MILNOR K_2 AND FIELD HOMOMORPHISMS

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ABSTRACT. We prove that the function field of an algebraic variety of dimension ≥ 2 over an algebraically closed field is completely determined by its first and second Milnor K-groups.

CONTENTS

1. INTRODUCTION

In this paper we study the problem of reconstruction of field homomorphisms from group-theoretic data. A prototypical example is the reconstruction of function fields of algebraic varieties from their absolute Galois group, a central problem in "anabelian geometry" (see [9], [6], [5], [7]). Within this theory, an important question is the "section conjecture", i.e., the problem of detecting homomorphisms of fields on the level of homomorphisms of their Galois groups. In the language of algebraic geometry, one is interested in obstructions to the existence of points of algebraic varieties over higher-dimensional function fields, or equivalently, rational sections of fibrations. Here we study group theoretic objects which are dual, in some sense, to small pieces of the Galois group, obtained from the *abelianization* of the absolute Galois group and its canonical central extension. This connection will be explained in Section 2.

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We now formulate the main results. In this paper, we work in characteristic zero. An element of an abelian group is called primitive, if it cannot be written as a nontrivial multiple in this group.

Definition 1. Let k be an infinite field. A field K will be called *geometric* over k if

(1) $k \subset K$;
(2) for each $f \in K^* \setminus k^*$, the set $\{f + \kappa\}_{\kappa \in k}$ has at most finitely many elements whose image in K^*/k^* is nonprimitive.

If X is an algebraic variety over an algebraically closed field k of characteristic zero then its function field $K = k(X)$ is geometric over k. There exist other examples, e.g., some *infinite* algebraic extensions of $k(X)$ are also geometric over k.

Theorem 2. *Let K, resp. L, be a geometric field of transcendence degree ≥ 2 over an algebraically closed field k, resp. l, of characteristic zero. Assume that there exists an injective homomorphism of abelian groups*

$$\psi_1 \,:\, K^*/k^* \to L^*/l^*$$

such that

(1) *the image of ψ_1 contains one primitive element in L^*/l^* and two elements whose lifts to L^* are algebraically independent over l;*
(2) *for each $f \in K^* \setminus k^*$ there exists a $g \in L$ such that*

$$\psi_1\left(\overline{k(f)}^*/k^* \cap K^*/k^*\right) \subseteq \overline{l(g)}^*/l^* \cap L/l^*.$$

Then there exists a field embedding

$$\psi \,:\, K \to L$$

which induces either ψ_1 or ψ_1^{-1}.

Remark 3. An analogous statement holds in positive characteristic. The final steps of the proof in Section 4 are more technical due to the presence of p^n-powers of "projective lines".

Let K be a field. Denote by $\mathrm{K}_i^M(K)$ the i-th Milnor K-group of K. Recall that

$$\mathrm{K}_1^M(K) = K^*$$

and that there is a canonical surjective homomorphism

$$\sigma_K \,:\, \mathrm{K}_1^M(K) \otimes \mathrm{K}_1^M(K) \to \mathrm{K}_2^M(K)$$

whose kernel is generated by $x \otimes (1-x)$, for $x \in K^* \setminus 1$ (see [4] for more background on K-theory). Put

$$\bar{\mathrm{K}}_i^M(K) := \mathrm{K}_i^M(K)/\text{infinitely divisible}, \quad i = 1, 2.$$

The homomorphism σ_K is compatible with reduction modulo infinitely divisible elements. As an application of Theorem 2 we prove the following result.

Theorem 4. *Let K and L be function fields of algebraic varieties of dimension ≥ 2 over an algebraically closed field k, resp. l. Let*

$$\psi_1 \,:\, \bar{\mathrm{K}}_1^M(K) \to \bar{\mathrm{K}}_1^M(L) \tag{1.1}$$

be an injective homomorphism of abelian groups such that the following diagram of abelian group homomorphisms is commutative

$$\begin{CD}
\bar{\mathrm{K}}_1^M(K) \otimes \bar{\mathrm{K}}_1^M(K) @>{\psi_1 \otimes \psi_1}>> \bar{\mathrm{K}}_1^M(L) \otimes \bar{\mathrm{K}}_1^M(L) \\
@V{\sigma_K}VV @VV{\sigma_L}V \\
\bar{\mathrm{K}}_2^M(K) @>>{\psi_2}> \bar{\mathrm{K}}_2^M(L).
\end{CD}$$

Assume further that $\psi_1(K^*/k^*)$ is not contained in E^*/k^* for any 1-dimensional subfield $E \subset L$. Then there exist a homomorphism of fields

$$\psi \,:\, K \to L,$$

and an $r \in \mathbb{Q}$ such that the induced map on K^*/k^* coincides with the r-th power of ψ_1.

In particular, the assumptions are satisfied when ψ_1 is an isomorphism of abelian groups. In this case, Theorem 4 states that a function field of transcendence degree ≥ 2 over an algebraically closed ground field of characteristic zero is determined by its first and second Milnor K-groups.

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2. Background

The problem considered in this paper has the appearance of an abstract algebraic question. However, it is intrinsically related to our program to develop a skew-symmetric version of the theory of fields, and especially, function fields of algebraic varieties.

Let K be a field and $\mathcal{G}_K$ its absolute Galois group, i.e., the Galois group of a maximal separable extension of K. It is a compact profinite group. We are interested in the quotient

$$\mathcal{G}_K^c = \mathcal{G}_K/[\mathcal{G}_K, [\mathcal{G}_K, \mathcal{G}_K]]$$

and its maximal topological pro-ℓ-completion

$$\mathcal{G}_{K,\ell}^c, \quad \ell \neq \mathrm{char}(K).$$

The group $\mathcal{G}_{K,\ell}^c$ is a central pro-ℓ-extension of the pro-ℓ-completion of the abelianization $\mathcal{G}_K^a$ of $\mathcal{G}_K$.

We now assume that K is the function field of an algebraic variety over an algebraically closed ground field k. In this case, $\mathcal{G}_{K,\ell}^a$ is a torsion-free topological pro-ℓ-group which is dual to the torsion-free abelian group K^*/k^*, i.e., there is a canonical identification

$$\mathcal{G}_{K,\ell}^a = \mathrm{Hom}(K^*/k^*, \mathbb{Z}_\ell(1)),$$

via Kummer theory. The group $\mathcal{G}_{K,\ell}^c$ admits a simple description in terms of one-dimensional subfields of K, i.e., subfields of transcendence degree 1 over k. For each such subfield $E \subset K$, which is normally closed in K, we have a surjective homomorphism $\mathcal{G}_{K,\ell}^c \to \mathcal{G}_{E,\ell}^c$, where the image is a *free* central pro-ℓ-extension of the group $\mathcal{G}_{E,\ell}^a$.

