Sect}(\mathcal{X}/B, h) = \{\sigma : B \to \mathcal{X} : h_{\omega_\pi^{-1}}(\sigma) = h\}.$$

If k is algebraically closed then the Brauer group of $k(B)$ for any smooth curve B is trivial, thus there exists a line bundle H on $\mathcal{X}$ restricting to the hyperplane class on each fiber of π. The sheaf $\pi_* H$ is locally free of rank $n+2$ and we have an embedding $\mathcal{X} \hookrightarrow \mathbb{P}((\pi_* H)^\vee)$. The defining equation is given by a section

$$q \in \mathrm{Sym}^2((\pi_* H)) \otimes I,$$

where I is an invertible sheaf of B. Note that H and I can be rescaled; for each invertible sheaf L on B, we may replace H by $H \otimes L$ and I by $I \otimes L^{\otimes 2}$ without altering q. Therefore, we will often normalize H so that $\deg(I) = 0$ or 1; when using this convention, we write $E = (\pi_* H)^\vee$. The parity

$$\epsilon(\pi) := \deg(I) \pmod 2$$

is an invariant of the fibration $\pi : \mathcal{X} \to B$.

We may interpret the defining quadratic form q as a homomorphism

$$q : E \to E^\vee \otimes I,$$

self-dual under the application of $\text{Hom}(-, I)$. The *discriminant* $\mathfrak{d}$ is defined as the divisor where q drops rank, which gives [HT84]

$$\Delta = \deg(\mathfrak{d}) = \deg(E^{\vee} \otimes I) - \deg(E) = -2\deg(E) + (n+2)\deg(I),$$

so in particular

$$\Delta \equiv \begin{cases} -2\deg(E) \pmod{2(n+2)} & \text{if } \epsilon(\pi) \equiv 0 \pmod 2 \\ -2\deg(E) + n + 2 \pmod{2(n+2)} & \text{if } \epsilon(\pi) \equiv 1 \pmod 2. \end{cases}$$

The fibration $\pi : \mathcal{X} \to B$ has *square-free discriminant* if the divisor $\mathfrak{d}$ is reduced; a local computation shows this is equivalent to the total space $\mathcal{X}$ being smooth.

Proposition 1. *If $\pi : \mathcal{X} \to B$ is a quadric hypersurface fibration of relative dimension n with square-free discriminant then*

$$h(\mathcal{X}) = n^n \Delta.$$

Proof. Let $C \to B$ be a simply branched double cover whose branch locus contains the discriminant. As we have seen, pulling back to C increases the height:

$$h(\mathcal{X} \times_B C) = 2h(\mathcal{X}).$$

Let $x_1, \ldots, x_\Delta$ denote the singularities of the fibers of $\mathcal{X} \times_B C \to C$. We have a modification

$$\begin{array}{ccc} \tilde{\mathcal{Y}} & \to & \mathcal{Y} \\ \downarrow & & \\ \mathcal{X} \times_B C & & \end{array}$$

obtained by blowing up the x_j and then blowing down the proper transforms of the fibers of $\mathcal{X} \times_B C \to C$ containing these points. The resulting $\mathcal{Y} \to C$ is a smooth quadric fibration.

Let $E_1, \ldots, E_\Delta$ be the exceptional divisors of $\beta : \tilde{\mathcal{Y}} \to \mathcal{X} \times_B C$ over the ordinary singularities $x_1, \ldots, x_\Delta$; in particular, E_j is a smooth quadric of dimension n and $E_j^{n+1} = (-1)^n 2$. The discrepancy formula

$$\omega_{\tilde{\mathcal{Y}}} = \beta^* \omega_{\mathcal{X} \times_B C} + (n-1) \sum_{j=1}^{\Delta} E_j$$

implies

$$c_1(\omega_{\tilde{\mathcal{Y}}/C})^{n+1} = c_1(\omega_{\mathcal{X} \times_B C/C})^{n+1} + \Delta(n-1)^{n+1}(-1)^n 2.$$

On the other hand, let $F_1, \ldots, F_\Delta$ denote the exceptional divisors of $\gamma : \tilde{\mathcal{Y}} \to \mathcal{Y}$; $F_j \simeq \mathbb{P}(\mathcal{O}_Q \oplus \mathcal{O}_Q(1))$, where Q is a smooth quadric of dimension $n-1$ (two points when $n = 1$). It follows that

$$\gamma^* c_1(\omega_{\mathcal{Y}/C})^r \cdot F_j^{n+1-r} = (-1)^{r+1} n^r 2,$$

for $r = 0, \ldots, n-1$; we get zero when $r \geq n$. Here the discrepancy formula is

$$\omega_{\tilde{\mathcal{Y}}} = \gamma^* \omega_{\mathcal{Y}} + \sum_{j=1}^{\Delta} F_j.$$

Thus we find

$$
\begin{aligned}
c_1(\omega_{\tilde{\mathcal{Y}}/C})^{n+1} &= c_1(\omega_{\mathcal{Y}/C})^{n+1} + \Delta \sum_{r=0}^{n-1} \binom{n+1}{r}(-1)^{r+1} n^r 2 \\
&= c_1(\omega_{\mathcal{Y}/C})^{n+1} - 2\Delta \sum_{r=0}^{n-1} \binom{n+1}{r}(-n)^r \\
&= c_1(\omega_{\mathcal{Y}/C})^{n+1} - 2\Delta[(1-n)^{n+1} - ((-n)^{n+1} + (n+1)(-n)^n)] \\
&= c_1(\omega_{\mathcal{Y}/C})^{n+1} - 2\Delta((1-n)^{n+1} - (-n)^n)
\end{aligned}
$$

Note that $c_1(\omega_{\mathcal{Y}/C})^{n+1} = 0$ as $\mathcal{Y} \to C$ is smooth. Combining the results of our discrepancy computations, we obtain

$$
\begin{aligned}
c_1(\omega_{\mathcal{X} \times_B C/C})^{n+1} &= 2\Delta((n-1)^{n+1}(-1)^{n+1} - (1-n)^{n+1} + (-n)^n) \\
&= 2\Delta(-n)^n,
\end{aligned}
$$

which yields our formula. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. Reduction to the discriminant for quadric surface fibrations

We recall the standard argument of 'reduction to the discriminant', in geometric terms. Let $\mathcal{X} \to B$ be a quadric surface fibration with square-free discriminant and generic fiber X. These fibrations were studied by Bhosle [BD84], especially when $B = \mathbb{P}^1$.

The basic construction. Let $\mathcal{F} := F_1(\mathcal{X}) \to B$ denote the space of lines in fibers of π; its Stein factorization

$$\mathcal{F} \to C \xrightarrow{g} B$$

is the composition of a *smooth* $\mathbb{P}^1$-bundle and a double cover branched along the discriminant divisor $\mathfrak{d}$. Let $\iota : C \to C$ denote the covering involution.

Each section of $\pi : \mathcal{X} \to B$ yields a section of $\mathcal{F} \to C$ and vice versa. Indeed, for $\sigma : B \to \mathcal{X}$ consider the pair of lines containing $\sigma(B)$, which

is a section of $\mathcal{F} \to C$. Conversely, for each section $\tau : C \to \mathcal{F}$ we can take the intersection of lines

$$\ell_{\tau(c)} \cap \ell_{\tau(\iota(c))} \in \mathcal{X}_{g(c)},$$

which is a section. Note that the universal line over $\mathcal{F}$ is a double cover of $\mathcal{X}$.

Reversing the construction. Suppose that $g : C \to B$ is a flat morphism of smooth projective curves of degree two; we assumed the characteristic is different from two, so g is tamely ramified over a divisor $\mathfrak{d} \subset B$. Fix a $\mathbb{P}^1$-bundle $\mathcal{F} \to C$, which can be expressed as the projectivization of a vector bundle. Restriction of scalars (Weil restriction) gives a projective morphism

$$\varpi : \mathrm{Res}_{C/B}(\mathcal{F}) \to B;$$

this can be interpreted as the Hilbert scheme of length-two punctual subschemes of fibers of $\mathcal{F} \to C$. Thus for $b \in (B \setminus \mathfrak{d})(\bar{k})$ we have

$$\varpi^{-1}(b) = \mathcal{F}_{c_1} \times \mathcal{F}_{c_2} \simeq \mathbb{P}^1 \times \mathbb{P}^1, \quad g^{-1}(b) = \{c_1, c_2\},$$

geometrically a smooth quadric surface.

