Integral points of bounded height on toric varieties

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ABSTRACT. — We establish asymptotic formulas for the number of integral points of bounded height on toric varieties.

RÉSUMÉ. — Nous établissons un développement asymptotique du nombre de points entiers de hauteur bornée dans les variétés toriques.

Contents

1. Introduction

In this paper we study the distribution of integral points of bounded height on toric varieties, *i.e.*, quasi-projective algebraic varieties defined over number fields, equipped with an action of an algebraic torus T and containing T as an open dense orbit.

The case of projective compactifications has been the subject of intense study. It has been treated completely over number fields via *adelic harmonic analysis* by Batyrev and the second author in a series of papers, see *e.g.*, [2, 3, 4]. Subsequently, Salberger [30] and de la Bretèche [10] provided an alternative proof which relies on the parametrization of rational points by integral points on certain descent varieties called *universal torsors.* These papers use a canonical height on toric varieties which reduces to the standard Weil height (maximum of absolute values of coordinates) in the case when X is a projective space. Some other choices of heights have also been considered, at least for projective spaces, *e.g.*, [32, 20]. Both methods, harmonic analysis and passage to universal torsors, have been applied in the function field case by Bourqui [8, 9].

These results were motivated by conjectures of Batyrev, Manin, Peyre and others concerning the asymptotic behavior of the number of points of bounded height in algebraic varieties over number fields [21, 1, 28, 5]. They stimulated the study of height zeta functions of equivariant compactifications of other algebraic groups and homogeneous spaces [34, 13, 33], as well as the study of universal torsors over Del Pezzo surfaces.

A related, classical, problem in number theory is the study of *integral points* on algebraic varieties, for example complete intersections of low degree (circle method, [6]), algebraic groups or homogeneous spaces of semisimple groups (via ergodic theory or spectral methods, [17, 18, 19, 7]).

In this paper, as well as in [15], we apply the geometric and analytic framework proposed in [14] to "interpolate" between these two counting problems. Precisely, let X be a smooth projective toric variety over a number field F, let T be the underlying torus, and let $U \subseteq X$ be the complement of a T-stable divisor D in X. We establish an asymptotic formula for the number of integral points of bounded height on U. The notion of integral points depends on the choice of a model of U over the ring $\mathfrak{o}_F$ of integers of F, while the normalization of the height is given by the log-anticanonical divisor $-(K_X + D)$ of the pair (X, D); we assume that this log-anticanonical divisor belongs to the interior of the effective cone of X. The asymptotic formula that we establish in Theorem 3.10.3 takes the form

$$N(B) \sim \Theta B (\log B)^{b-1}, \quad B \to \infty,$$

where b is a positive integer and Θ is a positive real number. To define these numbers we need the notion of the analytic Clemens complex $\mathscr{C}_v^{\mathrm{an}}(D)$, for a place v of F, introduced in [14]. It is a simplicial complex which encodes the incidence properties of the v-adic manifolds given by the irreducible components of D. In this language, the integer b is given by

$$b = \mathrm{rank}(\mathrm{Pic}(U)) + \sum_{v|\infty}(1 + \dim \mathscr{C}_v^{\mathrm{an}}(D));$$

The definition of Θ (see Theorem 3.11.3) involves the following analytic and geometric constants:

   – volumes of adelic subsets with respect to suitable Tamagawa measures;
   – local volumes (at archimedean places) of minimal strata of boundary components of D;
   – characteristic functions of certain variants of the effective cones of X attached to these strata and to the Picard group of U;
   – orders of Galois cohomology groups.

Moreover, we establish an equidistribution theorem for integral points of U of bounded height. This is already new for $U = X$ where we obtain that rational points of bounded height in $T(F)$ equidistribute to Peyre's Tamagawa measure on $X(\mathbb{A}_F)^{\mathrm{Br}}$, the subset of $X(\mathbb{A}_F)$ where the Brauer–Manin obstruction vanishes. This refines the classical result that rational points are dense in this subset.

In a series of papers, [23, 22, 24, 25], Moroz proved similar, though less precise, results for certain affine toric varieties over $\mathbf{Q}$.

Here is the roadmap of the paper. In Section 2, we recall basic facts concerning algebraic tori, toric varieties, heights, and Tamagawa measures. The proof of Theorem 3.10.3 is presented in Section 3. It relies on the Poisson summation formula on the adelic torus attached to T, and follows the strategy of [2]. As was already the case for equivariant compactifications of additive groups in [15], new technical complications arise from the presence of poles of the local Fourier transforms at archimedean places, which contribute to the main term in the asymptotic formula.

2. Toolbox

We now recall basic facts concerning algebraic tori, toric varieties, heights and measures.

2.1. Algebraic numbers

Let F be a number field and $\overline{F}$ a fixed algebraic closure of F. Let $\mathrm{Val}(F)$ be the set of normalized absolute values of F. For $v \in \mathrm{Val}(F)$, we write $|\cdot|_v$ for the corresponding absolute value, F_v for the completion of F at v. If v is ultrametric, we also put $\mathfrak{o}_v$, ϖ_v, k_v for the ring of integers, a chosen local uniformizing element and the residue field at v, respectively; we write p_v for the characteristic of the field k_v and q_v for its cardinality.

For any $v \in \mathrm{Val}(F)$, we fix a decomposition group Γ_v at v and write Γ_v^0 for its inertia subgroup. If v is finite, we fix a geometric Frobenius element $\mathrm{Fr}_v \in \Gamma_v/\Gamma_v^0$.

Let R be a ring, let $\overline{M}$ be an $R[\Gamma]$-module which is free of finite rank as an R-module. The Artin L-function of $\overline{M}$ is defined as the Euler product

$$\mathrm{L}(s,\overline{M}) = \prod_{v \nmid \infty} \mathrm{L}_v(s,\overline{M}), \qquad \mathrm{L}_v(s,\overline{M}) = \det\left(1 - q_v^{-s}\,\mathrm{Fr}_v\,\big|\overline{M}^{\Gamma_v^0}\right)^{-1}.$$

We normalize the Haar measure of a local field E as in [35] (p. 310) so that the unit ball has measure

  – 2 if $E = \mathbf{R}$;
  – 2π if $E = \mathbf{C}$;
  – $|\mathrm{disc}(E/\mathbf{Q}_p)|^{-1/2}$ if E is a finite extension of $\mathbf{Q}_p$.

We also define a real number c_E by the following formula:

  – $c_{\mathbf{R}} = 2$;
  – $c_{\mathbf{C}} = 2\pi$;
  – $c_E = |\mathrm{disc}(E/\mathbf{Q}_p)|^{-1/2}\,(1 - q^{-1})/\log(q))$ if E is a finite extension of $\mathbf{Q}_p$ and q is the norm of a uniformizer.

2.2. Algebraic tori

Let T be an algebraic torus of dimension d over F, *i.e.*, an algebraic F-group scheme such that $T_{\overline{F}} \simeq \mathbf{G}_m^d$. A character of T is a morphism of algebraic groups $T \to \mathbf{G}_m$; and a cocharacter a morphism of algebraic groups $\mathbf{G}_m \to T$. Let E be a finite Galois extension of F such that $T_E \simeq \mathbf{G}_m^d$ and Γ its Galois group. Let $\overline{M} = \mathrm{X}^*(T_{\overline{F}})$ be the group of $\overline{F}$-rational characters of T, it is a torsion-free $\mathbf{Z}$-module of rank d endowed with an action of Γ. The group $\overline{N}$ dual to $\overline{M}$ is the group of cocharacters of $T_{\overline{F}}$.

The group $M = \overline{M}^\Gamma$ is the group of F-rational characters. The group $\overline{N}^\Gamma$ of F-rational cocharacters maps naturally into the space of coinvariants $N = \overline{N}_\Gamma$ which identifies with the dual of M. The map $\overline{N}^\Gamma \to N$ is not an isomorphism in general.

For any place $v \in \mathrm{Val}(F)$, we put

$$M_v = \begin{cases} \overline{M}^{\Gamma_v} & \text{for } v \nmid \infty \\ \overline{M}^{\Gamma_v} \otimes \mathbf{R} & \text{for } v \mid \infty \end{cases},$$

and define N_v similarly. For $v \mid \infty$, the perfect duality between $\overline{M}$ and $\overline{N}$ induces a perfect duality $M_v \times N_v \to \mathbf{R}$. If v is nonarchimedean, there is a natural bilinear map $M_v \times N_v \to \mathbf{Z}$ which, however, is not a perfect pairing in general.

For any nonarchimedean $v \in \mathrm{Val}(F)$, the bilinear map

$$T(F_v) \times M_v \to \mathbf{Z}, \qquad (t, m) \mapsto -\log(|m(t)|)/\log(q_v)$$

induces a homomorphism $\log_v \colon T(F_v) \to N_v$ whose kernel K_v is the maximal compact subgroup of $T(F_v)$ and whose image has finite index. Moreover, $\log_v$ is surjective for all ultrametric places v which are unramified in the splitting field E (see, *e.g.*, [16, p. 449]). Similarly, for any archimedean $v \in \mathrm{Val}(F)$, the bilinear map

$$T(F_v) \times \overline{M}^{\Gamma_v} \to \mathbf{R}, \qquad (t, m) \mapsto \log(|m(t)|)$$

induces a surjective homomorphism $\log_v \colon T(F_v) \to N_v$ whose kernel K_v is the maximal compact subgroup of $T(F_v)$.

2.3. Description of the adelic group

Let $\mathbb{A}_F$ be the ring of adeles of F. The bilinear map

$$T(\mathbb{A}_F) \times M_{\mathbf{R}} \to \mathbf{R}, \qquad ((t_v), m) \mapsto \sum_{v \in \mathrm{Val}(F)} \log(|m(t_v)|)$$

induces a surjective continuous morphism $T(\mathbb{A}_F) \to N_{\mathbf{R}}$. This morphism admits a section, *e.g.*, given by $n \mapsto (t_v(n))$, where $t_v(n) = 1$ if v is finite, $t_v(n) = \exp(n/[F:\mathbf{Q}])$ if v is real, and $t_v(n) = \exp(2n/[F:\mathbf{Q}])$ if v is complex.

Let $T(\mathbb{A}_F)^1$ be its kernel. By the product formula, $T(F) \subset T(\mathbb{A}_F)^1$ as a discrete subgroup; moreover, the quotient $T(\mathbb{A}_F)^1/T(F)$ is compact. This induces a decomposition

$$T(\mathbb{A}_F)/T(F) \simeq N_{\mathbf{R}} \times (T(\mathbb{A}_F)^1/T(F))$$

of $T(\mathbb{A}_F)/T(F)$. The group $K_T = \prod_{v \in \mathrm{Val}(F)} K_v$ is the maximal compact subgroup of $T(\mathbb{A}_F)$; it is contained in $T(\mathbb{A}_F)^1$.

