CONSTRUCTING RATIONAL CURVES ON K3 SURFACES

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Abstract. We develop a mixed-characteristic version of the Mori-Mukai technique for producing rational curves on K3 surfaces. We reduce modulo p, produce rational curves on the resulting K3 surface over a finite field, and lift to characteristic zero. As an application, we prove that all complex K3 surfaces with Picard group generated by a class of degree two have an infinite number of rational curves.

Contents

1. Introduction

Let K be an algebraically closed field and S a K3 surface defined over K. It is known that S contains rational curves–see Mori-Mukai [18], as well as Theorem 8 and Corollary 18 below. In fact, an extension of the argument in [18] shows that the *general* K3 surface of given degree has infinitely many rational curves; we sketch this below in Theorem 10 (cf. [7]). The idea is to specialize the K3 surface S to a K3 surface S_0 with Picard group of rank 2, where some multiple of the polarization can be expressed as a sum of linearly independent classes of smooth rational curves. The union of these rational curves deforms to an irreducible

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rational curve on S. This idea applies to K3 surfaces parametrized by points outside a countable union of subvarieties of the moduli space. In particular, *a priori* it doesn't apply to K3 surfaces over countable fields, such as $\overline{\mathbb{F}}_p$ and $\overline{\mathbb{Q}}$. Of course, there are also other techniques proving density of rational curves on special K3 surfaces, e.g., certain Kummer surfaces [18], surfaces with infinite automorphisms [5, proof of Thm. 4.10], or with elliptic fibrations (see Remark 6 below). These K3 surfaces have Picard rank ≥ 2, and all except finitely many lattices in rank ≥ 3 correspond to K3 surfaces with infinite automorphisms or elliptic fibrations [19, 29].

Moreover, in [6] it is proved that, over $k = \overline{\mathbb{F}}_p$, every algebraic point on a Kummer K3 surface lies on an irreducible rational curve. The proof of this result uses the Frobenius endomorphism on the covering abelian surface.

Theorem 1. *Let S be a K3 surface over an algebraically closed field of characteristic zero with* $\mathrm{Pic}(S) = \mathbb{Z}$, *generated by a divisor of degree two. Then S contains infinitely many rational curves.*

The motivation for our argument comes from a result of Bogomolov and Mumford [18]: Let (S, f) be a general K3 surface of degree $2g-2$. We can degenerate S to a Kummer K3 surface (S_0, f), which has infinitely many rational curves. Indeed, we can produce examples where there are infinitely many (reducible) rational curves in $|Nf|, N \geq 1$, consisting of unions of smooth components meeting transversally. A deformation argument shows that these deform to infinitely many (irreducible) rational curves in nearby fibers. However, on subsequent specializations, distinct rational curves might collapse onto each other. If there were an infinite number of such collisions, the specialized K3 surfaces might only have a finite number of rational curves.

Here we emulate the argument in [18] in mixed characteristic. K3 surfaces over finite fields play the rôle of the Kummer surface; the 'general' K3 surface is a K3 surface over a number field with Picard group of rank one. The main technical issue is that we cannot assume *a priori* that the rational curves on the reduction $\pmod{\mathfrak{p}}$ have mild singularities. Thus we are forced to use more sophisticated deformation techniques.

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2. Guiding questions and examples

The following is well-known but hard to trace in the literature:

Conjecture 2 (Main conjecture)**.** Let K be an algebraically closed field of arbitrary characteristic and S a projective K3 surface over K. There exist infinitely many rational curves on S.

In characteristic zero, we can reduce this to the case of number fields:

Theorem 3. *Assume that for every K3 surface S_0 defined over a number field K_0, there are infinitely many rational curves in*

$$\overline{S_0} := S_0 \times_{\mathrm{Spec}(K_0)} \mathrm{Spec}(\overline{\mathbb{Q}}).$$

Then Conjecture 2 holds over fields of characteristic zero.

Proof. Let S be a K3 surface defined over a field of characteristic zero, which we may assume is the function field of a variety B defined over a number field F. Shrinking B as necessary, we obtain a smooth projective morphism

$$\pi : \mathcal{S} \to B$$

with generic fiber S.

We claim there exists a point $b \in B(\overline{\mathbb{Q}})$ such that the specialization map to the fiber $S_b = \pi^{-1}(b)$

$$\mathrm{Pic}(S) \to \mathrm{Pic}(S_b)$$

is surjective. The argument is essentially the same as the proof of the main result of [10]. The only difference is that Ellenberg considers the Galois representation on the full primitive cohomology of a polarized K3 surface surface, whereas here we restrict to the representation on the transcendental cohomology of S, i.e., the orthogonal complement to $\mathrm{Pic}(S) \subset H^2(S, \mathbb{Z})$.

Our assumption is that S_b admits infinitely many rational curves. We claim each of these lifts to a rational curve of S, perhaps after a generically-finite base-change $\tilde{B} \to B$. Suppose we have a morphism $\phi_b : \mathbb{P}^1 \to S_b$, birational onto its image. The class $\phi_*[\mathbb{P}^1]$ remains algebraic in the fibers of $\mathcal{S} \to B$. Thus we can apply a result of Ran [24, Cor. 3.2 and 3.3], which builds on earlier work of Voisin [30] and Bloch [3], to conclude that ϕ_b lifts to a morphism $\phi : \mathbb{P}^1 \to S$. $\square$

Remark 4. We do not know whether the positive-characteristic case of Conjecture 2 can be reduced to the case of finite fields.

