RECONSTRUCTION OF FUNCTION FIELDS

by

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ABSTRACT. — For k an algebraic closure of the finite field $\mathbb{F}_p$, ℓ prime distinct from p and X a surface over k, we prove that the field of rational functions $k(X)$ can be recovered from the maximal pro-ℓ-quotient $\mathcal{G}_K$ of its absolute Galois group - in fact already from the second central descending series quotient of $\mathcal{G}_K$.

Contents

KEY WORDS AND PHRASES. — Galois groups, function fields.

Introduction

We fix two distinct primes p and ℓ. Let $k = \bar{\mathbb{F}}_p$ be an algebraic closure of the finite field $\mathbb{F}_p$. Let K be a finite type extension of k. It is the function field of a (non-unique) normal projective variety X. Let $\mathcal{G}^a_K$ be the abelianization of the pro-ℓ-quotient $\mathcal{G}_K$ of the absolute Galois group of K. Under our assumptions on k, $\mathcal{G}^a_K$ is a torsion-free $\mathbb{Z}_\ell$-module. Let $\mathcal{G}^c_K$ be the second lower central series quotient of $\mathcal{G}_K$. It is a central extension of $\mathcal{G}^a_K$ It determines the following structure: the set Σ^0_K of closed subgroups of $\mathcal{G}^a_K$ whose inverse image in $\mathcal{G}^c_K$ is abelian. We define Σ_K to be the set of maximal non topologically cyclic elements of Σ^0_K. We show in Proposition 8.3 that for surfaces X, every element in Σ_K has rank 2.

THEOREM 1. — *Let K and L be function fields over algebraic closures of finite fields of characteristic $\neq \ell$. Assume that $K = k(X)$ is a function field of a surface X/k and that there exists an isomorphism*

$$\Psi = \Psi_{K,L} \, : \, \mathcal{G}^a_K \simeq \mathcal{G}^a_L$$

of abelian pro-ℓ-groups inducing a bijection of sets

$$\Sigma_K = \Sigma_L.$$

Then, for some $c \in \mathbb{Z}^*_\ell$, $c\Psi$ is induced by an isomorphism $\bar{\Psi}$ of the perfect closure of K with the perfect closure of L.

We implement the program outlined in [1] and [2] describing the correspondence between higher-dimensional function fields and their abelianized Galois groups. For results concerning the reconstruction of function fields from their (full) Galois groups (the birational Grothendieck program) we refer to the works of Pop, Mochizuki and Efrat (see [9], [8],[5]).

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2. Basic algebra and geometry of fields

NOTATIONS 2.1. — Throughout, k is an algebraic closure of the finite field $\mathbb{F}_p$ and K a finitely generated extension of k. It is the function field of a normal projective algebraic variety X over k. Such a variety X is called a *model* of K. Its set of k-rational points is denoted by $X(k)$, the Picard group by $\mathrm{Pic}(X)$ and Néron-Severi group by $\mathrm{NS}(X)$. We write $\mathrm{Pic}^0(X)$ for the kernel of the natural projection $\mathrm{Pic}(X) \to \mathrm{NS}(X)$.

In this paper we use the fact that two-dimensional function fields K have "nice" models: smooth projective surfaces X over k with $K = k(X)$, whose geometric properties play an important role in the recognition procedure. In this section we collect some technical results about function fields of curves and surfaces and their models.

LEMMA 2.2. — *Let C/k be a smooth curve and $Q \subset C(k)$ a finite set. Then there exists an $n_Q \in \mathbb{N}$ such that for every degree zero divisor D with support in Q the divisor $n_Q D$ is principal.*

Proof. — Finitely generated subgroups of torsion groups are finite. The group of degree zero divisors $\mathrm{Pic}^0(C)$ (over any finite field) is torsion and every subgroup of divisors with support in a finite set $Q \subset C(k)$ is finitely generated. □

LEMMA 2.3. — *Let $K/\mathfrak{K}$ be a purely inseparable extension. Then*

- *$\mathfrak{K} \supset k$;*
- *$K/\mathfrak{K}$ is a finite extension;*
- *$\mathfrak{K}$ is finitely generated over k.*

DEFINITION 2.4. — *We write $\overline{E}^K \subset K$ for the normal closure of a subfield $E \subset K$ (elements in K which are algebraic over E). We say that $x \in K \setminus k$ is* generating, *if $E = k(x)$ is normally closed, meaning that $\overline{E}^K = E$.*

REMARK 2.5. — If $E \subset K$ is one-dimensional then for all $x \in E \setminus k$ one has $\overline{k(x)}^K = \overline{E}^K$ (a finite extension of E).

LEMMA 2.6. — *For any one-dimensional subfield $E \subset K$ there is a model X of k and a sequence*

$$X \xrightarrow{\pi_E} C' \to C,$$

where C and C' are the projective nonsingular curves with function fields $E = k(C)$ and $\overline{E}^K = k(C')$, and a general fiber of π_E is irreducible and reduced.

Proof. — Since $\overline{k(C')}^K = k(C')$ the general fiber of π_E is irreducible. There are no purely inseparable extensions of $k(C')$ contained in K, which implies that the differential of π_E is surjective at the generic point of a general fiber. □

For generating $x \in K$ we write

$$\pi_x : X \to C$$

for the morphism from Lemma 2.6, with $k(C') = k(C) = k(x)$. For $y \in K \setminus k(x)$ define $\deg_x(y)$ (the degree of y on the general fiber of π_x) as the degree of the corresponding surjective map from the general fiber of π_x under π_y (By Lemma 2.6, the general fiber of π_x is reduced and irreducible, for generating x).

LEMMA 2.7. — *Let $K = k(X)$ be the function field of a surface, $x \in K \setminus k$ be generating, and $y \in K \setminus k(x)$ be such that*

$$\deg_x(y) = \min_{f \in K \setminus \overline{k(x)}^K} (\deg_x(f))$$

and $\overline{k(y)}^K = k(y')$ for some $y' \in K^*$. Then y is generating: $k(y) = \overline{k(y)}^K$.

Proof. — If y is not generating then $y = z(y')$ for some $y' \in K$ and some function $z \in k(y')^*$ of degree ≥ 2. This implies that $\deg_x(y) \geq 2\deg_x(y')$, contradicting minimality. □

LEMMA 2.8. — *Let X be a model of K containing a rational curve R and $x \in K^*$ a function such that its restriction x_R to R is defined and such that $k(R) = k(x_R)$. Then x is generating: $\overline{k(x)}^K = k(x)$.*

Proof. — The restriction map extends to $\overline{k(x)}^K$ and hence is an isomorphism between $k(x_R)$ and $k(x) = \overline{k(x)}^K$. □

The next proposition characterizes multiplicative groups of fields $\mathfrak{K} \subset K$ such that $K/\mathfrak{K}$ is a purely inseparable extension. Notice that for a one-dimensional field $k(C)$ such subfields are always of the form $k(C)^{p^n}$, for some $n \in \mathbb{N}$. Thus for any one-dimensional subfield $E \subset K$ there is an

$r(E) \in \mathbb{N}$ such that the intersection of $\mathfrak{K}^*$ with E^* consists exactly of $r(E)$-powers of the elements of E^*. Below we show that this property of intersection with subfields of the special form $k(x) = \overline{k(x)}^K$ already characterizes multiplicative groups of such $\mathfrak{K}$ among multiplicative subgroups of K^*.

DEFINITION 2.9. — *Let $\mathfrak{K}^* \subset K^*$ be a (multiplicative) subgroup such that for any rational normally closed subfield $E = k(x) \subset K$ there exists an $r = r(E)$ with the property that $\mathfrak{K}^* \cap E^* = (E^*)^r$ (r-powers of elements of E^*). For every $t \in E^* \setminus k^*$ we define $r(t) = r(E)$.*

REMARK 2.10. — Note that $r(t)$ is not defined for $t \in K^* \setminus k^*$ iff $\overline{k(t)}^K$ is the function field of a curve of genus ≥ 1.

DEFINITION 2.11. — *We will say that $y \in K^* \setminus k^*$ is a* power *if y is not primitive in the multiplicative group, i.e., there exist an $x \in K^*$ and an integer $n \geq 2$ such that $y = x^n$.*

LEMMA 2.12. — *Let X be a smooth projective surface with $k(X) = K$. Let $f \in K^*$ and assume that $\mathrm{div}_0(f) \cap \mathrm{div}_\infty(f) \neq \emptyset$. Then $\overline{k(f)}^K = k(\mathbb{P}^1)$.*

Proof. — Let $q \in \mathrm{div}_0(f) \cap \mathrm{div}_\infty(f)$ be a point in the intersection. Then f defines a composition map $\pi_f : \tilde{X} \to C \to \mathbb{P}^1$ from a blowup of X with support at finitely many points, including q. Since all the components of the preimage of q are rational curves, there is a surjective map $\mathbb{P}^1 \to C$ from at least one of them. Thus C is also rational and $\overline{k(f)}^K = k(C) = k(\mathbb{P}^1)$. □

PROPOSITION 2.13. — *Let $K = k(X)$ be the function field of a surface and $\mathfrak{K}^* \subset K^*$ a subset such that*

(1) *$\mathfrak{K}^*$ is a multiplicative subgroup of K^*;*
(2) *for every rational normally closed $E = k(x) = \overline{k(x)}^K \subset K$ there exists an $r = r(E) \in \mathbb{N}$ with*
$$\mathfrak{K}^* \cap E^* = (E^*)^r;$$
(3) *there exists a $y \in K \setminus k$ with $r(y) = 1$.*

Then $\mathfrak{K} := \mathfrak{K}^* \cup 0$ is a field, whose multiplicative group is $\mathfrak{K}^*$ and $K/\mathfrak{K}$ is a purely inseparable finite extension.

Proof. — By (2), $k \subset \mathfrak{K}$. To prove that $\mathfrak{K}$ is a field, it suffices to show that for every $x \in \mathfrak{K}$ one has $x+1 \in \mathfrak{K}$ (and then use multiplicativity). For every $x \in \mathfrak{K} \setminus k$ with $r(x) = 1$ we have $\mathfrak{K}^* \cap k(x)^* = k(x)^*$ and

$$x + \kappa \in \mathfrak{K}^*, \quad \text{for all } \kappa \in k.$$

In particular, this holds for y, and we can choose y to be a generating element of K.

Consider $x \in \mathfrak{K}^*$ with $r(x) > 1$ or not defined. In particular, $x \notin k(y)$, and $k(x, y)$ has transcendence degree two over k. Assume first that K is a separable extension of $k(x, y)$. Then there is a smooth projective model X of K admitting proper maps $\pi_x : X \to \mathbb{P}^1_x$ and $\pi_y : X \to \mathbb{P}^1_y = (y : 1)$. The functions x and y define coordinates on $\mathbb{P}^1_x$, resp. $\mathbb{P}^1_y$. The restriction of π_y to any fiber $\pi_x^{-1}(b_x)$, for $b_x \in \mathbb{P}^1_x$, maps surjectively onto $\mathbb{P}^1_y$. In particular, the fiber $\pi_x^{-1}(0)$ has an irreducible component D_0 surjecting onto $\mathbb{P}^1_y$. Let $D_0^\circ \subset D_0$ be the open subset obtained by removing all intersections points with other components of this fiber. Let $\kappa \in k$ be such that

(i) both fibers $\pi_y^{-1}(-\kappa)$ and $\pi_y^{-1}(1-\kappa)$ are transversal to the generic fiber of π_x;
(ii) $\pi_y^{-1}(-\kappa) \cap D_0^\circ \neq \emptyset$.

These properties are satisfied for all but finitely many $(\kappa : 1) \in \mathbb{P}^1_y$.

Consider $t := (y+\kappa)/x$. It is in $\mathfrak{K}^*$. It satisfies the assumptions of Lemma 2.12, i.e., the divisor of zeroes has nontrivial intersection with the divisor of poles on X. It follows that $r(t)$ is defined, by assumption (2). Note that t is not a power in K, by the transversality condition (i). Hence, $r(t) = 1$. If follows that

$$(1/t) + 1 = (x+y+\kappa)/(y+\kappa) \in \mathfrak{K}^*.$$

In particular,

$$z := \frac{x+y+\kappa}{y+\kappa-1} \in \mathfrak{K}^*.$$

By (i), it is also not a power in K, and since $\mathrm{div}_\infty(z) \cap \mathrm{div}_0(z) \neq \emptyset$, $r(z)$ is defined and hence equal to 1. This implies that

$$z - 1 = (x+1)/(y+\kappa-1) \in \mathfrak{K}^* \quad \text{and } x+1 \in \mathfrak{K}^*,$$

(by multiplicativity).

Consider the case when the extension $K/k(x,y)$ is finite but not separable. Let $\tilde{K}$ be the maximal separable extension of $k(x,y)$ contained in K, so that $K/\tilde{K}$ is purely inseparable. Then, for all rational normally closed subfields $E \subset K$, the intersection $\tilde{K}^* \cap E^* = (E^*)^{p^m}$, for $m \in \mathbb{N}$ (here we use the freeness of the group E^*/k^*). Hence the intersection $\tilde{\mathfrak{K}}^* := \tilde{K}^* \cap \mathfrak{K}^*$ satisfies conditions (1) and (2) of the proposition. Since $y \in \tilde{\mathfrak{K}}^*$, condition (3) holds as well. We conclude that $\tilde{\mathfrak{K}}^* \cup 0$ is a field, and hence contains $x+1$.

It follows that $\mathfrak{K}^*$ is the multiplicative group of a subfield $\mathfrak{K} \subset K$ with the additional property that $\mathfrak{K}^*$ intersects the multiplicative group of every one-dimensional normally closed subfield $k(x) = \overline{k(x)}^K$ in a subgroup of the form $(k(x)^*)^{r(x)}$. If $r(x)$ is a power of p, for all x, then K is a finite purely inseparable extension of $\mathfrak{K}$, since the multiplicative groups of such fields generate K^*. Otherwise, there exist a $y \in K \setminus \mathfrak{K}^*$ and a (minimal) positive integer s, coprime to p, such that $y^s \in \mathfrak{K}^*$. Consider $E := k(y)$. Since $\mathfrak{K}$ is a subfield of K its intersection with E is a proper subfield $E' \subset E$ not containing y. The field $E = k(y)$ is a separable cyclic s-extension of E'.

By assumption on $\mathfrak{K}$ some power of any element in $E = k(y)$ is contained in E'. Let us show that this contradicts the separability of E/E'. Indeed, if $z \in E$ has norm $N_{E/E'}(z) = 1$ then the same holds for any power of z. Hence none of the powers of z are contained in E. Since the group of elements of norm 1 is nontrivial, for a separable extension, the extension E/E' must be inseparble, contradicting the condition that $s > 1$ and s coprime to p. Thus s is a power of p, for any $y \in K$, and hence $K/\mathfrak{K}$ is purely inseparable. □

LEMMA 2.14. — *Let $K/\mathfrak{K}$ be a finite purely inseparable extension. Assume that $\mathfrak{K} \not\subset K^p$. Then the multiplicative group $\mathfrak{K}^*$ satisfies the assumptions Proposition 2.13.*

Proof. — Let $E \subset \mathfrak{K}$ be a normally closed subfield in $\mathfrak{K}$ of transcendence degree 1. Since $K/\mathfrak{K}$ is purely inseparable, the normal closure $\bar{E}^K$ in K is equal to E^{1/p^m}, for some $m \in \mathbb{N}$. The intersection of multiplicative groups of $\bar{E}^K$ and $\mathfrak{K}$ is equal to $E^* = ((E^*)^{1/p^m})^{p^m}$.

Thus for any $k(x) = \overline{k(x)}^K$ we have $k(x) \cap \mathfrak{K}^* = (k(x)^*)^{p^m}$. Hence $\mathfrak{K}^*$ satisfies the first two assumptions of the proposition.