Our main goal is to establish a functorial correspondence between function fields of algebraic varieties K and L, over algebraically closed ground fields k and l, respectively, and corresponding topological groups $\mathcal{G}_K^c$, resp. $\mathcal{G}_{K,\ell}^c$. We are aiming at a (conjectural) equivalence between homomorphisms of function fields

$$\bar{\Psi} \,:\, K \to L$$

and homomorphisms of topological groups

$$\Psi_\ell^c \,:\, \mathcal{G}_{L,\ell}^c \to \mathcal{G}_{K,\ell}^c.$$

It is clear that $\bar{\Psi}$ induces (but not uniquely) a homomorphism Ψ_ℓ^c as above. The problem is to find conditions on Ψ_ℓ^c such that it corresponds to some $\bar{\Psi}$. In particular, Ψ_ℓ^c would give rise to homomorphisms of the full Galois groups $\mathcal{G}_L \to \mathcal{G}_K$.

Remark 5. By a theorem of Stallings [8], a group homomorphism that induces an isomorphism on $H_1(-,\mathbb{Z})$ and an epimorphism on $H_2(-,\mathbb{Z})$ induces an isomorphism on the lower central series.

Thus we expect that $\mathcal{G}_{K,\ell}$ is in some sense the maximal pro-ℓ-group with given H_1 and H_2.

Consider the diagram

$$\begin{tikzcd} \mathcal{G}^c_{L,\ell} \arrow[r] \arrow[d] & \mathcal{G}^c_{K,\ell} \arrow[d] \\ \mathcal{G}^a_{L,\ell} \arrow[r] & \mathcal{G}^a_{K,\ell} \end{tikzcd}$$

The group $\mathcal{G}^c_{L,\ell}$ can be identified with a closed subgroup in the direct product of free central pro-ℓ-extensions

$$\prod_E \mathcal{G}^c_{E,\ell},$$

where the product runs over all normally closed one-dimensional subfields E of L. The homomorphisms $\mathcal{G}^c_{L,\ell} \to \mathcal{G}^c_{E,\ell}$ are induced from certain homomorphisms of abelian quotients $\mathcal{G}^a_{L,\ell} \to \mathcal{G}^a_{K,\ell}$, namely those which commute with surjective maps of $\mathcal{G}^a_{L,\ell}$ and $\mathcal{G}^a_{K,\ell}$ to the abelian groups of one-dimensional subfields of L and K, respectively.

It is shown in [2] that in the case of functional fields of transcendence degree 2 over $k = \bar{\mathbb{F}}_p$ and $\ell \neq p$, any isomorphism Ψ^c_ℓ defines an isomorphism between K and some finite purely inseparable extension of L. In this paper we treat the first problem which arises when we try to extend the result to general homomorphisms. By the description above, it suffices to treat the corresponding homomorphisms of abelian groups

$$\Psi^a_\ell \,:\, \mathcal{G}^a_{L,\ell} \to \mathcal{G}^a_{K,\ell}.$$

By Kummer theory, these can be identified with homomorphisms

$$\Psi^*_\ell \,:\, \mathrm{Hom}(L^*/l^*,\mathbb{Z}_\ell) \to \mathrm{Hom}(K^*/k^*,\mathbb{Z}_\ell).$$

The condition that Ψ^c_ℓ commutes with projections onto Galois groups of one-dimensional fields is the same as commuting with projections

$$\mathrm{Hom}(L^*/l^*,\mathbb{Z}_\ell(1)) \to \mathrm{Hom}(E^*,\mathbb{Z}_\ell(1)).$$

If it were possible to dualize the picture we would have a homomorphism

$$\Psi^* : K^*/k^* \to L^*/l^*,$$

mapping multiplicative groups of one-dimensional subfields in K to multiplicative groups of one-dimensional subfields of L. This is the problem that we consider in the paper.

In order to solve the problem for Galois groups we need to consider the maps

$$\hat{\Psi}^*_\ell \,:\, \hat{K}^* \to \hat{L}^*,$$

between ℓ-completions of the dual spaces (as in [2]) and to find conditions which would allow to reconstruct Ψ^* from $\hat{\Psi}^*_\ell$. This problem will be addressed in a future publication.

3. Functional equations

Lemma 6. *Let $x, y \in K$ be algebraically independent elements and $z \in k(x, y)$ a nonconstant rational function. Let $f, g \in k(t)^*$ be nonconstant functions such that $f(x)/g(y) \in k(z)$. Then there exist $\tilde{f}, \tilde{g} \in k(t)^*$ such that*

$$k(z) = k(\tilde{f}(x)/\tilde{g}(y)).$$

Proof. Write $z = p(x, y)/q(x, y)$, with coprime $p, q \in k[x, y]$. Then

$$f(x)/g(y) = \prod_i (p/q - c_i)^{n_i} = q^{-\sum_i n_i} \prod_i (p - c_i q)^{n_i},$$

modulo k^*, for pairwise distinct $c_i \in k$ and some $n_i \in \mathbb{Z}$. The factors on the right are pairwise coprime, i.e., their divisors have no common components. Thus the divisors of $q(x, y)$ and $p(x, y) - c_i q(x, y)$ are either "vertical" or "horizontal", i.e.,

$$q(x, y) = t(x)u(y) \quad \text{and} \quad p(x, y) - c_i q(x, y) = v_i(x)w_i(y),$$

for some $t, u, v_i, w_i \in k(t)$. It follows that

$$z(x, y) - c_i = v_i(x)w_i(y)/t(x)u(y)$$

and we can put $\tilde{g} = v_i(x)/t(x)$ and $\tilde{f} = z(y)/w_i(y)$. □

A rational function $f \in k(x, y)^*$ is called *homogeneous of degree r* if

$$\lambda^r f(x, y) = f(\lambda x, \lambda y), \quad \text{for all } \lambda \in k^*. \tag{3.1}$$

A function f is homogeneous of degree 0 iff $f \in k(x/y)^*$.

Lemma 7. *Let $p_1, p_2 \in k(x, y)^*$ be rational functions with disjoint divisors. Assume that $p_1(x, y) \cdot p_2(x, y)$ is homogeneous of degree r. Then p_1 is homogeneous of degree r_1, p_2 is homogeneous of degree r_2 and $r_1 + r_2 = r$.*

Corollary 8. *Let $f, g \in k[t]$ be nonzero polynomials. Assume that $p(x,y) := g(x)f(y)$ is homogeneous of degree $d \in \mathbb{N}$. Then*

$$g(x) = ax^n$$
$$f(y) = by^{d-n},$$

for some $n \in \mathbb{N}$ and $a, b \in k^*$.

Lemma 9. *Let $f, g \in k[t]$ be polynomials such that*

$$p(x,y) = ax^r f(y) - cy^r g(x) \in k[x,y] \tag{3.2}$$

is homogeneous of degree $r \in \mathbb{N}$. Then

$$g(x) = a_d x^r + a_0,$$
$$f(y) = c_d y^r + c_0,$$

and $ac_d - ca_d = 0$.