Over points of the branch divisor $b \in \mathfrak{d} \subset B$, the fiber $\varpi^{-1}(b)$ is set-theoretically $\mathrm{Sym}^2(\mathcal{F}_b) \simeq \mathbb{P}^2$, but non-reduced of multiplicity two. However, the restriction of scalars can be modified as follows:

$$
\begin{array}{ccc}
 & \tilde{\mathcal{X}} & \\
 \beta \swarrow & & \searrow \gamma \\
\mathrm{Res}_{C/B}(\mathcal{F}) & & \mathcal{X} \\
\varpi \searrow & & \swarrow \pi \\
 & B &
\end{array}
$$

where the arrows have the following definitions:

- β is obtained by blowing up the diagonal in $\mathrm{Sym}^2(\mathcal{F}_b)$ over each point $b \in \mathfrak{d}$;
- γ is obtained by blowing down the proper transform of $\varpi^{-1}(b)$ in $\tilde{\mathcal{X}}$ over each point $b \in \mathfrak{d}$;
- π is the induced morphism back to B.

A local computation over each $b \in \mathfrak{d}$ shows that the fiber $\mathcal{X}_b$ is isomorphic to a quadric surface with isolated singularity.

Riemann-Roch computations. We first analyze the space of sections $\mathrm{Sect}(\mathcal{X}/B, h)$ directly. Regarding $\mathrm{Sect}(\mathcal{X}/B, h)$ as an open subscheme of the Hilbert scheme of $\mathcal{X}$, we can compute its tangent space at $\sigma : B \to \mathcal{X}$

$$T_{[\sigma]}\mathrm{Sect}(\mathcal{X}/B, h) = \Gamma(N_\sigma).$$

The Riemann-Roch formula gives

$$\chi(N_\sigma) = h_{\omega_\pi^{-1}}(\sigma) + 2(1 - \mathbf{g}(B)),$$

which implies

$$\dim_\sigma \mathrm{Sect}(\mathcal{X}/B, h) \leq \dim T_\sigma \mathrm{Sect}(\mathcal{X}/B, h) = h^0(N_\sigma) \geq h + 2(1 - \mathbf{g}(B)),$$

with equality when N_σ has no higher cohomology. It is possible for $h_{\omega_\pi^{-1}}(\sigma) < 0$, but such sections are typically confined to subvarieties of $\mathcal{X}$ (see Remark 15). In characteristic zero, sections with deformations dominating $\mathcal{X}$ have normal bundles that are globally generated at the generic point, and thus have positive degree.

The discriminant construction gives an alternate approach. When k is algebraically closed or finite, we may interpret $\mathcal{F} \simeq \mathbb{P}(V)$ for a rank-two vector bundle $V \to C$. We are using the fact that the Brauer group of a projective curve over a finite field is trivial, essentially by class field theory. Let $\mathcal{O}_{\mathbb{P}(V)}(1)$ denote the resulting polarization. Let $\mathrm{Sect}(\mathcal{F}/C)$ denote the space of sections $\tau : C \to \mathcal{F} \simeq \mathbb{P}(V)$, again regarded as an open subscheme of the Hilbert scheme of $\mathcal{F}$. We have a morphism

$$\begin{aligned}\alpha : \mathrm{Sect}(\mathcal{F}/C) &\to \mathrm{Pic}(C) \\ \tau &\mapsto \tau^*\mathcal{O}_{\mathbb{P}(V)}(1)\end{aligned}$$

with fibers corresponding to extensions

$$0 \to N \to V^\vee \to \tau^*\mathcal{O}_{\mathbb{P}(V)}(1) \to 0.$$

These yield elements of

$$\mathrm{Hom}(N, V^\vee) = N^\vee \otimes V^\vee = N^\vee \otimes \bigwedge^2 V^\vee \otimes V = \tau^*\mathcal{O}_{\mathbb{P}(V)}(1) \otimes V.$$

Given $L \in \mathrm{Pic}(C)$, the sections with $\tau^*\mathcal{O}_{\mathbb{P}(V)}(1) = L$ lie in the projectivization $\mathbb{P}(\Gamma(V \otimes L))$. Thus for $d := \deg(L)$ sufficiently large, the sections form a Zariski-open dense subset of a projective bundle over $\mathrm{Pic}^d(C)$. The boundary points correspond to unions of sections with fibers of $\mathbb{P}(V) \to C$, reflecting homomorphisms $V^\vee \to \tau^*\mathcal{O}_{\mathbb{P}(V)}(1)$ with non-vanishing cokernel. The Riemann-Roch formula implies

$$\chi(V \otimes L) = 2d + \deg(V) + 2(1 - \mathbf{g}(C)) = 2d + \deg(V) - 2\mathbf{g}(C) + 2.$$

We summarize this as follows:

Proposition 2. *Retain the notation introduced above, including the choice of a vector bundle V such that $\mathcal{F} \simeq \mathbb{P}(V)$. For each $h \in \mathbb{Z}$, there exists a $d \in \mathbb{Z}$ and a morphism*

$$\gamma_h : \mathrm{Sect}(\mathcal{X}/B, h) \to \mathrm{Pic}^d(C).$$

For $h \gg 0$ this is the composition of an open immersion with a projective bundle of relative dimension

$$(3.1) \qquad\qquad 2d + \deg(V) - 4\mathsf{g}(B) - \Delta + 3.$$

The morphism γ_h and the integer d are not canonical, but depend on the choice of V. Nevertheless, comparing the expected dimensions for $\mathrm{Sect}(\mathcal{X}/B)$ and $\mathrm{Sect}(\mathcal{F}/C)$ we find

$$h + 2(1 - \mathsf{g}(B)) = 2d + \deg(V) - 4\mathsf{g}(B) - \Delta + 3 + \dim \mathrm{Pic}^d(C),$$

which yields the relation

$$(3.2) \qquad\qquad h = 2d + \deg(V) - \frac{\Delta}{2}.$$

A useful congruence. Assume k is algebraically closed. Recall the set-up in the proof of Proposition 1: We have the base-changed family $\mathcal{X} \times_B C \to C$, the singularities $x_1, \ldots, x_\Delta \in \mathcal{X} \times_B C$, and the modification:

$$
\begin{array}{ccc}
\tilde{\mathcal{Y}} & \to & \mathcal{Y} \\
\downarrow & & \\
\mathcal{X} \times_B C & &
\end{array}
$$

Consider the elementary transformation of $g^* \pi_* H$ associated with the ordinary double points [Mar82]

$$(3.3) \qquad\qquad 0 \to W \to g^* \pi_* H \to Q \to 0,$$

where Q is a skyscraper sheaf supported at $\{x_1, \ldots, x_\Delta\}$ with length one at each point. We can compute

$$
\begin{aligned}
\deg(W) &= \deg(g^* \pi_* H) - \deg(Q) = -2 \deg(E) - \Delta \\
&= \begin{cases} 0 & \text{if } \epsilon(\pi) \equiv 0 \pmod 2 \\ -4 & \text{if } \epsilon(\pi) \equiv 1 \pmod 2. \end{cases}
\end{aligned}
$$

The geometric interpretation of the elementary transformation gives an embedding

$$\mathcal{Y} \hookrightarrow \mathbb{P}(W^\vee).$$

The Fano variety of lines is a disjoint union

$$\mathcal{F}(\mathcal{Y}/C) = \mathcal{F} \sqcup \iota^* \mathcal{F} = \mathbb{P}(V) \sqcup \mathbb{P}(\iota^* V).$$

Indeed, $\mathcal{F} \times_B C$ is non-normal over the discriminant, reflecting the fact that the two rulings of a smooth quadric surface both specialize to the rulings of the quadric cone (see [HVAV11, §3] for further details). In particular, $\mathcal{Y} \to C$ is the Segre embedding of a product of two copies of $\mathbb{P}^1$, isomorphic to $\mathbb{P}(V)$ and $\mathbb{P}(\iota^* V)$. Rescaling V by tensoring with a suitable line bundle L, we can express

$$(V \otimes L) \otimes \iota^*(V \otimes L) \simeq W^\vee,$$

i.e., $\deg(V \otimes L) = 0$ or 1 depending on the parity of $\epsilon(\pi)$:

(3.4) $$\deg(V \otimes L) \equiv \epsilon(\pi) \pmod{2}.$$

Eliminating $\epsilon(\pi)$ from the expressions for $\deg(E)$ and $\deg(V)$, we find

(3.5) $$4 \deg(V) \equiv \Delta - 2 \deg(\pi_* H) = \Delta + 2 \deg(E) \pmod 8$$

This is true regardless of how we normalize H or V.