To show condition (3) we need to find an $y \in \mathfrak{K}^*$ such that $r(y) = 1$. Let $x' \in \mathfrak{K}^*, x' \notin \mathfrak{K}^p$. We have an equality of normal closures

$$E = \overline{k(x')}^K = \overline{k(x')}^{\mathfrak{K}}.$$

It contains an element x which is not a power in K. We have a surjective morphism $\pi_x : X \to \mathbb{P}^1$. The divisor of zeroes $\mathrm{div}_0(x)$ contains a reduced irreducible component D_0. Let $t \in K^*$ be a generating element in K such that D_0 is not a component of the support of t such the supports of the intersections of D_0 with the three divisors

$$\mathrm{div}_0(t), \mathrm{div}_\infty(t), \mathrm{div}_\infty(x)$$

are nontrivial and pairwise distinct. We are done if $t \in \mathfrak{K}^*$. Otherwise, there is an $m > 0$ such that $t^{p^m} \in \mathfrak{K}^*$. Put $z := xt^{p^m}$. The $\mathrm{div}_0(z)$ contains a reduced irreducible component D_0 which intersects $\mathrm{div}_\infty(z)$. Thus $z \in \mathfrak{K}^*$ and is not a power in K. By Lemma 2.12, $\overline{k(z)}^K = k(\mathbb{P}^1)$ and it contained in $\mathfrak{K}$. Hence there is an element $y \in \overline{k(z)}^K \subset \mathfrak{K}^*$ with $r(y) = 1$. □

LEMMA 2.15. — *Assume that $\mathfrak{K}^*$ satisfies the conditions (1) and (2) of Proposition 2.13. Then there is a maximal $r_0 \in \mathbb{N}$ such that for all $x \in \mathfrak{K}^*$ such that $r(x)$ is defined, $r(x)$ is divisible by r_0. Moreover, $r(x)/r_0$ is a power of p.*

Proof. — Let $x \in \mathfrak{K}^* \subset K^*$ be such that $r(x)$ is defined. Consider the subgroup $\tilde{\mathfrak{K}}^* := (\mathfrak{K}^*)^{1/r(x)} \cap K^*$. For any subfield $k(y) = \overline{k(y)}^K \subset K$ the intersection $k(y)^* \cap \mathfrak{K}^*$ is equal to $(k(y)^*)^{(r(y)/\gcd(r(x),r(y)))}$ since $k(y)^*/k^*$ is a free abelian group and k^* is infinitely divisble. The subgroup $\tilde{\mathfrak{K}}^* \subset K^*$ satisfies assumptions (1) and (2) of Proposition 2.13, and we can apply Definition 2.9 to introduce the function $\tilde{r}$ on the subset of K^* where r was defined. It also satisfies (3) since by construction $\tilde{r}(x) = 1$. Applying Proposition 2.13, we conclude that there is a subfield $K_x \subset K$ with $\tilde{\mathfrak{K}}^* \cup 0 = K_x$, where K/K_x is a finite purely inseparable extension. In particular, for all $y \in \mathfrak{K}^*$ one has $(r(y)/\gcd(r(x), r(y)) = p^m$ for some integer m and any $y \in K^*$. Applying the same construction to $r(y)$ we see that $r(x)/r(y)$ is a power of p for any two elements $x, y \in \mathfrak{K}^*$, where the function r is defined. □

LEMMA 2.16. — *Assume that $\mathfrak{K}^* \subset K^*$ satisfies assumptions (1) and (2) of Proposition 2.13. Let m be the maximal common divisor of $r(x), x \in K^*$. Then $(\mathfrak{K}^*)^{1/m} \subset K^*$. Moreover, there exists a subfield $\tilde{K} \subset K$, such that*

– *$\tilde{K} \not\subset K^p$,*
– *$K/\tilde{K}$ is finite purely inseparable,*
– *$\tilde{K}^* = (\mathfrak{K}^*)^{1/m}$.*

Proof. — The group $(\mathfrak{K}^*)^{1/m} \cap K^*$ satisfies all assumptions of Proposition 2.13. Hence it coincides with the multiplicative subgroup of a field $\tilde{K}$ as in the statement. Note that m is a minimal possible $r(x)$ and hence $\tilde{K} \notin K^p$. It remains to show that $x^{1/m} \in K^*$ for $x \in \mathfrak{K}^*$ and hence $(\mathfrak{K}^*)^{1/m} \subset K^*$. By Lemma 2.15, this holds for any x with $\overline{k(x)}^K = k(\mathbb{P}^1)$, i.e., when $r(x)$ is defined. It remains to consider $z \in K$ with $r(z)$ not defined. Assume that $z^{1/m} \notin K^*$. That means there exist an element $u \in K^*$ which is not a power and an integer $n \geq 1, n < m$ and n dividing m such that $u^n = z \in \mathfrak{K}^*$. We claim that there exist elements $x, y \in \mathfrak{K}^*$, with $y^m \in \mathfrak{K}^*$,

$$\overline{k(x)}^K = \overline{k(y)}^K = k(\mathbb{P}^1)$$

and such that $uy^{m/n} = x$. Then $(uy^{m/n})^n \in \mathfrak{K}^*$ and hence $x^n \in \mathfrak{K}^*$. Since $n < m$ this would give a contradiction.

Consider again a smooth model X of K where u defines a proper surjective map $X \to \mathbb{P}^1$. The field $\tilde{K}$ is not contained in K^p. Using the argument in the proof of Lemma 2.14. we can find a $y \in \tilde{K} \setminus \tilde{K} \cap K^p$ with a divisor $\mathrm{div}_\infty(y)$ which intersects $\mathrm{div}_0(z)$ and $\mathrm{div}_0(y)$. Thus $\overline{k(x)}^K = \overline{k(y)}^K = k(\mathbb{P}^1)$. □

Let $\mathrm{Alb}(X)$ be the Albanese variety of X. In our terminology, $\mathrm{Alb}(X)$ is a principal homogeneous space for an abelian variety $A^0(X)$ and there is a universal map $X \to \mathrm{Alb}(X)$. In our analysis of Galois groups we need to keep track of rational curves on surfaces.

LEMMA 2.17. — *Let X be a smooth projective surface over k with function field $K = k(X)$. There are three mutually disjoint possibilities:*

(1) $\mathrm{Pic}^0(X) \neq 0$ *and X contains finitely many rational curves;*
(2) $\mathrm{Pic}^0(X) \neq 0$ *and the corresponding surface admits a surjective morphism onto a curve C of genus* $\mathrm{g}(C) \geq 1$ *with generic fiber a (possibly singular) rational curve;*
(3) $\mathrm{Pic}^0(X) = 0$.

Proof. — If X is smooth and $\mathrm{Pic}^0(X) \neq 0$ then there is a nontrivial map to the Albanese variety of X, and all rational curves lie in fibers. The generic fiber of this map is either rational or there are only finitely many rational curves on X. □

LEMMA 2.18. — *Let $\mathcal{D} := \{D_j\}_{j\in J}$ be a finite set of irreducible divisors on X. Assume that there is a nonconstant $f \in k(X)^*$ whose divisor is supported in $\mathcal{D}$. Let $B \subset A^0(X)$ be the smallest abelian subvariety such that the image of D_j under the map $\alpha : \mathrm{Alb}(X) \to A := \mathrm{Alb}(X)/B$ is a point, for all $j \in J$.*

Assume that $B \neq A^0(X)$. Then the image of X in A is a curve C and A is isomorphic to the Jacobian $\mathrm{Jac}(C)$ of degree 1 zero-cycles on C.

Proof. — First of all, $\dim \alpha(X) \geq 1$: the surface X is connected and $\alpha(X)$ generates A. Further, $\alpha(X)$ is not a surface: otherwise if $X' \to \alpha(X)$ is the normalization, then there is a map $\mu : X \to X'$ and the image of $\{D_j\}_{j\in J}$ is a finite set of points on X'. The intersection matrix of the set of irreducible components in the divisorial support of $\mu^{-1}(x')$, for any $x' \in X'$, is negative definite, contradicting the assumption that there is a function supported in $\mathcal{D}$.

Let $C := \alpha(X) \subset A$, we have $k(C) \subset K$. Let C' be a curve with function field $k(C') = \overline{k(C)}^K \subset K$. The map $C' \to C$ is finite. The map $\alpha : X \to A$ factors through the Jacobian $\mathrm{Jac}(C')$: we have

$$\begin{array}{ccc} X & \xrightarrow{\alpha_{C'}} & \mathrm{Jac}(C') \\ & & \downarrow \\ & & A \end{array}$$

The image of $\{D_j\}_{j\in J}$ under $\alpha_{C'}$ is a finite set of points in $\mathrm{Jac}(C')$. We have surjections $\mathrm{Jac}(C') \to \mathrm{Jac}(C) \to A$ and a canonical map $\mathrm{Alb}(X) \to \mathrm{Jac}(C')$. Then $B = \mathrm{Ker}(\alpha_{C'})$ and $C' = C$. □

REMARK 2.19. — Let X' be a model of a purely inseparable extension of $K = k(X)$ and assume that X' admits a dominant map onto a curve C' of genus ≥ 1. Then X also admits a dominant map onto a curve C which has the same genus as C', and corresponds to a purely inseparable extension of $k(C')$. In particular, Lemma 2.18 describes all such maps.

DEFINITION 2.20. — *An ℓ-Lefschetz pencil on a smooth surface X is a surjective morphism $\pi : X \to C$ onto a smooth curve and a point $c_0 \in C$ such that*

1. *the generic fiber C_η is smooth;*
2. *all fibers over $C\backslash c_0$ contain a unique nonrational irreducible component with* $\mathrm{g} > 1$*;*
3. $\ell \nmid \mathrm{g}(C_\eta) - 1$.

Let X be a smooth model of K and write ω_X for its canonical class.

LEMMA 2.21. — *The field K admits a smooth projective model X with the following property: there exists a polarization* H *on X such that*

$$(\mathrm{H}\cdot\omega_X, \mathrm{H}^2)\begin{cases} \neq (0,0) & \mod \ell \quad \textit{if } \ell > 2 \\ = (1,1) & \mod 4 \quad \textit{if } \ell = 2. \end{cases} \tag{2.1}$$

Proof. — Assume otherwise. Let X' be the blow-up of X in a point $q \in X$ and E the exceptional divisor. Assuming H sufficiently ample, we have a polarization $\mathrm{H}' = \mathrm{H} - \mathrm{E}$ on X'. Then

$$(\mathrm{H}-\mathrm{E})^2 = \mathrm{H}^2 - 1 \neq 0 \mod \ell,$$

for $\ell > 2$. This process transforms $(H\omega_X, H^2)$ by adding $(1,-1)$. For $\ell = 2$ an iterated application of this process gives $(1,1) \mod 4$, provided $H(H+\omega_X) \neq 0 \mod 4$. When $H(H+\omega_X) = 0 \mod 4$, put $H' = H - 2E$, with E as above. Then

$$H'^2 = H^2 \mod 4 \quad \text{and} \quad H'(H'+\omega_{X'}) = H(H+\omega_X) - 2 \mod 4,$$

hence equal to $2 \mod 4$, and we can proceed replacing H by H'. □

PROPOSITION 2.22. — *For every $f \in K^*/k^*$ there exist functions $f_1, f_2 \in K^*/k^*$ such that $f = f_1/f_2$ and f_1, f_2 induce ℓ-Lefschetz pencils on some model of K.*

In particular, the group $K^*/k^* \otimes \mathbb{Z}_{(\ell)}$ is generated by subgroups of the form $E^*/k^* \otimes \mathbb{Z}_{(\ell)}$ such that

- *$E = k(x)$ and E is normally closed in K;*
- *the induced fibration $X \to C$ is an ℓ-Lefschetz pencil.*

Proof. — Choose a smooth model X of and a polarization H on X satisfying the conditions of Lemma 2.21. Let f be a function with a divisor $\mathrm{div}(f) = D_0 - D_\infty$. Consider elements $n\mathrm{H}, n \gg 0$ such that n and $n(n\mathrm{H}^2 + \mathrm{H}\omega_X)/2$ are prime to ℓ. There is an infinite sequence of such n - for $\ell > 2$ odd we take any n prime to ℓ with $n\mathrm{H}^2 + \mathrm{H}\omega_X$ prime to ℓ, an arithmetic progression.

Similarly, for $\ell = 2$ we have $\mathrm{H}\omega_X = \mathrm{H}^2 \mod 2$ and we may choose $n = 1 \mod 4$ to safisfy the condition.

For $n \gg 0$ there is an effective divisor $D \subset X$ such that $D_0 + D \sim n\mathrm{H}$. Consider the projective space $\mathbb{P}(H^0(X, n\mathrm{H}))$. Let $\mathrm{Sing} \subset \mathbb{P}(H^0(X, n\mathrm{H}))$ be the subvariety corresponding to singular divisors. For sufficiently large n in this sequence the locus Sing is a reduced irreducible hypersurface whose generic point corresponds to a curve with one singular double point. A generic point $x \in \mathbb{P}(H^0(X, n\mathrm{H})) \setminus \mathrm{Sing}$ corresponds to a smooth divisor R of genus $\mathsf{g} = \mathsf{g}(R)$ with $\ell \nmid \mathsf{g} - 1$. For such points the lines $\langle R, D_0 + D_0\rangle, \langle R, D_\infty + D\rangle$ intersect Sing transversally outside of $D_0 + D, D_\infty + D$, respectively. Then the functions $f_1, f_2 \in K^*/k^*$ corresponding to $\langle R, D_0 + D\rangle, \langle R, D_\infty + D\rangle$ define ℓ-Lefschetz pencils and $f = f_1/f_2$. □

3. Projective structures

In this section we explain the connection between fields and axiomatic projective geometry. We follow closely the exposition in [7].

DEFINITION 3.1. — *A* projective structure *is a pair $(S, \mathfrak{L})$ where S is a (nonempty) set (of points) and $\mathfrak{L}$ a collection of subsets $\mathfrak{l} \subset S$ (lines) such that*

P1 *there exist an $s \in S$ and an $\mathfrak{l} \in \mathfrak{L}$ such that $s \notin \mathfrak{l}$;*
P2 *for every $\mathfrak{l} \in \mathfrak{L}$ there exist at least three distinct $s, s', s'' \in \mathfrak{l}$;*
P3 *for every pair of distinct $s, s' \in S$ there exists exactly one*

$$\mathfrak{l} = \mathfrak{l}(s, s') \in \mathfrak{L}$$

such that $s, s' \in \mathfrak{l}$;
P4 *for every quadruple of pairwise distinct $s, s', t, t' \in S$ one has*

$$\mathfrak{l}(s, s') \cap \mathfrak{l}(t, t') \neq \emptyset \;\Rightarrow\; \mathfrak{l}(s, t) \cap \mathfrak{l}(s', t') \neq \emptyset.$$

For $s \in S$ and $S' \subset S$ define the *join*

$$s \vee S' := \{s'' \in S \mid s'' \in \mathfrak{l}(s, s') \text{ for some } s' \in S'\}.$$

For any finite set of points $s_1, \dots, s_n$ define

$$\langle s_1, \dots s_n\rangle := s_1 \vee \langle s_2 \vee \cdots \vee s_n\rangle$$

(this does not depend on the order of the points). Write $\langle S' \rangle$ for the join of a *finite* set $S' \subset S$. A finite set $S' \subset S$ of pairwise distinct points is called *independent* if for all $s' \in S'$ one has

$$s' \notin \langle S' \setminus \{s'\} \rangle.$$

A set of points $S' \subset S$ *spans* a set of points $T \subset S$ if

- $\langle S'' \rangle \subset T$ for every finite set $S'' \subset S'$;
- for every $t \in T$ there exists a finite set of points $S_t \subset S'$ such that $t \in \langle S_t \rangle$.

A set $T \subset S$ spanned by an independent set S' of points of cardinality ≥ 1 is called a projective *subspace* of dimension $|S'| - 1$.

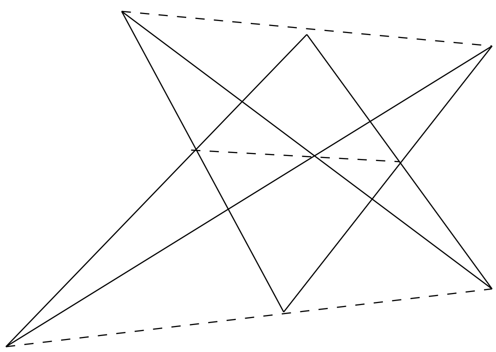
The axioms imply that projective subspaces of a given projective space S form a lattice and that the dimension function is well defined, i.e.,

$$\dim(T \cup T') + \dim(T \cap T') = \dim(T) + \dim(T')$$

for all pairs of projective subspaces $T, T' \subset S$. Here we put $\dim(\emptyset) := -1$.