Proof. Write $g(x) = \sum_i a_i x^i$ and $f(y) = \sum_j c_j y^j$, substitute into the equation (3.2), and use homogeneity. $\square$

Lemma 10. *Let $f_1, f_2, g_1, g_2 \in k[t]$ be polynomials such that*

$$\gcd(g_1, g_2) = \gcd(f_1, f_2) = 1 \in k[t]/k^*.$$

Let

$$p(x,y) = g_1(x)f_2(y) - g_2(x)f_1(y) \in k[x,y]$$

be a polynomial, homogeneous of degree $r \in \mathbb{N}$. Then

$$g_i(x) = a_i x^r + b_i,$$
$$f_i(y) = c_i y^r + d_i,$$

for some $a_i, b_i, c_i, d_i \in k$, for $i = 1, 2$, with

$$b_1 d_2 - b_2 d_1 = 0,$$
$$a_1 c_2 - a_2 c_1 = 0.,$$

Proof. By homogeneity, $p(0,0) = 0$, i.e.,

$$g_1(0)f_2(0) - g_2(0)f_1(0) = 0.$$

Rescaling, using the symmetry and coprimality of f_1, f_2, resp. g_1, g_2, we may assume that

$$\begin{pmatrix} f_1(0) & f_2(0) \\ g_1(0) & g_2(0) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

In the first case, restricting to $x = 0$, resp. $y = 0$, we find

$$g_1(x) - g_2(x) = ax^r,$$
$$f_1(y) - f_2(y) = cy^r,$$

for some constants $a, c \in k^*$. Solving for f_2, g_2 and substituting we obtain

$$p(x, y) = ax^r f_1(y) - cy^r g_1(x).$$

In the second case, we have directly

$$g_1(x) = ax^r,$$
$$f_1(y) = cy^r,$$

for some $a, c \in k^*$, and

$$p(x, y) = ax^r f_2(y) - cy^r g_2(x).$$

It suffices to apply Lemma 9. □

Proposition 11. *Let $x, y \in K^*$ be algebraically independent elements. Fix nonzero integers r and s and consider the equation*

$$Ry^r = Sq^s, \tag{3.3}$$

with

$$R \in k(x/y), \quad p \in k(x), \quad q \in k(y), \quad S \in k(p/q),$$

where $p \in k(x)$ and $q \in k(y)$ are nonconstant rational functions. Assume that

(i) *x, y, p, q are multiplicatively independent;*
(ii) *R, S are nonconstant.*

Then

$$p(x) = \frac{x^{r_1}}{p_{2,1}x^{r_1} + p_2(0)}, \quad q(y) = \frac{y^{r_1}}{q_{2,1}y^{r_1} + q_1(0)},$$

or

$$p(x) = \frac{p_{1,1}x^{r_1} + p_1(0)}{x^{r_1}}, \quad q(y) = \frac{q_{1,1}y^{r_1} + q_2(0)}{y^{r_1}},$$

with

$$r_1 \in \mathbb{N}, \quad p_{1,1}, p_{2,1}, p_1(0), p_2(0), q_{1,1}, q_{2,1}, q_1(0), q_2(0) \in k^*.$$

We have

$$Sq^s = \left(\frac{x^{r_1}y^{r_1}}{q_1(0)x^{r_1} - d_1p_2(0)y^{r_1}}\right)^s$$

with $d_1 = q_{2,1}/p_{2,1}$ and $r = r_1 s$ in the first case and

$$Sq^s = \left(\frac{p_1(0)y^{r_1} - d_1q_2(0)x^{r_1}}{x^{r_1}y^{r_1}}\right)^s,$$

with $d_1 = p_{1,1}/q_{1,1}$ and $r = -r_1 s$ in the second case.

Conversely, every pair (p, q) as above leads to a solution of (3.3)*.*

Proof. Equation (3.3) gives, modulo constants,

$$y^r \prod_{i=0}^{I} (x/y - c_i)^{n_i} = q^s \prod_{j=0}^{J} (p/q - d_j)^{m_j}, \tag{3.4}$$

for pairwise distinct constants $c_i, d_j \in k$, and some $n_i, m_j \in \mathbb{Z}$. We assume that $c_0 = d_0 = 0$ and that $c_i, d_j \in k^*$, for $i, j \geq 1$. Expanding, we obtain

$$x^{n_0} y^{r - \sum_{i \geq 0} n_i} \prod_{i>0} (x - c_i y)^{n_i} =$$
$$p_1^{m_0} p_2^{-\sum_{j \geq 0} m_j} q_2^{m_0 - s} q_1^{s - \sum_{j \geq 0} m_j} \prod_{j>0} (p_1 q_2 - d_j p_2 q_1)^{m_j},$$

where $p = p_1/p_2$ and $q = q_1/q_2$, with p_1, p_2 and q_1, q_2 coprime polynomials in x, resp. y. It follows that:

$(A1)$ $\quad x^{n_0} = p_1^{m_0}(x) p_2^{-m_0 - \sum_{j>0} m_j}(x),$

$(A2)$ $\quad y^{r - n_0 - \sum_{i>0} n_i} = q_2(y)^{m_0 - s} q_1(y)^{s - m_0 - \sum_{j>0} m_j},$

$(A3)$ $\quad \prod_{i=1}^{I} (x - c_i y)^{n_i} = \prod_{j=1}^{J} (p_1(x) q_2(y) - d_j p_2(x) q_1(y))^{m_j}.$

Lemma 12. *If $n_1 \neq 0$ then the exponents n_i, m_j have the same sign, for all $i, j \geq 1$.*

Proof. Assume otherwise. Collecting terms in (A3) with exponent of the same sign we obtain:

$$\prod_{i>0, n_i>0} (x - c_i y)^{n_i} = \prod_{j>0, m_j>0} (p_1 q_2 - d_j p_2 q_1)^{m_j},$$
$$\prod_{i>0, n_i<0} (x - c_i y)^{n_i} = \prod_{j>0, m_j<0} (p_1 q_2 - d_j p_2 q_1)^{m_j}$$

Thus there are $a, b \in \mathbb{N}$ such that

$$\Big(\prod_{i>0, n_i>0} (x - c_i y)^{n_i}\Big)^a \Big(\prod_{i>0, n_i<0} (x - c_i y)^{n_i}\Big)^b$$

is a nontrivial rational function of x/y with trivial divisor at infinity in $\mathbb{P}^1 \times \mathbb{P}^1$, with standard coordinates x, y. The same holds for

$$\Big(\prod_{j>0, m_j>0} (p_1 q_2 - d_j p_2 q_1)^{m_j}\Big)^a \Big(\prod_{j>0, m_j<0} (p_1 q_2 - d_j p_2 q_1)^{m_j}\Big)^b,$$

a nontrivial rational function of p/q. Thus $k(p/q) \cap k(x/y) \neq k$, which contradicts the assumption that p/q and x/y are multiplicatively independent. Indeed, the functions p/q and x/y generate a subgroup of rank 2 in K^*/k^* and hence belong to fields intersecting by constants only. $\square$

By Lemma 12, if $\sum_{i>0} n_i = 0$ or $\sum_{j>0} m_j = 0$ then $n_i = m_j = 0$ for all $i, j \geq 1$. By (A1),

$$x^{n_0} = p_1^{m_0} p_2^{-m_0}.$$

By assumption (ii), R is nonconstant. Hence $n_0 \neq 0$. It follows that p is a power of x, contradicting (i).