Remark 3. The key here is the coincidence of Lie theory

$$\mathfrak{so}(4, \mathbb{C}) = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}),$$

reflecting the equivalence of Dynkin diagrams

$$D_2 = A_1 \cup A_1.$$

Bichsel and Knus [BK94] compute Clifford algebras for rank-four quadratic forms taking values in invertible sheaves. This gives an alternate approach to the varieties of maximal isotropic subspaces of $\pi : \mathcal{X} \to B$. Knus, Parimala, and Sridharan [KPS86] develop the dictionary discussed here using the language of quadratic forms over an affine base. Auel [Aue11, §5.3] and Auel-Bernardara-Bolognesi [ABB11, Thm. 2.24] address this over more general base schemes.

4. Census of quadric fibrations over $\mathbb{P}^1$

Our approach here has connections to the work of Ramanan and Bhosle on vector bundles over hyperelliptic curves [DR77, Bho84, Bho98, Bho02, Bho10]. It would be very interesting to work out a complete dictionary between their work and our approach, with particular attention to degenerations of the hyperelliptic curves.

We assume $B \simeq \mathbb{P}^1$ and the discriminant curve C has genus g. Equation 3.5 is equivalent to

(4.1) $$\deg(\pi_* H) \equiv \mathsf{g} + 1 - 2 \deg(V) \pmod 4.$$

The expected dimension of the space of sections over a fixed $L \in \mathrm{Pic}(C)$ (Equation 3.1) can be written

(4.2) $$\chi(V \otimes L) - 1 = \deg(V \otimes L) - 2\mathbf{g} + 1.$$

The normalized bundle $E \sim (\pi_* H)^\vee$ satisfies

$$\deg(E) = \begin{cases} -\mathbf{g} - 1 & \text{if } \epsilon(\pi) \equiv 0 \pmod 2 \\ -\mathbf{g} + 1 & \text{if } \epsilon(\pi) \equiv 1 \pmod 2. \end{cases}$$

We can decompose

$$\pi_* H = \mathcal{O}_{\mathbb{P}^1}(-a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-a_2) \oplus \mathcal{O}_{\mathbb{P}^1}(-a_3) \oplus \mathcal{O}_{\mathbb{P}^1}(-a_4),$$
$$a_1 \leq a_2 \leq a_3 \leq a_4;$$

for simplicity, from now on assume this is as 'balanced' as possible, i.e., $a_4 - a_1 \leq 1$. We refer the reader to Section 1 of [LPS97] for a more thorough classification.

Remark 4. Any vector bundle E_0 on $\mathbb{P}^1$ admits a small deformation to a balanced vector bundle E, i.e., $E \simeq \oplus_{j=1}^r \mathcal{O}_{\mathbb{P}^1}(-m_j)$ where $|m_i - m_j| \leq 1$ for each $i, j = 1, \ldots, r$. Indeed, the splitting $E_0 \simeq \oplus_{j=1}^r \mathcal{O}_{\mathbb{P}^1}(-n_j)$ can be deformed to a non-trivial extension that is balanced [Sha78]. If $\mathrm{Sym}^2(E_0^\vee) \otimes I$ is globally generated (here I is invertible of degree 0 or 1, as in Section 2) then $H^1(\mathrm{Sym}^2(E_0^\vee) \otimes I) = 0$. Cohomology-and-base-change implies that sections in $\Gamma(\mathbb{P}^1, \mathrm{Sym}^2(E_0^\vee) \otimes I)$ arise as specializations of sections in $\Gamma(\mathbb{P}^1, \mathrm{Sym}^2(E^\vee) \otimes I)$. Thus the balanced bundles we consider are generic for large classes of quadric surface fibrations.

Case 1. Here we have $a_1 = a_2 = a_3 = a_4$ so that

$$\mathbb{P}((\pi_* H)^\vee) \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 4}) \simeq \mathbb{P}^1 \times \mathbb{P}^3.$$

The equation of $\mathcal{X}$ is a form of bidegree $(n, 2)$, thus $h(\mathcal{X}) = 16n$, $\Delta = 4n$, and $\mathbf{g} = 2n - 1$. The normalized rank-four bundle is

$$E = \begin{cases} \mathcal{O}_{\mathbb{P}^1}(-m)^{\oplus 4} & \text{if } n = 2m + 1 \text{ odd} \\ \mathcal{O}_{\mathbb{P}^1}(-m)^{\oplus 4} & \text{if } n = 2m \text{ even}. \end{cases}$$

Equation 4.1 implies that $\deg(V) \equiv n \pmod 2$. In light of (4.2), we take

$$\deg(V) = \begin{cases} 4n - 3 = 2\mathbf{g} - 1 & \text{if } n \text{ odd} \\ 4n - 2 = 2\mathbf{g} & \text{if } n \text{ even}. \end{cases}$$

A form of bidegree $(n, 2)$ depends on $(n + 1) \cdot 10 - 1 = 10n + 9$ parameters; taking into account the automorphisms of $\mathbb{P}^1 \times \mathbb{P}^3$, we are left with $10n - 9 = 5\mathbf{g} - 4$ free parameters.

Case 2. Here we have $a_1 + 1 = a_2 = a_3 = a_4$ so that

$$\mathbb{P}((\pi_* H)^\vee) \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 3}) \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^4.$$

The $\mathbb{P}^3$-bundle is given by a form of degree $(1, 1)$, and the second equation defining $\mathcal{X}$ has bidegree $(n, 2)$. We have $\omega_{\mathcal{X}/\mathbb{P}^1} = \mathcal{O}_{\mathcal{X}}(n + 1, -2)$ and $h(\mathcal{X}) = 16n + 8$; then $\Delta = 4n + 2$ and $\mathbf{g} = 2n$. The normalized twist of $(\pi_* H)^\vee$ is:

$$E = \begin{cases} \mathcal{O}_{\mathbb{P}^1}(-m - 1) \oplus \mathcal{O}_{\mathbb{P}^1}(-m)^{\oplus 3} & \text{if } n = 2m + 1 \text{ odd} \\ \mathcal{O}_{\mathbb{P}^1}(-m - 1) \oplus \mathcal{O}_{\mathbb{P}^1}(-m)^{\oplus 3} & \text{if } n = 2m \text{ even.} \end{cases}$$

Formula (4.1) implies $\deg(V) \equiv n \pmod 2$. In light of (4.2), we take

$$\deg(V) = \begin{cases} 4n - 1 = 2\mathbf{g} - 1 & \text{if } n \text{ odd} \\ 4n = 2\mathbf{g} & \text{if } n \text{ even.} \end{cases}$$

Note that $\mathcal{X}$ is cut out by a form of bidegree $(1, 1)$ and a form of bidegree $(n, 2)$, determined modulo multiples of the first form. The former depends on nine parameters, the latter on $10n + 14$ parameters (even when $n = 0$). Taking automorphisms of $\mathbb{P}^1 \times \mathbb{P}^4$ into account, we are left with a total of

$$10n - 4 = 5\mathbf{g} - 4$$

free parameters. When $n = 0$, this should be understood to mean that the families admit positive-dimensional automorphism groups.