Let $S \subset \mathrm{Val}(F)$ be a finite subset containing the archimedean places. The map

$$T(\mathbb{A}_F) \to \prod_{v \in S} T(F_v) \to \prod_{v \in S} N_v$$

induces an isomorphism

$$T(\mathbb{A}_F)/T(F)K_T \simeq \prod_{v \in S} N_v/T(\mathfrak{o}_{F,S}),$$

where $T(\mathfrak{o}_{F,S}) = T(F) \cap \bigcap_{v \notin S} K_v$. The map $T(\mathfrak{o}_{F,S}) \to \prod_{v \in S} N_v$ has finite kernel. Its image is a cocompact lattice in the subspace of $\prod_{v \in S} N_v$ consisting of tuples $(n_v)_{v \in S}$ such that $\sum_{v \in S} \langle n_v, m \rangle = 0$ for any $m \in M$.

2.4. Characters

Recall that the characters of a topological group G are the continuous homomorphisms to the group $\mathbf{S}^1$ of complex numbers of absolute value 1. They form a topological group G^*.

A character χ of $T(\mathbb{A}_F)$ is the product (χ_v) of its local components: for any $v \in \text{Val}(F)$, χ_v is a character of $T(F_v)$. A local character χ_v is called unramified if it is trivial on K_v; then there exists a unique element $m(\chi_v) \in M_v$ such that

$$\chi_v(t) = \exp(i\langle m(\chi_v), \log_v(t)\rangle), \qquad \text{for all } t \in T(F_v).$$

A global character χ is called unramified if all of its local components are unramified, equivalently if it is trivial on K_T; it is called automorphic if it is trivial on $T(F)$.

The description of $T(\mathbb{A}_F)/T(F)K_T$ in Section 2.3 identifies an automorphic unramified character of $T(\mathbb{A}_F)$ as a character of $\prod_{v \in S} N_v/T(\mathfrak{o}_{F,S})$. Then, $(T(\mathbb{A}_F)/T(F)K_T)^*$ is the product of the continuous group $M_{\mathbf{R}}$ and the dual $\text{Hom}(T(\mathfrak{o}_{F,S}), \mathbf{Z})$ of the discrete group $T(\mathfrak{o}_{F,S})/\text{torsion}$.

2.5. Toric varieties

Let X be a smooth projective equivariant compactification of T, *i.e.*, a smooth projective variety X over F endowed with an action of T, and containing T as a dense open orbit. The boundary divisor is the complementary closed subset $X \setminus T$; it is the opposite $-K_X$ of a canonical divisor.

By the general theory of toric varieties over algebraically closed fields, the boundary divisor $X_{\overline{F}} \setminus T_{\overline{F}}$ is reduced, its irreducible components are smooth and meet transversally. Let $\overline{\mathscr{A}}$ be the set of these irreducible boundary components. Since $X_{\overline{F}} \setminus T_{\overline{F}}$ is defined over F, the set $\overline{\mathscr{A}}$ admits a natural action of the Galois group Γ, as well as of its subgroups Γ_v, for $v \in \text{Val}(F)$. We write $\mathscr{A}$, resp. $\mathscr{A}_v$ for the sets of orbits; the corresponding elements label F-irreducible, respectively F_v-irreducible, boundary components of $X \setminus T$.

For any $\alpha \in \overline{\mathscr{A}}$, we write F_α for the subfield of E fixed by the stabilizer of D_α in Γ, and Δ_α for the sum of all irreducible components $D_{\alpha'}$, for $\alpha' \in \Gamma\alpha$. If α and α' belong to the same orbit, the fields F_α and $F_{\alpha'}$ are conjugate. For any finite place $v \in \text{Val}(F)$, the choice of a decomposition subgroup Γ_v induces a specific place of F_α, still denoted v, and we write f_α for the degree of $F_{\alpha,v}$ over F_v.

The closed cone of effective divisors $\Lambda_{\text{eff}}(X_{\overline{F}}) \subset \text{Pic}(X_{\overline{F}})_{\mathbf{R}}$ on $X_{\overline{F}}$ is spanned by the classes of boundary components D_α, for $\alpha \in \overline{\mathscr{A}}$. Similarly, the closed cone of effective divisors $\Lambda_{\text{eff}}(X) \subset \text{Pic}(X)_{\mathbf{R}}$ on X is spanned by the classes of the divisors Δ_α.

Viewing a character of $T_{\overline{F}}$ as a rational function on $X_{\overline{F}}$ and taking its divisor defines a canonical exact sequence of torsion-free Γ-modules

$$(2.5.1) \qquad\qquad 0 \to \overline{M} \to \text{Pic}^T(X_{\overline{F}}) \xrightarrow{\pi} \text{Pic}(X_{\overline{F}}) \to 0,$$

where $\text{Pic}^T(X_{\overline{F}}) \simeq \mathbf{Z}^{\overline{\mathscr{A}}}$ is the group of equivalence classes of $T_{\overline{F}}$-linearized line bundles on $X_{\overline{F}}$. (Linearized line bundles are in canonical correspondence with $T_{\overline{F}}$-invariant divisors in $X_{\overline{F}}$, that is, linear combinations of boundary components.) The injectivity on the left follows from the fact that $X_{\overline{F}}$ is normal and projective: if a character of $T_{\overline{F}}$ has neither zeroes nor poles, then it is a regular invertible function on $X_{\overline{F}}$, hence a constant. Taking Galois cohomology and using the fact that $\text{Pic}^T(X_{\overline{F}})$ is a permutation module, we obtain

the following exact sequences:

$$(2.5.2) \qquad 0 \to M \to \mathrm{Pic}^T(X) \to \mathrm{Pic}(X) \to \mathrm{H}^1(\Gamma, M) \to 0$$

$$(2.5.3) \qquad 0 \to M_v \to \mathrm{Pic}^T(X_{F_v}) \to \mathrm{Pic}(X_{F_v}) \to \mathrm{H}^1(\Gamma_v, \overline{M}) \to 0.$$

Moreover, the isomorphism $\mathrm{Pic}^T(X_{\overline{F}}) \simeq \mathbf{Z}^{\overline{\mathscr{A}}}$ induces similar isomorphisms

$$\mathrm{Pic}^T(X) \simeq \mathbf{Z}^{\mathscr{A}}, \quad \mathrm{Pic}^T(X_{F_v}) \simeq \mathbf{Z}^{\mathscr{A}_v}.$$

By duality, the map $\overline{M} \to \mathbf{Z}^{\overline{\mathscr{A}}}$ gives rise to a morphism of tori $\prod_{\alpha \in \mathscr{A}} T_\alpha \to T$, where, for $\alpha \in \mathscr{A}$, $T_\alpha = \mathrm{Res}_{F_\alpha/F} \mathbf{G}_m$ is the Weil restriction of scalars of the multiplicative group from F_α to F. Using this morphism, any automorphic character $\chi \in (T(\mathbb{A}_F)/T(F))^*$ induces an automorphic character of $T_\alpha(\mathbb{A}_F)$, *i.e.*, a Hecke character χ_α of F_α.

2.6. Quasi-projective toric varieties

Let D be a reduced divisor in X disjoint from T and let $U = X \setminus D$. It is a (non-projective for $D \neq \varnothing$) toric variety. Let $\overline{\mathscr{A}}_D \subset \overline{\mathscr{A}}$ be the set of irreducible components of $D_{\overline{F}}$. The irreducible components of the divisor $U_{\overline{F}} \setminus T_{\overline{F}}$ are indexed by the traces on $U_{\overline{F}}$ of the D_α, for $\alpha \in \overline{\mathscr{A}} \setminus \overline{\mathscr{A}}_D$. There is a similar Γ-equivariant exact sequence

$$(2.6.1) \qquad 0 \to \mathrm{H}^0(U_{\overline{F}}, \mathscr{O}_U^\times)/\overline{F}^\times \to \overline{M} \to \mathbf{Z}^{\overline{\mathscr{A}} \setminus \overline{\mathscr{A}}_D} \xrightarrow{\pi} \mathrm{Pic}(U_{\overline{F}}) \to 0.$$

Let $\rho = (\rho_\alpha)$ with $\rho_\alpha = 0$ if $\alpha \in \mathscr{A}_D$ and 1 otherwise. Throughout we shall assume that $\rho \in \Lambda_{\mathrm{eff}}(X)^\circ$, *i.e.*, is contained in interior of the image under π of the simplicial cone $\mathbf{R}_{\geqslant 0}^{\mathscr{A}}$. In more geometric terms, this means that the line bundle $-(K_X + D)$ on X is big; this includes the particular case where (X, D) is log-Fano, *i.e.*, $-(K_X + D)$ is ample.

2.7. Metrized line bundles

Each boundary divisor D_α, $\alpha \in \overline{\mathscr{A}}$, defines a T_E-linearized line bundle on X_E. We fix smooth adelic metrics on these line bundles: by definition these are collections of metrics, at all places w of E, almost all of which come from a model of X_E defined over the ring of integers of E; the smoothness condition means locally constant at finite places, and $\mathscr{C}^\infty$ at archimedean places. We assume that these metrics are invariant under the natural action the Galois group Γ.

For each $\alpha \in \overline{\mathscr{A}}$, let f_α be the canonical section of the line bundle $\mathscr{O}(D_\alpha)$ with divisor D_α. Then the resulting height pairing is defined by

$$H : T(\mathbb{A}_E) \times \mathrm{Pic}^T(X_E)_{\mathbf{C}} \to \mathbf{C}^*, \quad ((x_w); \sum s_\alpha D_\alpha)) \mapsto \prod_{\alpha \in \overline{\mathscr{A}}} \prod_{w \in \mathrm{Val}(E)} \|\mathsf{f}_\alpha(x_w)\|^{s_\alpha/[E:F]}.$$

It is Γ-equivariant, smooth in the first variable and linear in the second variable. (If $\mathbf{s} \in \mathbf{C}^{\overline{\mathscr{A}}}$, we simply write $H(x; \mathbf{s})$ for $H(x; \sum s_\alpha D_\alpha)$.)

We shall also restrict the height pairing to line bundles defined over F and points in $T(\mathbb{A}_F)$. This is compatible with a corresponding theory of adelic metrics over F. Indeed, a component of $X \setminus T$ decomposes over E as a sum of some divisors D_α and this furnishes a canonical adelic metric on every line bundle on X.

2.8. Volume forms and measures

Our analysis of the number of points of bounded height makes use of certain Radon measures on local analytic manifolds and on adelic spaces. Here we recall the main definitions, referring to [14] for a detailed account of the constructions of these measures in a general geometric context.