We can now assume

$$\sum_{i>0} n_i \neq 0, \quad \text{and} \quad \sum_{i>0} m_j \neq 0. \tag{3.5}$$

It follows that

$$(m_0, -m_0 - \sum_{j>0} m_j) \neq (0,0) \quad \text{and} \quad (m_0 - s, s - m_0 - \sum_{j>0} m_j) \neq (0,0).$$

On the other hand, by (i), combined with (A1) and (A2), one of the terms in each pair is zero. We have the following cases:

(1) $m_0 \neq 0$, $m_0 = -\sum_{j>0} m_j$, $\quad m_0 = s$ and

$$x^{n_0} = p_1^{m_0}, \quad q_1^s = y^{r - n_0 - \sum_{i>0} n_i};$$

(2) $m_0 = 0$, $s = \sum_{j>0} m_j$ and

$$x^{n_0} = p_2^{-\sum_{j>0} m_j} = p_2^{-s}, \quad q_2^{-s} = y^{r - n_0 - \sum_{i>0} n_i}.$$

We turn to (A3), with $J \geq 1$ and n_i, m_j replaced by $|n_i|, |m_j|$. From (A1) we know that $p_1(x) = x^a$ or $p_2(x) = x^a$, for some $a \in \mathbb{N}$. Similarly, from (A2) we have $q_1(y) = y^b$ or $q_2(y) = y^b$, for some $b \in \mathbb{N}$. All irreducible components of the divisor of

$$f_j := p_1(x) q_2(y) - d_j p_2(x) q_1(y)$$

are of the form $x = c_i y$, i.e., these divisors are homogeneous with respect to

$$(x, y) \mapsto (\lambda x, \lambda y), \quad \lambda \in k^*.$$

It follows that f_j is homogeneous, of some degree $r_j \in \mathbb{N}$. If

$$p_1(x) q_2(y) = x^a y^b, \quad \text{or} \quad p_2(x) q_1(y) = x^a y^b,$$

then f_j has a nonzero constant term, contradiction. Lemma 10 implies that either

$$p_1(x) = x^{r_j} \quad \text{and} \quad q_1(y) = y^{r_j}, \tag{3.6}$$

or

$$p_2(x) = x^{r_j} \quad \text{and} \quad q_2(y) = y^{r_j}. \tag{3.7}$$

It follows that all r_j are equal, for $j \geq 1$.

The cases are symmetric, and we first consider (3.6). Note that equation (3.6) is incompatible with Case $m_0 = 0$ and equation (3.7) with the Case $m \neq 0$. By Lemma 10,

$$\begin{array}{rcl} p_2(x) & = & p_{2,j}x^{r_j} + p_2(0) \\ q_2(y) & = & q_{2,j}y^{r_j} + q_2(0), \end{array}$$

with

$$p_2(0), q_2(0) \neq 0, \quad \text{and} \quad q_{2,j} - d_j p_{2,j} = 0. \tag{3.8}$$

By assumptions (i), $q_{2,j}$ and $p_{2,j}$ are nonzero. The coefficients d_j were distinct, thus there can be at most one one such equation, i.e., $J = 1$.

To summarize, we have the following cases:

(1) $m_0 \neq 0, m_0 = -m_1 = s$ and

$$p(x) = \frac{x^{r_1}}{p_{2,1}x^{r_1} + p_2(0)}, \quad q(y) = \frac{y^{r_1}}{q_{2,1}y^{r_1} + q_1(0)},$$

with coefficients satisfying $q_{2,1} - d_1 p_{2,1} = 0$,

$$x^{n_0} = x^{r_1 s}, \quad y^{r_1 s} = y^{r - n_0 - \sum_i n_i},$$

$$\prod_{i \geq 1} (x - c_i y)^{n_i} = (q_1(0)x^{r_1} - d_1 p_2(0) y^{r_1})^{-s}.$$

It follows that $I = r_1$ and that $n_i = m_1 = -s$, for $i \geq 1$. We have

$$c_i = \zeta_{r_1}^i d^{1/r_1},$$

with $d = -d_1/p_2(0)/q_1(0)$.

This yields $r = n_0 = r_1 s$. We can rewrite equation (3.4) as

$$y^{r_1} \left(\frac{x}{y}\right)^{r_1} \prod_{i=1}^{r_1} (\frac{x}{y} - c_i)^{-1} = \frac{p}{q} \left(\frac{p}{q} - d_1\right)^{-1} q,$$

which is the same as (3.3) with $s = 1$ and $r = r_1$. We have

$$\begin{aligned} Sq^s &= (q^{-1} - d_1 p^{-1})^{-s} \\ &= \left(\frac{x^{r_1} y^{r_1}}{q_1(0) x^{r_1} - d_1 p_2(0) y^{r_1}} \right)^s . \end{aligned}$$

(2) $m_0 = 0$, $m_1 = s$, and

$$p(x) = \frac{p_{1,1} x^{r_1} + p_1(0)}{x^{r_1}}, \quad q(y) = \frac{q_{1,1} y^{r_1} + q_2(0)}{y^{r_1}},$$

with $p_{1,1} - d_1 q_{1,1} = 0$,

$$x^{n_0} = x^{-r_1 s}, \quad y^{-r_1 s} = y^{r - n_0 - \sum_{i>0} n_i}$$

$$\prod_{i \geq 1} (x - c_i y)^{n_i} = (p_1(0) y^{r_1} - d_1 q_2(0) x^{r_1})^s .$$

We obtain $I = r_1$, $n_i = s$, for $i \geq 1$, $n_0 = -r_1 s = r$, and

$$c_i = \zeta_{r_1}^i d^{1/r_1},$$

with $d = d_1 q_2(0)/p_1(0)$. We can rewrite equation (3.4) as

$$y^{-r_1} \left(\frac{x}{y} \right)^{-r_1} \prod_{i=1}^{r_1} (\frac{x}{y} - c_i) = \left(\frac{p}{q} - d_1 \right) q.$$

We have

$$\begin{aligned} Sq^s &= (p - d_1 q)^s \\ &= \left(\frac{p_1(0) y^{r_1} - d_1 q_2(0) x^{r_1}}{x^{r_1} y^{r_1}} \right)^s . \end{aligned}$$

This concludes the proof of Proposition 11. □

Lemma 13. *Let $x_1, x_2 \in K^*$ be algebraically independent elements and let $f_i \in \overline{k(x_i)}$, $i = 1, 2$. Assume that $f_1 f_2 \in \overline{k(x_1 x_2)}$. Then there exists an $a \in \mathbb{Q}$ such that $f_i(x_i) = x_i^a$, in K^*/k^*.*

Proof. Assume first that $f_i \in k(x_i)$ and write

$$f_i(x_i) = \prod_j (x_i - c_{ij})^{n_{ij}}.$$

By assumption,

$$\prod_{i,j} (x_i - c_{ij})^{n_{ij}} = \prod_r (x_1 x_2 - d_r)^{m_r}.$$

However, the factors are coprime, unless $c_{ij} = 0, d_r = 0$, for all i, j, r.