Case 3. Here we have $a_1 + 1 = a_2 + 1 = a_3 = a_4$ hence

$$\mathbb{P}((\pi_* H)^\vee) \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}) \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^5.$$

The $\mathbb{P}^3$-bundle is given by two forms of degree $(1, 1)$, with $\mathcal{X}$ defined by one additional equation of bidegree $(n, 2)$. The dualizing sheaf $\omega_{\mathcal{X}/\mathbb{P}^1} = \mathcal{O}_{\mathcal{X}}(n + 2, -2)$, $\Delta = 4n + 4$, $\mathbf{g} = 2n + 1$, and

$$h(\mathcal{X}) = 16n + 16.$$

Here we have

$$E = \begin{cases} \mathcal{O}_{\mathbb{P}^1}(-m - 1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-m)^{\oplus 2} & \text{if } n = 2m + 1 \text{ odd} \\ \mathcal{O}_{\mathbb{P}^1}(-m - 1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-m)^{\oplus 2} & \text{if } n = 2m \text{ even.} \end{cases}$$

Formula (4.1) implies $\deg(V) \equiv n \pmod 2$. To get smallest possible non-negative dimensions in (4.2), we take

$$\deg(V) = \begin{cases} 4n + 1 = 2\mathbf{g} - 1 & \text{if } n \text{ odd} \\ 4n + 2 = 2\mathbf{g} & \text{if } n \text{ even} \end{cases}.$$

We compute the number of free parameters: The forms of bidegree $(1, 1)$ correspond to a point of $\mathrm{Gr}(2, \Gamma(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^5}(1, 1)))$, which has dimension 20. When $n > 0$, the form of bidegree $(n, 2)$ modulo the first two forms depends on $(n+1)21 - 2(n)6 + (n-1) - 1 = 10n + 19$ parameters. Taking automorphisms into account, we obtain

$$20 + (10n + 19) - 38 = 10n + 1 = 5(2n + 1) - 4 = 5\mathbf{g} - 4$$

parameters. When $n = 0$ and $\mathbf{g} = 1$, the construction depends on two parameters.

Case 4. In this case $a_1 = a_2 = a_3 = a_4 - 1$ hence

$$\mathbb{P}((\pi_* H)^\vee) \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 3}(-1) \oplus \mathcal{O}_{\mathbb{P}^1})) \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^6.$$

The $\mathbb{P}^3$-bundle is given by three forms of degree $(1, 1)$, with $\mathcal{X}$ defined by one additional equation of bidegree $(n, 2)$. The dualizing sheaf $\omega_{\mathcal{X}/\mathbb{P}^1} = \mathcal{O}_{\mathcal{X}}(n + 3, -2)$, $\Delta = 4n + 6$, $\mathbf{g} = 2n + 2$, and

$$h(\mathcal{X}) = 16n + 24.$$

In this case

$$E = \begin{cases} \mathcal{O}_{\mathbb{P}^1}(-m - 1)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(-m) & \text{if } n = 2m + 1 \text{ odd} \\ \mathcal{O}_{\mathbb{P}^1}(-m - 1)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(-m) & \text{if } n = 2m \text{ even.} \end{cases}$$

Formula (4.1) implies $\deg(V) \equiv n \pmod 2$. In light of (4.2), we take

$$\deg(V) = \begin{cases} 4n + 3 = 2\mathbf{g} - 1 & \text{if } n \text{ odd} \\ 4n + 4 = 2\mathbf{g} & \text{if } n \text{ even.} \end{cases}$$

We compute free parameters: The forms of bidegree $(1, 1)$ correspond to a point of $\mathrm{Gr}(3, \Gamma(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^6}(1, 1)))$, which has dimension 33. The form of bidegree $(n, 2)$, modulo the first three forms, depends on

$$(n + 1)28 - 3(n)7 + 3(n - 1) - 1 = 10n + 24$$

parameters. Taking automorphisms into account, we obtain

$$33 + (10n + 24) - 51 = 10n + 6 = 5(2n + 2) - 4 = 5\mathbf{g} - 4$$

parameters.

Remark 5. In our analysis the case where the discriminant curve has genus zero stands out; we have yet to exhibit an example where $\mathbf{g} = 0$ and $\deg(V)$ is odd. This may be interpreted as the $n = -1$ instance of Case 4 above.
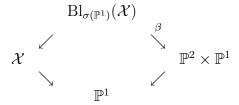
Specifically, there are quadric surface fibrations

$$\mathcal{X} \subset \mathbb{P}((\pi_* H)^\vee) \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 3}(-1) \oplus \mathcal{O}_{\mathbb{P}^1})$$

that do not arise as restrictions of hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^6$. These correspond to global sections of

$$\mathrm{Sym}^2(\pi_* H) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 6} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(-1),$$

which necessarily contain the distinguished section $\sigma : \mathbb{P}^1 \to \mathbb{P}((\pi_* H)^\vee)$. Projecting from σ, we obtain

$$\mathrm{Bl}_{\sigma(\mathbb{P}^1)}(\mathcal{X})$$

$$\mathcal{X} \qquad \overset{\beta}{\searrow} \qquad \mathbb{P}^2 \times \mathbb{P}^1$$

$$\mathbb{P}^1$$

where β blows up a genus-zero bisection $\mathcal{Z} \subset \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^1$. The bisection $\mathcal{Z}$ is a complete intersection of hypersurfaces of bidegrees $(1,0)$ and $(2,1)$ in $\mathbb{P}^2 \times \mathbb{P}^1$. The former takes the form $\ell \times \mathbb{P}^1$, where $\ell \subset \mathbb{P}^2$ is a line, and coincides with the proper transform of the exceptional divisor of $\mathrm{Bl}_{\sigma(\mathbb{P}^1)}(\mathcal{X}) \to \mathcal{X}$. Constant sections of $\mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^1$ induce sections of $\pi : \mathcal{X} \to \mathbb{P}^1$; points of ℓ give rise to reducible curves, consisting of the union of $\sigma(\mathbb{P}^1)$ and a line in a fiber of π incident to $\sigma(\mathbb{P}^1)$.

The families constructed here admit positive-dimensional automorphism groups.

We summarize our computations in the following table:

Case	Δ (mod 8)	$n \equiv \deg(V)$ (mod 2)	$\mathbf{g}$ (mod 4)
1	0	0	-1
	4	1	1
2	2	0	0
	6	1	2
3	4	0	1
	0	1	-1
4	6	0	2
	2	1	0

Parameter counts and relations to moduli spaces of bundles.
When V is a rank-two vector bundle over C, we have

$$\chi(\text{End}(V)) = 4(1 - \mathbf{g}).$$

When V is simple, the moduli space has dimension $4\mathbf{g} - 3$; fixing the determinant gives a moduli space of dimension $3\mathbf{g} - 3$. Taking into account the fact that $\mathbb{P}(V \otimes L) \simeq \mathbb{P}(V)$ for each $L \in \text{Pic}(C)$, the corresponding moduli space of $\mathbb{P}^1$-bundles over C also depends on $3\mathbf{g}-3$ parameters. Hyperelliptic curves depend on $2\mathbf{g} - 1$ parameters so the total number of free parameters is

$$3\mathbf{g} - 3 + 2\mathbf{g} - 1 = 5\mathbf{g} - 4,$$

the number of free parameters we observed in each case.

5. Hecke correspondences and elementary transformations

The data tabulated above exhibit an involution preserving Δ (mod 8) and $\mathbf{g}$ (mod 4) but reversing the parity of $\deg(V)$ and altering $\deg(\pi_*H)$ (mod 4) by two. This can be explained geometrically via elementary transformations.

Fix a smooth fiber $\mathcal{X}_p$ of π and a line $\ell \subset \mathcal{X}_p$. Applying an elementary transformation along ℓ converts $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 4})$ to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2})$ (resp. $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 3})$ to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1})$). The proper transform $\tilde{\pi} : \tilde{\mathcal{X}} \to \mathbb{P}^1$ of $\mathcal{X}$ is still a quadric surface fibration with the same degenerate fibers. This also induces an elementary transformation of $\mathcal{F}(X) = \mathbb{P}(V) \to C$ at the point ℓ, which is $\mathcal{F}(\tilde{X})$; this changes the parity of the degree of this rank-two bundle.

This process does change the heights of sections of $\pi : \mathcal{X} \to \mathbb{P}^1$. Suppose that $\sigma : \mathbb{P}^1 \to \mathcal{X}$ is a section disjoint from ℓ, with proper transform $\tilde{\sigma} : \mathbb{P}^1 \to \tilde{\mathcal{X}}$. The birational map $\mathcal{X} \dashrightarrow \tilde{\mathcal{X}}$ factors

$$\text{Bl}_\ell(\mathcal{X})$$

$$\swarrow \qquad \searrow$$

$$\mathcal{X} \qquad\qquad\qquad \tilde{\mathcal{X}},$$

where the right arrow blows down the proper transform of ℓ. Thus we find

$$\deg(\tilde{\sigma}^* \omega_{\tilde{\mathcal{X}}/\mathbb{P}^1}) = \deg(\sigma^* \omega_{\mathcal{X}/\mathbb{P}^1}) - 1$$

and

$$h_{\omega_{\tilde{\pi}}^{-1}}(\tilde{\sigma}) = h_{\omega_{\pi}^{-1}}(\sigma) + 1.$$

Thus taking elementary transformations along lines incident to a section reduces the height of that section.