Example 5. Here we show that any Kummer K3 surface over an arbitrary algebraically-closed field of characteristic $\neq 2$ admits an infinite number of rational curves.

Let A be an abelian surface with Kummer surface S:

$$\begin{array}{ccc} & & S \\ & & \downarrow \\ A & \rightarrow & A/\pm \end{array}$$

Note that A is isogenous to the Jacobian J of a genus two curve C. (Every abelian surface is isogenous to a principally-polarized surface, which is either a Jacobian or a product $E_1 \times E_2$ of elliptic curves. In the latter case, if we express E_1 and E_2 as branched coverings on $\mathbb{P}^1$ at $\{0, \infty, \alpha_1, \beta_1\}$ and $\{0, \infty, \alpha_2, \beta_2\}$ with the α_i and β_i distinct, then the genus-two double cover $C \to \mathbb{P}^1$ branched at $\{0, \infty, \alpha_1, \beta_1, \alpha_2, \beta_2\}$ will work. Indeed, E_1 and E_2 are Prym varieties of C.)

We choose the embedding $C \hookrightarrow J$ such that a Weierstrass point is mapped to zero. Then the images of $n \cdot C$ in $A/\pm$ are distinct rational curves. Indeed, multiplication-by-n commutes with $\pm$, and acts on C via the hyperelliptic involution.

Remark 6. Elliptic complex K3 surfaces always have infinitely many rational curves: see [5, Thm. 1.8] or [13, Cor. 8.12, Prop. 9.10, Rem. 9.7] and Example 5 above for the degenerate case where the elliptic surface arises from a Kummer construction.

Example 7. Let S be a K3 surface over an algebraically-closed field such that the Picard group is generated over $\mathbb{Q}$ by smooth rational curves C_1 and C_2 satisfying

$$\begin{array}{c|cc} & C_1 & C_2 \\ \hline C_1 & -2 & 6 \\ C_2 & 6 & -2 \end{array} \tag{2.1}$$

and generated over $\mathbb{Z}$ by C_1 and $f = (C_1 + C_2)/2$. Note that (S, f) can be realized geometrically as the double cover of $\mathbb{P}^2$ branched over a plane sextic curve that admits a six-tangent conic.

Surfaces of this type can be defined over $\mathbb{F}_3$ [11, Ex. 6.1], e.g.,

$$w^2 = (y^3 - x^2 y)^2 + (x^2 + y^2 + z^2)(2x^3 y + x^3 z + 2x^2 yz + x^2 z^2 + 2xy^3 + 2y^4 + z^4).$$

The technique of [10] can be used to obtain examples over $\mathbb{Q}$. Indeed, the moduli space of lattice-polarized K3 surfaces of type (2.1) is unirational: The sextic plane curves six-tangent to a fixed conic plane curve D are parametrized by a $\mathbb{P}^{15}$-bundle over $\mathbb{P}^6$, and these dominate our moduli space. We can apply Ellenberg's Hilbert irreducibility argument [10] directly to this rational variety.

We do not know how to construct infinitely many rational curves on K3 surfaces of this type, over $\overline{\mathbb{Q}}$ or $\overline{\mathbb{F}}_p$.

3. Background results

A polarized K3 surface (S, f) consists of a K3 surface and an ample divisor f that is primitive in the Picard group. Its degree is the positive even integer $f \cdot f$. Let $\mathcal{K}_g, g \geq 2$ denote the moduli space (stack) of complex polarized K3 surfaces of degree $2g-2$, which is smooth and connected of dimension 19.

The following result was initially presented by the first author in October 1981 at Mori's seminar at IAS; the proof was based on deformation-theoretic ideas developed several years earlier. A different argument was presented in [18]; Mori and Mukai indicate that Mumford also had a proof.

Theorem 8. [18] *Every complex projective K3 surface contains at least one rational curve.*

For our purposes, it is useful to recall the Mori-Mukai argument for the existence of rational curves in the generic K3 surface of degree $2g-2$.

Proof. We exhibit a K3 surface S_0 containing two smooth rational curves C_1 and C_2 meeting transversally at $g+1$ points, such that the class $f = [C_1 + C_2]$ is primitive. We then deform $C_1 \cup C_2$ to an irreducible rational curve in a nearby polarized K3 surface.

Let E_1 and E_2 be elliptic curves admitting an isogeny $E_1 \to E_2$ of degree $2g+3$ with graph $\Gamma \subset E_1 \times E_2$; $p \in E_2$ a 2-torsion point. Take the associated Kummer surface S_0, i.e., the minimal desingularization of

$$(E_1 \times E_2)/\langle \pm 1 \rangle .$$

Note that Γ intersects $E_1 \times p$ transversally in $2g+3$ points, one of which is 2-torsion in $E_1 \times E_2$

Take C_1 and C_2 to be the images of Γ and $E_1 \times p$ in S_0, smooth rational curves meeting transversally in $g+1$ points. The sublattice of

$\mathrm{Pic}(S_0)$ determined by C_1 and C_2 is:

$$\begin{array}{c|cc} & C_1 & C_2 \\ \hline C_1 & -2 & g+1 \\ C_2 & g+1 & -2 \end{array} \tag{3.1}$$