DEFINITION 3.2. — *A projective structure* $(S, \mathfrak{L})$ *satisfies* Pappus' axiom *if*

PA *for all 2-dimensional subspaces and every configuration of six points and lines in these subspaces as below*



the intersections are collinear.

The main theorem of abstract projective geometry is:

THEOREM 3.3. — *Let* $(S, \mathfrak{L})$ *be a projective structure of dimension* $n \geq 2$ *which satisfies Pappus' axiom. Then there exists a vector space* V *over a field*

L and an isomorphism

$$\sigma\,:\,\mathbb{P}_L(V)\xrightarrow{\sim} S.$$

Moreover, for any two such triples (V,L,σ) and (V',L',σ') there is an isomorphism

$$V/L\xrightarrow{\sim} V'/L'$$

compatible with σ,σ' and unique up to homothety $v\mapsto \lambda v$, $\lambda\in L^*$.

Proof. — See [7], Chapter 6. □

DEFINITION 3.4. — *A morphism of projective structures*

$$\rho\,:\,(S,\mathfrak{L})\to(S',\mathfrak{L}')$$

is an injection of sets $\rho\,:\,S\hookrightarrow S'$ such that $\rho(\mathfrak{l})\in\mathfrak{L}'$ for all $\mathfrak{l}\in\mathfrak{L}$.

EXAMPLE 3.5. — Let k be a field and $\mathbb{P}^n_k$ the usual projective space over k of dimension $n\geq 2$. Then $\mathbb{P}^n_k$ carries a projective structure: the set of lines is the set of usual projective lines $\mathbb{P}^1_k\subset\mathbb{P}^n_k$.

Let K/k be an extension of fields (not necessarily finite). Then the set

$$S:=\mathbb{P}_k(K)=(K\setminus 0)/k^*$$

carries a natural (possibly, infinite-dimensional) projective structure. Moreover, multiplication by elements in the group K^*/k^* preserves this structure.

THEOREM 3.6. — *Let K/L and K'/L' be field extensions of degree ≥ 3 and*

$$\bar\phi\,:\,S=\mathbb{P}_L(K)\to\mathbb{P}_{L'}(K')=S'$$

a bijection of sets which is an isomorphism of abelian groups and of projective structures. Then there is a unique isomorphism

$$L/K\xrightarrow{\sim} L'/K'$$

inducing $\bar\phi$.

Proof. — Consider $V:=K$ as a vector space over L. By Theorem 3.3, to S there are canonically attached the L-algebra $\mathrm{End}(V)$ and $\mathrm{GL}(V)\subset\mathrm{End}(V)$, as the set of elements preserving the collineations of the projective space S (because the action of homotheties on $\mathrm{End}(V)$ is trivial). This allows us to canonically recover the field K (resp. K') as the subfield of the L-algebra $\mathrm{End}(V)$ given by

$$\{0\}\cup\{x\in\mathrm{GL}(V)\subset\mathrm{End}(V)\,|\,x\ \text{induces a group-translation}\ \ \text{on}\ \ S\},$$

and the claimed isomorphisms.

□

DEFINITION 3.7. — *Let K/k be the function field of an algebraic variety X of dimension ≥ 2 and $S = \mathbb{P}_k(K)$ the associated projective structure from Example 3.5. The lines passing through 1 and a generating element of K (see Definition 2.4) and their multiplicative translations by elements in K^*/k^* will be called* primary.

LEMMA 3.8. — *Let $K = k(X)$ be the function field of a surface. For every line $\mathfrak{l} = \mathfrak{l}(1, x)$ there exists a $\mathbb{P}^2 \subset \mathbb{P}_k(K)$ containing $\mathfrak{l}$ such that all other lines in this $\mathbb{P}^2$ are primary.*

Proof. — Choose a smooth model X of K and two points $q_1, q_2 \in X$ such that $x(q_1) = 0, x(q_2) = 1$. Blow up q_1, q_2 and let $\mathbb{P}^1_i$ be the corresponding exceptional curves. Let $y \in K^*$ be an element restricting to a generator of $k(\mathbb{P}^1_i)$. The restriction map extends to the normal closure $\overline{k(y)} \subset K$. Hence the normal closure $\overline{k(y)} \subset K$ coincides with $k(y)$.

To prove that every line $\mathfrak{l} \neq \mathfrak{l}(1, x) \subset \mathbb{P}^2 = \mathbb{P}_k(k \oplus kx \oplus ky)$ is primary we need to show that $(y + a + bx)/(y + c + dx)$ is generating, provided $(a, b) \neq (c, d)$. If $a \neq c$ then the restriction of $(y+a+bx)/(y+c+dx)$ to $\mathbb{P}^1_{q_1}$ is equal to $(y+a)/(y+c)$ and hence is a generator of $k(\mathbb{P}^1_{q_1})$. By the argument of the previous lemma, $(y+a+bx)/(y+c+dx)$ is generating. If $a = c, b \neq d$ then $(y+a+bx)/(y+c+dx)$ on $\mathbb{P}^1_{q_2}$ coincides with $(y+a+b)/(y+c+d)$ and is also generating since $a + b \neq c + d$, by assumption. □

LEMMA 3.9. — *Assume that a set S has two projective structures $(S, \mathfrak{L}_1)$ and $(S, \mathfrak{L}_2)$, both of dimension ≥ 2, and that for some $\mathbb{P}^2_1$ (in the first projective structure) every line $\mathfrak{l}_1$ of this $\mathbb{P}^2_1$, except possibly one line, is also a line in the second structure. Then the set $\mathbb{P}^2_1$ is a projective plane in the second structure $(S, \mathfrak{L}_2)$ and the sets of lines $\mathfrak{L}_1$ and $\mathfrak{L}_2$ on this $\mathbb{P}^2_1$ are equal.*

Proof. — Let $\hat{\mathbb{P}}^2_1$ be the set of all lines in $\mathbb{P}^2_1$. Let $\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3$ be three lines from $\hat{\mathbb{P}}^2_1 \setminus \mathfrak{l}$ which don't have a common intersection point and which are projective in $\mathfrak{L}_2$. Then $\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3$ lie in the same plane $\mathbb{P}^2_2$. Since every other line $\mathfrak{l}' \in \hat{\mathbb{P}}^2_1 \setminus \mathfrak{l}$ intersects $\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3$ then $\mathfrak{l}' \subset \mathbb{P}^2_2$. Thus $\hat{\mathbb{P}}^2_1 \setminus \mathfrak{l} \subset \hat{\mathbb{P}}^2_2$ and $\mathbb{P}^2_2 \supset \mathbb{P}^2_1$, as a subset, since the union of all points contained in lines $\mathfrak{l}' \subset \mathbb{P}^2_1, \mathfrak{l}' \neq \mathfrak{l}$ coincides with $\mathbb{P}^2_1$.

Consider the complementary $\hat{\mathbb{P}}_2^2 \setminus (\hat{\mathbb{P}}_1^2 \setminus \mathfrak{l})$. Let $\mathfrak{l}(x,y) \subset \mathbb{P}_2^2$ be a line containing two distinct points $x, y \in \mathfrak{l} \subset \mathbb{P}_1^2 \subset \mathbb{P}_2^2$. Assume that $\mathfrak{l}(x,y)$ contains a point $q \in \mathbb{P}_1^2 \setminus \mathfrak{l}$. Then $\mathfrak{l}(x,y) = \mathfrak{l}(q,x)$ which is one of the lines from $\hat{\mathbb{P}}_1^2 \setminus \mathfrak{l}$, contradiction. Thus any such line $\mathfrak{l}(x,y) \subset \mathbb{P}_2^2$ intersects $\mathbb{P}_1^2$ only in points of $\mathfrak{l}$. On the other hand, $\mathfrak{l}(x,y)$ intersects all the lines $\mathfrak{l}' \subset \mathbb{P}_1^2, \mathfrak{l}' \neq \mathfrak{l}$, and hence contains all points of $\mathfrak{l}$. Thus $\mathfrak{l}(x,y)$ does not depend on $x, y \in \mathfrak{l}$. There is a unique such line in $\mathbb{P}_2^2$. If $\mathfrak{l}'$ is another line in $\hat{\mathbb{P}}_2^2 \setminus (\hat{\mathbb{P}}_1^2 \setminus \mathfrak{l})$ then it contains at least two points q, q' in the intersection with $\mathbb{P}_1^2 \subset \mathbb{P}_2^2$ since $\mathfrak{l}'$ intersects any $\mathfrak{l}'' \neq \mathfrak{l}, \mathfrak{l}'' \subset \mathbb{P}_1^2$. However the line through q, q' is either one of the lines $\mathfrak{l}''$ or is equal to $\mathfrak{l}(x,y)$ if $q, q' \in \mathfrak{l}$. Thus $\hat{\mathbb{P}}_2^2 \setminus (\hat{\mathbb{P}}_1^2 \setminus \mathfrak{l})$ consists only of $\mathfrak{l}(x,y)$. Note that $\mathfrak{l}(x,y)$ coincides with $\mathfrak{l}$ as a subset of $\mathbb{P}_2^2$, since any point of $\mathfrak{l}(x,y)$ is the intersection of $\mathfrak{l}(x,y)$ with a line $\mathfrak{l}'' \in \hat{\mathbb{P}}_1^2 \setminus \mathfrak{l}$.

Thus $\mathbb{P}_1^2 = \mathbb{P}_2^2$ set-theoretically and this bijection identifies lines in $\mathbb{P}_1^2$ and $\mathbb{P}_2^2$. □

COROLLARY 3.10. — *Let K/k and K'/k' be function fields of algebraic surfaces*

$$\bar{\phi} \,:\, S = \mathbb{P}_k(K) \to S' = \mathbb{P}_{k'}(K')$$

an isomorphism of (multiplicative) abelian groups inducing a bijection on the set of primary lines in the corresponding projective structures. Then $\bar{\phi}$ is an isomorphism of projective structures and

$$k \simeq k' \quad \text{and} \quad K \simeq K'.$$

Proof. — By Lemma 3.8 and Lemma 3.9 $\bar{\phi}$ induces an isomorphism of projective structures. It remains to apply Theorem 3.6. □

4. Flag maps

NOTATIONS 4.1. — We fix two distinct prime numbers ℓ and p. Let

- $\mathbb{F} = \mathbb{F}_q$ be a finite field with $q = p^n$ and $\mathbb{F}^*$ its multiplicative group;
- $\mathrm{Vect}_{\mathbb{F}}$ - the set of finite-dimensional $\mathbb{F}$-vector spaces;
- A a vector space over $\mathbb{F}$ and $\mathbb{P}(A) = \mathbb{P}_{\mathbb{F}}(A) = (A \setminus 0)/\mathbb{F}^*$;
- $\mathcal{M}(A)$ the set of maps from $A \setminus 0$ to $\mathbb{Z}_\ell$;
- for $\mu \in \mathcal{M}(A)$ and $B \subset A$ an $\mathbb{F}$-linear subspace, μ_B the restriction of μ to $B \setminus 0$.

DEFINITION 4.2. — *A map* $\mu \in \mathcal{M}(A)$ *will be called* $\mathbb{F}^*$*-invariant if for all* $a \in A \setminus 0$ *and all* $\kappa \in \mathbb{F}^*$ *one has*

$$\mu(\kappa \cdot a) = \mu(a).$$

DEFINITION 4.3. — *A map* μ *on* $A \setminus 0$*, for a (possibly infinite-dimensional) vector space* A*, will be called an* $\mathbb{F}$*-flag map, if*

- μ *is* $\mathbb{F}^*$*-invariant;*
- *every finite-dimensional* $\mathbb{F}$*-vector space* $B \subset A$ *has a flag of* $\mathbb{F}$*-subspaces*

$$B = B_0 \supset B_1 \supset \ldots \supset B_d = 0$$

such that μ_B *is constant on* $B_n \setminus B_{n+1}$*, for all* $n = 0, \ldots, d-1$.

The value of μ *on* $B = B_0 \setminus B_1$ *is called the* generic *value of* μ *on* B*; we denote it by* $\mu^{\mathrm{gen}}(B)$*. The set of* $\mathbb{F}$*-flag maps will be denoted by* $\Phi_{\mathbb{F}}(A)$.

EXAMPLE 4.4. — Let $K = k(X)$ be a function field. We can consider it as a vector space over k or over any of the finite subfields $\mathbb{F} \subset k$. Let ν be a nonarchimedean valuation on K and $\chi : \Gamma_\nu \to \mathbb{Z}_\ell$ a homomorphism from the value group of ν (see Section 6). Then $\chi \circ \nu \in \Phi_k(K)$.

DEFINITION 4.5. — *Let* A *be an* $\mathbb{F}$*-algebra (without zero-divisors). A map* $\mu \in \mathcal{M}(A)$ *will be called logarithmic if*

$$\mu(a \cdot a') = \mu(a) + \mu(a'), \text{ for all } a, a' \in A \setminus 0.$$

The set of such maps will be denoted by $\mathcal{L}_{\mathbb{F}}(A)$.

Since $\mathbb{F}^*$ is torsion, a logarithmic map to $\mathbb{Z}_\ell$ is $\mathbb{F}^*$-invariant.

DEFINITION 4.6. — *Let* A *be an* $\mathbb{F}$*-vector space. Two maps* $\mu, \mu' \in \mathcal{M}(A)$ *will be called a* c*-pair (commuting pair) if for all two-dimensional* $\mathbb{F}$*-subspaces* $B \subset A$ *there exist constants* $\lambda, \lambda', \lambda'' \in \mathbb{Z}_\ell$ *(depending on* B*) with* $(\lambda, \lambda') \neq (0,0)$ *such that for all* $b \in B \setminus 0$ *one has*

$$\lambda\mu_B(b) + \lambda'\mu'_B(b) = \lambda''.$$

THEOREM 4.7. — *Let* $\mathbb{F} \subset k$ *be a finite field with* $\#\mathbb{F} \geq 11$*,* $K = k(X)$ *and* $\mu, \mu' \in \mathcal{L}_{\mathbb{F}}(K)$ *nonproportional maps forming a* c*-pair. Then there exists a pair* $(\lambda, \lambda') \in \mathbb{Z}_\ell^2 \setminus (0,0)$ *such that* $\lambda\mu + \lambda'\mu' \in \Phi_{\mathbb{F}}(K)$.

Proof. — This is a special case of the main theorem of [3], where it is proved over general ground fields k. However, the case when $k = \bar{\mathbb{F}}_q$ is easier. Following the request of the referee, we now give a complete proof in this special case. The main steps in the proof are:

- characterization of flag maps by their restriction to 2-dimensional $\mathbb{F}$-linear subspaces, for $\#\mathbb{F} \geq 11$ (see Lemma 4.16);
- reduction to linear spaces over prime fields, resp. $\mathbb{F}_4$, see Lemma 4.18: if $\mu \notin \Phi_{\mathbb{F}'}(A)$, for a finite field $\mathbb{F}'$, and μ is $\mathbb{F}^*$-invariant with respect to a large finite extension $\mathbb{F}/\mathbb{F}'$ then there is a subgroup $C \simeq \mathbb{F}_p^2 \subset A$, (resp. $\mathbb{F}_4^2$), so that $\mu_C \notin \Phi_{\mathbb{F}_p}(C)$.
- reduction to dimension 3: if the rank two $\mathbb{Z}_\ell$-module $\sigma := \langle \mu, \mu' \rangle$ does not contain a flag map then there is a subgroup $B \simeq \mathbb{F}_p^3 \subset A$ (resp. $\mathbb{F}_4^3$), such that for any nontrivial $\mu'' \in \sigma$ there is a proper subspace $C = C_{\mu''} \subsetneq B$ where $\mu''_C \notin \Phi_{\mathbb{F}_p}(C)$ (this step uses the logarithmic property);
- geometry of collineations on $\mathbb{P}^2 = \mathbb{P}_{\mathbb{F}}(B)$ over prime fields $\mathbb{F} = \mathbb{F}_p$ (resp. $\mathbb{F}_4$): such subgroups B cannot exist. This shows the existence of the desired flag map on A.

□

LEMMA 4.8. — *If $A \in \mathrm{Vect}_{\mathbb{F}}$ and $\mu \in \Phi_{\mathbb{F}}(A)$ then there exists a canonical $\mathbb{F}$-flag $(A_n)_{n=0,\dots,d}$ such that*

$$\mu^{\mathrm{gen}}(A_n) \neq \mu^{\mathrm{gen}}(A_{n+1}),$$

for all $n = 0, \dots, d-1$.