If we apply *two* elementary transformations to $\mathcal{X} \to \mathbb{P}^1$, the resulting quadric fibration has the same numerical invariants and discriminant curve $C \to B$. And of course, the resulting fibrations are birational over $\mathbb{P}^1$ but not isomorphic, as the corresponding vector bundles are related by an elementary transformation. This is an instance of a *Hecke correspondence* on the moduli space of vector bundles over C; these have been studied by many authors, e.g., [NR78].

6. Stable bundles and cohomology

In Section 3, we saw how to translate the existence of sections of quadric surface bundles $\mathcal{X} \to B$ to the existence of sections of a ruled surface $\mathcal{F} \simeq \mathbb{P}(V) \to C$, where C is the discriminant double cover of B and V is a rank-two vector bundle over C. The behavior of the sections of a ruled surface depends on the characteristics of this vector bundle; in general, there is little we can say uniformly without making some assumptions on the vector bundle.

Throughout this section, C is a smooth projective curve of genus g over an algebraically closed field. Recall that a locally-free sheaf V over C is *stable* (resp. *semistable*) if, for every locally-free quotient

$$V \to W \to 0$$

we have

$$\deg(W)/\mathrm{rank}(W) > (\text{ resp. } \geq) \deg(V)/\mathrm{rank}(V).$$

A vector bundle is *stable* if the associated locally-free sheaf of sections is stable. The stability of a vector bundle is not affected by tensoring it by a line bundle, or by taking its dual.

General facts on rank-two bundles. Here we collect more refined vanishing results for stable bundles of rank two, which will be useful for effective estimates for the numbers of sections with prescribed properties.

Let V be a rank-two vector bundle over a C and $M \subset V$ an invertible subsheaf of maximal degree. Recall that V/M is invertible and

$$\Gamma(V \otimes M_1^{-1}) = 0$$

for all invertible M_1 with $\deg(M_1) > \deg(M)$ [Har77, V.2]. This implies that $\dim \Gamma(V \otimes M^{-1}) \leq 2$.

Proposition 6. *Let V be a vector bundle on C of rank two.*

- *An invertible subsheaf of maximal degree $M \subset V$ satisfies* [Nag70]

$$\frac{\deg(V) - \mathsf{g}}{2} \leq \deg(M).$$

  If V is semistable then $\deg(M) \leq \deg(V)/2$.
- *If V is semistable then for a generic line bundle L on C of degree zero we have* [Ray82, Prop. 1.6.2]

$$h^0(V \otimes L) = \max(0, \chi(V)).$$

Remark 7. The original formulation of Nagata's theorem referenced above is worth mentioning: Let $\mathbb{P}(V) \to C$, where V is a rank-two vector bundle over C (not necessarily semistable). Then there exists a section $\tau : C \to \mathbb{P}(V)$ such that

$$\tau(C) \cdot \tau(C) \leq \mathsf{g}.$$

Corollary 8. *Let V be a semistable vector bundle of rank two over C. If $\deg(V) \geq 3\mathsf{g} + 2$ then $V \otimes L$ is globally generated with vanishing higher cohomology for generic $L \in \mathrm{Pic}^0(C)$. When $\deg(V) \geq 4\mathsf{g} - 1$ then V itself is globally generated with vanishing higher cohomology.*

Proof. For the first assertion, Proposition 6 implies we may express V as an extension of invertible sheaves

$$0 \to M \to V \to V/M \to 0,$$

where $\deg(M) \geq \mathsf{g} + 1$ and $\deg(V/M) \geq \frac{3}{2}\mathsf{g} + 1$ if $\deg(V) \geq 3\mathsf{g} + 2$. For generic L, $M \otimes L$ and $(V/M) \otimes L$ are globally generated with vanishing higher cohomology, so the same is true for $V \otimes L$.

We are grateful to N. Hoffmann for suggesting improvements on the bound for the second assertion. To prove that V is globally generated, it suffices to show that for each expression

$$0 \to U \to V \to Q \to 0$$

with Q a torsion sheaf of length one, we have $\Gamma(U) \subsetneq \Gamma(V)$. If we can show that $H^1(U) = 0$, it will follow that $H^1(V) = 0$ and

$$\dim \Gamma(U) = \chi(U) = \chi(V) - 1 < \dim \Gamma(V).$$

However, if $H^1(U)$ were non-vanishing then $\Gamma(\omega_C \otimes U^\vee) \neq 0$ by Serre duality. Let $N \subset \omega_C \otimes U^\vee$ denote the saturation of some non-vanishing section, whence $\deg(N) \geq 0$; consider the resulting extension

$$0 \to N \to \omega_C \otimes U^\vee \to \omega_C^2 \otimes N^{-1} \otimes \det(U)^{-1} \to 0.$$

On dualizing, we obtain

$$0 \to N \otimes \det(U) \otimes \omega_C^{-1} \to U \to N^{-1} \otimes \omega_C \to 0.$$

We may regard the first term as a subsheaf of V, so semistability implies

$$\deg(N \otimes \det(U) \otimes \omega_C^{-1}) \le \deg(V)/2$$

and

$$\deg(N) + \deg(V) - 1 - (2\mathbf{g} - 2) \le \deg(V)/2.$$

Thus we conclude

$$\deg(N) \le \frac{-\deg(V)}{2} + 2\mathbf{g} - 1 < 0,$$

a contradiction. $\square$

7. Projective bundles over limits of hyperelliptic curves

The purpose of this and the subsequent section is to analyze how sections of quadric surface fibrations specialize as the base of the fibration degenerates to a nodal curve. Essentially, the excellent *a priori* control we have for sections of quadric surface fibrations gives structure to how sections 'break' as the fibration breaks into a union of two fibrations of smaller height. We carry out this analysis with a view toward understanding the behavior of sections of del Pezzo fibrations of smaller degree over $\mathbb{P}^1$.

Sections of projective bundles over nodal curves. Let C be a nodal projective curve of arithmetic genus $\mathbf{g}$ over an algebraically closed field.

Lemma 9. *If $P \to C$ is a projective bundle then there exists a vector bundle V over C such that $P = \mathbb{P}(V)$. Sections $t : C \to P$ with $t^* \mathcal{O}_{\mathbb{P}(V)}(1) \simeq L$ correspond to short exact sequences*

$$0 \to N \to V^\vee \to L \to 0,$$

or equivalently, elements of

$$\mathbb{P}(\Gamma(C, V \otimes L))$$

such that the induced $V^\vee \to L$ is surjective.

Proof. The obstruction to lifting a cocycle in PGL_r to GL_r lies in the Brauer group, which is trivial on a curve. This gives the first assertion. The second assertion is the standard characterization of morphisms into projective space. $\square$

Let e denote an integer-valued function from the set of irreducible components of C and $|e|$ the sum of this function over these components. Consider the irreducible component of the Hilbert scheme $\mathrm{Sect}(P/C, e)$ containing the sections

$$\{\tau : C \to P : \deg(\tau^* \mathcal{O}_{\mathbb{P}(V)}(1)) = e\}.$$

We have a rational map

$$\begin{aligned}
\alpha_e : \mathrm{Sect}(P/C, e) &\dashrightarrow \mathrm{Pic}^e(C) \\
\tau &\mapsto \tau^*(\mathcal{O}_{\mathbb{P}(V)}(1)).
\end{aligned}$$

We are interested in those e such that α_e is dominant. By Lemma 9, these include all e such that, for generic $L \in \mathrm{Pic}^e(C)$, we have

(7.1) $$\Gamma(V \otimes I \otimes L) \subsetneq \Gamma(V \otimes L)$$

for each ideal sheaf $I \subsetneq \mathcal{O}_C$. Indeed, if $V^\vee \to L$ fails to be surjective then its image is isomorphic to $L \otimes I$ for some ideal sheaf I.