Now we consider the general case: $f_i \in \overline{k(x_i)}$. We have a diagram of field extensions

$$\begin{tikzcd}
k(x_1x_2) \arrow[r, dash] & \overline{k(x_1x_2)} \arrow[r, dash] & \overline{k(x_1,x_2)} \arrow[d, dash] \\
& \overline{k(x_1)}\,k(x_2) \arrow[r, dash] \arrow[d, dash] & \overline{k(x_1)}\,\overline{k(x_2)} \arrow[d, dash] \\
& k(x_1,x_2) \arrow[r, dash] & k(x_1)\,\overline{k(x_2)}
\end{tikzcd}$$

The Galois group $\mathrm{Gal}(\overline{k(x_1,x_2)}/k(x_1,x_2))$ preserves $\overline{k(x_1x_2)}$. We have

$$\Gamma := \mathrm{Gal}(\overline{k(x_1)}\,\overline{k(x_2)}/k(x_1,x_2)) = \Gamma_1 \times \Gamma_2,$$

with Γ_i acting trivially on $\overline{k(x_i)}$. Put $f_3 := f_1f_2$ and consider the action of $\gamma_1 := (\gamma_1, 1) \in \Gamma$ on

$$(f_1, f_2, f_3) \mapsto (f_1, \gamma_1(f_2), \gamma_1(f_3)).$$

It follows that

$$f_1\gamma_1(f_2) = \gamma_1(f_3),$$

and

$$\overline{k(x_1)} \ni f_2/\gamma_1(f_2) = f_3/\gamma_1(f_3) \in \overline{k(x_3)}.$$

Hence each side is in k. The action of γ_1 has finite orbit, so that $\gamma_1(f_3) = \zeta_n f_3$ and $\gamma_1(f_2) = \zeta_n' f_2$ for some n-th roots of 1. Note that Γ acts on f_1, f_2, and f_3 through a finite quotient. It follows that for some $m \in \mathbb{N}$, we have $f_i^m \in k(x_i)$, for $i = 1, 2, 3$, and we can apply the argument above. $\square$

Let $x, y \in K^*$ be algebraically independent over k. We want to determine the set of solutions of the equation

$$Ry = Sq, \tag{3.9}$$

where

$$R \in \overline{k(x/y)}, \quad q \in \overline{k(y)}, \quad p \in \overline{k(x)}, \quad S \in \overline{k(p/q)}.$$

We assume that x, p, y, q are multiplicatively independent in K^*/k^* and that S and R are nonconstant. We will reduce the problem to the one solved in Proposition 11.

Lemma 14. *There exists an $n(p) \in \mathbb{N}$ such that $p^{n(p)} \in \overline{k(x/y)}\,\overline{k(y)}$.*

Proof. The function $S \in \overline{k(p/q)} \cap \overline{k(x/y)}\,\overline{k(y)}$ is nonconstant. The Galois group

$$\Gamma := \mathrm{Gal}(\overline{k(x,y)}/\overline{k(x/y)}\,\overline{k(y)})$$

acts trivially on $q \in \overline{k(y)}$ and S. Thus $\overline{k(p/q)} = \overline{k(\gamma(p)/q)}$. Assume that $\gamma \in \Gamma$ acts nontrivially on $p \in \overline{k(x)}$. It follows that

$$\gamma(p)/p \in \overline{k(p/q)} \cap \overline{k(x)} = k,$$

by assumption on these 1-dimensional fields. Thus $\gamma(p) = \zeta p$, where ζ is a root of 1. Since Γ acts on p via a finite quotient and since each $\gamma \in \Gamma$ acts by multiplication by a root of 1, $p^{n(p)} \in \overline{k(x/y)}\,\overline{k(y)}$, for some $n(p) \in \mathbb{N}$. □

Lemma 15. *There exists an $N = N(p) \in \mathbb{N}$ such that*

$$p^{n(p)} \in k(x^{1/N}).$$

Proof. The intersection $\overline{k(x)} \cap \overline{k(x/y)}\,\overline{k(y)}$ is preserved by action of $\Gamma = \Gamma_{x/y} \times \Gamma_y$. Its elements are fixed by any lift of

$$\sigma\ : y \mapsto x/y.$$

to the Galois group Γ. All such lifts are obtained by conjugation in $\Gamma_{x/y} \times \Gamma_y$. Hence $(1,\gamma)$ acts as $(\sigma(\gamma),1)$. The group homomorphism

$$\Gamma_{x/y} \times \Gamma_y \to \Gamma_x := \mathrm{Gal}(\overline{k(x)}/k(x))$$

has abelian image since $(\gamma_1, 1)$ and $(1, \gamma_2)$ commute and generate Γ. Every abelian extension of $k(x)$ is described by the ramification divisor. It remains to observe that the only common irreducible divisors of $\overline{k(y)}$, $\overline{k(x/y)}$ and $\overline{k(x)}$ are $x = 0$ or $x = \infty$. □

Lemma 16. *There exists an $n \in \mathbb{N}$ such that*

$$S^n \in k(x^{1/N}, y) \quad \text{and} \quad q^n \in k(y).$$

Proof. Let

$$\Gamma'_x \subset \Gamma_x = \mathrm{Gal}(\overline{k(x)}/k(x^{1/N}))$$

be the subgroup of elements acting trivially on $k(x^{1/N})$. Let

$$\gamma = (\gamma'_1, 1) \in \Gamma_x \times \Gamma_{x/y}, \quad \gamma'_1 \in \Gamma'_x.$$

Then

$$Ry = Sq = \gamma(S)\gamma(q) \quad \text{and} \quad S/\gamma(S) = \gamma(q)/q.$$

We also have

$$\frac{p/\gamma(q)}{p/q} = q/\gamma(q)$$

with

$$S \in \overline{k(p/q)}, \quad p/\gamma(q),\ \gamma(S) \in \overline{k(p/\gamma(q))}, \quad q/\gamma(q) \in \overline{k(y)}.$$

By Lemma 13, if we had $\overline{k(p/q)} \cap \overline{k(p/\gamma(q))} = k$ then $S = p/q$. However, equation $Ry = p$ and Lemma 13 imply that $R = x/y$, contradicting the assumption that x and p are multiplicatively independent. Thus we have $\overline{k(p/q)} = \overline{k(p/\gamma(q))}$. The equality $S/\gamma(S) = (q/\gamma(q))^{-1}$ implies that both sides are constant. Hence there exists an $n \in \mathbb{N}$ such that $S^n \in k(x^{1/N}, y)$, and $q^n \in k(y)$. $\square$