Let v be a place of F. We fix a Haar measure on each completion F_v of F, in such a way that $\mu_v(\mathfrak{o}_v) = 1$ for almost all finite places v. Recall that the divisor on $\overline{X}$ of the invariant n-form $\mathrm{d}x$ on T (which is well-defined up to sign) is given by

$$\mathrm{div}(\mathrm{d}x) = -\sum_{\alpha \in \overline{\mathscr{A}}} D_\alpha.$$

We now define several measures on $X(F_v)$. The first is a Haar measure for the torus $T(F_v)$. It is defined "à la Weil" by

$$\mu'_{T,v} = |\mathrm{d}x|_v\,,$$

considering the invariant form $\mathrm{d}x$ as a gauge form on T. Let $\tau_v(T) = \mu'_{T,v}(K_v)$ be the measure of the maximal compact subgroup K_v of $T(F_v)$. If v is unramified in E, then T has good reduction at v and

$$\tau_v(T) = \#T(\mathfrak{o}_v)/q_v^{\dim T} = \mathrm{L}_v(1, \overline{M})^{-1}$$

(see [26], 3.3), where we extend T as a torus group scheme over $\mathrm{Spec}(\mathfrak{o}_v)$. We shall use the renormalized measure

$$\mu_{T,v} = \tau_v(T)^{-1}\,|\mathrm{d}x|_v\,.$$

The local Peyre-Tamagawa measure on $X(F_v)$ is defined by

$$\mu'_{X,v} = |dx|_v\,/\,\|dx\|_v\,.$$

Since $\mathrm{Pic}(X_{\overline{F}})$ is a free **Z**-module of finite rank, two other normalizations are possible:

$$\mu_{X,v} = \mathrm{L}_v(1, \mathrm{Pic}(X_{\overline{F}}))^{-1}\mu'_{X,v},$$
$$\mu_{U,v} = \mathrm{L}_v(1, \mathrm{EP}(U))\mu'_{X,v},$$

where $\mathrm{EP}(U)$ is the virtual Galois module

$$\left[\mathrm{H}^0(\overline{U}, \mathscr{O}_X^*)/\overline{F}^*\right] - \left[\mathrm{H}^1(\overline{U}, \mathscr{O}_X^*)/\mathrm{torsion}\right].$$

With these normalizations, the products of local measures converge and define measures on suitable adelic spaces: $\prod_v \mu_{T,v}$ is a Haar measure on $T(\mathbb{A}_F)$, $\prod_v \mu_{X,v}$ and $\prod_v \mu_{U,v}$ are Radon measures on $X(\mathbb{A}_F)$ and $U(\mathbb{A}_F)$, respectively ([14], Theorem 2.4.7). For any finite $S \subset \mathrm{Val}(F)$, we define Radon measures on $T(\mathbb{A}_F^S)$, $X(\mathbb{A}_F^S)$, and $U(\mathbb{A}_F^S)$, by

$$\mu_T = \mathrm{L}_*^S(1, \overline{M})^{-1}\prod_{v \notin S} \mu_{T,v}, \quad \mu_X = \mathrm{L}_*^S(1, \mathrm{Pic}\,X_{\overline{F}})\prod_{v \notin S} \mu_{X,v}, \quad \mu_U = \mathrm{L}_*^S(1, \mathrm{EP}(U))^{-1}\prod_{v \notin S} \mu_{U,v},$$

where $\mathrm{L}_*^S(1, \cdot)$ denotes the principal value of the Artin L-function at 1, with the finite Euler factors in S removed.

In [14], we have also introduced *residue measures* which are Radon measures on $X(F_v)$ with support on $D(F_v)$. Recall that the F_v-analytic Clemens complex $\mathscr{C}_v^{\mathrm{an}}(X, D)$ is a poset whose faces are pairs (A, Z) where A is a nonempty subset of $\mathscr{A}_v$ and Z is an F_v-irreducible component of $D_A = \bigcap_{\alpha \in A} D_\alpha$ such that $Z(F_v) \neq \varnothing$. Its order relation is the one opposite to inclusion.

For each $\alpha \in \mathscr{A}_v$, we let $\mathbf{A}_{F_{\alpha,v}}$ be the Weil restriction of scalars of the affine line from $F_{\alpha,v}$ to F_v; it is an affine space of dimension $[F_{\alpha,v} : F_\alpha] = |\alpha|$. The norm map $\mathrm{N} \colon F_{\alpha,v} \to F_v$ induces a polynomial function N on $\mathbf{A}_{F_{\alpha,v}}$ which defines the origin on the level of F_v-rational points. By abuse of notation, we write $\mathrm{d}x_\alpha$ for the top differential form on $\mathbf{A}_{F_{\alpha,v}}$ deduced from the one-form $\mathrm{d}x$ on $\mathbf{A}^1$.

Let $x \in X(F_v)$ and let A_x be the set of $\alpha \in \mathscr{A}_v$ such that $x \in D_\alpha$. There exists a neighborhood U_x of x in $X(F_v)$ and a smooth map $(x_\alpha)_{\alpha \in A} \colon U_x \to \prod_{\alpha \in A} \mathbf{A}_{F_{\alpha,v}}$ which defines $D_A(F_v)$ in a neighborhood of x.

Fix a pair (A, Z) in $\mathscr{C}_v^{\mathrm{an}}(X, D)$. The description above shows that $Z(F_v)$ is a smooth v-adic submanifold of $X(F_v)$ of codimension $\sum_{\alpha \in \mathscr{A}_v} |\alpha|$. Moreover, its canonical bundle admits a metric, defined inductively via the adjunction formula, in such a way that for any local top differential form ω on $Z(F_v)$,

$$\|\omega\| = \left\| \tilde{\omega} \wedge \bigwedge_{\alpha \in A} \mathrm{d}x_\alpha \right\| \prod_{\alpha \in \mathscr{A}_v} \lim_x \frac{\|\mathsf{f}_\alpha\|}{|\mathrm{N}(x_\alpha)|},$$

where $\tilde{\omega}$ is any local differential form on $X(F_v)$ which extends ω. This gives rise to a measure $\tau_{(A,Z)}$ on $Z(F_v)$. As in [14], we normalize this measure further, multiplying it by the finite product $\prod_{\alpha \in A} c_{F_\alpha}$ of constants defined as in Section 2.1.

3. Integral points

3.1. Setup

Let F be a number field, T an algebraic torus defined over F, and X a smooth projective equivariant compactification of T. Let D be a reduced effective divisor in $X \setminus T$ and let $U = X \setminus D$. We assume that the divisor $-(K_X + D)$ on X is big.

Let $\mathscr{U}$ be a fixed flat $\mathfrak{o}_F$-scheme of finite type with generic fiber U. A rational point $x \in T(F)$ will be called $\mathfrak{o}_F$-integral if there exists a section $\varepsilon_x \colon \mathrm{Spec}\, \mathfrak{o}_F \to \mathscr{U}$ which extends x. Similarly, for any finite place $v \in \mathrm{Val}(F)$, a point $x \in T(F_v)$ will be called $\mathfrak{o}_v$-integral if it extends to a section $\mathrm{Spec}\, \mathfrak{o}_v \to \mathscr{U}$. For any finite place v, we write δ_v for the set-theoretic characteristic function of the set of $\mathfrak{o}_v$-integral points in $T(F_v)$. It is a locally constant function on $T(F_v)$ whose support is relatively compact in $U(F_v)$. For any archimedean place v, we put $\delta_v = 1$ and write

$$\delta = \prod_{v \in \mathrm{Val}(F)} \delta_v.$$

The generating Dirichlet series of integral points is called the *height zeta function*; it takes the form

$$Z(\mathbf{s}) = \sum_{x \in T(F) \cap \mathcal{U}(\mathfrak{o}_F)} H(x;\mathbf{s})^{-1} = \sum_{x \in T(F)} H(x;\mathbf{s})^{-1} \delta(x).$$

This series converges absolutely and uniformly when all coordinates of $\mathbf{s}$ have a sufficiently large real part, and defines a holomorphic function in that domain. Formally, we have the Poisson formula

$$Z(\mathbf{s}) = \int \hat{H}(\chi;\mathbf{s}) \mathrm{d}\chi,$$

where the integral is over the locally compact abelian group of characters of $T(\mathbb{A}_F)/T(F)$ with respect to an appropriate Haar measure $\mathrm{d}\chi$, and

$$\hat{H}(\chi;\mathbf{s}) = \int_{T(\mathbb{A}_F)} H(x;\mathbf{s})^{-1} \delta(x) \chi(x) \mathrm{d}^{\times} x$$

is the corresponding Fourier transform of the height function with respect to the fixed Haar measure $\mathrm{d}^{\times} x$. As in the study of rational points in [2, 3, 4], we investigate the height zeta function by proving first that the Poisson formula applies; its right-hand-side provides a meromorphic continuation for the height zeta function. A Tauberian theorem will then imply an asymptotic expansion for the number of integral points of bounded height.

3.2. Fourier transforms at finite places

LEMMA 3.2.1. — *For any finite place v of F and any character $\chi_v \in T(F_v)^*$ the local Fourier transform $\hat{H}_v(\chi_v;\mathbf{s})$ converges absolutely if $\mathrm{Re}(s_\alpha) > 0$ for all $\alpha \notin \mathscr{A}_U$ and defines a holomorphic function of $\mathbf{s}$ in the tube domain defined by these inequalities.*

Moreover, there exists a compact open subgroup K_v of $T(F_v)$, equal to the maximal compact subgroup for almost all v, such that $\hat{H}_v(\chi_v;\mathbf{s}) = 0$ for any character χ_v which is nontrivial on K_v.

Proof. — The first part is a special case of our results concerning geometric Igusa functions. For the second, observe that we assumed the metrics to be locally constant at finite places, and the same holds for the characteristic function of the set of local integral points. As a consequence, there exists a compact open subgroup K_v ot $T(F_v)$ such that the height function $H_v(\mathbf{s};\cdot)$ is K_v-invariant. It follows that the Fourier transform vanishes at any character which is not trivial on K_v. Moreover, the adelic condition on the metrics, and the fact that the chosen integral model of the toric varieties U and X are toric schemes over a dense open subset of $\mathrm{Spec}\,\mathfrak{o}_F$, imply that for almost all v, one can take K_v to be the maximal compact subgroup of $T(F_v)$. $\square$

LEMMA 3.2.2. — *For almost all finite places v, and all $\mathbf{s}$ such that $\mathrm{Re}(s_\alpha) > 0$ for all $\alpha \notin \mathscr{A}_U$, one has*

$$\hat{H}_v(\mathbf{1}; \mathbf{s}) = \tau_v(T)^{-1} q_v^{-\dim X} \sum_{A \subset \mathscr{A}_U} \#D_A^\circ(k_v) \prod_{\alpha \in A} \frac{q_v^{f_{\alpha,v}} - 1}{q_v^{f_{\alpha,v} s_\alpha} - 1}.$$

Proof. — Let $\mathscr{X}$ be a flat projective $\mathfrak{o}_F$-scheme with generic fiber X and $\mathscr{D}$ the Zariski closure of D in $\mathscr{X}$. There exists a finite set of places S in $\mathrm{Val}(F)$ such that, after restriction to $\mathfrak{o}_{F,S}$, $\mathscr{X}$ is a smooth toric scheme, $\mathscr{X} \setminus \mathscr{D}$ is equal to $\mathscr{U}$, and all local metrics are defined by the given model.