Applications to degenerate quadric fibrations. Here, a *degenerate quadric surface fibration* consists of

- a connected nodal curve

$$B := B_1 \cup_p B_2$$

  with a single node p;
- a flat morphism from a projective scheme

$$\pi : \mathcal{X} \to B,$$

  such that the restrictions

$$\pi_j := \pi|\mathcal{X}_j = \mathcal{X} \times_B B_j \to B_j, \quad j = 1, 2,$$

  are quadric surface fibrations smooth over p with square-free discriminant elsewhere.

Let $g : C \to B$ denote the discriminant curve; note that

$$C = C_1 \cup_{q,r} C_2, \quad g(q, r) = p$$

where $g|C_i : C_i \to B_i$ is a double cover. The Fano variety $\mathcal{F}$ of lines on $\mathcal{X}$ remains a $\mathbb{P}^1$-bundle over C. We can express $\mathcal{F} = \mathbb{P}(V)$, where V is a rank-two vector bundle on C by Lemma 9. The argument of Section 3 still yields a natural identification between $\mathrm{Sect}(\mathcal{X}/B)$ and $\mathrm{Sect}(\mathcal{F}/C)$. As before, we define

$$\epsilon(\pi) \equiv \deg(V) \pmod 2,$$

so that $\epsilon(\pi) \equiv \epsilon(\pi_1) + \epsilon(\pi_2)$. This is invariant under deformations of π, including smoothings to quadric surface fibrations over smooth curves.

Proposition 10. *Let $\pi : \mathcal{X} \to B$ be a degenerate quadric surface fibration as described above, over a curve of genus zero. Assume that the discriminant curve $C = C_1 \cup C_2 \to B$ has genus g and admits a component $C_1 \simeq \mathbb{P}^1$. Set*

$$h = \begin{cases} \mathsf{g} - 1 & \text{if } \epsilon(\pi) \equiv 0 \pmod 2 \\ \mathsf{g} - 2 & \text{if } \epsilon(\pi) \equiv 1 \pmod 2 \end{cases}$$

and consider

$$\gamma_h : \mathrm{Sect}(\mathcal{X}/B, h) \to \mathrm{Pic}(C).$$

(1) If $\epsilon(\pi_1) \equiv \epsilon(\pi_2) \equiv 0$ then γ_h dominates two components of the Picard variety, over which the generic fiber $\simeq \mathbb{P}^1$.

(2) If $\epsilon(\pi_1) \equiv \epsilon(\pi_2) \equiv 1$ then γ_h dominates three components of the Picard variety, over which the generic fiber $\simeq \mathbb{P}^1$.

(3) If $\epsilon(\pi_1) \not\equiv \epsilon(\pi_2)$ then γ_h dominates two components of the Picard variety, over which it is birational.

Proof. In light of the analysis in Section 4, we normalize

$$\deg(V) = \begin{cases} 2\mathsf{g} & \epsilon(\pi) \equiv 0 \pmod 2 \\ 2\mathsf{g} - 1 & \epsilon(\pi) \equiv 1 \pmod 2 \end{cases}$$

which means that $\chi(V) = 2$ in the even case and 1 in the odd case. Note that a generic vector bundle of this degree on a smooth projective curve of genus g has no higher cohomology (see Proposition 6).

Odd case. Our first subcase is

$$\deg(V|C_1) \equiv 0 \pmod 2, \quad \deg(V|C_2) \equiv 1 \pmod 2.$$

The possibilities compatible with (7.1) are:

- $\deg(V \otimes L|C_1) = 2, \quad \deg(V \otimes L|C_2) = 2\mathsf{g} - 3$
- $\deg(V \otimes L|C_1) = 0, \quad \deg(V \otimes L|C_2) = 2\mathsf{g} - 1$

In the first case, $V \otimes L|C_1 \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$ and $\Gamma(V \otimes L|C_2)$ admits a unique non-zero section, up to scalar. On gluing we see that $V \otimes L$ admits a unique section as well. Otherwise, $V \otimes L|C_1 \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$ which is globally generated by two sections. Since $V \otimes L|C_2$ has a three-dimensional space of sections, after gluing we have a unique section up to scalar.

The other subcase is

$$\deg(V|C_1) \equiv 1 \pmod 2, \quad \deg(V|C_2) \equiv 0 \pmod 2,$$

which leads to the possibilities:

- $\deg(V \otimes L|C_1) = 1, \quad \deg(V \otimes L|C_2) = 2\mathsf{g} - 2$

- $\deg(V \otimes L|C_1) = -1, \quad \deg(V \otimes L|C_2) = 2g$

In the first instance, $V \otimes L|C_1 \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ which admits three sections, and $V \otimes L|C_2$ generally has a two-dimensional space of sections. In the second instance, $V \otimes L|C_1 \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ which admits a single section and $V \otimes L|C_2$ generally has a four-dimensional space of sections. In both instances, we have a unique section up to scalar.

Even case. Our first subcase is

$$\deg(V|C_1) \equiv \deg(V|C_2) \equiv 0 \pmod 2.$$

The possibilities consistent with (7.1) are limited to:

- $\deg(V \otimes L|C_1) = 2, \quad \deg(V \otimes L|C_2) = 2(g-1)$
- $\deg(V \otimes L|C_1) = 0, \quad \deg(V \otimes L|C_2) = 2g$.

In the former case, we have $V \otimes L|C_1 \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$, which is globally generated with four sections. For generic L, we find that $V \otimes L|C_2$ admits a two-dimensional space of sections. Overall, we find that $V \otimes L$ has a two-dimensional space of sections. In the latter case, we have $V \otimes L|C_1 \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$ which is globally generated with two sections. For generic L, $V \otimes L|C_2$ admits a four-dimensional space of sections. After gluing we find that $\Gamma(V \otimes L)$ is two-dimensional.

Our second subcase is

(7.2) $$\deg(V|C_1) \equiv \deg(V|C_2) \equiv 1 \pmod 2.$$

Condition (7.1) allows the following *three* possibilities:

- $\deg(V \otimes L|C_1) = 1, \quad \deg(V \otimes L|C_2) = 2g - 1$
- $\deg(V \otimes L|C_1) = -1, \quad \deg(V \otimes L|C_2) = 2g + 1$
- $\deg(V \otimes L|C_1) = 3, \quad \deg(V \otimes L|C_2) = 2g - 3$

Again, in each case we find that $\Gamma(V \otimes L)$ is two-dimensional. $\qquad\square$

8. Limits of sections and Néron models of intermediate Jacobians

We retain the notation of Section 7. Let

$$\mathbb{D} = \{t \in \mathbb{C} : 0 < t < 1\}$$

denote a complex disc, $\mathcal{X}(t)$ a family of quadric surface fibrations specializing to $\mathcal{X} = \mathcal{X}(0)$ over $\mathbb{D}$, and $\mathcal{C}(t)$ the corresponding family of discriminant curves specializing to C. Note that this family is *not* stable, as the component $C_1 \simeq \mathbb{P}^1 \subset C$ must be contracted in a stable reduction.

The intermediate Jacobian $IJ(\mathcal{X}(t))$ is isomorphic to the Jacobian $J(\mathcal{C}(t))$ of the discriminant curve $\mathcal{C}(t)$. Here we compute the special fiber $\tilde{J}_e(0)$ of the Néron model

$$\tilde{J}_e \to \mathbb{D}$$

of the intermediate Jacobians $IJ(\mathcal{X}(t))$, following the exposition of [GGK10, pp.313-314], which draws on previous work of I. Nakamura [Nak77].