Consider deformations of S_0 in $\mathcal{K}_g$,

$$\pi : (\mathcal{S}, f) \to \Delta, \quad \Delta = \{t : |t| < 1\},$$

i.e., deformations for which $f = [C_1] + [C_2]$ remains algebraic. Since $\mathcal{O}_{S_0}(C_1 + C_2)$ is nef and big, it has trivial higher cohomology (by Kawamata-Viehweg vanishing) and $\Gamma(\mathcal{O}_{S_t}(f))$ is constant in t. Thus $C_1 \cup C_2$ deforms to divisors in nearby fibers in a smooth family of relative dimension g. We can assume that the general fiber of π has Picard group generated by f, so that $C_1 \cup C_2$ deforms to *irreducible curves* in the general fiber.

Consider the rational map

$$\begin{array}{ccc} \mathbb{P}(\pi_* \mathcal{O}_{\mathcal{S}}(f)) & \stackrel{\mu}{\dashrightarrow} & \overline{M}_g \\ \downarrow & & \\ \Delta & & \end{array}$$

assigning to each curve the corresponding point in moduli. Note that μ is regular at $[C_1 \cup C_2]$.

Let T denote the union of two smooth rational curves at a node, i.e., the singular conic curve $\{xy = 0\}$. Choose a birational morphism $\phi : T \to C_1 \cup C_2$, i.e., one that normalizes all but one of the nodes. The preimages of the remaining g nodes yield $2g$ smooth points $p_1, \ldots, p_{2g} \in T$, numbered so that p_{2i-1} and p_{2i} are identified. More generally, consider the morphism

$$\iota : \overline{M}_{0,2g} \to \overline{M}_g$$

which identifies p_{2i-1} and p_{2i} for $i = 1, \ldots, g$, so that $\iota(T, p_1, \ldots, p_{2g}) = C_1 \cup C_2$. The image of ι has dimension $2g-3$ and parametrizes curves with g nodes. Finally, let $\delta \subset \overline{M}_{0,2g}$ denote the irreducible boundary divisor parametrizing curves with combinatorial type equal to the type of T.

Let Z be the closed image of μ, which has dimension $\leq g+1$. Note that $\mu^{-1}(C_1 \cup C_2)$ is zero-dimensional, in fact, $\iota^{-1}(\mu^{-1}(\delta))$ is zero-dimensional. Indeed, the generic fibers admit only irreducible curves in $|f|$, and the special fiber admits only finitely-many rational curves. Consequently, $\iota(\overline{M}_{0,2g})$ and Z meet properly and thus in dimension one. These are stable reductions of curves appearing in the fibers of

π, which are necessarily of genus zero. It follows that $\mathcal{S}$ contains a one-parameter family of genus-zero curves containing $C_1 \cup C_2$. Again, since each fiber admits only finitely-many rational curves, there exist rational deformations of $C_1 \cup C_2$ in the generic fiber of π. $\square$

Remark 9. This argument requires that the curve in the special K3 surface S_0 be nodal. Without this, we cannot carry out the dimension estimates.

There exists an (irreducible) eighteen-dimensional family of lattice-polarized K3 surfaces of type (3.1). *A posteriori*, we know that for the generic such K3, the rational curves C_1 and C_2 meet transversally. The argument above is applicable to any such K3 surface, not just the Kummer surfaces.

Moreover, even when C_1 and C_2 fail to meet transversally, $C_1 \cup C_2$ is still the limit of irreducible rational curves.

A fairly straightforward specialization argument allows us to deduce that every indecomposable effective class in a K3 surface contains rational curves. Any cycle of curves that is a specialization of a rational curve is a union of rational curves, perhaps with multiplicities. We next show that the general K3 surface in $\mathcal{K}_g$ admits an infinite number of rational curves.

The following theorem has been known to experts; the first published proof is in [7]. (Xi Chen attributes a special case to S. Nakatani.)

Theorem 10. [7] *Fix $N \geq 1$. Then for a generic $(S, f) \in \mathcal{K}_g$ there exists an irreducible rational curve in $|Nf|$.*

Corollary 11. *A very general K3 surface of degree $2g-2$ contains an infinite number of rational curves.*

The proof in [7] involves specializing the K3 surface to a union of two rational normal scrolls, meeting transversely along an elliptic curve. Xi Chen identifies reducible rational curves on this surface that can be deformed back to irreducible rational curves on a general K3 surface.

For our purposes this argument is not sufficiently flexible. Our technique entails analyzing the reductions $\pmod p$ of a K3 surface defined over a number field. We cannot expect these reductions to be unions of rational normal scrolls.

We sketch an alternative proof for Theorem 10, with a view towards highlighting the geometric ideas behind our main theorem:

Proof. Let (S_0, f) be a polarized K3 surface with

$$\mathrm{Pic}(S_0)_{\mathbb{Q}} = \mathbb{Q}C_1 + \mathbb{Q}C_2,$$

where C_1 and C_2 are smooth rational curves such that

$$\begin{array}{c|cc} & C_1 & C_2 \\ \hline C_1 & -2 & N^2(g-1)+2 \\ C_2 & N^2(g-1)+2 & -2 \end{array}.$$

Furthermore, we assume that

$$\mathrm{Pic}(S_0) = \mathbb{Z}C_1 + \mathbb{Z}C_2 + \mathbb{Z}f$$

where

$$Nf = C_1 + C_2.$$

The existence of these can be deduced from surjectivity of Torelli.