For $v \notin S$, one may compute $\hat{H}_v(\mathbf{1}; \mathbf{s})$ using Denef's formula. Using the good reduction hypothesis, the set of integral points in $T(F_v)$ is equal to $T(F_v) \cap \mathscr{U}(\mathfrak{o}_v)$. Moreover, $U(\mathfrak{o}_v) \cap (U \setminus T)$ has measure zero with respect to the measure $\mu_{X,v}$. We thus can split the integral over the residue classes and write

$$\begin{aligned}
\hat{H}_v(\mathbf{1}; \mathbf{s}) &= \int_{T(F_v)} H_v(x; \mathbf{s})^{-1} \delta_v(x) \mathrm{d}\mu_{T,v} \\
&= \tau_v(T)^{-1} \int_{\mathscr{U}(\mathfrak{o}_v)} H_v(x; \mathbf{1} - \mathbf{s}) \mathrm{d}\mu_{X,v} \\
&= \tau_v(T)^{-1} \sum_{A \subset \mathscr{A}_U} \#D_A^\circ(k_v) q_v^{|A| - \dim(X)} \prod_{\alpha \in A} \int_{\mathfrak{m}_{\alpha,v}} \left| \mathrm{N}_{F_{\alpha,v}/F_\alpha}(x_\alpha) \right|^{s_\alpha - 1} \mathrm{d}x_\alpha \\
&= \tau_v(T)^{-1} q_v^{-\dim(X)} \sum_{A \subset \mathscr{A}_U} \#D_A^\circ(k_v) \prod_{\alpha \in A} \frac{q_v^{f_{\alpha,v}} - 1}{q_v^{f_{\alpha,v} s_\alpha} - 1}.
\end{aligned}$$

(See [14], 4.1.6, for more details.) $\qquad\square$

3.3. The product of local Fourier transforms at finite places

Let $K_{H,\mathrm{fin}} = \prod_{v \nmid \infty} K_v$ be the product of compact open subgroups from by Lemma 3.2.1 and $T(\mathbb{A}_F)_K^* \subset T(\mathbb{A}_F)^*$ the subgroup of characters whose restriction to $K_{H,\mathrm{fin}}$ is trivial. Let S be the finite set of those finite places v such that either K_v is distinct from the maximal compact subgroup of $T(F_v)$ or Lemma 3.2.2 fails for v.

For any character $\chi \in T(\mathbb{A}_F)_K^*$ and $\mathbf{s} \in \mathbf{C}^{\mathscr{A}}$ such that $\mathrm{Re}(s_\alpha) > 0$ for $\alpha \notin \mathscr{A}_U$, define

$$\hat{H}_{\mathrm{fin}}(\chi; \mathbf{s}) = \prod_{v \nmid \infty} \hat{H}_v(\chi_v; \mathbf{s}) = \prod_{v \nmid \infty} \int_{T(F_v)} H_v(x_v; \mathbf{s})^{-1} \delta_v(x_v) \chi_v(x_v) \mathrm{d}^\times x_v.$$

In this section, we study the convergence of this infinite product and its analytic properties with respect to $\mathbf{s}$ and χ.

LEMMA 3.3.1. — *Let $\Omega \subset \mathbf{C}^{\mathscr{A}}$ be the tube domain of all $\mathbf{s} \in \mathbf{C}^{\mathscr{A}}$ such that $\mathrm{Re}(s_\alpha) > 1/2$ for $\alpha \in \mathscr{A}_U$. The infinite product $\hat{H}_{\mathrm{fin}}(\chi; \mathbf{s})$ converges whenever $\mathrm{Re}(s_\alpha) > 1$ for all $\alpha \in \mathscr{A}_U$ and extends to a meromorphic function of $\mathbf{s} \in \Omega$.*

More precisely, for each $\chi \in T(\mathbb{A}_F)^*_K$, there exists a holomorphic function $\varphi(\chi; \cdot)$ on Ω such that

$$\hat{H}_{\mathrm{fin}}(\chi; \mathbf{s}) = \varphi(\chi; \mathbf{s}) \prod_{\alpha \in \mathscr{A}_U} \mathrm{L}(s_\alpha, \chi_\alpha).$$

Moreover, for any positive real number ε, there exists $C(\varepsilon)$ such that $|\varphi(\chi; \mathbf{s})| \leqslant C(\varepsilon)$ for any character χ and any $\mathbf{s} \in \Omega$ such that $\mathrm{Re}(s_\alpha) > \frac{1}{2} + \varepsilon$ for all $\alpha \in \mathscr{A}_U$.

(In that formula, χ_α is the Hecke character of F_α deduced from χ, as in Section 2.5.)

Proof. — This is a slight modification of the proof provided in [2], in the projective case. For any finite place $v \in \mathrm{Val}(F)$, let us define a function φ_v on Ω by the formula

$$\hat{H}_v(\chi_v; \mathbf{s}) = \varphi_v(\chi_v; \mathbf{s}) \prod_{\alpha \in \mathscr{A}_U} \mathrm{L}_v(s_\alpha, \chi_\alpha),$$

where χ_v is the local component at v of a character $\chi \in T(\mathbb{A}_F)^*_K$.

For $v \notin S$, χ_v is unramified and hence takes the form

$$\chi_v(x) = H_v(x; -\mathrm{i}m(\chi_v)),$$

for some $m(\chi_v) \in M_v$, where we used the injection (2.5.3) to embed M_v into $\mathbf{Z}^{\mathscr{A}_v}$. Consequently, for any such v, one has

$$\hat{H}_v(\chi_v; \mathbf{s}) = \int_{T(F_v)} H_v(x; \mathbf{s})^{-1} \chi_v(x) \delta_v(x) \, \mathrm{d}x$$

$$= \int_{T(F_v)} H_v(x; \mathbf{s} + \mathrm{i}m(\chi_v))^{-1} \delta_v(x) \, \mathrm{d}x$$

$$= \hat{H}_v(\mathbf{1}; \mathbf{s} + \mathrm{i}m(\chi_v)).$$

Observe that

$$\mathrm{L}_v(s_\alpha, \chi_\alpha) = \mathrm{L}_v(s_\alpha + \mathrm{i}m(\chi_\alpha), \mathbf{1}) = \zeta_{F_\alpha, v}(s_\alpha + \mathrm{i}m(\chi_\alpha)).$$

Lemma 3.2.2 implies that φ_v is holomorphic on its domain. Moreover, for any positive real number ε, there is an upper bound of the form

$$|\varphi_v(\chi; \mathbf{s}) - 1| \ll q_v^{-\min(1 + 2\varepsilon, 3/2)}$$

for all $\mathbf{s}$ such that $\mathrm{Re}(s_\alpha) > \frac{1}{2} + \varepsilon$ for $\alpha \in \mathscr{A}_U$.

For $v \in S$, Lemma 3.2.1 implies that $\hat{H}_v(\chi_v; \cdot)$ is holomorphic and uniformly bounded in this domain, independently of χ; the same holds for $\varphi_v(\chi_v; \cdot)$.

These estimates imply the uniform and absolute convergence of the infinite product $\prod \varphi_v(\chi; \cdot)$ on the tube domain Ω; it defines a holomorphic function $\varphi(\chi; \cdot)$ on this domain. Moreover, for any positive real number ε, there exists a constant $C(\varepsilon)$ such that

$$|\varphi(\chi; \mathbf{s})| \ll C(\varepsilon)$$

whenever $\mathrm{Re}(s_\alpha) > \frac{1}{2} + \varepsilon$ for $\alpha \in \mathscr{A}_U$.

Since Hecke L-functions converge when their parameter has real part > 1, the infinite product $\hat{H}_{\mathrm{fin}}(\chi; \mathbf{s})$ converges absolutely on the subset Ω_1 of the tube domain Ω defined by

the inequalities $\mathrm{Re}(s_\alpha) > 1$ for all $\alpha \in \mathscr{A}_U$ and defines a holomorphic function on Ω_1 such that

$$\hat{H}_{\mathrm{fin}}(\chi; \mathbf{s}) = \varphi(\chi; \mathbf{s}) \prod_{\alpha \in \mathscr{A}_U} \mathrm{L}(s_\alpha, \chi_\alpha).$$

This provides the asserted meromorphic continuation of $\hat{H}_{\mathrm{fin}}$. □

3.4. Fourier transforms at archimedean places

To establish analytic properties of Fourier-transforms at archimedean places, we will extend the technique of geometric Igusa integrals developed in [14].

Fix an archimedean place v of F. Since the F-rational divisors D_α may decompose over F_v, it is natural to consider the local height function H_v and its Fourier transform as functions of the complex parameter $\mathbf{s} \in \mathbf{C}^{\mathscr{A}_v}$. This generalization will in fact be required in the following sections.

Fix a splitting of the exact sequence

$$0 \to T(F_v)^1 \to T(F_v) \to M_v \to 0.$$

Each character of $T(F_v)$ can now be viewed as a pair (χ_1, m) of a character of the compact torus $T(F_v)^1$ and an element $m \in M_v$. Similarly, for each $\alpha \in \mathscr{A}_v$, let $F_{\alpha,1}$ be the subgroup of $F_\alpha^\times$ consisting of elements of absolute value 1. We decompose the field $F_\alpha^\times = \mathbf{R}_+^\times \times F_{\alpha,1}$ and decompose the character χ_α accordingly, writing

$$\chi_\alpha(x_\alpha) = |x_\alpha|^{-im_\alpha} \chi_{\alpha,1}(x_\alpha),$$

where $\chi_{\alpha,1}$ is a character of $F_{\alpha,1}$.

LEMMA 3.4.1. — *Let v be an archimedean place of F. Then, for each face A of the analytic Clemens complex $\mathscr{C}_v^{\mathrm{an}}(X, D)$, there exists a function $\varphi_{v,A}(\mathbf{s}, \chi_v)$ holomorphic on the tube domain of $\mathbf{C}^{\mathscr{A}_v}$ defined by $\mathrm{Re}(s_\alpha) > -1/2$ for all $\alpha \in \mathscr{A}_v$, such that*

$$\hat{H}_v(\chi_v; \mathbf{s}) = \sum_{A \in \mathscr{C}_v^{\mathrm{an}}(X,D)} \varphi_{v,A}(\chi_v; \mathbf{s}) \prod_{\alpha \in A} \frac{1}{s_\alpha + im_\alpha}.$$

Moreover, each function $\varphi_{v,A}$ is rapidly decreasing in vertical strips; namely, for any positive κ, one has

$$|\varphi_{v,A}(\chi_v; \mathbf{s})| \ll (1 + \|\mathrm{Im}(\mathbf{s})\| + \|m(\chi_v)\| + \|\chi_{v,1}\|)^{-\kappa},$$

provided $\mathrm{Re}(s_\alpha) > -\frac{1}{2}$ for all $\alpha \in \mathscr{A}_v$.

Assume that $s_\alpha = 0$ for $\alpha \in A$. If there exists an $\alpha \in A$ such that χ_α is ramified, then $\varphi_{v,A}(\chi_v; \mathbf{s}) = 0$.