Proof. — Put $A_0 = A$ and let A_{n+1} be the additive subgroup of A_n spanned by a with $\mu(a) \neq \mu^{\mathrm{gen}}(A_n)$. Since μ is $\mathbb{F}^*$-invariant, A_{n+1} is an $\mathbb{F}$-vector space. Indeed, for $a, a' \in A_{n+1}$ and $\kappa, \kappa' \in \mathbb{F}^*$ write

$$a = \sum_{i \in I} b_i, \quad a' = \sum_{j \in J} b'_j$$

with finite I, J. Since

$$\mu(b_i) \neq \mu^{\mathrm{gen}}(A_n), \quad \mu(b'_j) \neq \mu^{\mathrm{gen}}(A_n),$$

for all $i \in I, j \in J$, we have

$$\mu(\kappa b_i) = \mu(b_i) \neq \mu^{\mathrm{gen}}(A_n) \quad \text{and} \quad \mu(\kappa' b'_j) = \mu(b'_j) \neq \mu^{\mathrm{gen}}(A_n)$$

so that $\kappa a + \kappa' a' \in A_{n+1}$. $\square$

REMARK 4.9. — Since a flag map μ is $\mathbb{F}^*$-invariant, it defines a unique map on $(A \setminus 0)/\mathbb{F}^* = \mathbb{P}_{\mathbb{F}}(A)$. Conversely, a map μ on $\mathbb{P}_{\mathbb{F}}(A)$ gives rise to an $\mathbb{F}^*$-invariant maps on $A \setminus 0$. An $\mathbb{F}$-flag map on $A \in \mathrm{Vect}_{\mathbb{F}}$ defines a flag by projective subspaces on $\mathbb{P}_{\mathbb{F}}(A)$. We denote by *generic* elements of $\mathbb{P}_{\mathbb{F}}(A)$ the image of generic elements from A.

NOTATIONS 4.10. — We denote by $\hat{\mathbb{P}}(A) = \hat{\mathbb{P}}_{\mathbb{F}}(A)$ the set of codimension one projective $\mathbb{F}$-subspaces of $\mathbb{P}(A)$.

DEFINITION 4.11. — *Assume that $A \in \mathrm{Vect}_{\mathbb{F}}$, and for all codimension one $\mathbb{F}$-subspaces $B \subset A$ one has $\mu_B \in \Phi_{\mathbb{F}}(B)$. Define $\hat{\mu}$ by*

$$\begin{array}{rcl} \hat{\mathbb{P}}(A) & \to & \mathbb{Z}_\ell \\ \mathbb{P}(B) & \mapsto & \hat{\mu}(\mathbb{P}(B)) := \mu^{\mathrm{gen}}(B). \end{array}$$

LEMMA 4.12. — *If $A \in \mathrm{Vect}_{\mathbb{F}}$ and $\mu \in \Phi_{\mathbb{F}}(A)$ then either $\hat{\mu}$ is constant on $\hat{\mathbb{P}}(A)$ or it is constant on the complement to one point.*

Proof. — Consider the canonical flag $(A_n)_{n=0,\ldots,d}$. If $\mathrm{codim}(A_1) \geq 2$ then for every $\mathbb{P}(B) \in \hat{\mathbb{P}}(A)$ one has $\mu^{\mathrm{gen}}(B) = \mu^{\mathrm{gen}}(A)$ and $\hat{\mu}$ is constant. Otherwise, $\mu^{\mathrm{gen}}(B) = \mu^{\mathrm{gen}}(A)$, on any $B \neq A_1$ (and differs at $\mathbb{P}(A_1) \in \hat{\mathbb{P}}(A)$). $\square$

LEMMA 4.13. — *Let $\mathbb{F} = \mathbb{F}_q$ be a finite field with $q \geq 11$ and $\mathbb{P}^m = \mathbb{P}^m_{\mathbb{F}}$, $m \geq 2$ a projective space over $\mathbb{F}$. For any four distinct projective hyperplanes H_i, $i = 1, \ldots, 4$ and any projective hyperplane $B_i \subset H_i$ there exists a line $L = \mathbb{P}^1$ over $\mathbb{F}$ which does not intersect any of the B_i or $H_{ij} := H_i \cap H_j$, $i, j = 1, \ldots, 4$.*

Proof. — Consider first the case $m = 2$. Then H_i are lines and B_i, H_{ij} are points, all defined over $\mathbb{F}$. Consider the dual plane $\hat{\mathbb{P}}^2$. The line L in the lemma corresponds to a point $\hat{L} \in \hat{\mathbb{P}}^2$ which is not contained in any of the lines $\hat{H}_{ij}, \hat{B}_i$. The total number of points in the union of at most ten lines $\hat{H}_{ij}, \hat{B}_i$ is $\leq 10q + 1$. Such a point $\hat{L}$ exists if

$$q^2 + q + 1 - (10q + 1) = q^2 - 9q > 0,$$

e.g., $q \geq 11$. A similar estimate holds for $m > 2$ since we can reduce the case $\mathbb{P}^m, n > 2$ to $\mathbb{P}^{m-1}$. For this we need to find a hyperplane $\mathbb{P}^{m-1}$ which does not contain B_i and H_{ij}. By dualizing we need to find a point $\hat{L} \in \hat{\mathbb{P}}^m$ which is not contained in any of the lines $\hat{B}_i, \hat{H}_{ij}$. To satisfy the induction hypothesis we need $q^m + \ldots + q^2 - 9(q+1) > 0$ which holds for $m > 2$ and $q \geq 4$. The hypersurface $\mathbb{P}^{m-1}$ intersects H_i by different hyperplanes and any B_i or H_{ij} by a proper subspace of codimension 1, so that the obtained configuration on $\mathbb{P}^{m-1}$ satisfies the conditions of the lemma. □

LEMMA 4.14. — *Let $\mathbb{F} = \mathbb{F}_q$ be a finite field with $q \geq 11$, $A \in \mathrm{Vect}_{\mathbb{F}}$ and $\mu \in \mathcal{M}(A)$ an $\mathbb{F}^*$-invariant map. Assume that there exist $\mathbb{F}$-subspaces $B_i \subset A$, $\mathrm{codim}(B_i) = 1$, for $i = 1, \ldots, 4$ such that $\mu_{B_i} \in \Phi_{\mathbb{F}}(B_i)$, for all i, and*

(1) *either $\#\{\mu^{\mathrm{gen}}(B_i)\} \geq 3$ or*
(2) $\mu^{\mathrm{gen}}(B_1) = \mu^{\mathrm{gen}}(B_2) \neq \mu^{\mathrm{gen}}(B_3) = \mu^{\mathrm{gen}}(B_4)$.

Then there exists an $\mathbb{F}$-subspace $C \subset A$, $\dim_{\mathbb{F}}(C) = 2$ such that $\mu_C \notin \Phi_{\mathbb{F}}(C)$.

Proof. — By Lemma 4.13, there exists a $\mathbb{P}^1 = \mathbb{P}(C) \in \mathbb{P}(A)$ such that its intersection points with $\mathbb{P}(B_i)$ are pairwise distinct and generic in the corresponding $\mathbb{P}(B_i)$ (the nongeneric points of $\mathbb{P}(B_i)$ are contained in 4 subspaces in $\mathrm{codim}_{\mathbb{F}} \geq 2$, the intersections of B_i give rise to 6 more subspaces). Then either μ takes at least three distinct values on $\mathbb{P}(C)$ or has distinct values in at least two pairs of points. In both cases $\mu \notin \Phi_{\mathbb{F}}(C)$. □

COROLLARY 4.15. — *Assume that $\mu_B \in \Phi_{\mathbb{F}}(B)$ for all $\mathbb{P}(B) \in \hat{\mathbb{P}}(A)$ (and $\#\mathbb{F} \geq 11$). Then $\hat{\mu}$ is constant outside of one point.*

Proof. — The map $\hat{\mu}$ takes two different values on $\hat{\mathbb{P}}(B)$. By Lemma 4.14, among any three hyperplanes two have the same generic value, so that there can be at most two such values. If there are hyperplanes $h_1, h_2, h_3 \in \hat{\mathbb{P}}(A)$, where $\hat{\mu}(h_1) = \hat{\mu}(h_2) \neq \hat{\mu}(h_3)$ then for any other $h \in \hat{\mathbb{P}}(A)$ we have $\hat{\mu}(h) = \hat{\mu}(h_1)$ and $\hat{\mu}$ is constant outside of h_3. □

LEMMA 4.16. — *Let $A \in \mathrm{Vect}_{\mathbb{F}}$, with $\#\mathbb{F} \geq 11$, and $\mu \in \mathcal{M}(A)$ be an $\mathbb{F}^*$-invariant map such that for every two-dimensional $\mathbb{F}$-subspace $B \subset A$, $\mu_B \in \Phi_{\mathbb{F}}(B)$. Then $\mu \in \Phi_{\mathbb{F}}(A)$.*

Proof. — Assume the statement holds if $\dim(A) \leq n - 1$. Then $\hat{\mu}$ is defined and, by Corollary 4.15, either $\hat{\mu}$ is constant on $\hat{\mathbb{P}}(A)$ or constant on the complement to one point.

If $\hat{\mu}$ is constant, then the $\mathbb{F}$-linear envelope of points $b \in A$ such that $\mu(b) \neq \hat{\mu}$ has codimension at least two. Indeed, if there is a codimension one subspace $B \subset A$ generated by such b then by assumption $\mu \in \Phi_{\mathbb{F}}(B)$ and $\mu^{\mathrm{gen}}(B) \neq \hat{\mu}$, contradicting the assumption that $\hat{\mu}$ is constant. Otherwise, put $B := A_1$. By induction, $\mu \in \Phi_{\mathbb{F}}(B)$ and is constant on $A \setminus B$. Hence $\mu \in \Phi_{\mathbb{F}}(A)$.

Assume that $\hat{\mu}$ is nonconstant and let $B \subset A$ be the unique codimension one subspace with differing $\mu^{\mathrm{gen}}(B)$. Choose an $\mathbb{F}$-basis $b_1, \dots, b_{n-1}$ in B such that $\mu(b_i) = \mu^{\mathrm{gen}}(B)$. Assume that there is a point $a \in A \setminus B$ such that $\mu(a) \neq$ the generic value of $\hat{\mu}$ and let B' be the codimension one $\mathbb{F}$-subspace spanned $b_1, \dots, b_{n-2}, a$. Then $\mu^{\mathrm{gen}}(B') \neq$ the generic value of $\hat{\mu}$, contradicting the uniqueness of B. It follows that μ is constant on $A \setminus B$. □

REMARK 4.17. — Let $\mathbb{F}/\mathbb{F}'$ be a finite extension, $A \in \mathrm{Vect}_{\mathbb{F}}$, considered as an $\mathbb{F}'$-vector space, and $\mu \in \Phi_{\mathbb{F}'}(A)$. If μ is $\mathbb{F}^*$-invariant, then $\mu \in \Phi_{\mathbb{F}}(A)$. Indeed, by the proof of Lemma 4.8, the canonical $\mathbb{F}'$-flag is a flag of $\mathbb{F}$-vector spaces. We use this observation to reduce our problem to prime fields (resp. $\mathbb{F}_4$).

LEMMA 4.18. — *Let $\mathbb{F}/\mathbb{F}'$ be a quadratic extension, with $\#\mathbb{F}' > 2$. Let A be an $\mathbb{F}$-vector space of dimension 2, considered as an $\mathbb{F}'$-vector space of dimension 4. Let $\mu \in \mathcal{M}(A)$ be an $\mathbb{F}^*$-invariant map such that for every $\mathbb{F}'$-subspace $C \subset A$, $\dim_{\mathbb{F}'}(C) = 2$, one has $\mu_C \in \Phi_{\mathbb{F}'}(C)$. Then $\mu \in \Phi_{\mathbb{F}}(A)$.*

Proof. — First assume that μ takes only two values on $A \setminus 0$, say $0, 1$, and that $\mu \notin \Phi_{\mathbb{F}}(A)$. Since $\mathbb{P}_{\mathbb{F}}(A) = \mathbb{P}^1_{\mathbb{F}}$ there exist elements $a_1, a_2, a_3, a_4 \in A \setminus 0$ such that the orbits $\mathbb{F}^* \cdot a_i$ do not intersect and

$$0 = \mu(a_1) = \mu(a_2) \neq \mu(a_3) = \mu(a_4) = 1.$$

Then $\mathbb{F}^* \cdot a_i = \Lambda_i \setminus 0$, where Λ_i is a linear subspace over $\mathbb{F}'$. The $\mathbb{F}'$-span Λ_{12} of two nonzero vectors $x_1 \in \Lambda_1, x_2, \in \Lambda_2$ has $\mu^{\mathrm{gen}}(\Lambda_{12}) = 0$. Hence Λ_{12} contains at most one $\mathbb{F}'$-subspace $\langle b \rangle$ of $\mathbb{F}'$-dimension 1 with generic value 1. The union of the spaces Λ_{12}, for different choices of x_1, x_2, covers A and

$$\#\{b \in \mathbb{P}_{\mathbb{F}'}(A) \,|\, \mu(b) = 1\} \leq (q+1)^2,$$

where $\#\mathbb{F}' = q$. Similarly, there are at most $(q+1)^2$ such nongeneric $c \in \mathbb{P}_{\mathbb{F}'}(A)$ with $\mu(c) = 0$. Since $\#\mathbb{P}^3(\mathbb{F}') = q^3 + q^2 + q + 1 > 2(q^2 + 2q + 1)$, for $q > 2$, we get a contradiction.

Assume now that μ takes at least 3 distinct values on $A \setminus 0$, say $0, 1, 2$, and that there are two vectors $a_1, a_2 \subset A$ such that the orbits $\mathbb{F}^* \cdot a_1, \mathbb{F}^* \cdot a_2$

don't intersect and $0 = \mu(a_1) = \mu(a_2)$. Such a configuration must exist (take two $\mathbb{F}'$-spaces of $\mathbb{F}'$-dimension two spanned by $\mathbb{F}^*$-orbits; the $\mathbb{F}'$ span of two generic vectors in these spaces contains elements whose μ-value coincides with the value of μ on one of the two orbits). The modified map, given by

$$\tilde{\mu}(a) := \begin{cases} 0 & \text{if } \mu(a) = 0 \\ 1 & \text{otherwise} \end{cases},$$

satisfies the conditions of the Lemma, and by the above argument $\tilde{\mu} \in \Phi_{\mathbb{F}}(A)$. In particular, $\tilde{\mu} = 0$ outside one $\mathbb{F}^*$-orbit on $A \setminus 0$. Since μ is $\mathbb{F}^*$-invariant it follows that μ takes two values, and not three as we assumed. Contradiction. □

LEMMA 4.19. — *Let $\mathbb{F}' = \mathbb{F}_p$ (resp. $\mathbb{F}_4$), and $\mathbb{F}/\mathbb{F}'$ be an extension of degree divisible by 4. Consider $K = k(X)$ as an $\mathbb{F}$-vector space and let $\mu, \mu' \in \mathcal{L}_{\mathbb{F}}(K)$ be a c-pair such that the linear span $\sigma = \langle \mu, \mu', 1 \rangle_{\mathbb{Z}_\ell}$ does not contain a nonconstant $\Phi_{\mathbb{F}}$-map. Then there exist an $\mathbb{F}'$-subspace $B \subset K$ with $\dim_{\mathbb{F}'}(B) = 3$, two distinct $\mathbb{F}'$-subspaces $C, C' \subset B$ of dimension 2 and maps $\tilde{\mu}, \tilde{\mu}' \in \sigma$ such that*

- *$\tilde{\mu}_C \notin \Phi_{\mathbb{F}'}(C)$ and $\tilde{\mu}_{C'}$ is constant;*
- *$\tilde{\mu}'_{C'} \notin \Phi_{\mathbb{F}'}(C')$ and $\tilde{\mu}'_C$ is constant;*

In particular, for every (nonzero) map $\mu'' \in \sigma$ there exists an $\mathbb{F}'$-subspace $C'' \subset B$, $\dim_{\mathbb{F}'} C'' = 2$ with the property that $\mu''_{C''} \notin \Phi_{\mathbb{F}'}(C'')$.