There is a basis for the homology of $\mathcal{C}(t)$ such that the monodromy matrix takes the form

$$\begin{pmatrix} I_{2g-2} & 0 \\ 0 & T \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$

where I_{2g-2} is the identity matrix of the indicated size. The logarithm of this matrix takes the form

$$\begin{pmatrix} 0 & 0 \\ 0 & N \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

Applying formula II.C.1 of [GGK10], we obtain an exact sequence

$$0 \to J(C) \to \tilde{J}_e(0) \to G \to 0,$$

where $G = \mathbb{Z}/2\mathbb{Z}$ is the group of connected components. Note that $J(C)$ is itself an extension

$$0 \to \mathbb{G}_m \to J(C) \to J(C_2) \to 0.$$

This extension is important because it is the target of the cycle class map for *limits* of one-cycles homologous to zero. Let $\mathcal{X}(t)$ be a family of quadric fibrations with $\mathcal{X} \simeq \mathcal{X}(0)$ and $\mathcal{X}(t)$ non-singular for $t \neq 0$. Let $Z^2(\mathcal{X}(t))$ denote the codimension-two cycles of $\mathcal{X}(t)$ homologous to zero, e.g., differences of two sections of $\pi(t) : \mathcal{X}(t) \to \mathbb{P}^1$. Let $\mathcal{Z}(t)$ denote a family of such cycles in $\mathcal{X}(t)$, with $\mathcal{Z}(t)$ homologous to zero for $t \neq 0$. Note however that $\mathcal{Z}(0)$ need not be homologous to zero, e.g., when it is a difference of two sections of

$$\pi : \mathcal{X} \to B = B_1 \cup_p B_2$$

whose heights are equal but are allocated differently between the components of $\mathcal{X}$. Nevertheless, the Abel-Jacobi images of the $\mathcal{Z}(t)$ yield a section of

$$\tilde{J}_e \to \mathbb{D}$$

and thus an element $\gamma(\mathcal{Z}(0)) \in \tilde{J}_e(0)$.

This is visibly consistent with the description in Section 7, except in the case (7.2) where there are *three* kinds of sections of the projective

bundle but only *two* components of the Néron model. We explain the geometry of the induced mapping

$$\gamma : \mathrm{Sect}(\mathcal{X}/B, h) \to \tilde{\mathrm{J}}_e(0),$$

where the height is chosen so that the sections correspond to elements of $\Gamma(V \otimes L)$ with $\deg(V \otimes L) = 2\mathbf{g}$.

Proposition 11. *Recall the notation and assertions of Proposition 10. For assertions 1 and 3, the two components of $\mathrm{Pic}(C)$ dominated by γ_h correspond to the two components of $\tilde{\mathrm{J}}_e(0)$. For assertion 2, sections corresponding to the cases*

$$\deg(V \otimes L|C_1) = -1, \quad \deg(V \otimes L|C_2) = 2\mathbf{g} + 1$$

and

$$\deg(V \otimes L|C_1) = 3, \quad \deg(V \otimes L|C_2) = 2\mathbf{g} - 3$$

are mapped onto the same connected component of $\tilde{\mathrm{J}}_e(0)$.

Proof. Only the last statement requires proof. Indeed, the fact that these are related by an involution can be seen by tensoring L in the first case by $\mathcal{O}(C_2)|C_2$, where C_2 is regarded as a Cartier divisor on the total space of C_t. This increases the degree of the vector bundle on C_1 (and decreases the degree on C_2) by four.   □

The functorial properties of Néron-models allow us to compactify

$$\tilde{\mathrm{J}}_e(0) \subset \overline{\mathrm{J}}_e(0),$$

where the latter is a $\mathbf{g}$-dimensional toroidal compactification over the $\mathbf{g} - 1$-dimensional abelian variety $\mathrm{J}(C_2)$. Thus its fibers consist of pairs of $\mathbb{P}^1$'s meeting in two nodes. The fibers of

$$\overline{\mathrm{Sect}}(\mathcal{X}/B, h) \dashrightarrow \overline{\mathrm{J}}_e(0) \to \mathrm{J}(C_2),$$

from the irreducible component of the Hilbert scheme compactifying $\mathrm{Sect}(\mathcal{X}/B, h)$, do have three components corresponding to the three cases of (7.2).

Remark 12. The analysis of limiting intermediate Jacobians here is also reminiscent of Caporaso's compactification of the relative Picard scheme over the moduli space of stable curves [Cap94], especially Section 7.3 which addresses curves with two components. The two cases of (7.2) identified to the same component of $\tilde{\mathrm{J}}_e(0)$ correspond to strictly semistable line bundles that are identified by the Geometric Invariant Theory.

Remark 13. An alternative approach to limits of intermediate Jacobians via log geometry can be found in [KNU10].

9. STABILITY AND DISCRIMINANT CURVES

Let $\pi : \mathcal{X} \to B$ be a quadric surface fibration with square-free discriminant. Let $\mathcal{F} \to C \xrightarrow{g} B$ be the Fano variety of lines, realized as a $\mathbb{P}^1$-bundle over the discriminant, and $\iota : C \to C$ the covering involution over B. Consider a section $\sigma : B \to \mathcal{X}$ of π and the corresponding section $\tau : C \to \mathcal{F}$ described in Section 3.

If π is smooth then the restriction of scalars $\varpi : \mathrm{Res}_{C/B}(\mathcal{F}) \to B$ is isomorphic to $\mathcal{X} \to B$; this yields an isomorphism of normal bundles

$$g_* N_\tau = N_\sigma.$$

Furthermore, expressing $\mathcal{F}$ as the projectivization $\mathbb{P}(V)$ of a vector bundle on C, we can write

$$g^*((\pi_* H)^\vee) \simeq V \otimes \iota^* V$$

for a suitable normalization of H. This reflects the fact that $\mathcal{X} \hookrightarrow \mathbb{P}((\pi_* H)^\vee)$ is the Segre embedding of $\mathrm{Res}_{C/B}(\mathcal{F})$.

We extend these formulas to quadric surface fibrations with square-free discriminant, with a view toward comparing various notions of stability and applying the results of Section 6. As in Section 3, there exists a vector bundle W on C such that

- $W^\vee \simeq V \otimes \iota^* V$, which yields an involution

$$
\begin{array}{ccc}
W^\vee & \xrightarrow{i} & W^\vee \\
\downarrow & & \downarrow \\
C & \xrightarrow{\iota} & C
\end{array}
$$

  given by $i(v_c \otimes v'_{\iota(c)}) = v'_{\iota(c)} \otimes v_c$.
- W arises as an extension (see 3.3))

$$0 \to W \to g^* \pi_* H \to Q \to 0,$$

  where Q is a skyscraper sheaf supported at the singularities of the fibers of $\mathcal{X} \times_B C \to C$.

The extension above dualizes to

$$0 \to g^*(\pi_* H)^\vee \to W^\vee \to Q' \to 0,$$

where Q' is also a skyscraper sheaf supported where the fibration fails to be smooth.

Now V is stable provided $W^\vee$ is stable, or even if $(W^\vee, i)$ is stable as a bundle with involution, i.e., we only test against quotient bundles compatible with the action of i. Bhosle [BD84, §1] shows that this is equivalent to the following form of *parabolic semistability* for $(\pi_* H)^\vee$: For every isotropic subbundle $F \subset (\pi_* H)^\vee$ we have

$$\frac{\deg(\pi_* H)^\vee + \frac{1}{2}\Delta}{4} \geq \frac{\deg(F) + \frac{1}{2}\#\{x_j \in \mathbb{P}(F)\}}{r}$$

where $r = \operatorname{rank}(F)$. For stability, we impose strict inequality. (There appears to be a notational inconsistency between Definition 1.1 and Proposition 1.2 of [BD84]; here we rely on the proof of the proposition.)

If the covering $C \to B$ is non-trivial then $\mathcal{F} \to B$ has no sections, i.e., there are no isotropic subbundles or rank two. Thus isotropic subbundles $F \neq 0$ have rank one, so $\mathbb{P}(F) = \sigma(B)$ for some section $\sigma : B \to \mathcal{X}$. And if $\pi : \mathcal{X} \to B$ has square-free discriminant, sections avoid singularities in the fibers. Then the condition takes the form

$$(9.1) \qquad \deg((\pi_* H)^\vee) + \frac{1}{2}\Delta \geq 4 \deg(F).$$

We can express this in terms of the heights of the fibrations, using the formulas of Section 2. We may normalize $(\pi_* H)^\vee = E$ where

$$\Delta = -2 \deg(E) + 4 \deg(I), \quad \deg(I) = \begin{cases} 0 & \text{if } \epsilon(\pi) \equiv 0 \pmod 2 \\ 1 & \text{if } \epsilon(\pi) \equiv 1 \pmod 2 \end{cases}.$$