Proof. — The proof is a variant of the analysis conducted in our paper [14]. Let us consider a partition of unity (h_A), indexed by the faces $A \subset \mathscr{A}_v$ of the analytic Clemens complex $\mathscr{C}_v^{\mathrm{an}}(X, D)$ at v such that the only divisors D_α which intersect the support of h_A are those with index $\alpha \in A$. Up to refining this partition of unity, we also assume that on the support of h_A, there is a smooth map (x_α) to $\prod_{\alpha \in A} F_\alpha$ such that for each $\alpha \in A$, $x_\alpha = 0$ is a local equation of D_α. By the theory of toric varieties, we can moreover assume that

the restriction of the map $\mathbf{x} \mapsto x_\alpha$ to $T(F)$ is an algebraic character. Considering a complement torus, we obtain a system of local analytic coordinates $(\mathbf{x}, \mathbf{y}) = (x_\alpha)_{\alpha \in A}, (y_\beta)_{\beta \in B}$. In these coordinates, the character χ_v can be expressed as

$$\chi_v(\mathbf{x}) = \prod_{\alpha \in A} \chi_\alpha(x_\alpha) \times \chi^A(\mathbf{y}),$$

where χ^A is a character of $T(F)$.

After the corresponding change of variables, the integral, localized around the stratum $D_A(F_v)$, takes the form

$$\mathscr{I}_A(\chi_v; \mathbf{s}) = \int \prod_{\alpha \in A} \chi_\alpha(x_\alpha) |x_\alpha|^{s_\alpha - 1} \theta(\mathbf{s}, \mathbf{x}, \mathbf{y}) \chi^A(\mathbf{y}) \mathrm{d}\mathbf{x} \mathrm{d}\mathbf{y},$$

where θ is a smooth function with compact support around the origin in $\prod_{\alpha \in A} F_\alpha \times F_v^B$. The local integral $\mathscr{I}_A$ takes the form

$$\mathscr{I}_A(\chi_v; \mathbf{s}) = \int \prod_{\alpha \in A} |x_\alpha|^{s_\alpha + im_\alpha - 1} \left(\int \theta(\mathbf{s}, \mathbf{x}, \mathbf{y}) \chi^A(\mathbf{y}) \prod_{\alpha \in A} \chi_{\alpha,1}(x_{\alpha,1}) \prod_{\alpha \in A} \mathrm{d}x_{\alpha,1} \mathrm{d}\mathbf{y} \right) \prod \mathrm{d}|x_\alpha|.$$

In the inner integral, the variables $x_{\alpha,1}$ run over $F_{\alpha,1}$, *i.e.*, $\{\pm 1\}$ or $\mathbf{S}_1$, according to whether $F_\alpha = \mathbf{R}$ or $F_\alpha = \mathbf{C}$. In the latter case, we first perform integration by parts to establish the rapid decay of the inner integral with respect to the discrete part $\chi_{\alpha,1}$ of the character χ_α. Observe also that this inner integral tends to 0 when $|x_\alpha| \to 0$ if the character $\chi_{\alpha,1}$ is nontrivial, *i.e.*, the character χ_α is unramified.

The stated meromorphic continuation can then be established, *e.g.*, iteratively integrating by parts with respect to the variables $|x_\alpha|$ and writing

$$t^{s+im} = \frac{1}{s + im} \frac{\partial}{\partial t} \left(t^{s+im} \right).$$

This gives a formula as indicated, except for the rapid decay in $\mathbf{s}$. To obtain this, it suffices to perform integration by parts with respect to invariant vector fields in the definition of $\hat{H}_v$. The point is that for any element $\mathfrak{d}$ of $\mathrm{Lie}(T(F_v))$, there exists a vector field $\mathfrak{d}_X$ on $X(F_v)$ whose restriction to $T(F_v)$ is invariant; moreover, $\mathfrak{d}_X(H(\mathbf{x}; \mathbf{s})) H(\mathbf{x}; \mathbf{s})^{-1}$ is a linear form in $\mathbf{s}$ times a regular function on $X(F_v)$ (see [13], Proposition 2.2). $\square$

3.5. Integrating Fourier transforms

We now have to integrate the Fourier transform of the height function over the group of automorphic characters. For the analysis, it will be necessary to first enlarge the set of variables and then restrict to a suitable subspace. We thus consider a variant of the height zeta function depending on a variable

$$\tilde{\mathbf{s}} = (\mathbf{s}, (\mathbf{s}_v)_{v | \infty}) \in V_{\mathbf{C}},$$

where V is the real vector space

$$V = \mathrm{Pic}^T(X)_{\mathbf{R}} \oplus \bigoplus_{v | \infty} \mathrm{Pic}^T(X_v)_{\mathbf{R}}.$$

For $\tilde{\mathbf{s}} \in V_{\mathbf{C}}$ such that the series converges, we set:

$$\tilde{Z}(\tilde{\mathbf{s}}) = \sum_{x \in T(F) \cap \mathscr{U}(\mathfrak{o}_F)} \prod_{v \nmid \infty} H_v(\mathbf{s}, x)^{-1} \times \prod_{v \mid \infty} H_v(\mathbf{s}_v, x)^{-1}.$$

Formally, we again have the Poisson formula

$$\tilde{Z}(\tilde{\mathbf{s}}) = c_T \int \hat{H}(\tilde{\mathbf{s}}; \chi) \mathrm{d}\chi,$$

the integral over the locally compact abelian group of characters of $T(\mathbb{A}_F)/T(F)$ with respect to a chosen Haar measure $\mathrm{d}\chi$ on $(T(\mathbb{A}_F)/T(F))^*$, here, for $\tilde{\mathbf{s}} = (\mathbf{s}, (\mathbf{s}_v))$,

$$\hat{H}(\tilde{\mathbf{s}}; \chi) = \hat{H}_{\mathrm{fin}}(\mathbf{s}; \chi) \prod_{v \mid \infty} \hat{H}_v(\mathbf{s}_v; \chi_v)$$

is the corresponding Fourier transform of the height function. The constant c_T depends on the actual choice of measures, which we now make explicit.

Fix a section of the surjective homomorphism $T(\mathbb{A}_F) \to M_{\mathbf{R}}^{\vee}$, whose kernel $T(\mathbb{A}_F)^1$ contains $T(F)$. If $M_{\mathbf{R}}^{\vee}$ is endowed with the Lebesgue measure normalized by the lattice $M^{\vee}$, this gives rise to a Haar measure on $T(\mathbb{A}_F)^1$. The section decomposes the group of automorphic characters as $M_{\mathbf{R}} \oplus \mathscr{U}_T$. Let $\mathscr{U}_T^K = T(\mathbb{A}_F)_K^* \cap \mathscr{U}_T$ be the subgroup of $\mathscr{U}_T$ consisting of characters whose restriction to the compact open subgroup $K_{H,\mathrm{fin}}$ is trivial. By Lemma 3.2.1 (see also the beginning of Section 3.3), the Fourier transform vanishes at any character $\chi \in \mathscr{U}_T$ such that $\chi \notin \mathscr{U}_T^K$. We normalize the Haar measure $\mathrm{d}m$ on $M_{\mathbf{R}}$ by the lattice M and define a Haar measure on $(T(\mathbb{A}_F)/T(F))^*$ as the product of $\mathrm{d}m$ by the counting measure on $\mathscr{U}_T$. Provided the expression in the right hand side converges absolutely, one can apply the Poisson summation formula and obtain

$$\tilde{Z}(\tilde{\mathbf{s}}) = \frac{c_T}{(2\pi)^{\mathrm{rank}\,M}} \sum_{\chi \in \mathscr{U}_T} \int_{M_{\mathbf{R}}} \hat{H}(\tilde{\mathbf{s}} + im; \chi)\, \mathrm{d}m,$$

with

$$c_T = \mathrm{vol}(T(\mathbb{A}_F)^1/T(F))^{-1}.$$

In the domain of $\tilde{\mathbf{s}}$ defined by the inequalities $\mathrm{Re}(s_\alpha) > 1$ for $\alpha \in \mathscr{A}_U$ and $\mathrm{Re}(s_{v,\alpha}) > 0$ for $\alpha \in \mathscr{A}$, Lemmas 3.3.1, 3.4.1, and the moderate growth of Hecke L-functions in vertical strips imply that the integrand decays rapidly, hence the validity of the Poisson formula. For any $\chi \in \mathscr{U}_T^K$ we set

$$\tilde{Z}(\tilde{\mathbf{s}}; \chi) = \frac{1}{(2\pi)^{\mathrm{rank}\,M}} \int_{M_{\mathbf{R}}} \hat{H}(\tilde{\mathbf{s}} + im; \chi)\, \mathrm{d}m,$$

so that

$$\tilde{Z}(\tilde{\mathbf{s}}) = c_T \sum_{\chi \in \mathscr{U}_T^K} \tilde{Z}(\tilde{\mathbf{s}}; \chi).$$

We first analyze individually the functions $\tilde{Z}(\tilde{\mathbf{s}}; \chi)$, for a fixed χ. By Lemmas 3.4.1 and 3.3.1, one can write

$$\hat{H}(\tilde{\mathbf{s}} + \mathrm{i}m; \chi)$$

$$= \varphi(\mathbf{s} + \mathrm{i}m; \chi_{\mathrm{fin}}) \prod_{\alpha \in \mathscr{A}_U} \mathrm{L}(s_\alpha + \mathrm{i}m_\alpha; \chi_\alpha) \prod_{v | \infty} \sum_{A \in \mathscr{C}_v^{\mathrm{an,max}}(X,D)} \frac{\varphi_{v,A}(\mathbf{s}_v + \mathrm{i}m + \mathrm{i}m(\chi_v); \chi_{v,1})}{\prod_{\alpha \in A_v}(s_{v,\alpha} + \mathrm{i}m + \mathrm{i}m(\chi_{v,\alpha}))}.$$

Let

$$\mathscr{C}_\infty^{\mathrm{an,max}}(X, D) = \prod_{v | \infty} \mathscr{C}_v^{\mathrm{an,max}}(X, D).$$

For any $A = (A_v) \in \mathscr{C}_\infty^{\mathrm{an,max}}(X, D)$, set

$$\hat{H}_A(\tilde{\mathbf{s}}; \chi) = \varphi(\mathbf{s}; \chi_{\mathrm{fin}}) \prod_{\alpha \in \mathscr{A}_U} \mathrm{L}(s_\alpha; \chi_\alpha) \prod_{v | \infty} \frac{\varphi_{v,A_v}(\mathbf{s}_v + \mathrm{i}m(\chi_v); \chi_{v,1})}{\prod_{\alpha \in A_v}(s_{v,\alpha} + \mathrm{i}m(\chi_{v,\alpha}))}$$

so that

$$\hat{H}(\tilde{\mathbf{s}}; \chi) = \sum_{(A_v) \in \mathscr{C}_\infty^{\mathrm{an,max}}(X,D)} \hat{H}_A(\tilde{\mathbf{s}}; \chi).$$

For each $A \in \mathscr{C}_\infty^{\mathrm{an,max}}(X, D)$, the function $\hat{H}_A(\tilde{\mathbf{s}}; \chi)$ is a meromorphic function on a tube domain, with poles given by affine linear forms whose vector parts are real and linearly independent. Now we apply a straightforward generalization of the integration theorem 3.1.14 from [11], where we only assume that the linear forms describing the poles are linearly independent, rather than a basis of the dual vector space. The convergence is guaranteed by the rapid decay of the functions φ and φ_{v,A_v} in vertical strips.