Proof. — We consider K as an $\mathbb{F}$-vector space as well as an $\mathbb{F}'$-vector space. Let μ be an $\mathbb{F}^*$-invariant map on K. If μ were an $\mathbb{F}'$-flag map on every two-dimensional $\mathbb{F}'$-subspace of K then, by Lemma 4.18, μ would be an $\mathbb{F}$-flag map on every $\mathbb{F}$-subspace $B \subset K$ of $\dim_{\mathbb{F}} B = 2$. Since $\#\mathbb{F} \geq 11$ we could apply Lemma 4.16 and conclude that $\mu \in \Phi_{\mathbb{F}}(K)$.

Thus, since $\mu \notin \Phi_{\mathbb{F}}(K)$, there is an $\mathbb{F}'$-subspace $C \subset K$, $\dim_{\mathbb{F}'}(C) = 2$ such that $\mu_C \notin \Phi_{\mathbb{F}'}(C)$. If μ'_C is constant, put $\tilde{\mu}' := \mu$. Otherwise, using the c-pair property on C we find constants d_C, d'_C, d''_C, with $d'_C \neq 0$, such that

$$d_C\mu + d'_C\mu'_C = d''_C \ \text{ and put } \tilde{\mu}' = \mu' - \frac{d''_C - d_C\mu}{d'_C}.$$

Then $\tilde{\mu}'_C = 0$. Since the linear combination $\tilde{\mu}'$ is not a flag map, there exists a C', $\dim_{\mathbb{F}'}(C') = 2$, where $\tilde{\mu}' \notin \Phi_{\mathbb{F}'}(C')$. If $\mu_{C'}$ is constant, put $\tilde{\mu} := \mu$.

Otherwise, using the c-pair property on C' we find constants $d_{C'}, d'_{C'}, d''_{C'}$, with $d'_{C'} \neq 0$, such that

$$d_{C'}\mu + d'_{C'}\mu'_{C'} = d''_{C'} \ \text{ and put } \tilde{\mu} = \mu - \frac{d''_{C'} - d'_{C'}\tilde{\mu}'}{d_{C'}}.$$

Then $\tilde{\mu}_{C'} = 0$ and $\tilde{\mu}_C \notin \Phi_{\mathbb{F}'}(C)$ (since $\tilde{\mu}'_C$ is constant). Now put

$$B := C + \frac{c}{c'} \cdot C',$$

for some nonzero $c \in C$ and $c' \in C'$. Then $\dim_{\mathbb{F}'}(B) = 3$, the maps $\tilde{\mu}_B, \tilde{\mu}'_B$ are linearly independent, and they satisfy the required conditions. For $s \neq 0$, we have $s\tilde{\mu} + s'\tilde{\mu}' \notin \Phi_{\mathbb{F}'}(C)$. Otherwise, $s\tilde{\mu} + s'\tilde{\mu}' \notin \Phi_{\mathbb{F}'}(\frac{c}{c'} \cdot C')$. Note that the logarithmic property of the maps is used to reduce to dimension 3. □

A detailed analysis of c-pairs on projective planes as above shows that such planes cannot exist. This will complete the proof of the main theorem.

LEMMA 4.20 (Lemma 4.3.2 in [3]). — *Let $V \subset \mathbb{Z}_\ell^2$ be such that for any two pairs of distinct points the affine line through one pair and the affine line through the other have a common point and that this point of intersection is contained in V. Then V is contained in a line union one point.*

Proof. — Otherwise, V contains four points in general position. Embed $\mathbb{Z}_\ell^2$ into $\mathbb{P}^2(\mathbb{Q}_\ell)$, choose coordinates for these four points

$$(1:0:0), (0:1:0), (0:0:1) \text{ and } (1:1:1)$$

and close V for the operation

$$x, y, z, t \mapsto \mathfrak{l}(x,y) \cap \mathfrak{l}(z,t), \quad \text{when } \ x \neq y, z \neq t, \mathfrak{l}(x,y) \neq \mathfrak{l}(z,t).$$

The closure $\bar{V}$ of V satisfies the axioms of a projective plane (see Definition 3.1). For example, to verify that any "line" in $\bar{V}$ contains at least three points it suffices to pick one of the four initial points not on this line and to draw lines through this point and the remaining three points in the initial set.

By the fundamental theorem of projective geometry, $\bar{V} = \mathbb{P}^2(\mathbb{Q})$. On the other hand, $\mathbb{P}^2(\mathbb{Q})$ is dense in $\mathbb{P}^2(\mathbb{Q}_\ell)$. In particular, it cannot be contained in $\mathbb{Z}_\ell^2$. Contradiction. □

COROLLARY 4.21. — *Let $B = \mathbb{F}^3$ and $\mu, \mu' \in \mathcal{M}(B)$ be a c-pair of $\mathbb{F}^*$-invariant maps. Then the image of $\mathbb{P}(B)$ under the map*

$$\begin{array}{rcccc} \varphi & : & \mathbb{P}(B) & \to & \mathbb{A}^2(\mathbb{Z}_\ell) \\ & & b & \mapsto & (\mu(b), \mu'(b)) \end{array}$$

is contained in a union of an affine line and (possibly) one more point.

Proof. — The c-pair condition for μ, μ' implies that the image of every $\mathbb{P}^1 \subset \mathbb{P}(B)$ is contained in an affine line in $\mathbb{Z}_\ell^2$. Next, for any two pairs of distinct points $(a, b), (a', b')$ in $\varphi(\mathbb{P}(B))$ the affine lines $\mathfrak{l} = \mathfrak{l}(a, b), \mathfrak{l}' = \mathfrak{l}'(a', b')$ in $\mathbb{A}^2 = \mathbb{Z}_\ell^2$ through these pairs of points must intersect. (Choose $\tilde{a}, \tilde{b}, \tilde{a}', \tilde{b}'$ in the preimages of a, b, a', b'; the projective lines $\tilde{\mathfrak{l}}, \tilde{\mathfrak{l}}' \subset \mathbb{P}(B) = \mathbb{P}^2$ through these points intersect in some x and, by the first observation, $\varphi(x)$ must lie on both $\mathfrak{l}$ and $\mathfrak{l}'$). Now it suffices to apply Lemma 4.20. □

ASSUMPTION 4.22. — *We may now assume that*

- *$\mathbb{F} = \mathbb{F}_p$ or $\mathbb{F}_4$;*
- *$\mu, \mu' \in \mathcal{L}_{\mathbb{F}}(A)$ is a c-pair of linearly independent maps as in Lemma 4.19,*
- *B is as in Lemma 4.19: for every two-dimensional $C'' \subset B$ there exists a $\mu'' \in \langle \mu, \mu' \rangle$ such that $\mu''_{C''} \notin \Phi_{\mathbb{F}}(C'')$.*

We can exclude the following degenerate cases, which contradict our assumption that no linear combination of μ, μ' is a flag map on B:

1. *$\varphi(\mathbb{P}(B))$ is contained in a line; this means that μ, μ' are linearly dependent (modulo constants);*
2. *$\varphi(\mathfrak{l})$ is a point, for some $\mathfrak{l} \subset \mathbb{P}(B)$; this implies that $\varphi(\mathfrak{l}) \in \varphi(\mathfrak{l}')$, for all $\mathfrak{l}' \subset \mathbb{P}(B)$ and $\varphi(\mathbb{P}(B))$ is contained in a line, contradiction to (1);*
3. *φ is constant outside one line; here the affine map $\mathbb{Z}_\ell^2 \to \mathbb{Z}_\ell$ projecting $\varphi(\mathfrak{l})$ to one point gives a nontrivial flag map in the span of μ, μ'.*

LEMMA 4.23. — *Let $\mathfrak{l}, \mathfrak{l}' \subset \mathbb{P}^2$ be distinct lines. Let $x \in \mathbb{P}^2$ be a point such that $\varphi(x) \notin (\varphi(\mathfrak{l}) \cup \varphi(\mathfrak{l}'))$. Then there is a natural projective isomorphism $\pi_{x,\mathfrak{l}'} : \mathfrak{l} \to \mathfrak{l}'$ respecting the level sets of φ. Namely, for every pair $y_1, y_2 \in \mathfrak{l}$ with $\varphi(y_1) = \varphi(y_2)$ one has*

$$\varphi(\pi_{x,\mathfrak{l}'}(y_1)) = \varphi(\pi_{x,\mathfrak{l}'}(y_2))$$

(and vice versa). In particular, if $\varphi(\mathfrak{l}) \subset \varphi(\mathfrak{l}')$ then $\varphi(\mathfrak{l}) = \varphi(\mathfrak{l}')$.

Proof. — The images $\varphi(\mathfrak{l}(x, y_1))$ and $\varphi(\mathfrak{l}(x, y_2))$ span the same affine line L_x. We have $\varphi(\mathfrak{l}') \not\subset L_x$. Define $\pi_{x,\mathfrak{l}'}(y_i) := \mathfrak{l}(x, y_i) \cap \mathfrak{l}'$. By Corollary 4.21, $\varphi(\pi_{x,\mathfrak{l}'}(y_i))$ are contained in the intersection of $\varphi(\mathfrak{l}')$ and L_x, so that $\varphi(\pi_{x,\mathfrak{l}'}(y_1)) = \varphi(\pi_{x,\mathfrak{l}'}(y_2))$. □

COROLLARY 4.24. — *If there exist a line $\mathfrak{l} \subset \mathbb{P}^2$ and a point $x \in \mathfrak{l}$ such that φ is constant on $\mathfrak{l} \setminus x$ then there is a nontrivial flag map in the span of μ, μ'.*

Proof. — By Assumption 4.22, φ is nonconstant on every line. Assume that there exists a point $a \in \varphi(\mathbb{P}^2)$ such that $\varphi^{-1}(a)$ consists exactly of x. Then for all $\mathfrak{l}', \mathfrak{l}''$ not containing x one has $\varphi(\mathfrak{l}') = \varphi(\mathfrak{l}'')$ and φ is constant on the complement to x on every line through x. Then a linear combination of μ, μ' is constant on $\mathbb{P}^2 \setminus x$, thus a flag map, contradicting the assumption.

Let x' be a point in $\mathbb{P}^2 \setminus \mathfrak{l}$ with $\varphi(x') = \varphi(x)$. The lines $\mathfrak{l}$ and $\mathfrak{l}(x, x')$ are not equivalent, $\varphi(\mathfrak{l}) \neq \varphi(\mathfrak{l}')$. For any line $\mathfrak{l}'' \neq \mathfrak{l}(x, x')$ through x' we have $\varphi(\mathfrak{l} \cap \mathfrak{l}'') \neq \varphi(x)$. Using a point on $y \in \mathfrak{l}'$ with $\varphi(y) \neq \varphi(x)$ and applying Lemma 4.23 we find that $\varphi(\mathfrak{l}'') = \varphi(\mathfrak{l})$. For any $y \notin (\mathfrak{l} \cup \mathfrak{l}')$ consider the line $\mathfrak{l}(x', y)$. It follows that $\varphi(y)$ equals the value of φ on $\mathfrak{l} \setminus x$, thus φ is constant on the complement to $\mathfrak{l}'$, contradicting Assumption 4.22(3). □

COROLLARY 4.25. — *Let $x, y \in \mathbb{P}^2_{\mathbb{F}}$ be distinct points so that $\varphi(x), \varphi(y) \notin (\varphi(\mathfrak{l}) \cup \varphi(\mathfrak{l}'))$ and the line $\mathfrak{l}(x, y)$ through x, y passes through the intersection $q_0 := \mathfrak{l} \cap \mathfrak{l}'$. Then the composition*

$$\pi_{x,\mathfrak{l}} \circ \pi_{y,\mathfrak{l}}^{-1} \; : \; \mathfrak{l} \to \mathfrak{l}$$

induces a nontrivial translation on $\mathfrak{l}$, with fixed point q_0, preserving the level sets of φ. (By symmetry we have a similar translation on $\mathfrak{l}'$.)

In particular, if $\mathbb{F} = \mathbb{F}_p$ (the prime field) then the group generated by this translation is transitive on $\mathfrak{l} \setminus (\mathfrak{l} \cap \mathfrak{l}')$ and φ is constant on this complement. If $\mathbb{F} = \mathbb{F}_4$ then the complement $\mathfrak{l} \setminus (\mathfrak{l} \cap \mathfrak{l}')$ is a union of two (two point) orbits of this translation and φ is constant on each orbit.

Proof of Theorem 4.7. — We keep the Assumptions 4.22.

For every point $x \in \mathbb{P}^2$ and every line $\mathfrak{l}$ through x there exist lines $\mathfrak{l}', \mathfrak{l}''$ through x such that $\varphi(\mathfrak{l}) = \varphi(\mathfrak{l}')$ and $\varphi(\mathfrak{l}') \neq \varphi(\mathfrak{l}'')$. Indeed, consider a line $\tilde{\mathfrak{l}}$ with $\varphi(x) \notin \varphi(\tilde{\mathfrak{l}})$. If on all such lines φ takes more than two values, then all these lines are equivalent and φ is constant on the complement to x on every line through x, contradiction to Corollary 4.24. Otherwise, each value on $\tilde{\mathfrak{l}}$ will be taken at least twice, hence the claim.

Corollary 4.25 gives a translation on $\mathfrak{l} \setminus x$ preserving the level sets of φ. Over the prime field $\mathbb{F}_p$, $p > 2$, φ restricted to $\mathfrak{l}$ is constant on the complement to x and we can apply Corollary 4.24.

Over $\mathbb{F}_4$, φ is either constant on $\mathfrak{l} \setminus x$, contradicting Corollary 4.24, or the level sets of φ on $\mathfrak{l} \setminus x$ fall into two orbits of cardinality two. Since we can pick x on $\mathfrak{l}$ arbitrarily, φ must be constant on $\mathfrak{l}$, contradicting Assumption 4.22(2). □

5. Galois groups

Let k be an algebraic closure of a finite field of characteristic $\neq \ell$, K the function field of an algebraic variety X over k, $\mathcal{G}^a_K$ the abelianization of the pro-ℓ-quotient $\mathcal{G}_K$ of the Galois group G_K of a separable closure of K,

$$\mathcal{G}^c_K = \mathcal{G}_K/[[\mathcal{G}_K, \mathcal{G}_K], \mathcal{G}_K] \xrightarrow{\mathrm{pr}} \mathcal{G}^a_K$$

its canonical central extension and pr the natural projection.

DEFINITION 5.1. — *We say that $\gamma, \gamma' \in \mathcal{G}^a_K$ form a* commuting pair *if for some (and therefore any) of their preimages $\tilde{\gamma}, \tilde{\gamma}'$ in $\mathcal{G}^c_K$ one has $[\tilde{\gamma}, \tilde{\gamma}'] = 0$. A subgroup $\mathcal{H}$ of $\mathcal{G}^a$ is called* liftable *if any two elements in $\mathcal{H}$ form a commuting pair.*

DEFINITION 5.2. — *The* fan *$\Sigma_K = \{\sigma\}$ on $\mathcal{G}^a_K$ is the set of all topologically noncyclic liftable subgroups $\sigma \subset \mathcal{G}^a_K$ which are not properly contained in any other liftable subgroup of $\mathcal{G}^a_K$.*

REMARK 5.3. — For function fields K/k of surfaces, all groups $\sigma \in \Sigma_K$ are primitive $\mathbb{Z}_\ell$-submodules of rank 2 of the torsion free $\mathbb{Z}_\ell$-module $\mathcal{G}^a_K$ (see Proposition 8.3).

NOTATIONS 5.4. — Let

$$\boldsymbol{\mu}_{\ell^n} := \{ \sqrt[\ell^n]{1} \} \quad \text{and} \quad \mathbb{Z}_\ell(1) = \varprojlim \boldsymbol{\mu}_{\ell^n}.$$

We often identify $\mathbb{Z}_\ell$ and $\mathbb{Z}_\ell(1)$ (since k is algebraically closed). Write

$$\hat{K}^* := \varprojlim K^*/(K^*)^{\ell^n}$$

for the multiplicative group of (formal) rational functions on X.