Lemma 17. *There exists an $n(R)$ such that $R^{n(R)} \in k(\sqrt[N]{x/y})$.*

Proof. We have that

$$R^n y^n = S^n q^n$$

with $q^n \in k(y)$ and $S^n \in k(x^{1/N}, y)$. Thus

$$R^n \in \overline{k(x/y)} \cap k(x^{1/N})k(y).$$

Applying a nontrivial element $\gamma \in \mathrm{Gal}(\overline{k(x^{1/N}, y)}/k(x^{1/N}, y))$ we find that $R^n/\gamma(R^n) \in k^*$, and is thus a root of 1. As in the proofs above, we find that there is a multiple $n(R)$ of n such that $R^{n(R)} \in k(\sqrt[N]{x/y})$. $\square$

We change the coordinates

$$\tilde{x} := x^{1/N}, \quad \tilde{y} := y^{1/N}.$$

Lemma 18. *There exist*

$$\tilde{p} \in k(\tilde{x}), \tilde{q} \in k(\tilde{y})$$

such that

$$F := \overline{k(p/q)} \cap k(\tilde{x}, \tilde{y}) = k(\tilde{p}/\tilde{q}). \tag{3.10}$$

Proof. Every subfield of a rational field is rational. In particular, $F = k(\tilde{s})$ for some $\tilde{s} \in k(\tilde{x}, \tilde{y})$. Since $p \in k(x), q \in k(y)$ they are both in $k(\tilde{x}, \tilde{y})$ so that $p(x)/q(x) \in F = k(\tilde{s})$. By Lemma 6, $F = k(\tilde{p}/\tilde{q})$, as claimed. $\square$

Corollary 19. *There exists an $m \in \mathbb{N}$ such that*

$$S^m \in k(\tilde{p}/\tilde{q}),$$

with

$$\tilde{p} \in k(\tilde{x}) \quad \text{and} \quad \tilde{q} \in k(\tilde{y}).$$

Moreover, $\tilde{q} = q^r$, for some $r \in \mathbb{Q}$.

Proof. We apply Lemma 13: since

$$\tilde{p} \in k(\tilde{x}) \subset \overline{k(x)} = \overline{k(p)}, \quad 1/\tilde{q} \in \overline{k(y)} = \overline{k(1/q)}$$

and

$$\tilde{p}/\tilde{q} \in \overline{k(p/q)},$$

by (3.10),

$$\overline{k(\tilde{p}/\tilde{q})} = \overline{k(S)} = \overline{k(p/q)},$$

we have

$$p/q = (\tilde{p}/\tilde{q})^a,$$

for some $a \in \mathbb{Q}$. □

We have shown that if R, S satisfy equation (3.9) then for all sufficiently divisible $m \in \mathbb{N}$ we have

$$R^m \tilde{y}^{mN} = S^m \tilde{q}^{m/a}, \tag{3.11}$$

with

$$\tilde{S} := S^m \in k(\tilde{p}/\tilde{q}), \quad \tilde{R} := R^m \in k(\tilde{x}/\tilde{y}) \text{ and } \tilde{q} := q^m \in k(y) \subset k(\tilde{y}).$$

Choose a smallest possible m such that $s := m/a \in \mathbb{Z}$ and put $r = mN$. Equation 3.11 transforms to

$$\tilde{R}\tilde{y}^r = \tilde{S}\tilde{q}^s.$$

In the proof of Proposition 11 we have shown that $s \mid r$ and that either

$$\tilde{R} = \left(\frac{\tilde{x}}{\tilde{y}}\right)^{r_1 s} \prod_{i=1}^{r_1} \left(\frac{\tilde{x}}{\tilde{y}} - c_i\right)^{-s}, \quad \tilde{S} = \left(\frac{\tilde{p}}{\tilde{q}}\right)^{s} \left(\frac{\tilde{p}}{\tilde{q}} - d_1\right)^{-s} \tilde{q}^s$$

with $r_1 s = r$ or

$$\tilde{R} = \left(\frac{\tilde{x}}{\tilde{y}}\right)^{-r_1 s} \prod_{i=1}^{r_1} \left(\frac{\tilde{x}}{\tilde{y}} - c_i\right)^{s}, \quad \tilde{S} = \left(\frac{\tilde{p}}{\tilde{q}} - d_1\right)^{s} \tilde{q}^s$$

with $-r_1 s = r$.

We have obtained that every nonconstant element in the intersection

$$\overline{k(x/y)}^* \cdot y \cap \overline{k(p/q)}^* \cdot q, \tag{3.12}$$

is of the form

$$\left(\frac{x^b y^b}{x^b - \kappa y^b}\right)^s, s \in \mathbb{N}, \quad \text{or} \quad \left(\frac{x^b - \kappa' y^b}{x^b y^b}\right)^s, -s \in \mathbb{N}, \tag{3.13}$$

with $b = r_1/N$, $N \in \mathbb{N}$, and $\kappa, \kappa' \in k^*$. The corresponding solutions, modulo k^*, are

$$p_{\kappa_x,b,m}(x) = \left(\frac{x^b}{x^b+\kappa_x}\right)^{1/m}, \quad q_{\kappa_y,b,m}(y) = \left(\frac{y^b}{y^b+\kappa_y}\right)^{1/m},$$

with

$$\kappa = \kappa_x/\kappa_y$$

respectively,

$$p_{\kappa_x,b,m}(x) = \left(\frac{x^b+\kappa'_x}{x^b}\right)^{1/m}, \quad q_{\kappa_y,b,m}(y) = \left(\frac{y^b+\kappa'_y}{y^b}\right)^{1/m},$$

with

$$\kappa' = \kappa'_y/\kappa'_x.$$

By equation (3.9), we have (for $s \in \mathbb{Z}$)

$$\left(\frac{x^b y^b}{x^b - \kappa y^b}\right)^s \cdot y^{-1} \in \overline{k(x/y)}^*$$

It follows that $bs = 1$.

Assumption 20. *The pair (x, y) satisfies the following condition: if both $x^b, y^b \in K^*$ then $b \in \mathbb{Z}$.*

This assumption holds e.g., when either x, y or xy is primitive in K^*/k^*.