Let $\psi : \mathbb{P}(E) \to B$ be the structure map and $\xi = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$ the relative hyperplane class; it follows that

$$c_1(T_{\mathbb{P}(E)/B}) = c_1(E) + 4\xi.$$

The quadratic form defining $\mathcal{X}$ is a symmetric homomorphism $q : E \to E^\vee \otimes I$, so

$$[\mathcal{X}] = 2\xi + c_1(I).$$

The standard exact sequence

$$0 \to T_{\mathcal{X}/B} \to T_{\mathbb{P}(E)/B}|\mathcal{X} \to N_{\mathcal{X}/\mathbb{P}(E)} \to 0$$

implies

$$c_1(T_{\mathcal{X}/B}) = c_1(E) + 2\xi - c_1(I).$$

Pulling back via σ, we obtain

$$h_{\omega_\pi^{-1}}(\sigma) = \deg(E) + 2 \deg(\sigma^* \xi) - \deg(I) = -\frac{1}{4}\Delta + \frac{1}{2} \deg(E) + 2 \deg(\sigma^* \xi).$$

On the other hand, we can express

$$F = \sigma^* \mathcal{O}_{\mathbb{P}(E)}(-1),$$

as the latter is the tautological subbundle for $\mathbb{P}(E)$. Thus Equation 9.1 translates into

$$\deg(E) + \frac{1}{2}\Delta \geq -4 \deg \sigma^* \xi = -2h_{\omega_\pi^{-1}}(\sigma) - \frac{1}{2}\Delta + \deg(E)$$

which simplifies to

$$h_{\omega_\pi^{-1}}(\sigma) \geq -\frac{\Delta}{2}.$$

We summarize this computation:

Proposition 14. *Let $\pi : \mathcal{X} \to B$ be a quadric surface fibration with square-free discriminant of degree Δ. Assume that*

- *the discriminant double cover $C \to B$ is non-trivial;*
- *all sections $\sigma : B \to \mathcal{X}$ satisfy the inequality*

$$h_{\omega_\pi^{-1}}(\sigma) \geq -\frac{\Delta}{2}.$$

Then the Fano variety of lines $\mathcal{F} \to C$ is the projectivization of a semistable vector bundle.

Note that the first condition holds provided $\operatorname{Pic}(X) \simeq \mathbb{Z}$, i.e., the monodromy/Galois action exchanges the two rulings of the geometric generic fiber of π.

Remark 15. The inequality can be verified effectively. Since ω_π^{-1} is ample relative to π, there exists a line bundle A on B such that $\omega_\pi^{-1} \otimes A$ is ample on $\mathcal{X}$. The sections violating this condition have degree less than $\deg(A) - \frac{\Delta}{2}$, and thus are bounded.

10. ARITHMETIC APPLICATIONS

From now on, we assume k is a finite field of odd characteristic and $B \simeq \mathbb{P}^1$. We retain the notation of Sections 3 and 4, so $\mathcal{X} \to \mathbb{P}^1$ is a quadric surface fibration with square-free discriminant of degree $\Delta = 2g + 2$. Let C be the discriminant curve and $\mathcal{F} \to C$ the Fano variety of lines. The geometric analysis of Section 3 applies, giving a bijection between sections $\sigma : \mathbb{P}^1 \to \mathcal{X}$ and sections $\tau : C \to \mathcal{F}$.

Effective existence results.

Proposition 16. *There exists a section* $\sigma : \mathbb{P}^1 \to \mathcal{X}$ *defined over* k *with*

$$(10.1) \qquad h_{\omega_\pi^{-1}}(\sigma) \leq \begin{cases} \frac{\Delta}{2} - 2 & \text{if } \epsilon(\pi) \equiv 0 \pmod{2} \\ \frac{\Delta}{2} - 3 & \text{if } \epsilon(\pi) \equiv 1 \pmod{2}. \end{cases}$$

Proof. Recall Equations 3.4 and 3.2

$$\deg(V) \equiv \epsilon(\pi) \pmod{2}, \quad h_{\omega_\pi^{-1}}(\sigma) = \deg(V \otimes L) - \Delta/2.$$

The morphism $\lambda : \mathrm{Sect}(\mathcal{F}/C) \to \mathrm{Pic}(C)$ is dominant over $\mathrm{Pic}^d(C)$ provided $\Gamma(V \otimes L) \neq 0$ for generic L of degree d. This is guaranteed to be the case if $\chi(V \otimes L) > 0$; our hypothesis implies $\deg(V \otimes L) \geq 2\mathsf{g} - 1$, which yields the necessary positivity.

The generic fiber of λ consists of a <u>non-empty</u> open subspace of the projective space $\mathbb{P}(\Gamma(V \otimes L))$. Let $\overline{\mathrm{Sect}(\mathcal{F}/C)}$ denote the closure of $\mathrm{Sect}(\mathcal{F}/C)$ in the Hilbert scheme parametrizing divisors in $\mathcal{F}$; λ extends to $\overline{\mathrm{Sect}(\mathcal{F}/C)}$. Its fibers are projective spaces $\mathbb{P}(\Gamma(V \otimes L))$ parametrizing linear series on $\mathcal{F} = \mathbb{P}(V)$, whose members are 'broken sections', consisting of one section of $\mathcal{F} \to C$ together with a configuration of fibers.

Since $\mathrm{Pic}^d(C)$ is a principal homogeneous space over an abelian variety, $\mathrm{Pic}^d(C)(k) \neq \emptyset$ by Lang's Theorem. Consequently

$$\overline{\mathrm{Sect}(\mathcal{F}/C)}(k) \neq \emptyset$$

corresponding to a broken section of height bounded by (10.1); the (unique) horizontal component satisfies the same inequality. $\square$

Effective weak approximation.

Theorem 17. *Let* $\pi : \mathcal{X} \to \mathbb{P}^1$ *be a quadric surface fibration with square-free discriminant of degree* Δ, *defined over a finite field* k. *Assume that*

- *the discriminant double cover* $C \to \mathbb{P}^1$ *is non-trivial over* $\bar{k}$;
- *all sections* $\sigma : \mathbb{P}^1 \to \mathcal{X}$ *over* $\bar{k}$ *satisfy*

$$h_{\omega_\pi^{-1}}(\sigma) \geq -\Delta/2.$$

Fix a positive integer N, *distinct geometric points* $b_1, \ldots, b_N \in \mathbb{P}^1 \setminus \mathfrak{d}$ *and* $x_j \in \mathcal{X}_{b_j} = \pi^{-1}(b_j)$ *for* $j = 1, \ldots, N$, *such that* $\{x_1, \ldots, x_N\}$ *is defined over* k. *Then there exists a section* $\sigma : \mathbb{P}^1 \to \mathcal{X}$ *satisfying*

- $\sigma(b_j) = x_j$ *for* $j = 1, \ldots, N$;

- $h_{\omega_\pi^{-1}}(\sigma) \leq \frac{3}{2}\Delta + 2N$.

Proof. Let $g : C \to \mathbb{P}^1$ denote the discriminant cover and write $g^{-1}(b_j) = \{c_j', c_j''\}$; let $R_j' \subset V_{c_j'}$ and $R_j'' \subset V_{c_j''}$ denote the one-dimensional subspaces corresponding to the lines of $\mathcal{X}_{b_j}$ containing x_j. We seek a section $\tau : C \to \mathcal{F}$ such that $\tau(c_j') \in \mathbb{P}(R_j')$ and $\tau(c_j'') \in \mathbb{P}(R_j'')$. This imposes $2N$ independent linear conditions on the sections of V, which together are defined over k.

Proposition 14 implies $\mathcal{F} = \mathbb{P}(V)$ for V semistable. Corollary 8 gives the existence of $t \in \Gamma(C, V)$ satisfying

$$0 \neq t(c_j') \in R_j', \quad 0 \neq t(c_j'') \in R_j''$$

provided

$$\deg(V) \geq 4\mathrm{g} - 1 + (2N - 1).$$

Since

$$h_{\omega_\pi^{-1}}(\sigma) = \deg(V) - \Delta/2 = \deg(V) - (\mathrm{g} + 1)$$

we obtain

$$h_{\omega_\pi^{-1}}(\sigma) = 3\mathrm{g} - 3 + 2N = \frac{3}{2}\Delta + 2N.$$

$\square$

Remark 18. Our argument yields two variants:

- we can approximate any collection of jet data over places of good reduction, defined over k, with lengths summing to N;
- if the ground field k is algebraically closed we can improve the bound to
$$h_{\omega_\pi^{-1}}(\sigma) \leq \Delta + 2N - 2$$
by tensoring V with a generic $L \in \mathrm{Pic}(C)$.

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