THEOREM 5.5 (Kummer theory). — *The group K^*/k^* is a free $\mathbb{Z}$-module. One has*

- $K^*/(K^*)^{\ell^n} = (K^*/k^*)/\ell^n$, *for all* $n \in \mathbb{N}$ *;*
- *the discrete group* $K^*/(K^*)^{\ell^n}$ *and the compact profinite group* $\mathcal{G}^a_K/\ell^n$ *are Pontryagin dual to each other, for a* $\boldsymbol{\mu}_{\ell^n}$*-duality;*
- *for* $K^*/k^* \xrightarrow{\sim} \mathbb{Z}^{(I)}$, *one has* $K^*/(K^*)^{\ell^n} \xrightarrow{\sim} (\mathbb{Z}/\ell^n)^{(I)}$ *and*

$$\mathcal{G}^a_K/\ell^n \xrightarrow{\sim} (\mathbb{Z}/\ell^n(1))^I,$$

hence the duality between $\hat{K}^* = \widehat{K^*/k^*}$ *and* $\mathcal{G}^a_K$ *is modeled on that between*

$$\{\textit{functions } I \to \mathbb{Z}_\ell \textit{ tending to } 0 \textit{ at } \infty\} \textit{ and } \mathbb{Z}_\ell^I.$$

LEMMA 5.6. — *Let E/k be the function field of a curve. Then $\Sigma_E = \emptyset$.*

Proof. — By a result of Grothendieck, the pro-ℓ fundamental group $\hat{\pi}_{1,\ell}$ of a curve punctured in finitely many points is free. We have

$$\mathcal{G}^a_E = \lim_{\substack{\longleftarrow \\ J \subset I}} \mathbb{Z}_\ell^J, \quad \mathrm{Ker}(\mathcal{G}^c_E \to \mathcal{G}^a_E) = \lim_{\substack{\longleftarrow \\ J \subset I}} \wedge^2(\mathbb{Z}_\ell^J),$$

with the commutation map equal to $\wedge$. This implies that a liftable subgroup of $\mathcal{G}^a_E$ is topologically cyclic. □

6. Valuations

In this section we recall basic results concerning valuations and valued fields (we follow [4]). Most of this material is an adaptation of well-known facts to our context.

NOTATIONS 6.1. — A *value group*, denoted by Γ, is a totally ordered (torsion-free) abelian group. We use the additive notation "+" for the group law and $\geq$ for the order. We have

$$\Gamma = \Gamma^+ \cup \Gamma^-, \ \ \Gamma^+ \cap \Gamma^- = \{0\} \ \text{ and } \ \gamma \geq \gamma' \ \text{ iff } \ \gamma - \gamma' \in \Gamma^+.$$

Then $\Gamma_\infty = \Gamma \cup \{\infty\}$ is a totally ordered monoid, by the conventions

$$\gamma < \infty, \ \gamma + \infty = \infty + \infty = \infty, \ \ \forall \gamma \in \Gamma.$$

DEFINITION 6.2. — *A (nonarchimedean)* valuation *on a field K is a pair $\nu = (\nu, \Gamma_\nu)$ consisting of a value group Γ_ν and a map*

$$\nu : K \to \Gamma_{\nu,\infty}$$

such that

- $\nu : K^* \to \Gamma_\nu$ *is a surjective homomorphism;*
- $\nu(\kappa + \kappa') \geq \min(\nu(\kappa), \nu(\kappa'))$ *for all $\kappa, \kappa' \in K$;*
- $\nu(0) = \infty$.

REMARK 6.3. — In particular, since Γ_ν has no torsion, $\nu(\zeta) = 0$ for every element ζ of finite order in K^*.

A valuation is called *trivial* if $\Gamma = \{0\}$. If $K = k(X)$ is a function field over an algebraic closure k of a finite field then every valuation of K restricts to a trivial valuation on k (every element in k^* is torsion).

LEMMA 6.4. — *Let $K = k(X)$ and ν be a nonarchimedean valuation on $k(X)$. Then $\mathrm{Hom}(\Gamma_\nu, \mathbb{Z}_\ell)$ is a finitely generated $\mathbb{Z}_\ell$-module.*

Proof. — Note that the $\mathbb{Q}$-rank of ν is bounded by $\dim(X)$ (see [11]). □

NOTATIONS 6.5. — We denote by K_ν, $\mathfrak{o}_\nu, \mathfrak{m}_\nu$ and $\boldsymbol{K}_\nu$ the completion of K with respect to ν, the ring of ν-integers in K, the maximal ideal of $\mathfrak{o}_\nu$ and the residue field

$$\boldsymbol{K}_\nu := \mathfrak{o}_\nu / \mathfrak{m}_\nu.$$

If X (over k) is a model for K then the *center* $\mathfrak{c}(\nu)$ of a valuation is the irreducible subvariety Y whose trace on any affine chart $U \subset X$ for which $U \cap Y \neq \emptyset$ is defined by the prime ideal $\mathfrak{m}_\nu \cap k[U]$.

It is useful to keep in mind the following exact sequences:

$$1 \to \mathfrak{o}_\nu^* \cap K^* \to K^* \to \Gamma_\nu \to 1 \qquad (6.1)$$

and

$$1 \to (1 + \mathfrak{m}_\nu) \to \mathfrak{o}_\nu^* \to \boldsymbol{K}_\nu^* \to 1. \qquad (6.2)$$

NOTATIONS 6.6. — Write $\mathcal{I}_\nu^a \subset \mathcal{D}_\nu^a \subset \mathcal{G}_K^a$ for the images of the inertia and the decomposition group of the valuation ν in $\mathcal{G}_K^a$.

NOTATIONS 6.7. — If $\chi : \Gamma_\nu \to \mathbb{Z}_\ell(1)$ is a homomorphism then

$$\chi \circ \nu \; : \; K^* \to \mathbb{Z}_\ell(1)$$

defines an element of $\mathcal{G}^a_K$, called an inertia element of the valuation ν. The group of such elements is $\mathcal{I}^a_\nu \subset \mathcal{G}^a_K$.

NOTATIONS 6.8. — The decomposition group $\mathcal{D}^a_\nu$ is by definition equal to the image of $\mathcal{G}^a_{K_\nu}$ in $\mathcal{G}^a_K$.

LEMMA 6.9. — *There is a natural embedding $\mathcal{G}^a_{K_\nu} \hookrightarrow \mathcal{G}^a_K$ and a (canonical) isomorphism*

$$\mathcal{D}^a_\nu / \mathcal{I}^a_\nu \simeq \mathcal{G}^a_{\boldsymbol{K}_\nu}.$$

Proof. — See Theorem 19.6 in [6], for example. □

DEFINITION 6.10. — *Let $K = k(X)$ be a function field. The dimension of a valuation ν is defined as $\operatorname{tr\,deg}_k \boldsymbol{K}_\nu$. A valuation is called* divisorial *if* $\operatorname{tr\,deg}_k \boldsymbol{K}_\nu = \dim(X) - 1$.

NOTATIONS 6.11. — We let $\mathcal{V}_K$ be the set of all nontrivial (nonarchimedean) valuations of K and $\mathcal{DV}_K$ the subset of divisorial valuations. If $\nu \in \mathcal{DV}_K$ is realized by a divisor D on a model X of K (see Example 6.13) we sometimes write $\mathcal{I}^a_D$, resp. $\mathcal{D}^a_D$, for the corresponding inertia, resp. decomposition group.

EXAMPLE 6.12. — Let $E = k(C)$ be the function field of a smooth curve. Every point $q \in C(k)$ defines a nontrivial valuation ν_q on E (the order of a function $f \in E^*$ at q). Conversely, every nontrivial valuation ν on E defines a point $q := \mathfrak{c}(\nu)$ on C.

EXAMPLE 6.13. — Let $K = k(X)$ be the function field of a surface.

- Every positive-dimensional valuation is divisorial.
- Every (irreducible) curve $D \subset X$ defines a valuation ν_D on K with value group $\mathbb{Z}$. Conversely, every valuation on K with value group $\mathbb{Z}$ and algebraically nonclosed residue field defines a curve D on some model X of K.
- Every flag (D, q), (curve, point on its normalization), defines a valuation $\nu_{D,q}$ on K with value group $\mathbb{Z}^2$.
- There exist valuations on K with value group $\mathbb{Q}$ and center supported in a point (on every model).

LEMMA 6.14. — *Let $K = k(X)$ be the function field of a surface. If $\mathcal{D}^a_\nu/\mathcal{I}^a_\nu$ is nontrivial then ν is divisorial.*

Proof. — The only positive-dimensional valuations on function fields of surfaces are divisorial valuations. For other valuations, the residue field $\boldsymbol{K}_\nu = k$ is algebraically closed and $\mathcal{G}^a_{\boldsymbol{K}_\nu}$ trivial. □

7. A dictionary

Write

$$\begin{array}{lllll} \mathcal{L}_K & := & \mathcal{L}_k(K) & = & \{\text{ homomorphisms } K^* \to \mathbb{Z}_\ell(1)\} \\ \Phi_K & := & \Phi_k(K) & = & \{\text{ flag maps } K \to \mathbb{Z}_\ell(1)\} \end{array}$$

PROPOSITION 7.1. — *One has the following identifications:*

$$\begin{array}{lll} \mathcal{G}^a_K & = & \mathcal{L}_K, \\ \mathcal{D}^a_\nu & = & \{\mu \in \mathcal{L}_K \,|\, \mu \text{ trivial on } (1+\mathfrak{m}_\nu)\cap K^*\}, \\ \mathcal{I}^a_\nu & = & \{\mu \in \mathcal{L}_K \,|\, \mu \text{ trivial on } \mathfrak{o}^*_\nu \cap K^*\}. \end{array}$$

If two nonproportional $\mu, \mu' \in \mathcal{G}^a_K$ form a commuting pair then the corresponding maps $\mu, \mu' \in \mathcal{L}_K$ form a c-pair (in the sense of Definition 4.6).

Proof. — The first identification is a consequence of Kummer theory 5.5. The second identification can be checked on one-dimensional subfields of K, where it is evident. For this and the third identification we use (6.1) and (6.2). For the last statement, assume that $\mu, \mu' \in \mathcal{L}_K$ don't form a c-pair. Then there is an $x \in K$ such that the restrictions of $\mu, \mu' \in \mathcal{L}_K$ to the subgroup $\langle 1, x\rangle$ are linearly independent. Therefore, $\mu, \mu' \in \mathcal{G}^a_K$ define a rank 2 liftable subgroup in $\mathcal{G}^a_{k(x)}$. Such subgroups don't exist since $\mathcal{G}_{k(x)}$ is a free pro-ℓ-group (see [10]). □

COROLLARY 7.2. — *The subgroup $\mathcal{D}^a_\nu \subset \mathcal{G}^a_K$ is primitive.*

Proof. — Indeed, $\mathcal{D}^a_\nu$ is torsion free, and the description in Proposition 7.1 implies that if a multiple of an element is in $\mathcal{D}^a_\nu$ then so is the element itself. □

EXAMPLE 7.3. — If $\mu \in \mathcal{D}^a_\nu$ and $\alpha \in \mathcal{I}^a_\nu$ then μ, α form a commuting pair.

PROPOSITION 7.4. — *Let K be a field and $\alpha \in \Phi_K \cap \mathcal{L}_K$. Then there exists a unique valuation $\nu = (\nu_\alpha, \Gamma_{\nu_\alpha})$ (up to equivalence) and a homomorphism $\mathrm{pr} : \Gamma_{\nu_\alpha} \to \mathbb{Z}_\ell(1)$ such that*

$$\alpha(f) = \mathrm{pr}(\nu_\alpha(f))$$

for all $f \in K^*$. In particular, $\alpha \in \mathcal{I}^a_\nu$ (under the identification of Proposition 7.1).

Proof. — Let $\mathbb{F}$ be a finite subfield of k and assume that $\alpha(f) \neq \alpha(f')$ for some $f, f' \in K$ and consider the line $\mathbb{P}^1 = \mathbb{P}(\mathbb{F}f + \mathbb{F}f')$. Since α is a flag map, it is constant outside one point on this $\mathbb{P}^1$ so that either $\alpha(f+f') = \alpha(f)$ or $= \alpha(f')$. This defines a relation: $f' >_\alpha f$ (in the first case) and $f >_\alpha f'$ (otherwise). If $\alpha(f) = \alpha(f')$ and there exists an f'' such that $\alpha(f) \neq \alpha(f'')$ and $f >_\alpha f'' >_\alpha f'$ then we put $f >_\alpha f'$. Otherwise, we put $f =_\alpha f'$.

It was proved in [3], Section 2.4, that the above definitions are correct and that $>_\alpha$ is indeed an order which defines a filtration on the additive group K by subgroups $(K_\gamma)_{\gamma \in \Gamma}$ such that

- $K = \cup_{\gamma\in\Gamma} K_\gamma$ and
- $\cap_{\gamma\in\Gamma} K_\gamma = \emptyset$,

where Γ is the set of equivalence classes with respect to $=_\alpha$. Since $\alpha \in \mathcal{L}_K$ this order is compatible with multiplication in K^*, so that the map $K \to \Gamma$ is a valuation and α factors as $K^* \to \Gamma \to \mathbb{Z}_\ell \simeq \mathbb{Z}_\ell(1)$. By (6.1), $\alpha \in \mathcal{I}^a_\nu$. □

COROLLARY 7.5. — *Every (topologically) noncyclic liftable subgroup of $\mathcal{G}^a_K$ contains an inertia element of some valuation.*

Proof. — By Theorem 4.7, every such liftable subgroup contains an Φ-map, which by Proposition 7.4 belongs to some inertia group. □

8. Flag maps and valuations

In this section we give a Galois-theoretic description of inertia and decomposition subgroups of divisorial valuations.

LEMMA 8.1. — *Let $\alpha \in \Phi_K \cap \mathcal{L}_K$, $\nu = \nu_\alpha$ be the associated valuation and $\mu \in \mathcal{L}_K$. Assume that α, μ form a c-pair. Then*

$$\mu(1 + \mathfrak{m}_\nu) = \mu(1) = 0.$$

In particular, the restriction of μ to $\mathfrak{o}_\nu$ is induced from $\boldsymbol{K}_\nu$.

Proof. — First of all, $\mu(1) = 0$, since μ is logarithmic. We have

(1) $\alpha(\kappa) = 0$ for all $\kappa \in \mathfrak{o}_\nu \setminus \mathfrak{m}_\nu$;
(2) $\alpha(\kappa + m) = \alpha(\kappa)$ for all $\kappa \in \mathfrak{o}_\nu \setminus \mathfrak{m}_\nu$ and $m \in \mathfrak{m}_\nu$ as above;
(3) $\mathfrak{m}_\nu$ is additively generated by $m \in \mathfrak{o}_\nu$ such that $\alpha(m) \neq 0$.

If $m \in \mathfrak{m}_\nu$ is such that $\alpha(m) \neq 0$ and $\kappa \in \mathfrak{o}_\nu \setminus \mathfrak{m}_\nu$ then α is nonconstant on the subgroup $A := \langle \kappa, m \rangle$. Then

$$\mu(\kappa + m) = \mu(\kappa).$$

Indeed, if μ is nonconstant on A the restriction μ_A is proportional to α_A (by the c-pair property) and α satisfies (2). In particular, for such m we have $\mu(1 + m) = \mu(1) = 0$.

Assume that $\alpha(m) = 0$. Then there exists an $m' \in \mathfrak{m}_\nu$ such that $m > m' > 1$ and $\alpha(m') \neq \alpha(1) = 0$. Using the first step with $\kappa = 1$ and observing that $\alpha(m+m') \neq 0$ we have $\mu(1+m+m') = \mu(1) = 0$. On the other hand, putting $\kappa = 1 + m$ and using that $\alpha(m') \neq 0$ we see that $\mu(1 + m + m') = \mu(1 + m)$. Thus $0 = \mu(1) = \mu(1 + m)$ as claimed. □

COROLLARY 8.2. — *Nontrivial inertia elements $\alpha \in \mathcal{I}^a_\nu$ commute only with elements $\mu \in \mathcal{D}^a_\nu$.*

PROPOSITION 8.3. — *Let $K = k(X)$ be the function field of a surface. Every $\sigma \in \Sigma_K$ is a primitive $\mathbb{Z}_\ell$-lattice of rank 2. Moreover, it defines a unique valuation $\nu = \nu_\sigma$ of K so that either every element of σ is inertial for ν, or ν is divisorial and there is an element $\mu \in \sigma$ which is not inertial for ν, but $\mu \in \mathcal{D}^a_\nu$.*

If distinct $\sigma, \sigma' \in \Sigma_K$ have a nonzero intersection then there exists a divisorial valuation ν'' such that

– *$\sigma, \sigma' \in \mathcal{D}^a_{\nu''}$;*
– *$\sigma \cap \sigma' = \mathcal{I}^a_{\nu''}$.*

Conversely, if $\sigma \in \Sigma_K$ is not contained in a $\mathcal{D}^a_{\nu''}$ for any divisorial valuation ν'' then for all $\sigma' \in \Sigma_K$, $\sigma' \neq \sigma$, one has $\sigma \cap \sigma' = 0$.