Let us set

$$\mathrm{Pic}^T(U; A) = \mathrm{Pic}^T(U) \oplus \bigoplus_{v | \infty} \mathbf{Z}^{A_v}.$$

There is a natural homomorphism $M \to \mathrm{Pic}^T(U; A)$ and we define $\mathrm{Pic}(U; A)$ as the quotient $\mathrm{Pic}^T(U; A)/M$.

LEMMA 3.5.1. — *The abelian group* $\mathrm{Pic}(U; A)$ *is torsion-free.*

Proof. — [1] If $n(D, D_v) = m$, then $nD = m$ in $\mathrm{Pic}^T(U)$, so $D = m'$ in $\mathrm{Pic}^T(U)$ since $\mathrm{Pic}(U)$ is torsion-free (Fulton, p. 63). So we may assume that $D = 0$. Then $nD_v = m$ for all v. Pick up a maximal stratum of $\mathscr{C}_v^{\mathrm{an}}(X \setminus T)$ which contains A_v; necessarily, all components outside A_v belong to U. This shows that the character m is nm' on the open chart corresponding to the stratum, hence everywhere because no nontrivial character is invertible on an affine space... $\qquad\square$

Note that for $A = \varnothing$, one has $\mathrm{Pic}^T(U; \varnothing) = \mathrm{Pic}^T(U)$ but $\mathrm{Pic}(U; \varnothing)$ is a sublattice in $\mathrm{Pic}(U)$ of index $|\mathrm{H}^1(\Gamma_F, \overline{M})|$.[2]

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[1] AMELIORER LA REDACTION

[2] Trouver une meilleure structure entière sur $\mathrm{Pic}(U; A)_{\mathbf{R}}$ qui coïncide avec $\mathrm{Pic}(U)$ lorsque $A = \varnothing$.

Let V_A be the real vector space $V_A = \mathrm{Pic}^T(U;A)_{\mathbf{R}}$, and endowed with the measure normalized by the lattice $\mathrm{Pic}^T(U;A)$. We consider V_A as a *quotient* of V and let $r_A\colon V \to V_A$ be the map which forgets the missing components. Let Λ_A be the open simplicial cone in V_A consisting of all vectors $\tilde{\mathbf{s}} = (\mathbf{s},(\mathbf{s}_v))$ such that $s_\alpha > 0$ for all $\alpha \in \mathscr{A}_U$, and $s_{v,\alpha} > 0$ for all $v \mid \infty$ and $\alpha \in A_v$. Pulling-back via r_A the characteristic function of Λ_A and obtain a rational function $\mathscr{X}_{\Lambda_A}$ on $V_{\mathbf{C}}$; it is given by

$$\mathscr{X}_{\Lambda_A}(\tilde{\mathbf{s}}) = \left(\prod_{\alpha \in \mathscr{A}_U} s_\alpha \prod_{v\mid\infty} \prod_{\alpha \in A_v} s_{v,\alpha} \right)^{-1}.$$

Set $V' = V/M$ and let $\pi\colon V \to V'$ be the natural projection. By Proposition 3.8.3 below, the composition $M_{\mathbf{R}} \to V \xrightarrow{r_A} V_A$ is injective. Let $V'_A = V_A/M_{\mathbf{R}}$ and $\pi_A\colon V_A \to V'_A$ be the natural projection; one has $V'_A = \mathrm{Pic}(U;A)_{\mathbf{R}}$; let us endow V'_A with the Lebesgue measure normalized by $\mathrm{Pic}(U;A)$. There exists a unique map $r'_A\colon V' \to V'_A$ such that $\pi_A\colon r_A = r'_A\colon \pi$, so that V'_A is a quotient of V'. We let $\Lambda'_A = \pi(\Lambda_A)$ be the image of Λ_A in V'_A; it is an open cone whose closure $\overline{\Lambda}'_A$ is generated by the images of the generators of $\overline{\Lambda}_A$. We pull-back to V' the characteristic function of the cone Λ'_A and obtain a rational function which is given by the integral formula

$$\mathscr{X}_{\Lambda'_A}(\pi(\tilde{\mathbf{s}})) = \frac{1}{(2\pi)^{\mathrm{rank}\,M}} \int_{M_{\mathbf{R}}} \mathscr{X}_{\Lambda_A}(\tilde{\mathbf{s}} + \mathrm{i}m)\,\mathrm{d}m.$$

(See, *e.g.*, [12], prop. 3.1.9.)

We first conclude that for each $\chi \in \mathscr{U}_T^K$,

$$\tilde{Z}_A(\tilde{\mathbf{s}} + \tilde{\rho} + \mathrm{i}\tilde{m}(\chi);\chi) = \frac{1}{(2\pi)^{\mathrm{rank}\,M}} \int_{M_{\mathbf{R}}} \hat{H}_A(\tilde{\mathbf{s}} + \tilde{\rho} + \mathrm{i}\tilde{m}(\chi) + \mathrm{i}m;\chi)\,\mathrm{d}m$$

is a holomorphic function on the tube domain $\mathsf{T}((r'_A)^{-1}(\Lambda'_A))$ in $\mathsf{T}(V')$. Moreover, there exists an open neighborhood Ω_A of the origin in V' such that $\tilde{Z}_A(\tilde{\mathbf{s}}+\tilde{\rho}+\mathrm{i}\tilde{m}(\chi);\chi)$ extends to a meromorphic function on $\mathsf{T}(\Omega_A + (r'_A)^{-1}(\Lambda'_A))$ whose poles are given the linear forms on V' corresponding to the faces of $\overline{\Lambda}'_A$. Moreover, $\tilde{Z}_A$ decays rapidly in vertical strips and for any positive $\tilde{\mathbf{s}} \in V$,

$$(3.5.2)\quad \lim_{t\to 0} t^{\dim(\Lambda_A)}\tilde{Z}_A(t\tilde{\mathbf{s}} + \tilde{\rho} + \mathrm{i}\tilde{m}(\chi);\chi)$$

$$= \mathscr{X}_{\Lambda'_A}(\tilde{\mathbf{s}})\varphi(\rho;\chi) \prod_{\alpha \in \mathscr{A}_U} \mathrm{L}^*(1;\chi_\alpha) \prod_{v\mid\infty} \varphi_{v,A_v}(\mathrm{i}m(\chi_v);\chi_{v,1}).$$

We now sum these meromorphic functions $\tilde{Z}_A(\cdot;\chi)$ over all $\chi \in \mathscr{U}_T^K$. Due to the stated decay in vertical strips, this series converges and defines a meromorphic function with poles given by the translates of the cones Λ'_A by a discrete subgroup consisting of (the images of) imaginary vectors $\mathrm{i}\tilde{m}(\chi)$, for $\chi \in \mathscr{U}_T^K$.

LEMMA 3.5.3. — *Under the map $\chi \mapsto \pi(\tilde{m}(\chi))$, the group $\mathscr{U}_T^K$ is mapped to a discrete subgroup of V'_A.*

This furnishes the existence and holomorphy of $\tilde{Z}$ in the tube domain formed of $\tilde{\mathbf{s}}$ such that $\mathrm{Re}(\tilde{\mathbf{s}}) > \tilde{\rho}$. (This means $\mathrm{Re}(s_\alpha) > 1$ if $\alpha \in \mathscr{A}_U$, $\mathrm{Re}(s_{v,\alpha}) > 0$ for $v \mid \infty$ and α in a face of the analytic Clemens complex $\mathscr{C}_v^{\mathrm{an}}(X, D)$.)[3]

3.6. Restriction to a log-anticanonical line

Let us consider the particular case of the height zeta function with respect to the log-anticanonical line bundle. We assume that this line bundle belongs to the interior of the effective cone of X. Then there exists a λ in the interior of the effective cone of $\mathrm{Pic}^T(X)$ such that $\rho \sim \lambda$ in $\mathrm{Pic}(X)$. Since the height of a rational point only depends on the isomorphy class of the underlying line bundle, one has

$$Z(s\rho) = Z(\rho + (s-1)\lambda) = \tilde{Z}(\tilde{\rho} + (s-1)\tilde{\lambda}),$$

where $\tilde{\lambda}$ is the vector $(\lambda, (\lambda)) \in V$. Observe that all components of $\tilde{\lambda}$ are positive. It follows that $s \mapsto Z(s\rho)$ is holomorphic for $\mathrm{Re}(s) > 1$ and has a meromorphic continuation to the left of 1.

3.7. Poles on the boundary of the convergence domain

We now describe the poles of the function $s \mapsto Z(s\rho)$ which satisfy $\mathrm{Re}(s) = 1$.

We first claim that they lie in a finite union of arithmetic progressions. Indeed, according to the summation process above, there is a pole at $1 + i\tau$ whenever there exists $\chi \in \mathscr{U}_T$, $A = (A_v)$ a family of faces of maximal dimension of the analytic Clemens complexes, such that $r_A(\tau\tilde{\lambda} + \tilde{m}(\chi))$ belongs a face of the cone Λ_A. This means that there exists $\alpha \in \bigcup A_v$ such that $\tau = -m_v(\chi_\alpha)/\lambda_\alpha$. The result now follows from the fact that for each fixed (α, v), the image of $\mathscr{U}_T^K$ by the map $\chi \mapsto m_v(\chi_\alpha)$ is an arithmetic progression.

Fix such a character $\chi \in \mathscr{U}_T^K$ and $\tau \in \mathbf{R}$. According to the limit formula (3.5.2) and the vanishing of φ_v for ramified characters stated in Lemma 3.4.1, the order of the pole at $1 + i\tau$ is at most equal to the sum $b(\tau)$ of the following integers:

  – $-\mathrm{rank}\, M$;
  – if $\tau = 0$, the cardinality of $\mathscr{A}_U$;
  – for each $v \mid \infty$, the maximal cardinality of a face $A_v \in \mathscr{C}_v^{\mathrm{an}}(X, D)$ such that there exists an unramified character $\chi_v \in \mathscr{U}_T$ such that for any $\alpha \in A_v$, $m(\chi_{v,\alpha}) = -\tau$.

We set $b = b(0)$; observe that

$$b = -\mathrm{rank}\, M + |\mathscr{A}_U| + \sum_{v\mid\infty}(1 + \dim \mathscr{C}_v^{\mathrm{an}}(X, D))$$

$$(3.7.1) \qquad\qquad = \mathrm{rank}(\mathrm{Pic}(U)) + \sum_{v\mid\infty}(1 + \dim \mathscr{C}_v^{\mathrm{an}}(X, D)).$$

We shall prove later, by computing the constant term, that the order of the pole at $s = 0$ is indeed equal to b.

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[3] Il faudra inventer une bonne notation

Recall the assumption that the log-anticanonical divisor belongs to the interior of the effective cone; a fortiori that $T \neq U$, hence $\mathscr{A}_U \neq \varnothing$. Therefore,

$$b > \sum_{v|\infty} (\dim \mathscr{C}_v^{\mathrm{an}}(X, D) + 1) - \mathrm{rank}\, M \geqslant b(\tau).$$

3.8. Characters giving rise to the pole of maximal order

Let $A(T)^*$ be the group of all automorphic characters $\chi \in (T(\mathbb{A}_F)/T(F))^*$ such that $\chi_\alpha \equiv 1$ for all $\alpha \in \mathscr{A}$.