Lemma 21. *Assume that the pair (x, y) satisfies Assumption 20. Fix a solution (3.13) of Condition (3.12). Assume that the corresponding $p_{\kappa_x,b,m}$ is in K^*, for infinitely many κ_x, resp. κ'_x. Then $b = \pm 1$ and $m = \pm 1$.*

Proof. By the assumption on the pair (x, y) and K,

$$\frac{x^b}{x^b+\kappa_x}$$

is primitive in K^*/k^*, for infinitely many κ_x. It follows that $m = \pm 1$. To deduce that $b = \pm 1$ it suffices to recall the definitions: on the one hand, $b = r_1/N \in \mathbb{Z}$, with $N \in \mathbb{N}$, $r_1 \in \mathbb{N}$, and $r = \pm N$. Thus, $b = \pm r_1/r \in \mathbb{Z}$. On the other hand, $\pm r_1 s = r$, with $s \in \mathbb{N}$. □

After a further substitution $\delta = -b$, we obtain:

Theorem 22. *Let $x, y \in K^*$ be algebraically independent elements satisfying Assumption 20. Let $p \in \overline{k(x)}^*$, $q \in \overline{k(y)}^*$ be rational functions such that x, y, p, q are multiplicatively independent in K^*/k^*. Let $I \in \overline{k(x/y)}^* \cdot y$ be such that there exist infinitely many $p, q \in K^*/k^*$ with*

$$I \in \overline{k(x/y)}^* \cdot y \cap \overline{k(p/q)}^* \cdot q.$$

Then, modulo k^*,

$$I = I_{\kappa,\delta}(x, y) := (x^\delta - \kappa y^\delta)^\delta, \tag{3.14}$$

with $\kappa \in k^*$ and $\delta = \pm 1$. The corresponding p and q are given by

$$\begin{array}{rclrcl} p_{\kappa_x,1}(x) & = & x + \kappa_x, & q_{\kappa_y,1}(y) & = & y + \kappa_y \\ p_{\kappa_x,-1}(x) & = & (x^{-1} + \kappa_x)^{-1}, & q_{\kappa_x,-1}(y) & = & (y^{-1} + \kappa_y)^{-1} \end{array}$$

with

$$\kappa_x/\kappa_y = \kappa.$$

4. Reconstruction

In this section we prove Theorem 2. We start with an injective homomorphisms of abelian groups

$$\psi_1 : K^*/k^* \to L^*/l^*.$$

Assume that $z \in K^*$ is primitive in K^*/k^* and that its image under ψ_1 is also primitive. Let $x \in K^*$ be an element algebraically independent from z and put $y = z/x$. By Theorem 22, the intersection

$$\overline{k(x/y)}^* \cdot y \cap \overline{k(p/q)}^* \cdot q \subset K^*/k^*$$

with infinitely many corresponding pairs $(p, q) \subset K^* \times K^*$, consists of elements $I_{\kappa,\delta}(x, y)$ given in (3.14). Note that

$$I_{\kappa,\delta}(x, y) \neq I_{\kappa',\delta'}(x, y), \quad \text{for} \quad (\kappa, \delta) \neq (\kappa', \delta').$$

For $\delta = 1$, each $I_{\kappa,1}$ determines the infinite sets

$$\mathfrak{l}^\circ(1, x) = \{1, x + \kappa_x\}_{\kappa_x \in k^*}, \quad \mathfrak{l}^\circ(1, y) = \{1, y + \kappa_y\}_{\kappa_y \in k^*}$$

as the corresponding solutions (p, q). The set

$$\mathfrak{l}(1, x) := x \cup \mathfrak{l}^\circ(1, x) \subset \mathbb{P}_k(K)$$

forms a projective line. On the other hand, for $\delta = -1$, we get the set

$$\mathfrak{r}(1, x) = \left\{1, \frac{1}{x^{-1} + \kappa}\right\}_{\kappa \in k}.$$

Note that this set becomes a projective line in $\mathbb{P}_k(K)$, after applying the automorphism

$$\begin{array}{rcl} K^*/k^* & \to & K^*/k^* \\ f & \mapsto & f^{-1}. \end{array}$$

We can apply the same arguments to $\psi_1(x), \psi_1(y) = \psi_1(z)/\psi_1(x)$. Our assumption that ψ_1 maps multiplicative groups of 1-dimensional subfields of K into multiplicative groups of 1-dimensional subfields of L and Theorem 22 imply that ψ_1 maps the projective line $\mathfrak{l}(1,x) \subset \mathbb{P}_k(K)$ to either the projective line $\mathfrak{l}(1,\psi_1(x)) \subset \mathbb{P}_l(L)$ or to the set $\mathfrak{r}(1,\psi_1(x))$. Put

$$\begin{aligned} \mathcal{L} &:= \{x \in K^* \mid \psi_1(\mathfrak{l}(1,x)) = \mathfrak{l}(1,\psi_1(x))\} \\ \mathcal{R} &:= \{x \in K^* \mid \psi_1(\mathfrak{l}(1,x)) = \mathfrak{r}(1,\psi_1(x))\}. \end{aligned}$$

Note that these definitions are intrinsic, i.e., they don't depend on the choice of z.

By the assumption on K, both $\mathfrak{l}(1,\psi_1(x))$ and $\mathfrak{r}(1,\psi_1(x))$ contain infinitely many primitive elements in L^*/l^*, whose lifts to L^* are algebraically independent from lifts of $\psi_1(z)$. We can use these primitive elements as a basis for our constructions to determine the type of the image of $\mathfrak{l}(1,z')$ for every $z' \in \overline{k(z)}^* \cap K^*$. Thus

$$\mathcal{L} \cup \mathcal{R} = K^*/k^*, \quad \mathcal{L} \cap \mathcal{R} = 1 \in K^*/k^*.$$

Lemma 23. *Both sets $\mathcal{L}$ and $\mathcal{R}$ are subgroups of K^*/k^*. In particular, one of these is trivial and the other equal to K^*/k^*.*

Proof. Assume that x, y are algebraically independent and are both in $\mathcal{L}$. We have

$$\psi_1(I_{\kappa,1}(x,y)) = I_{\kappa,1}(\psi_1(x), \psi_1(y)).$$

Indeed, fix elements

$$p(x) = x + \kappa_x \in \mathfrak{l}(1,x) \quad \text{and} \quad q(y) = y + \kappa_y \in \mathfrak{l}(1,y)$$

so that x, y, p, q satisfy the assumptions of Theorem 22. Solutions of

$$R(x/y)y = S(p/q)q$$

map to solutions of a similar equation in L. These are exactly

$$I_{\kappa,1}(\psi_1(x), \psi_1(y)) = \psi_1(x) - \lambda\psi_1(y) \in L^*/l^*,$$

for some $\lambda \in l^*$. This implies that

$$\psi_1(x/y - \kappa) = \psi_1(x/y) - \lambda \in L^*/l^*,$$

i.e., $x/y \in \mathcal{L}$.

Now we show that if $x \in \mathcal{L}$ then every $x' \in \overline{k(x)}^*/k^* \cap K^*/k^*$ is also in $\mathcal{L}$. First of all, $1/x \in \mathcal{L}$. Next, elements in the ring $k[x]$, modulo k^*, can be written as products of linear terms $x + \kappa_i$. Hence

$$\psi_1(k[x]/k^*) \subset l[\psi_1(x)]/l^*.$$

Let f be integral over $k[x]$ and let

$$f^n + \ldots + a_0(x) \in k[x]$$

be the minimal polynomial for f, where $a_0(x) \notin k$. Replacing f by $f+\kappa$, if necessary, we may assume that f is not a unit in the ring $\overline{k[x]}$. Then $f \notin \mathcal{R}$, since otherwise we would have $a_0(x) \in \mathcal{R}$, contradiction. Finally, any element of $\overline{k(x)}^*$ is contained in the integral closure of some $k[1/g(x)]$, with $g(x) \in \overline{k[x]}$.