Proof. — We saw that $\sigma \in \Sigma_K$ contains an inertia element α for *some* valuation ν. Since σ is topologically noncyclic there is a $\mu \in \sigma$, $\mathbb{Z}_\ell$-independent on α, and commuting with α. If μ is not inertial, that is, $\mu \notin \Phi_K$, then μ gives a nontrivial element in the (abelianized) Galois group of the residue field $\boldsymbol{K}_\nu$ of ν. Thus ν is divisorial. Every rank two subgroup of σ containing μ also contains an inertia element, which must be an inertial element of *some* divisorial valuation. However, inertial elements of different divisorial valuations don't commute: the decomposition group of a divisorial valuation cannot contain nontrivial inertia elements of another divisorial valuation (geometrically, there exists a morphism from some model $X \to \mathbb{P}^1$ which maps the corresponding divisors to different points on $\mathbb{P}^1$ and hence a map from the ℓ-Galois group $\mathcal{G}_K$ onto a free pro-ℓ group). This proves that in this case $\mathrm{rk}_{\mathbb{Z}_\ell}\,\sigma = 2$, and that, by Corollary 8.2, $\mu \in \mathcal{D}^a_\nu$. Moreover, since $\mathcal{D}^a_\nu$ is primitive in $\mathcal{G}^a_K$ (see Corollary 7.2), a maximal σ as above, i.e., $\sigma \subset \mathcal{D}^a_\nu$, contains $\mathcal{I}^a_\nu$ and is a preimage of a primitive cyclic subgroup of $\mathcal{D}^a_\nu/\mathcal{I}^a_\nu$, hence a primitive subgroup of $\mathcal{G}^a_K$.

If σ contains *only* inertia elements, then there exists a unique valuation ν such that $\sigma \subset \mathcal{I}^a_\nu$ (note that maximal such σ are again primitive). Indeed, either $\mathfrak{m}_\nu + \mathfrak{m}_{\nu'} = K$ or we may assume that $\mathfrak{m}_\nu \subset \mathfrak{m}_{\nu'}$ (and $\mathfrak{o}_\nu \supset \mathfrak{o}_{\nu'}$). The first case is impossible since the corresponding inertia groups don't intersect. In the second case, $\mathcal{I}^a_\nu \subset \mathcal{I}^a_{\nu'}$, as claimed. Moreover, it follows that $\mathrm{rk}_{\mathbb{Z}_\ell}\,\sigma = 2$, since the $\mathbb{Q}$-rank of any valuation on a surface (over $\bar{\mathbb{F}}_q$) is at most two. This gives of $\nu = \nu_\sigma$ in this case.

If distinct σ, σ' have a nontrivial intersection, then the subgroup $\mathcal{D} \subset \mathcal{G}^a_K$ generated by σ, σ' is not the inertia group of any valuation (the rank of those is ≤ 2, as we have seen above). If the $\sigma \cap \sigma'$ contains a nontrivial inertia element α then $\mathcal{D}$ is contained in the decomposition group of this element (all elements of $\mathcal{D}$ commute with α) and the corresponding valuation is divisorial. If $\mu \in \sigma \cap \sigma'$ is not an inertia element then there exist inertia elements $\alpha \in \sigma$ and $\alpha' \in \sigma'$ corresponding to distinct *divisorial* valuations ν, ν'. The decomposition groups of distinct divisorial valuations don't intersect. □

Proposition 8.3 allows us to identify intrinsically (in terms of the Galois group) inertia subgroups of divisorial valuations as well as their decomposition groups as follows. Every pair of distinct groups $\sigma, \sigma' \in \Sigma_K$ with a nontrivial intersection defines a divisorial valuation ν, whose inertia group

$$\mathcal{I}^a_\nu = \sigma \cap \sigma'.$$

The corresponding decomposition subgroup is

$$\mathcal{D}^a_\nu = \cup_{\sigma \supset \mathcal{I}^a_\nu} \sigma.$$

DEFINITION 8.4. — *Let* $\tau : \mathcal{G}^a_K \to \mathcal{G}^{un}_K$ *be the maximal topological quotient of* $\mathcal{G}^a_K$ *with the property that* $\tau(\mathcal{I}^a_\nu) = 0 \in \mathcal{G}^{un}_K$*, for all* $\nu \in \mathcal{DV}_K$*. Let* $\mathcal{G}^{ram}_K := \ker(\tau)$ *be the kernel. We call these groups the* unramified*, resp.* ramified*, Galois groups.*

LEMMA 8.5. — *Let X be a smooth model of K. Then $\hat{\pi}^a_{1,\ell}(X)$, the ℓ-quotient of the abelianization of the algebraic fundamental group of X, is canonically isomorphic to $\mathcal{G}^{un}_K$. In particular, it is independent of the smooth model X of K.*

Proof. — Every finite ℓ-quotient of $\hat{\pi}^a_{1,\ell}(X)$ induces an abelian ℓ-extension of K, which is unramified for every divisorial valuation ν (i.e., the corresponding extension of K_ν is unramified). This gives a surjection in one direction.

Conversely, an ℓ-extension of K, unramified in divisorial valuations ν, induces an étale covering of $X \setminus \{q_1, \dots, q_r\}$, the complement of a finite number of points. Since X is smooth, this extends uniquely to X. □

9. Galois groups of curves

Here we give a Galois-theoretic characterization of subgroups $\sigma \in \Sigma_K$ which are inertia subgroups of rank two valuations of K arising from a flag (C, q), where C is a smooth irreducible curve (on some model of K) and $q \in C(k)$ is a point (see Example 6.13). We show that Galois-theoretic data determine the genus of C and all "points" on C, as special liftable subgroups of rank two inside $\mathcal{G}^a_{k(C)}$.

Throughout, $E = k(C)$ is the function field of a smooth curve of genus g. We have an exact sequence

$$0 \to E^*/k^* \to \mathrm{Div}(C) \to \mathrm{Pic}(C) \to 0 \qquad (9.1)$$

(where $\mathrm{Div}(C)$ can be identified with the free abelian group generated by points in $C(k)$). An element $\mu : E^*/k^* \to \mathbb{Z}_\ell$ extends uniquely to $\mathrm{Div}^0(C) \to \mathbb{Q}_\ell$, so that $\mathcal{G}^a_E = \mathrm{hom}(E^*/k^*, \mathbb{Z}_\ell)$ is the space of $\mu : \mathrm{Div}^0(C) \to \mathbb{Q}_\ell$, i.e., of $\tilde{\mu} : C(k) \to \mathbb{Q}_\ell$ modulo constants, which induce a map from E^*/k^* to $\mathbb{Z}_\ell$.

In detail, the sequence (9.1) gives a dual sequence

$$0 \to \mathbb{Z}_\ell \xrightarrow{\Delta} \mathcal{M}(C(k), \mathbb{Z}_\ell) \to \mathcal{G}^a_E \to \mathbb{Z}_\ell^{2\mathrm{g}} \to 0, \qquad (9.2)$$

with the identifications

- $\mathrm{Hom}(\mathrm{Pic}(C), \mathbb{Z}_\ell) = \Delta(\mathbb{Z}_\ell)$ (since $\mathrm{Pic}^0(C)$ is torsion);
- $\mathcal{M}(C(k), \mathbb{Z}_\ell) = \mathrm{Hom}(\mathrm{Div}(C), \mathbb{Z}_\ell)$ is the $\mathbb{Z}_\ell$-linear space of maps from $C(k) \to \mathbb{Z}_\ell$;
- $\mathbb{Z}_\ell^{2\mathrm{g}} = \mathrm{Ext}^1(\mathrm{Pic}^0(C), \mathbb{Z}_\ell)$.

Using this model and the results in Section 5, in particular the identification

$$\mathcal{G}^a_E = \mathrm{Hom}(E^*/k^*, \mathbb{Z}_\ell),$$

we can interpret

$$\mathcal{G}^a_E \subset \mathcal{M}(C(k), \mathbb{Q}_\ell)/\text{constant maps} \tag{9.3}$$

as the $\mathbb{Z}_\ell$-linear subspace of all maps $\mu : C(k) \to \mathbb{Q}_\ell$ (modulo constant maps) such that

$$[\mu, f] \in \mathbb{Z}_\ell \quad \text{for all} \quad f \in E^*/k^*.$$

Here $[\cdot, \cdot]$ is the pairing:

$$\begin{array}{rcl} \mathcal{M}(C(k), \mathbb{Q}_\ell) \times E^*/k^* & \to & \mathbb{Q}_\ell \\ (\mu, f) & \mapsto & [\mu, f] := \sum_q \mu(q) f_q, \end{array} \tag{9.4}$$

where $\mathrm{div}(f) = \sum_q f_q q$.

Let $\gamma \in \mathcal{G}^a_E$ be an element of the Galois group. By Kummer theory, γ is a homomorphism $K^*/k^* \to \mathbb{Z}_\ell(1) \simeq \mathbb{Z}_\ell$. Choose a point $c_0 \in C(k)$. For every point $c \in C(k)$, there is an $m_c \in \mathbb{N}$ such that the divisor $m_c(c - c_0)$ is principal (see Lemma 2.2). Define a map

$$\begin{array}{rccc} \mu_\gamma : & C(k) & \to & \mathbb{Q}_\ell, \\ & c & \mapsto & \gamma(m_c(c - c_0))/m_c. \end{array}$$

Changing c_0 we get maps differing by a constant map.

In this interpretation, an element of an inertia subgroup $\mathcal{I}^a_w \subset \mathcal{G}^a_E$ corresponds to a "delta"-map (constant outside the point q_w). Each $\mathcal{I}^a_w$ has a canonical (topological) generator δ_w, given by $\delta_w(f) = \nu_w(f)$, for all $f \in E^*/k^*$. The (diagonal) map $\Delta \in \mathcal{M}(C(k), \mathbb{Q}_\ell)$ from (9.2) is then given by

$$\Delta = \sum_{w \in \mathcal{V}_E} \delta_w = \sum_{q_w \in C(k)} \delta_{q_w}.$$

DEFINITION 9.1. — *We say that the support of a subgroup $\mathcal{I} \subset \mathcal{G}^a_E$ is $\leq s$ and write*

$$|\mathrm{supp}(\mathcal{I})| \leq s$$

if there exist valuations $w_1, ..., w_s \in \mathcal{V}_E$ such that

$$\mathcal{I} \subset \langle \mathcal{I}^a_{w_1}, ..., \mathcal{I}^a_{w_s} \rangle_{\mathbb{Z}_\ell} \subset \mathcal{G}^a_E.$$

Otherwise, we write $|\mathrm{supp}(\mathcal{I})| > s$.

LEMMA 9.2. — *Let $E = k(x)$ and let $\mathcal{I} \subset \mathcal{G}^a_E$ be a topologically cyclic subgroup which is not equal to $\mathcal{I}^a_w$ for some divisorial valuation (point) on E ($\mathbb{P}^1(k)$). Then for any nonzero $\iota \in \mathcal{I}$ there exist a finite group V and a homomorphism $\psi : \mathcal{G}^a_E \to V$ such that for all $w \in \mathcal{V}_E$ one has $\psi(\iota) \notin \psi(\mathcal{I}^a_w)$.*

Proof. — By the assumption on $\mathcal{I}$, the element $\iota \in \mathcal{I}$ corresponds to a $\mathbb{Z}_\ell$-map μ_ι on $\mathbb{P}^1(k)$ which is not a delta-map of a point (modulo addition of constants). If μ_ι takes at least three distinct values there are three distinct $q_1, q_2, q_3 \in \mathbb{P}^1(k)$ and $n \in \mathbb{N}$ so that the values $\mu_\iota(q_i) \mod \ell^n$ are pairwise distinct for $i = 1, 2, 3$. Consider a map $\psi : \mathcal{G}^a_E \to (\mathbb{Z}/\ell^n)^2$ defined by elements of $E = k(x)$ with divisors $(q_1 - q_2), (q_1 - q_3)$. Note that $\psi(\mathcal{I}^a_w) = 0,\ q_w \in \mathbb{P}^1(k)$ unless $q_w = q_1, q_2, q_3$ and $\psi(\iota) \notin \psi(\mathcal{I}^a_{w_i}), i = 1, 2, 3,$ as claimed.

Similarly, if μ_ι takes two values on $\mathbb{P}^1(k)$ there are points $q_i, i = 1, \dots, 4$ and $n \in \mathbb{N}$ so that

$$\mu_\iota(q_1) = \mu_\iota(q_2) \neq \mu_\iota(q_3) = \mu_\iota(q_4) \mod \ell^n.$$

Then $\psi : \mathcal{G}^a_E \to (\mathbb{Z}/\ell^n)^3$, given by elements of E with divisors

$$(q_1 - q_2), (q_1 - q_3), (q_3 - q_4),$$

satisfies the claim. □

The next step is an *intrinsic* definition of inertia subgroups

$$\mathcal{I}^a_w \subset \mathcal{D}^a_\nu / \mathcal{I}^a_\nu = \mathcal{G}^a_{k(C)}.$$

We have a projection

$$\pi_\nu : \mathcal{G}^a_K \to \mathcal{G}^a_K / \mathcal{I}^a_\nu$$

and an inclusion

$$\mathcal{G}^a_{\boldsymbol{K}_\nu} = \mathcal{D}^a_\nu / \mathcal{I}^a_\nu \hookrightarrow \mathcal{G}^a_K / \mathcal{I}^a_\nu$$

PROPOSITION 9.3. — *Let ν be a divisorial valuation of K. A topologically cyclic subgroup*

$$\mathcal{I} \subset \mathcal{D}^a_\nu / \mathcal{I}^a_\nu$$

is the inertia subgroup of a divisorial valuation of $k(C) = \boldsymbol{K}_\nu$ iff for every homomorphism

$$\psi : \mathcal{G}^a_K / \mathcal{I}^a_\nu \to V$$

onto a finite abelian group V there exists a divisorial valuation ν_ψ such that

$$\psi(\mathcal{I}) = \psi \circ \pi_\nu(\mathcal{I}^a_{\nu_\psi}).$$

Proof. — Let C be the smooth model for $\boldsymbol{K}_\nu = k(C)$,

$$\mathcal{I} = \mathcal{I}^a_w \subset \mathcal{D}^a_\nu/\mathcal{I}^a_\nu$$

the inertia subgroup of a divisorial valuation of $k(C)$ corresponding to a point $q = q_w \in C(k)$ and

$$\psi \,:\, \mathcal{G}^a_K/\mathcal{I}^a_\nu \to V$$

a homomorphism onto a finite abelian group. Since $\mathcal{G}^a_K$ is a pro-ℓ-group, we may assume that

$$V = \oplus_{j\in J}\mathbb{Z}/\ell^{n_j},$$

for some $n_j \in \mathbb{N}$. Let $n = \max_j(n_j)$. By Kummer theory,

$$\mathrm{Hom}(\mathcal{G}^a_K, \mathbb{Z}/\ell^n) = K^*/(K^*)^{\ell^n}$$

so that ψ determines elements

$$\bar{f}_j \in K^*/(K^*)^{\ell^n}$$

(for all $j \in J$). Choose functions f_j projecting to $\bar{f}_j$. They define a finite set of divisors D_{ij} on X, the irreducible components of the divisors of f_j. Changing the model $\tilde{X} \to X$, if necessary, we can ensure that the full preimage of a finite set of divisors becomes a divisor with normal crossings. In particular, we may assume that

– C is smooth (and irreducible);
– there exists exactly one irreducible component D in the full preimage of $\cup D_{ij}$ which intersects C in q. Moreover, this intersection is transversal.

Then the image of $\mathcal{I}^a_D$ under ψ is equal to the image of $\mathcal{I}^a_w$.