In this case, we lack general tools for producing examples where C_1 and C_2 meet transversally. This necessitates a modified approach to the deformation theory of the union $C_1 \cup C_2$.

Let T_0 again denote the nodal rational curve $\{xy = 0\} \subset \mathbb{P}^2$, and choose a birational morphism

$$T_0 \to C_1 \cup C_2,$$

which maps the node of T_0 to some singularity of $C_1 \cup C_2$. We regard the composition with the inclusion into S_0 as a stable map $\phi \in \overline{\mathcal{M}}_0(S_0, Nf)$. Again, consider deformations of S_0

$$\pi : (\mathcal{S}, f) \to \Delta,$$

where the generic fiber is a general K3 surface in $\mathcal{K}_g$, i.e., one with Picard group generated by f. This time, we consider the relative Kontsevich space of stable maps

$$\psi : \overline{\mathcal{M}}_0(\mathcal{S}/\Delta, Nf) \to \Delta,$$

whose fibers are Kontsevich spaces of the fibers of π. Recall that a *stable map* into a variety Y is a pair $(T, \phi : T \to Y)$ consisting of a connected nodal curve T and a morphism ϕ such that the dualizing sheaf ω_T is ample relative to T.

We claim that $\overline{\mathcal{M}}_0(\mathcal{S}/\Delta, Nf)$ is (at least) one-dimensional near $[\phi]$. Since every fiber of π is not uniruled, it follows that ψ is dominant. Furthermore, since C_1 and C_2 do not deform to algebraic classes in the generic fiber, the resulting genus zero stable maps have $\mathbb{P}^1$ as their domain. They are birational onto their images, which are therefore irreducible rational curves in $|Nf|$. $\square$

It remains to prove the claim, which requires some deformation theory, explained in the next section.

4. Deformation results for stable maps

In this section, we work over a field of arbitrary characteristic. Let Y be a smooth projective variety and β a curve class on Y.

Consider the open substack

$$\overline{\mathcal{M}}_0^{\circ}(Y,\beta) \subset \overline{\mathcal{M}}_0(Y,\beta)$$

corresponding to maps $\phi : T \to Y$ that are generic embeddings, i.e., for each irreducible component $T_i \subset T$ the restriction of ϕ to the generic point of T_i is an embedding.

This open set dominates the locus $\Xi \subset \mathrm{Hilb}$ consisting of curves C expressible as unions

$$C = C_1 \cup \ldots \cup C_r \subset Y, \quad \sum_{i=1}^{r} [C_i] = \beta,$$

of rational curves C_i with each component having multiplicity one; the induced morphism is finite-to-one. Indeed, there are a finite number of connected seminormal curves T through which the normalization factors

$$C^{\nu} = \coprod_{j=1}^{r} \mathbb{P}^1 \to T \to C.$$

The substack $\overline{\mathcal{M}}_0^{\circ}(Y,\beta)$ has several nice properties: First, its objects have trivial automorphisms, thus the substack is actually a scheme. (Indeed, it is a finite cover of Ξ, which is quasi-projective.) The obstruction theory over $\overline{\mathcal{M}}_{0,0}^{\circ}(Y,\beta)$ takes a particularly simple form: Given a stable map $\phi : T \to Y$, first-order deformations and obstructions are given by

$$\mathbb{H}^i(\mathbb{R}\mathcal{H}om_{\mathcal{O}_T}(\Omega_{\phi}^{\bullet}, \mathcal{O}_T)), i = 1, 2$$

where $\Omega_{\phi}^{\bullet}$ is the complex

$$d\phi^t : \phi^*\Omega_Y^1 \to \Omega_T^1$$

supported in degrees -1 and 0. We shall require the following result (cf. [12, p. 61]):

Lemma 12. *Let $\phi : T \to Y$ be a stable map to a smooth variety that is unramified at the generic point of each irreducible component of T. Then the complex*

$$\mathbb{R}\mathcal{H}om_{\mathcal{O}_T}(\Omega_{\phi}^{\bullet}, \mathcal{O}_T)$$

is quasi-isomorphic to $\mathcal{N}_{\phi}[-1]$, the normal sheaf shifted by -1.

In the special case where the domain T is smooth, $\mathcal{N}_\phi$ is the cokernel of the differential $d\phi : \mathcal{T}_T \to \phi^*\mathcal{T}_Y$. First order deformations of ϕ are given by $H^0(\mathcal{N}_\phi)$; obstructions are given by $H^1(\mathcal{N}_\phi)$.

Proof. Let T be a nodal projective curve. Then Ω^1_T admits a resolution

$$0 \to \mathcal{E}_1 \xrightarrow{f_1} \mathcal{E}_0 \to \Omega^1_T \to 0$$

where $\mathcal{E}_1$ is invertible and $\mathcal{E}_0$ is locally free of rank two. Locally, this takes the following form: Suppose that $p \in T$ is a node expressed as $xy = 0$ in local étale/analytic coordinates. Then locally

$$\Omega^1_T = (\mathcal{O}_T dx + \mathcal{O}_T dy)/ \langle y dx + x dy \rangle$$

which has a presentation of the specified form. More intrinsically, we may identify $\Omega^1_T = \omega_T \otimes \mathcal{I}_\Sigma$ where ω_T is the dualizing sheaf and $\mathcal{I}_\Sigma$ is the ideal sheaf of the nodes.