LEMMA 3.8.1. — *The group $A(T)^*$ is finite, and canonically identifies with the Pontryagin dual of the group $T(\mathbb{A}_F)/\overline{T(F)}^w$, quotient of $T(\mathbb{A}_F)$ by the closure of $T(F)$ for the product topology.*

Note that the product topology on $T(\mathbb{A}_F)$ is coarser than the adelic topology, so that $\overline{T(F)}^w$ is indeed a closed subgroup of $T(\mathbb{A}_F)$.

Proof. — Let P be the quasi-split torus dual to the permutation Galois module $\mathrm{Pic}^T(\overline{X})$; the map $\overline{M} \to \mathrm{Pic}^T(\overline{X})$ induces a morphism $\mu \colon P \to T$ of algebraic tori. By definition, $A(T)^*$ is the kernel of the morphism

$$\mu^* \colon (T(\mathbb{A}_F)/T(F))^* \to (P(\mathbb{A}_F)/P(F))^*.$$

By Pontryagin duality, $A(T)^*$ is the dual of the cokernel of the map

$$P(\mathbb{A}_F)/P(F) \to T(\mathbb{A}_F)/T(F)$$

induced by μ. Inspection of the proof of Theorem 6 of [36] then shows this cokernel is equal to the finite group $T(\mathbb{A}_F)/\overline{T(F)}^w$, hence the lemma. $\square$

The quotient $T(\mathbb{A}_F)/\overline{T(F)}^w$ is classically denoted $A(T)$, it measures the obstruction to weak approximation.

LEMMA 3.8.2. — *The closure $\overline{T(F)}$ of $T(F)$ in $X(\mathbb{A}_F)$ coincides with $X(\mathbb{A}_F)^{\mathrm{Br}}$, the locus in $X(\mathbb{A}_F)$ where the Brauer–Manin obstruction vanishes. In particular, it is open and closed in $X(\mathbb{A}_F)$.*

Proof. — Let us observe that $T(\mathbb{A}_F)$ is dense in $X(\mathbb{A}_F)$, so that $\overline{T(F)}^w = T(\mathbb{A}_F) \cap \overline{T(F)}$. According to [31], Theorem 8.12, $\overline{T(F)}^w$ coincides with the locus $T(\mathbb{A}_F)^{\mathrm{Br}}$ in $T(\mathbb{A}_F)$ where the Brauer–Manin obstruction vanishes. Moreover, by Corollary 9.4 of that paper, $T(\mathbb{A}_F)^{\mathrm{Br}} = T(\mathbb{A}_F) \cap X(\mathbb{A}_F)^{\mathrm{Br}}$. Since the Brauer–Manin pairing is continuous and $\mathrm{Br}(X)/\mathrm{Br}(F)$ is finite, $X(\mathbb{A}_F)^{\mathrm{Br}}$ is open and closed in $X(\mathbb{A}_F)$. An easy topological argument then shows that $\overline{T(F)} = X(\mathbb{A}_F)^{\mathrm{Br}}$. $\square$

A character $\chi \in \mathscr{U}_T^K$ contributes to a pole of order b at $s = 0$ if and only if the following properties hold:

– for any $\alpha \in \mathscr{A}_U$, the Hecke character χ_α is trivial;

– for any $v \mid \infty$, there exists a face A_v of maximal dimension of $\mathscr{C}_v^{\mathrm{an}}(X, D)$, such that for any $\alpha \in A_v$, the local character $\chi_{v,\alpha}$ is trivial.

According to the following proposition, this implies $\chi \in A(T)^*$.

PROPOSITION 3.8.3. — [4] *Let $A = (A_v)_{v \in S}$ be a family, where for each v, A_v is a maximal stratum of $\mathscr{C}_v^{\mathrm{an}}(X, D)$. Let $\chi \in (T(\mathbb{A}_F)/T(F))^*$ be any topological character satisfying the following assumptions:*

– For all $\alpha \in \mathscr{A}_U$, the adelic character χ_α is trivial;

– For any $v \in S$, the restriction of the analytic character χ_v to the stabilizer of the stratum D_{A_v} in $T(F_v)$ is trivial.

Then $\chi \in A(T)^*$.

Proof. — Let us first prove that for any place $v \in S$, the analytic character χ_v is trivial. Let us fix such a place. The description of the analytic Clemens of a smooth toric variety done in [14] implies that for each v, there exists a maximal stratum B_v of $\mathscr{C}_v^{\mathrm{an}}(X, X \setminus T)$ containing A_v, and this stratum is reduced to a point $b_v \in X(F_v)$. Moreover, there exists a maximal split torus T_v' in T_{F_v} such that the Zariski closure of T_v' in X_{F_v} contains b_v. The assumptions imply that $\chi_{v,\beta}$ is trivial for any $\beta \in B_v$. Since the corresponding cocharacters generate the group of cocharacters of T_v', the analytic character χ_v is trivial on T_v'. Moreover, the torus T_{F_v}/T_v' is anisotropic by construction, so that $T(F_v)/T_v'(F_v)$ is compact; moreover, $T(F_v)$ is contained in the product of $T'(F_v)$ and the stabilizer in $T(F_v)$ of the stratum D_{A_v}. It follows that χ_v is trivial, as claimed.

Returning to the adelic character χ, we conclude that it is trivial on $T(F) \prod_{v \in S} T(F_v)$.

Let us now consider the exact sequence of $\mathbf{Z}[\Gamma_F]$-modules

$$0 \to X^*(T_{\overline{F}}) \to \mathrm{Pic}^T(X_{\overline{F}}) \to \mathrm{Pic}(X_{\overline{F}}) \to 0$$

and the induced exact sequence of algebraic tori:

$$0 \to T_{NS} \to \prod_{\alpha \in \mathscr{A}} \mathbf{G}_{m, F_\alpha} \to T \to 0.$$

Fix $\alpha \in \mathscr{A}$. We need to prove that the adelic character χ_α is trivial. If $\alpha \in \mathscr{A}_U$, this holds by assumption. Let us now assume that $\alpha \in \mathscr{A}_D$; let S_α be the set of places of F_α which extend the places of S. The character χ_α of $\mathbf{G}_m(\mathbb{A}_{F_\alpha})$ is trivial on $\mathbf{G}_m(\mathbb{A}_{F_\alpha}) \prod_{w \in S_\alpha} F_{\alpha,w}^\times$. Moreover, since S_α contains the archimedean places of F_α and χ_α is continuous, the kernel of the restriction of χ_α to $\prod_{w \notin S_\alpha} \mathbf{G}_m(F_{\alpha,w})$ contains an open subgroup. By weak approximation in $F_\alpha^\times$, we conclude that χ_α is trivial, as was to be shown. □

3.9. The leading term

By the preceding analysis, with b defined as in Equation (3.7.1), one has

$$\lim_{t \to 1} (t-1)^b Z(t\rho) = \sum_{A \in \mathscr{C}_\infty^{\mathrm{an,max}}(X,D)} \Theta_A$$

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[4] Déjà utilisée plus haut!

where, for each $A \in \mathscr{C}_\infty^{\mathrm{an,max}}(X, D)$,

$$\Theta_A = c_T \sum_{\chi \in A(T)^*} \mathscr{X}_{\Lambda'_A}(\pi(\tilde{\lambda})) \varphi(\rho; \chi) \prod_{\alpha \in \mathscr{A}_U} \mathrm{L}^*(1; \chi_\alpha) \prod_{v | \infty} \varphi_{v, A_v}(\mathrm{im}(\chi_v); \chi_{v,1})$$

$$= c_T \, \mathscr{X}_{\Lambda'_A}(\pi(\tilde{\lambda})) \left(\lim_{t \to 1} (t-1)^{b+\mathrm{rank}\, M} \left(\sum_{\chi \in A(T)^*} \hat{H}_A(\tilde{\rho} + (t-1)\tilde{\lambda}; \chi) \right) \right).$$

Recall from Lemma 3.8.2 that $T(\mathbb{A}_F)^{\mathrm{Br}}$ is the orthogonal of $A(T)^*$ in $T(\mathbb{A}_F)$, it is an open subgroup of finite index in the adelic group $T(\mathbb{A}_F)$. We apply the Poisson summation formula again. For any positive $\tilde{\mathbf{s}}$, one has

$$\sum_{\chi \in A(T)^*} \hat{H}_A(\tilde{\rho} + \tilde{\mathbf{s}}; \chi) = |A(T)| \int_{T(\mathbb{A}_F)^{\mathrm{Br}}} H(\mathbf{x}; \tilde{\rho} + \tilde{\mathbf{s}}) \delta(\mathbf{x}) \theta_A(\mathbf{x}) \, \mathrm{d}\mathbf{x},$$

so that

$$(3.9.1) \quad \Theta_A = |A(T)| \, c_T \, \mathscr{X}_{\Lambda'_A}(\pi(\tilde{\lambda})) \times$$

$$\times \left(\lim_{t \to 1} (t-1)^{b+\mathrm{rank}\, M} \int_{T(\mathbb{A}_F)^{\mathrm{Br}}} H(\mathbf{x}; \rho + (t-1)\lambda s) \delta(\mathbf{x}) \theta_A(\mathbf{x}) \, \mathrm{d}\mathbf{x} \right).$$

3.10. Leading term and equidistribution for rational points

Assume that $U = X$, *i.e.*, we consider the distribution of rational points of bounded height in toric varieties. In this case, all analytic Clemens complexes are empty and $\mathscr{C}_\infty^{\mathrm{an,max}}(X, D)$ is understood as the set $\{\varnothing\}$. We have

$$\mathrm{Pic}(U; \varnothing) = \mathrm{Pic}(X);$$

because of the chosen normalizations for measures, and the characteristic function $\mathscr{X}_{\Lambda'_\varnothing}$ is the characteristic function of the effective cone $\Lambda_{\mathrm{eff}}(X)$ in $\mathrm{Pic}(X)$ multiplied by $|H^1(\Gamma_F, \overline{M})|$. The height zeta function has a single pole at $t = 1$ of multiplicity

$$b = |\mathscr{A}| - \mathrm{rank}\, M = \mathrm{rank}\, \mathrm{Pic}(X),$$

and

$$\Theta_\varnothing := \lim (t-1)^b Z(t\rho)$$

is given by

$$(3.10.1) \qquad \Theta_\varnothing = c_T \, |A(T)| \, |\mathrm{H}^1(\Gamma_F, \overline{M})| \, \mathscr{X}_{\Lambda_{\mathrm{eff}}(X)}(\rho) \lim_{t \to 1} (t-1)^{|\mathscr{A}|} \int_{T(\mathbb{A}_F)^{\mathrm{Br}}} H(\mathbf{x}; t\rho) \, \mathrm{d}\mathbf{x}.$$