The same argument applies to $\mathcal{R}$, once we composed with ψ_1^{-1}, to show that both $\mathcal{L}$ and $\mathcal{R}$ are subgroups of K^*/k^*. An abelian group cannot be a union of two subgroups intersecting only in the identity. Thus either $\mathcal{L}$ or $\mathcal{R}$ has to be trivial. $\square$

The set $\mathbb{P}(K) = K^*/k^*$ carries two compatible structures: of an abelian group and a projective space, with projective subspaces preserved by the multiplication. The projective structure on the multiplicative group $\mathbb{P}(K)$ encodes the field structure:

Proposition 24. [2, Section 3] *Let K/k and L/l be geometric fields over k, resp. l, of transcendence of degree ≥ 2. Assume that $\psi_1 : K^*/k^* \to L^*/l^*$ maps lines in $\mathbb{P}(K)$ into lines in $\mathbb{P}(L)$. Then ψ_1 is a morphism of projective structures, $\psi_1(\mathbb{P}(K))$ is a projective subspace in $\mathbb{P}(L)$, and there exist a subfield $L' \subset L$ and an isomorphism of fields*

$$\psi \,:\, K \to L',$$

which compatible with ψ_1.

Lemma 23 shows that either ψ_1 or ψ_1^{-1} satisfies the conditions of Proposition 24. This proves Theorem 2.

5. Milnor K-groups

Let $K = k(X)$ be a function field of an algebraic variety X over an algebraically closed field k. In this section we characterize intrinsically infinitely divisible elements in $\mathrm{K}_1^M(K)$ and $\mathrm{K}_2^M(K)$. For $f \in K^*$ put

$$\mathrm{Ker}_2(f) := \{\, g \in K^*/k^* = \bar{\mathrm{K}}_1^M(K) \mid (f,g) = 0 \in \bar{\mathrm{K}}_2^M(K)\,\}. \tag{5.1}$$

Lemma 25. *An element $f \in K^* = \mathrm{K}_1^M(K)$ is infinitely divisible if and only if $f \in k^*$. In particular,*

$$\bar{\mathrm{K}}_1^M(K) = K^*/k^*. \tag{5.2}$$

Proof. First of all, every element in k^* is infinitely divisible, since k is algebraically closed. We have an exact sequence

$$0 \to k^* \to K^* \to \mathrm{Div}(X).$$

The elements of $\mathrm{Div}(X)$ are not infinitely divisible. Hence every infinitely divisible element of K^* is in k^*. $\square$

Lemma 26. *Given a nonconstant $f_1 \in K^*/k^*$, we have*

$$\mathrm{Ker}_2(f_1) = E^*/k^*,$$

where $E = \overline{k(f_1)} \cap K$.

Proof. Let X be a normal projective model of K. Assume first that $f_1, f_2 \in K \setminus k$ lie in a 1-dimensional subfield $E \subset K$ that contains k and is normally closed in K. Such a field E defines a rational map $\pi : X \to C$, where C is a projective model of E.

By the Merkurjev–Suslin theorem [3], for any field F containing n-th roots of unity one has

$$\mathrm{Br}(F)[n] = \mathrm{K}_2^M(F))/(\mathrm{K}_2^M(F))^n,$$

where $\mathrm{Br}(F)[n]$ is the n-torsion subgroup of the Brauer group $\mathrm{Br}(F)$. On the other hand, by Tsen's theorem, $\mathrm{Br}(E) = 0$, since $E = k(C)$, and k is algebraically closed. Thus the symbol (f_1, f_2) is infinitely divisible in $\mathrm{K}_2^M(E)$ and hence in $\mathrm{K}_2^M(K)$.

Conversely, assume that the symbol (f_1, f_2) is infinitely divisible in $\mathrm{K}_2^M(K)$ and that the field $k(f_1, f_2)$ has transcendence degree two. Choosing an appropriate model of X, we may assume that the functions f_i define surjective morphisms $\pi_i : X \to \mathbb{P}_i^1 = \mathbb{P}^1$, and hence a proper surjective map $\pi : X \to \mathbb{P}_1^1 \times \mathbb{P}_2^1$.

For any irreducible divisor $D \subset X$ the restriction of the symbol (f_1, f_2) to D is well-defined, as an element of $\mathrm{K}_1^M(k(D))$. It has to be infinitely divisible in $\mathrm{K}_1^M(k(D))$, for each D.

For $j = 1, 2$, consider the divisors $\mathrm{div}(f_j) = \sum n_{ij} D_{ij}$, where D_{ij} are irreducible. Let D_{11} be a component surjecting onto $\mathbb{P}_1^1 \times 0$. The restriction of f_2 to D_{11} is nonconstant. Thus D_{11} is not a component in the divisor of f_2 and the residue

$$\varrho(f_1, f_2) \in \mathrm{K}_1^M(k(D_{11})^*) = (f_2|D_{11})^{n_{11}} \notin k^*.$$

It remains to apply Lemma 25 to conclude that the residue and hence the symbol are not divisible. This contradicts the assumption that $k(f_1, f_2)$ has transcendence degree two. □

Corollary 27. *Let K and L be function fields over k. Any group homomorphism*

$$\psi_1 : \bar{\mathrm{K}}_1^M(K) \to \bar{\mathrm{K}}_1^M(L)$$

satisfying the assumptions of Theorem 4 maps multiplicative subgroups of normally closed one-dimensional subfields of K to multiplicative subgroups of one-dimensional subfields of L.

We now prove Theorem 4.

Step 1. For each normally closed one-dimensional subfield $E \subset K$ there exists a one-dimensional subfield $\tilde{E} \subset L$ such that

$$\psi_1(E^*/k^*) \subset \tilde{E}^*/l^*$$

Indeed, Lemma 26 identifies multiplicative groups of 1-dimensional normally closed subfields in K: For $x \in K^* \setminus k^*$ the group $\overline{k(x)}^* \subset K^*$ is the set of all $y \in K^*/k^*$ such that the symbol $(x, y) \in \bar{\mathrm{K}}_2^M(K)$ is zero.

Step 2. There exists an $r \in \mathbb{N}$ such that $\psi_1^{1/r}(K^*/k^*)$ contains a primitive element of L^*/l^*. Note that L^*/l^* is torsion-free. For $f, g \in K^*/k^*$ assume that $\psi_1(f), \psi_1(g)$ are n_f, resp. n_g, powers of primitive, multiplicatively independent elements in L^*/l^*. Let $M := \langle \psi_1(f), \psi_1(g) \rangle$ and let $\mathrm{Prim}(M)$ be its primitivization. Then $\mathrm{Prim}(M)/M = \mathbb{Z}/n \oplus \mathbb{Z}/m$, with $n \mid m$, i.e., $n = \gcd(n_f, n_g)$. Thus, we can take r to be is the smallest nontrivial power of an element in $\psi_1(K^*/k^*) \subset L^*/l^*$.

Step 3. By Theorem 2 either $\psi_1^{1/r}$ or $\psi_1^{-1/r}$ extends to a homomorphism of fields.

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