Given a bounded complex of $\mathcal{O}_T$-modules

$$\mathcal{E}^\bullet = \{0 \cdots \mathcal{E}^{-p-1} \to \mathcal{E}^{-p} \to \mathcal{E}^{-p+1} \to \cdots 0\}$$

we compute $\mathbb{R}\mathcal{H}om_{\mathcal{O}_T}(\mathcal{E}^\bullet, \mathcal{O}_T)$ using the spectral sequence

$$E_1^{p,q} = \mathcal{E}xt^q_{\mathcal{O}_T}(\mathcal{E}^{-p}, \mathcal{O}_T) \Rightarrow \mathcal{E}xt^{p+q}_{\mathcal{O}_T}(\mathcal{E}^\bullet, \mathcal{O}_T).$$

Note that

- $\mathcal{E}xt^q_{\mathcal{O}_T}(\phi^*\Omega^1_Y, \mathcal{O}_T) = 0$ for $q > 0$ as Ω^1_Y is locally free;
- $\mathcal{E}xt^q_{\mathcal{O}_T}(\Omega^1_T, \mathcal{O}_T) = 0$ for $q > 1$ by the explicit resolution above.

In particular, only the following terms can be nonzero

$$\mathcal{H}om_{\mathcal{O}_T}(\phi^*\Omega^1_Y, \mathcal{O}_T), \mathcal{H}om_{\mathcal{O}_T}(\Omega^1_T, \mathcal{O}_T), \mathcal{E}xt^1_{\mathcal{O}_T}(\Omega^1_T, \mathcal{O}_T).$$

We focus on the unique interesting arrow

$$\begin{array}{rcl} d_1 : E_1^{0,0} & \to & E_1^{1,0} \\ d\phi : \mathcal{H}om_{\mathcal{O}_T}(\Omega^1_T, \mathcal{O}_T) & \to & \mathcal{H}om_{\mathcal{O}_T}(\Omega^1_Y, \mathcal{O}_T). \end{array}$$

Since $\mathcal{H}om_{\mathcal{O}_T}(\Omega^1_T, \mathcal{O}_T)$ is torsion-free, $d\phi$ is injective if and only if it is injective at generic points of T, which was one of our assumptions.

Thus we have

$$E_2^{1,0} = \mathcal{H}om_{\mathcal{O}_T}(\phi^*\Omega^1_Y, \mathcal{O}_T)/\mathcal{H}om_{\mathcal{O}_T}(\Omega^1_T, \mathcal{O}_T)$$

and

$$E_2^{0,1} = \mathcal{E}xt^1_{\mathcal{O}_T}(\Omega^1_T, \mathcal{O}_T).$$

Consequently

$$\mathbb{R}\mathcal{H}om_{\mathcal{O}_T}(\Omega^\bullet_\phi, \mathcal{O}_T)$$

is supported in degree one, and the associated sheaf $\mathcal{N}_\phi$ fits into an exact sequence

$$0 \to \mathcal{H}om_{\mathcal{O}_T}(\phi^*\Omega^1_Y, \mathcal{O}_T)/\mathcal{H}om_{\mathcal{O}_T}(\Omega^1_T, \mathcal{O}_T) \to \mathcal{N}_\phi \to \mathcal{E}xt^1_{\mathcal{O}_T}(\Omega^1_T, \mathcal{O}_T) \to 0.$$

Note that the first term corresponds to deformations that leave the nodes of T unchanged; the last term is the local versal deformation space of these nodes. □

Remark 13. In fact, $\mathcal{N}_\phi$ is locally-free if ϕ is unramified (see, for example, [12, §2])). Conversely, if ϕ is ramified at a smooth point then $\mathcal{N}_\phi$ necessarily has torsion.

Lemma 14. *Let $\phi : T \to S$ be an unramified morphism from a connected nodal curve of genus zero to a K3 surface. Then $h^0(T, \mathcal{N}_\phi) = 0$ and $h^1(T, \mathcal{N}_\phi) = 1$. In particular, the stable map $\phi : T \to S$ is rigid in S.*

Proof. We present an inductive argument on the number of components. Choose a decomposition

$$T = T' \cup_p T_0, \quad T_0 \simeq \mathbb{P}^1$$

where p is a node disconnecting the irreducible component T_0 from the rest of the curve. Let $\phi' : T' \to S$ be the induced map. We have exact sequences

$$0 \to \mathcal{N}_\phi \otimes \mathcal{O}_{T'}(-p) \to \mathcal{N}_\phi \to \mathcal{N}_\phi \otimes \mathcal{O}_{T_0} \to 0$$

and

$$0 \to \mathcal{N}_{\phi'} \to \mathcal{N}_\phi \otimes \mathcal{O}_{T'} \to Q \to 0,$$

where Q is a torsion sheaf of length one supported at p. Furthermore, $\mathcal{N}_\phi \otimes \mathcal{O}_{T_0} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$ so the first sequence gives

$$H^0(\mathcal{N}_\phi \otimes \mathcal{O}_{T'}(-p)) = H^0(\mathcal{N}_\phi), \quad H^1(\mathcal{N}_\phi \otimes \mathcal{O}_{T'}(-p)) = H^1(\mathcal{N}_\phi).$$