The boundary of $T(\mathbb{A}_F)^{\mathrm{Br}}$ in $X(\mathbb{A}_F)$ is contained in a countable union of spaces of the form $\prod_{w \neq v} X(F_w) \times (X \setminus T)(F_v)$, hence has measure 0 for the Tamagawa measure τ_X. In our paper [14], we have computed limits as in (**??**) (Proposition 4.4.4) and derived in

Theorem 4.4.5 an asymptotic formula for volumes, as well as the equidistribution property for height balls. This analysis implies that for any continuous function φ on $X(\mathbb{A}_F)$,

$$\lim_{t \to 1}(t-1)^{|\mathscr{A}|} \int_{T(\mathbb{A}_F)^{\mathrm{Br}}} H(\mathbf{x}; t\rho)\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} = \prod_{\alpha \in \mathscr{A}} \frac{1}{\rho_\alpha} \int_{X(\mathbb{A}_F)} \varphi(\mathbf{x})\,\mathrm{d}\tau_X(\mathbf{x}),$$

where τ_X is Peyre's Tamagawa measure on $X(\mathbb{A}_F)$. (The measure $\mathrm{d}\tau_{(X,D)}$ of that paper is $\mathrm{d}\mathbf{x}/\mathrm{L}^*(1,\overline{M})$.) Then, the same formula also holds for the characteristic function of $X(\mathbb{A}_F)^{\mathrm{Br}}$ since its boundary has measure 0. Therefore,

$$\Theta_\varnothing = c_T\,|A(T)|\,|H^1(\Gamma_F,\overline{M})|\,\mathscr{X}_{\Lambda_{\mathrm{eff}}(X)}(\rho)\,\tau_X(X(\mathbb{A}_F)^{\mathrm{Br}}).$$

LEMMA 3.10.2. — *One has*

$$c_T\,|A(T)|\,|H^1(\Gamma_F,\overline{M})| = |H^1(\Gamma_F,\mathrm{Pic}(\overline{X}))|.$$

Proof. — The measure $\mathrm{d}\tau(X, X \setminus T)$ is exactly the Haar measure of $T(\mathbb{A}_F)$ used by Ono in [27]. According to this paper,

$$\mathrm{vol}(T(\mathbb{A}_F)^1/T(F)) = \frac{|\mathrm{H}^1(\Gamma,\overline{M})|}{|\mathrm{III}(T)|}.$$

Moreover, Voskresenskii has shown ([36], Theorem 6) that

$$|A(T)|\,|\mathrm{III}(T)| = |\mathrm{H}^1(\Gamma,\mathrm{Pic}(\overline{X}))|.$$

The lemma follows. $\qquad\square$

Since this holds for an arbitrary choice of smooth adelic metrics on the involved line bundles, we also deduce that the rational points of bounded height are equidistributed in $X(\mathbb{A}_F)^{\mathrm{Br}}$ with respect to the restriction of the measure τ_X. Since the Tamagawa measure has full support on $X(\mathbb{A}_F)$, we see in particular that the closure of $T(F)$ contains $X(\mathbb{A}_F)^{\mathrm{Br}}$. However, $T(F)$ is contained in $T(\mathbb{A}_F)^{\mathrm{Br}}$ (product formula), hence in $X(\mathbb{A}_F)^{\mathrm{Br}}$, from which we conclude that

$$\overline{T(F)} = X(\mathbb{A}_F)^{\mathrm{Br}}.$$

We summarize our results:

THEOREM 3.10.3. — *Let X be a smooth projective toric variety over a number field F. Endow the canonical line bundle K_X with a smooth adelic metric and let H be the corresponding height function. Then*

$$\mathrm{Card}\{x \in T(F)\,;\, H(x) \leqslant B\} \sim \Theta B(\log B)^{r-1}, \qquad B \to \infty,$$

with

$$\Theta = |H^1(\Gamma_F,\mathrm{Pic}(\overline{X}))|\,\mathscr{X}_{\Lambda_{\mathrm{eff}}(X)}(\rho)\,\tau_X(X(\mathbb{A}_F)^{\mathrm{Br}}).$$

Moreover, rational points of height $\leqslant B$ equidistribute towards the probability measure

$$\frac{1}{\tau_X(X(\mathbb{A}_F)^{\mathrm{Br}}}\,\tau_X|_{X(\mathbb{A}_F)^{\mathrm{Br}}}$$

on $X(\mathbb{A}_F)^{\mathrm{Br}}$.

3.11. Leading term and equidistribution for integral points

We return to the case of integral points. By Lemma 3.8.2, the characteristic function of $X(\mathbb{A}_F)^{\mathrm{Br}}$ is continuous on $X(\mathbb{A}_F)$. Recall that

$$\Theta_A = |A(T)|\, c_T\, \mathscr{X}_{\Lambda'_A}(\pi(\tilde{\lambda}))\times$$
$$\times \left(\lim_{t\to 1}(t-1)^{b+\operatorname{rank} M} \int_{T(\mathbb{A}_F)^{\mathrm{Br}}} H(\mathbf{x};\rho+(t-1)\lambda s)\delta(\mathbf{x})\theta_A(\mathbf{x})\, \mathrm{d}\mathbf{x} \right).$$

Applying Equation (4.4.3) (rather, its straightforward generalization for finite adeles and in the quasi-projective case) and Proposition 4.2.4 (for places in v) of our paper [14], we obtain:

$$\lim_{t\to 1}(t-1)^{b+\operatorname{rank} M} \int_{T(\mathbb{A}_F)^{\mathrm{Br}}} H(\mathbf{x};\rho+(t-1)\lambda s)\delta(\mathbf{x})\theta_A(\mathbf{x})\, \mathrm{d}\mathbf{x}$$
$$= \int_{U(\mathbb{A}_{F,\mathrm{fin}})\prod D_{A_v}(F_v)} \mathbf{1}_{X(\mathbb{A}_F)^{\mathrm{Br}}}\delta(\mathbf{x})\theta_A(\mathbf{x})\mathrm{d}\tau_U^{\mathrm{fin}}(x_{\mathrm{fin}}) \prod_v \mathrm{d}\tau_{A_v}(x_v)$$
$$= \int_{U(\mathbb{A}_{F,\mathrm{fin}})\prod D_{A_v}(F_v)} \mathbf{1}_{X(\mathbb{A}_F)^{\mathrm{Br}}}\delta(\mathbf{x})\mathrm{d}\tau_U^{\mathrm{fin}}(x_{\mathrm{fin}}) \prod_v \mathrm{d}\tau_{A_v}(x_v)$$

since $\theta_{A_v} \equiv 1$ on $D_{A_v}(F_v)$. Moreover,

$$|A(T)|c_T = \frac{|\mathrm{H}^1(\Gamma_F, \operatorname{Pic}(\overline{X}))|}{|\mathrm{H}^1(\Gamma_F, \overline{M})|}$$

so that,

$$(3.11.1) \quad \Theta_A = |\mathrm{H}^1(\Gamma_F, \operatorname{Pic}(\overline{X}))||\mathrm{H}^1(\Gamma_F, \overline{M})|\mathscr{X}_{\Lambda'_A}(\rho)\times$$
$$\times \int_{U(\mathbb{A}_{F,\mathrm{fin}})\prod D_{A_v}(F_v)} \mathbf{1}_{X(\mathbb{A}_F)^{\mathrm{Br}}}\delta(\mathbf{x})\mathrm{d}\tau_U^{\mathrm{fin}}(x_{\mathrm{fin}}) \prod_v \mathrm{d}\tau_{A_v}(x_v).$$

Recall that

$$(3.11.2) \qquad\qquad \lim_{t\to 1}(t-1)^b \mathrm{Z}(t\rho) = \sum_{A\in\mathscr{C}_\infty^{\mathrm{an,max}}(X,D)} \Theta_A.$$

THEOREM 3.11.3. — *Let X be a smooth projective toric variety over a number field F. Let D be an invariant divisor such that $-(K_X + D)$ is big and let $U = X \setminus D$. Let $\mathscr{U}$ be a model of U over $\mathfrak{o}_F$. Let us endow the canonical line bundle K_X with a smooth adelic metric; let H be the corresponding height function. Then, when $B \to \infty$,*

$$\operatorname{Card}\{x \in T(F) \cap \mathscr{U}(\mathfrak{o}_F)\,;\, H(x) \leqslant B\} \sim \Theta B(\log B)^{b-1}$$

with

$$\Theta = \frac{|\mathrm{H}^1(\Gamma_F, \mathrm{Pic}(\overline{X}))|}{|\mathrm{H}^1(\Gamma_F, \overline{X})|} \times$$

$$\times \sum_{A \in \mathscr{C}_\infty^{\mathrm{an,max}}(X,D)} \mathscr{X}_{\Lambda_A'}(\rho) \int_{U(\mathbb{A}_{F,\mathrm{fin}}) \prod D_{A_v}(F_v)} \mathbf{1}_{X(\mathbb{A}_F)^{\mathrm{Br}}} \delta(\mathbf{x}) \mathrm{d}\tau_U^{\mathrm{fin}}(x_{\mathrm{fin}}) \prod_{v|\infty} \mathrm{d}\tau_{A_v}(x_v)$$

and

$$b = \mathrm{rank}\,\mathrm{Pic}(U) + \sum_{v|\infty} |A_v| = \mathrm{rank}\,\mathrm{Pic}(U) + \sum_{v|\infty} (1 + \dim \mathscr{C}_v^{\mathrm{an}}(X,D)).$$

Moreover, the integral points of height $\leqslant B$ equidistribute towards the probability measure proportional to

$$\sum_{A \in \mathscr{C}_\infty^{\mathrm{an,max}}(X,D)} \mathscr{X}_{\Lambda_A'}(\rho) \int_{U(\mathbb{A}_{F,\mathrm{fin}}) \prod D_{A_v}(F_v)} \mathbf{1}_{X(\mathbb{A}_F)^{\mathrm{Br}}} \delta(\mathbf{x}) \mathrm{d}\tau_U^{\mathrm{fin}}(x_{\mathrm{fin}}) \prod_{v|\infty} \mathrm{d}\tau_{A_v}(x_v)$$

on $X(\mathbb{A}_F)^{\mathrm{Br}} \cap U(\mathbb{A}_{F,\mathrm{fin}}) \times \prod_{v|\infty} |D|(F_v)$. In particular,

$$\overline{T(F) \cap \mathscr{U}(\mathfrak{o}_F)} = X(\mathbb{A}_F)^{\mathrm{Br}} \cap \left(\bigcup_{A=(A_v) \in \mathscr{C}_\infty^{\mathrm{an,max}}(X,D)} U(\mathbb{A}_{F,\mathrm{fin}})) \times \prod_{v|\infty} D_{A_v}(F_v) \right),$$

where $\overline{T(F) \cap \mathscr{U}(\mathfrak{o}_F)}$ is the closure of $T(F) \cap \mathscr{U}(\mathfrak{o}_F)$ in $X(\mathbb{A}_F)$.

In the theorem, note that the groups $\mathrm{Pic}(U; A)$ all have rank r.

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