The second sequence may be interpreted as $\mathcal{N}_\phi$ tensored by

$$0 \to \mathcal{O}_{T'}(-p) \to \mathcal{O}_{T'} \to \mathcal{O}_p \to 0,$$

thus we obtain $\mathcal{N}_{\phi'} = \mathcal{N}_\phi \otimes \mathcal{O}_{T'}(-p)$, hence

$$H^i(\mathcal{N}_{\phi'}) \simeq H^i(\mathcal{N}_\phi \otimes \mathcal{O}_{T'}(-p)) = H^i(\mathcal{N}_{\phi'}), \quad i = 0, 1.$$

□

5. K3 surfaces over finite fields

For general background and definitions, we refer the reader to [25].

Let S_0 be a K3 surface over a finite field $\mathbb{F}_q$, and $\overline{S_0}$ the resulting surface over $\overline{\mathbb{F}_q}$. Consider the Picard group $\mathrm{Pic}(\overline{S_0})$ and the ℓ-adic cohomology group $H^2(\overline{S_0}, \mathbb{Q}_\ell(1))$, which are related by the cycle-class map

$$\mathrm{Pic}(\overline{S_0}) \to H^2(\overline{S_0}, \mathbb{Q}_\ell(1)).$$

Frobenius acts on both these groups compatibly with this map, and preserving the intersection form.

The Frobenius action on $H^2(\overline{S_0}, \mathbb{Q}_\ell(1))$ is diagonalizable over $\overline{\mathbb{Q}}_\ell$ with eigenvalues $\alpha_1, \dots, \alpha_{22}$ [9]. Since this factors through the orthogonal group, if α appears as an eigenvalue then α^{-1} also appears. Consequently, we conclude that the following sets have an even number of elements

- the eigenvalues that are not roots of unity;
- the eigenvalues that are roots of unity but are not equal to ± 1;
- the total number of times ± 1 appears as an eigenvalue.

Using results of Nygaard and Ogus on the Tate conjecture for K3 surfaces [20, 21] we conclude

Theorem 15. *Let S_0 be a non-supersingular K3 surface over a finite field. Then* $\mathrm{Pic}(\overline{S_0})$ *has even rank.*

Suppose S is a K3 surface over a number field F with integral model

$$\mathcal{S} \to \mathrm{Spec}(\mathfrak{o}_F),$$

which is smooth away from a finite set of primes. For each finite extension F'/F, consider the set of primes

$$\mathrm{Ord}_{F'}(S) = \{\mathfrak{p} \in \mathrm{Spec}(\mathfrak{o}_{F'}) : S_{\mathfrak{p}} \text{ is smooth and ordinary } \}.$$

After passing to a suitably large finite extension F'/F, the set $\mathrm{Ord}_{F'}(S)$ has Dirichlet density one. This is due to Tankeev [28] in the special case where the Hodge group of $S_{\mathbb{C}}$ is semisimple, and to Joshi-Rajan [15, §6] and Bogomolov-Zarhin [4] in general.

Finally, ordinary K3 surfaces are not supersingular [26] (p. 1513 in translated version).

6. Curves on K3 surfaces in positive and mixed characteristic

Let k be an algebraically closed field of characteristic $p > 0$. We shall require some general results on lifting to characteristic zero. We first review deformation and lifting results for K3 surfaces:

Proposition 16. *Let S_0 be a K3 surface defined over k. Then the versal deformation space of S_0 is smooth over the Witt-vectors $W(k)$ of relative dimension twenty.*

Let (S_0, f) be a polarized K3 surface over k, such that self-intersection $f \cdot f$ is relatively prime to the characteristic. Then there exists a polarized K3 surface (S, f) over $W(k)$ reducing to (S_0, f).

Proof. This proof is a special case of the analysis in [8].

The deformations of S_0 are governed by the cohomology groups

$$H^0(S, \mathcal{T}_{S_0}), \quad H^1(S, \mathcal{T}_{S_0}), \quad H^2(S, \mathcal{T}_{S_0}),$$

which parametrize infinitesimal automorphisms, infinitesimal deformations and obstructions respectively. The deformation problem for pairs $(S_0, \mathcal{L})$, where S_0 is a smooth projective variety and $\mathcal{L}$ is an invertible sheaf. We have the Atiyah extension [14, p. 196] of the tangent sheaf

$$0 \to \mathcal{O}_{S_0} \to \mathcal{E}_{S_0,\mathcal{L}} \to \mathcal{T}_{S_0} \to 0, \tag{6.1}$$

classified (up to sign) by the Chern class

$$c_1(\mathcal{L}) \in H^1(S, \Omega_1^S) = \mathrm{Ext}^1(\mathcal{T}_S, \mathcal{O}_S),$$

which is the image of $[\mathcal{L}] \in H^1(S, \mathcal{O}_S^*)$ under the homomorphism induced by

$$d\log : \mathcal{O}_S^* \to \Omega_S^1.$$

Consider the formal deformation space of pairs $(S, \mathcal{L})$. The cohomology groups

$$\Gamma(S, \mathcal{E}_{S,\mathcal{L}}), \quad H^1(S, \mathcal{E}_{S,\mathcal{L}}), \quad H^2(S, \mathcal{E}_{S,\mathcal{L}})$$

parametrize infinitesimal automorphisms, infinitesimal deformations and obstructions respectively.

Now assume that S is a K3 surface and $\mathcal{L}$ is an ample class S with $[\mathcal{L}]^2$ relatively prime to the characteristic. It follows that [25, 17, 22]

$$\Gamma(S, \mathcal{T}_S) = 0.$$

Serre duality, combined with the isomorphism $\mathcal{T}_S \simeq \Omega_S^1$ arising from the symplectic form, yields

$$H^2(S, \mathcal{T}_S) = 0.$$

Thus

$$H^1(S, \mathcal{T}_S) = \chi(S, \mathcal{T}_S) = 20$$

and the deformation space is smooth of this dimension over the Witt vectors.

Furthermore, $c_1(\mathcal{L}) \neq 0$ since $c_1(\mathcal{L})^2 \neq 0 \in H^2(S, \Omega_S^2)$. Consequently, Extension (6.1) is nonsplit and the connecting homomorphism

$$H^1(S, \mathcal{T}_S) \to H^2(S, \mathcal{O}_S)$$

is nonvanishing. Indeed, this map is just cup-product with $c_1(\mathcal{L}) \in H^1(S, \Omega_S^1)$, which is non-zero by Serre-duality. It follows that

$$H^2(S, \mathcal{E}_{S,\mathcal{L}}) = H^2(S, \mathcal{T}_S) = 0,$$

so deformations of $(S, \mathcal{L})$ are unobstructed. □

Deligne [8] proves the following more general theorem:

Theorem 17. *Let (S_0, f) be a polarized K3 surface over an algebraically closed field k of characteristic p. Then S_0 admits a lifting to a possibly ramified extension of $W(k)$.*

In fact, he proves that the locus Σ_f in the formal versal deformation space corresponding to K3 surfaces admitting f as a polarization is a Cartier divisor, not contained in the fiber over the closed point of $\mathrm{Spf}(W(k))$. Ogus [23, §2] has more precise lifting results for ordinary K3 surfaces. These require finer analysis of Chern classes and crystalline cohomology.

Corollary 18. *Let (S_0, f) be a polarized K3 surface over an algebraically closed field of characteristic p. Then S_0 contains a rational curve.*

Indeed, the lifted K3 surface admits a rational curve by Theorem 8. This specializes to a cycle of rational curves in characteristic p.

Theorem 19. *Let (S_0, f) be a polarized K3 surface over k. Suppose that*

$$C = C_1 + \ldots + C_r$$

is a connected union of distinct rational curves $C_i \subset S_0$, such that $[C]$ is proportional to f. Let (S, f) be a polarized K3 surface over the Witt vectors $W(\overline{k})$ reducing to (S_0, f). Assume one of the following conditions:

- *S_0 is not supersingular; or*

- *the map from the normalization of C to S_0 is unramified, i.e., each branch of C is nonsingular.*

Then there exists a curve $R \subset S$, defined over a ramified finite extension of $W(k)$, such that R reduces to C and each irreducible component of R is rational.

Proof. Consider the formal versal deformation space of S_0

$$\mathcal{S} \to B,$$

where $B \simeq \mathrm{Spf}W(k)[[x_1, \ldots, x_{20}]]$, i.e., a smooth formal scheme of dimension 20 over $W(k)$. Let $b \in B$ denote the distinguished closed point. For each $N \geq 1$, consider the relative stable map space

$$\overline{\mathcal{M}}_{g,n}(\mathcal{S}/B, Nf) \to B.$$

This is a formal Artin stack with finite stabilizers, proper over B.

We digress to explain the construction of this object. There are at least two possible approaches. First, consider the category of Artinian local rings A with residue field k and morphisms

$$\Sigma := \mathrm{Spec}(A) \to B.$$

Each base-change

$$\mathcal{S}_\Sigma := \mathcal{S} \times_\Sigma B \to \Sigma$$

is proper and we can apply [1, §8.4] to show that $\overline{\mathcal{M}}_{g,n}(\mathcal{S}_\Sigma/\Sigma, Nf)$ exists. Taking inverse limits gives the desired formal stack.

However, we can offer a more explicit construction. Let $\overline{\mathcal{M}}^{ps}_{g,n}$ denote the moduli stack of prestable curves of genus g with n marked points, and $\mathcal{C} \to \overline{\mathcal{M}}^{ps}_{g,n}$ the universal curve (cf. [2, p. 602]). 'Prestable' means nodal and connected, but not necessarily satisfying the Deligne-Mumford stability condition; these form an Artin stack, locally of finite type. Standard deformation-theoretic results (for example [27, §3.2]) show that the relative Hilbert 'scheme' of a proper formal scheme $\mathcal{Y} \to B$ (parametrizing subschemes $\mathcal{Z} \subset \mathcal{Y}$ reducing to a closed subscheme $Z \subset \mathcal{Y}_b$) is prorepresentable. Given two proper formal schemes $\mathcal{W}, \mathcal{Y} \to B$, the morphism 'scheme' $\mathrm{Mor}_B(\mathcal{W}, \mathcal{Y})$ is also prorepresentable (cf. [27, §3.4].) A stable map

$$\phi_b : (C_b, c_1(b), \ldots, c_n(b)) \to S_b \simeq S_0$$

of genus g with n marked points such that $(\phi_b)_*[C_b] = Nf$, naturally corresponds to an element of

$$\mathrm{Mor}_B(\mathcal{C} \times B, \mathcal{S}) \to \overline{\mathcal{M}}^{ps}_{g,n} \times B.$$

(In general, we may have nontrivial stack structure when $\mathcal{C}_b$ has infinite automorphisms.) The proof of prorepresentability of

$$\overline{\mathcal{M}}_{g,n}(\mathcal{S}/B, Nf) \to B$$

reduces then to the case of the morphism 'scheme'.

General deformation-theoretic arguments (cf. [16, I.2.15]) show that the relative dimension of $\overline{\mathcal{M}}^{\circ}_{0,0}(\mathcal{S}/B, Nf)$ over B at ϕ is at least

$$\chi(T, \mathcal{N}_\phi) + \dim(B) = \dim(B) - 1.$$

When $\phi : T \to S_0$ is unramified, Lemma 14 implies that the fibers of $\overline{\mathcal{M}}^{\circ}_{0,0}(\mathcal{S}/B, Nf) \to B$ are zero-dimensional. Otherwise, we use the hypothesis that S is not supersingular and ϕ is *generically* unramified to conclude that ϕ does not deform to another genus-zero stable map. In either case, the dimensions of $\overline{\mathcal{M}}^{\circ}_{0,0}(\mathcal{S}/B, Nf)$ and its image in B are at least 20. On the other hand, this image is contained in the locus

$$\Sigma_{Nf} \subset B$$

parametrizing K3 surfaces admitting Nf as a polarization. Indeed, in each fiber

$$(\phi_t)_*\mathcal{C}_t = Nf.$$

By Theorem 17, the formal scheme Σ_{Nf} has dimension 20 and is not contained in the fiber over the closed point of $\mathrm{Spf}(W(k))$. The same must hold for $\overline{\mathcal{M}}^{\circ}_{0,0}(\mathcal{S}/B, Nf)$, so there are formal lifts of $\phi : T \to S$ to genus-zero maps in characteristic zero.

It remains to show these formal deformations are algebraic. For this purpose, we restrict to the polarized deformation space

$$\mathcal{S}_{\Sigma_{Nf}} \to \Sigma_{Nf},$$

which is projective in the sense that it admits a formal embedding into a projective space $\mathbb{P}^d_{\Sigma_{Nf}}$, $d = \chi(S, \mathcal{O}_S(Nf)) - 1$. This deformation is algebraizable by standard results of Grothendieck (see [27, 2.5.13], for example.) It follows that the associated moduli spaces of stable maps is algebraizable as well. Indeed, moduli spaces of stable maps into projective schemes are proper stacks with projective coarse moduli spaces. □

7. Proof of the Main Theorem

Assume $\mathrm{Pic}(S)$ is generated by an ample class f, of arbitrary degree. Suppose S admits a finite number of rational curves $R_1, \dots, R_s$ with classes $[R_i] = m_i f$ and write $m = \max\{m_1, \dots, m_s\}$.

Lemma 20. *There are only a finite number of primes p such that there exists a curve $C \subset S_p$ with $[C] \notin \mathbb{Z}f$ and*

$$C \cdot f \leq mf \cdot f.$$

Proof. There are a finite number of rank-two extensions

$$\mathbb{Z}f \subset \Lambda,$$

where Λ is an integral lattice of signature $(1,1)$ admitting a vector v linearly independent from f with $v \cdot f \leq mf \cdot f$. For each such lattice, there are at most a finite number of primes p such that $\Lambda \subset \operatorname{Pic}(S_p)$. $\square$

Now we assume S has degree two. Let $\iota : S \to S$ denote the involution associated to the branched double cover $S \to \mathbb{P}^2$. It acts on the primitive cohomology of S via multiplication by -1. Let ι_p denote its reduction mod p. We shall derive a contradiction by producing an irreducible rational curve in a class Nf for some $N > m$.

Choose p to a prime of good reduction that is not in the finite set of primes specified in Lemma 20. Let N be the the smallest positive integer such that Nf is decomposable

$$Nf = a_1[C_1] + \ldots + a_r[C_r],$$

where the a_i are positive integers and not all of the $[C_i]$ are proportional to f. We assume each class $[C_i]$ is indecomposable and the C_i is irreducible and rational. (This is possible by the Mori-Mukai argument.) By the minimality of N, none of the C_i is proportional to f.

Since

$$\iota_p^* C_i + C_i = C_i' + C_i$$

is invariant under ι_p it equals some multiple of f. We claim this lifts to an irreducible rational curve R over $\overline{\mathbb{Q}}$. Consider the chain of two $\mathbb{P}^1$'s

$$T = \{xy = 0\} \subset \mathbb{P}^2$$

and choose a birational morphism $\phi : W \to C_i' + C_i$. Let $j : T \to S_p$ be the induced morphism; then T is the specialization of a rational curve over the Witt-vectors by Theorem 19. Since this curve is rigid, it can be defined over $\overline{\mathbb{Q}}$ as well.

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