Conversely, we need to show that if $\mathcal{I} \neq \mathcal{I}^a_w$ (for any $w \in \mathcal{DV}_{\boldsymbol{K}_\nu}$), then there exists a homomorphism

$$\psi \,:\, \mathcal{G}^a_K/\mathcal{I}^a_\nu \to V$$

onto a finite abelian group V such that for all $\nu' \in \mathcal{DV}_K$ one has

$$\psi(\mathcal{I}) \neq \psi \circ \pi_\nu(\mathcal{I}^a_{\nu'}).$$

Let $\bar{\iota} \in \mathcal{I}$ be any nonzero element. Its lift ι to $\mathcal{G}^a_K$ is not a flag map on K^*. By Lemma 4.16 there exists a $\mathbb{P}^1_\iota = \mathbb{P}^1(k) \subset \mathbb{P}_k(K)$ such that the restriction of ι to $\mathbb{P}^1_\iota$ is is not a flag map. By the logarithmic property of ι we can assume that $\mathbb{P}^1_\iota$ is the projectivization of the k-span of $1, x$, for some $x \in K^*$. This defines a birational surjective map $\pi_x \,:\, X \to \mathbb{P}^1$ and a corresponding map of Galois groups $\pi_x^a : \mathcal{G}_K^a \to \mathcal{G}_{k(x)}^a$. Under this map, the image of $\mathcal{I}_\nu^a$ is zero (otherwise, C lies in a fiber of π_x and the whole group $\mathcal{G}_{\boldsymbol{K}_\nu}^a$ is mapped to the valuation group of $\pi_x(C) \subset \mathbb{P}^1$, contradicting the assumption that the image of ι is not a flag map on $\mathbb{P}_\iota^1 = \mathbb{P}_k(k \oplus k \cdot x) = \mathbb{P}^1(k)$).

This gives a homomorphism $\eta_\nu : \mathcal{D}_\nu^a/\mathcal{I}_\nu^a \to \mathcal{G}_{k(x)}^a$ so that $\eta_\nu(\bar{\iota})$ is not a flag map on $k(x)$. Let $\psi_x : \mathcal{G}_{k(x)}^a \to V$ be any homomorphism such that $\psi_x(\bar{\iota}) \notin \psi(\mathcal{I}_w^a)$ for every $\mathcal{I}_w^a \subset \mathcal{G}_{\boldsymbol{K}_\nu}^a$, as in Lemma 9.2. The composition $\psi := \psi_x \circ \eta_\iota$ has the required properties. □

LEMMA 9.4. — *Let $E = k(C)$ be the function field of a curve. Then $\mathrm{g}(C) \geq 1$ iff there exists a nonzero homomorphism from $\mathcal{G}_E^a$ to a finite (abelian) group which maps all inertia elements to* 0.

Proof. — Indeed, every curve of genus ≥ 1 over a finite field of characteristic p has unramified coverings of degree ℓ. These coverings define maps of Galois groups, which are trivial on all inertia elements. If C is rational then $\mathcal{G}_E^a$, and hence its image under every homomorphism (onto any finite group), is generated by inertia elements (see the exact sequence (9.2)). □

REMARK 9.5. — Combining this with Proposition 9.3 we can decide in purely Galois-theoretic terms which divisorial valuations of K correspond to nonrational (irreducible) curves C on some model X of K. We call such valuations *nonrational*. The genus of C is well-defined by Galois-data.

10. Valuations on surfaces

Next we are lead to the following problem: How to characterize subgroups $\widehat{k(C)}^* \subset \hat{K}^*$? We recall a geometric argument (from algebraic K-theory) characterizing pairs $f, g \in K^*$ which are contained in $k(C)^* \subset K^*$, for some curve C (such curves correspond to projections $X \to C$).

Let ν be a divisorial valuation of K and

$$\nu : K^* \to \mathbb{Z}$$

the valuation map. We have the residue map

$$\mathrm{res}_\nu : \mathrm{Ker}(\nu) \to \boldsymbol{K}_\nu^*$$

and a bilinear (with respect to multiplication) symbol

$$\begin{array}{ccc} K^* \times K^* & \xrightarrow{\varrho_\nu} & \boldsymbol{K}^*_\nu/k^* \\ f, g & \mapsto & f^{\nu(g)}/g^{\nu(f)}. \end{array} \tag{10.1}$$

On a smooth model X of K, where $\nu = \nu_D$ for a divisor $D \subset X$, we define

$$\varrho_\nu = \varrho_D \, : \, K^* \times K^* \to \boldsymbol{K}^*_\nu/k^* \tag{10.2}$$

as follows:

- $\varrho_\nu(f, g) = 1$ if both f, g are invertible on D;
- $\varrho_\nu(f, g) = f_D^m$ if f is invertible (f_D is the restriction to D) and g has multiplicity m along D;
- $\varrho_\nu(f, g) = (f^{m_g}/g^{m_f})_D$ in the general case, when f, g have multiplicities m_f, m_g, respectively.

The definition does not depend on the choice of the model.

The following is a standard result in K-theory. We include a proof since we will need its ℓ-adic version.

LEMMA 10.1. — *For* $f, g \in K^*$

$$\varrho_\nu(f, g) = 1 \ \forall \nu \in \mathcal{DV}_K \iff f, g \in E = k(C) \subset K \text{ for some curve } C.$$

Proof. — ($\Leftarrow$) On an appropriate model X we have $\nu = \nu_D$ for a divisor $D \subset X$ and $\pi \, : \, X \to C$ is regular and flat with irreducible generic fiber (and $f, g \in k(C)^*$). By definition, $\varrho_\nu(f, g) = 1$ if D is not in a fiber of π. If D is in a fiber then there is a $t \in k(C)^*, \nu_D(t) \neq 0$ such that both ft^{m_f}, gt^{m_g} are regular and constant on D (for some $m_f, m_g \in \mathbb{N}$) so that $\varrho_\nu(f, g) = 1$.

($\Rightarrow$) Assume that $\varrho_\nu(f, g) = 1$ for every $\nu \in \mathcal{DV}_K$. Every nonconstant function f defines a unique map (with irreducible generic fiber)

$$\pi_f \, : \, X \to C_f$$

which corresponds to the algebraic closure of $k(f)$ in K (we will say that f is induced from C_f). We claim that $\pi_f = \pi_g$.

Since f is induced from C_f, we have

$$\mathrm{div}(f) = \sum_{q \in Q} a_q D_q,$$

where $Q \subset C_f(k)$ is finite and $D_q = \pi^{-1}(q)$. Then $D_q^2 = 0$ and D_q is either a multiple of a fiber of π_g or it has an irreducible component $D \subset D_q$ which dominates C_g (under π_g). In the second case, the restriction of g to D_q is a nonconstant element in $k(D_q)$. Then $\nu_D(f) \neq 0$, while $\nu_D(g) = 0$. Hence $\varrho_D(f,g) \neq 1$ since it coincides with $g_D^{-\nu_D(f)} \neq 1$, a contradiction. Therefore, all D_q are contained in finitely many fibers S of π_g. That means $\mathrm{div}(f)$ does not intersect the fibers $R_t, t \in C_g, t \notin S$ which implies that f is constant on such R_t. Hence f belongs to the normal closure of $k(C_g)$ in K, and in fact $f \in k(C_g)$ since $k(C_g)$ is algebraically closed in K, by construction. Thus f is induced from C_g and hence $C_f = C_g$ and $\pi_f = \pi_g$. □

We will also use a finite version of Lemma 10.1.

LEMMA 10.2. — *For $f, g \in K^*$ let X be a smooth projective model of K such that*

$$\mathrm{div}(f) = \sum_i a_i D_i \text{ and } \mathrm{div}(g) = \sum_j a_j D_j,$$

where $D_i, D_j \subset X$ are smooth and irreducible. Let ν_i, resp. ν_j, be the corresponding divisorial valuations. Assume that

$$D := \bigcup_i D_i \cup \bigcup_j D_j$$

is a divisor with normal crossings and that

$$\varrho_\nu(f,g) = 1 \ \forall \nu \in (\{\nu_i\} \cup \{\nu_j\}).$$

Then

$$f, g \in E = k(C) \subset K \text{ for some curve } C.$$

Proof. — The assumption implies that f is constant on every D'_j. Hence each D'_j belongs to the fibers of f. If f does not define a morphism from X to a curve, then we can replace the model X by a blowup with centers at intersection points of the divisors D_i, i.e., in points not in the support of g. Same holds for g.

After passing to this model, we can assume that both f and g define morphisms, onto some curves. By assumption, the divisors D'_j belong to the fibers of f. Hence g is constant on fibers of f not containing D'_j. This implies that g is constant on all fibers of f and hence belongs to the normal closure of $k(f)$ in $K = k(X)$, and similarly, f is in the normal closure of $k(g)$. Thus the normal closures coincide and there is a curve C such that $f, g \in E = k(C)$ and $E \subset K$ is a normally closed subfield. □

11. ℓ-adic analysis: generalities

Let X be a smooth model of K. We have an exact sequence

$$0 \to K^*/k^* \xrightarrow{\rho_X} \mathrm{Div}^0(X) \xrightarrow{\varphi} \mathrm{Pic}^0(X) \to 0. \tag{11.1}$$

This gives rise to an exact sequence

$$0 \to K^*/k^* \otimes \mathbb{Z}_\ell \xrightarrow{\rho_{X,\ell}} \mathrm{Div}^0(X)_\ell \xrightarrow{\varphi_\ell} \mathrm{Pic}^0(X)_\ell \to 0, \tag{11.2}$$

where

$$\mathrm{Pic}^0(X)_\ell := \mathrm{Pic}^0(X)\{\ell\} = \mathrm{Pic}^0(X) \otimes \mathbb{Z}_\ell$$

is the ℓ-primary component of the torsion group $\mathrm{Pic}^0(X)$ and

$$\mathrm{Div}^0(X)_\ell := \mathrm{Div}^0(X) \otimes \mathbb{Z}_\ell.$$

Put

$$\mathcal{T}_\ell(X) := \varprojlim \mathrm{Tor}_1(\mathbb{Z}/\ell^n, \mathrm{Pic}^0(X)_\ell).$$

We have $\mathcal{T}_\ell(X) = \mathbb{Z}_\ell^{2\mathrm{g}}$, where g is the dimension of $\mathrm{Pic}^0(X)$. Passing to pro-ℓ-completions in (11.1) we obtain an exact sequence:

$$0 \to \mathcal{T}_\ell(X) \to \hat{K}^* \xrightarrow{\hat{\rho}_X} \widehat{\mathrm{Div}}^0(X) \longrightarrow 0, \tag{11.3}$$

since $\mathrm{Pic}^0(X)$ is an ℓ-divisible group. Note that all groups in this sequence are torsion free. We write $\widehat{\mathrm{Div}}^0(X)$, resp. $\widehat{\mathrm{Div}}(X)$ for the ℓ-completion of $\mathrm{Div}^0(X)$, resp. $\mathrm{Div}(X)$. Clearly, $\mathrm{Div}^0(X)_\ell \subset \widehat{\mathrm{Div}}^0(X)$ and we have a diagram

(11.4)

$$\begin{array}{ccccccccc} & & 0 & \to & K^*/k^* \otimes \mathbb{Z}_\ell & \xrightarrow{\rho_{X,\ell}} & \mathrm{Div}^0(X)_\ell & \xrightarrow{\varphi_\ell} & \mathrm{Pic}^0(X)_\ell & \to & 0 \\ & & & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{T}_\ell(X) & \to & \hat{K}^* & \xrightarrow{\hat{\rho}_X} & \widehat{\mathrm{Div}}^0(X) & \xrightarrow{\hat{\varphi}} & 0. & & \end{array}$$

For every element $\hat{f} \in \hat{K}^*$ its image $\hat{\rho}_X(\hat{f}) \in \widehat{\mathrm{Div}}^0(X) \subset \widehat{\mathrm{Div}}(X)$ has a representation

$$\hat{\rho}_X(\hat{f}) = \sum_{m \in M} \hat{a}_m D_m, \tag{11.5}$$

where M is the *countable* set of irreducible divisors on X and $\hat{a}_m \in \mathbb{Z}_\ell$ are coefficients converging to zero in ℓ-adic topology: for all $r > 0$, the set

$\{m \,|\, |\hat{a}_m|_\ell \geq r\}$ is finite. Let

$$\widehat{\mathrm{Div}_{\mathrm{nr}}}(X) \subset \widehat{\mathrm{Div}}(X),$$

be the direct factor of nonrational divisors: for $D \in \widehat{\mathrm{Div}_{\mathrm{nr}}}(X)$, all D_m in (11.5) with $\hat{a}_m \neq 0$ are nonrational. We write

$$\hat{\rho}_{X,\mathrm{nr}} \,:\, \hat{K}^* \to \widehat{\mathrm{Div}_{\mathrm{nr}}}(X)$$

for the corresponding projection.

LEMMA 11.1. — *Let X/k be a smooth projective surface, M a finite set and*

$$D = \sum_{m \in M} a_m D_m \in \mathrm{Div}^0(X)_\ell, \quad a_m \in \mathbb{Z}_\ell$$

a divisor such that $\varphi_\ell(D) = 0$. Then there exist a finite set I, functions $f_i \in K^*/k^*$ and numbers $a_i \in \mathbb{Z}_\ell$, linearly independent over $\mathbb{Z}$, such that for all $i \in I$

$$\mathrm{supp}_X(f_i) \subset \mathrm{supp}_X(D),$$

where $\mathrm{supp}_X(f_i)$ is the support of the divisor $\mathrm{div}(f_i)$ on X, and

$$D = \sum a_i \mathrm{div}(f_i).$$

Proof. — Any $\mathbb{Z}_\ell$-lattice of principal divisors with support in a finite set of divisors contains a generating $\mathbb{Z}$-lattice of principal divisors. □

LEMMA 11.2. — *For varieties over k we have*

(1) *a morphism of smooth varieties $\xi \,:\, X \to Y$ induces a homomorphism*

$$\xi_\ell^* \,:\, \mathcal{T}_\ell(Y) \to \mathcal{T}_\ell(X),$$

if ξ is birational then ξ_ℓ^* is an isomorphism;

(2) *the canonical morphism $\mathrm{alb} \,:\, X \to \mathrm{Alb}(X)$ to the Albanese variety induces a canonical isomorphism $\mathrm{alb}_\ell^* \,:\, \mathcal{T}_\ell(\mathrm{Alb}(X)) \to \mathcal{T}_\ell(X)$;*

(3) *an exact sequence of abelian varieties*

$$1 \to A' \to A \to A'' \to 1$$

induces an exact sequence

$$1 \to \mathcal{T}_\ell(A'') \to \mathcal{T}_\ell(A) \to \mathcal{T}_\ell(A') \to 1. \tag{11.6}$$

Proof. — This follows from the corresponding properties of the functor Pic^0 for smooth algebraic varieties over k. Note that the group $\mathrm{Pic}^0(X)_\ell$ is induced from the map $X \to \mathrm{Alb}(X)$ which implies (1) and (2). The proof of (3) follows from the fact that

$$\mathrm{Tor}_2(\mathbb{Z}/\ell^n, \mathrm{Pic}^0(X)_\ell) = \mathrm{Tor}_0(\mathbb{Z}/\ell^n, \mathrm{Pic}^0(X)_\ell) = 0$$

for all X. □

From now on we denote the corresponding group by $\mathcal{T}_\ell(K)$.

Let $E \subset K$ be a one-dimensional subfield. We know that (after a finite purely inseparable extension) there is a projection $\pi : X \to C$ onto a curve with $E = k(C)$. We assume, for simplicity, that E is normally closed in K, i.e., that the projection has irreducible fibers. We have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E^*/k^* & \xrightarrow{\rho_C} & \mathrm{Div}^0(C) & \xrightarrow{\varphi_C} & \mathrm{Pic}^0(C) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K^*/k^* & \xrightarrow[\rho_X]{} & \mathrm{Div}^0(X) & \xrightarrow[\varphi_X]{} & \mathrm{Pic}^0(X) & \longrightarrow & 0, \end{array}$$

we will write π^* for the vertical arrows. Our assumption impies that all π^* are injective. Passing to ℓ-completions we obtain the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T}_\ell(E) & \longrightarrow & \hat{E}^* & \xrightarrow{\hat{\rho}_C} & \widehat{\mathrm{Div}}^0(C) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{T}_\ell(K) & \longrightarrow & \hat{K}^* & \xrightarrow[\hat{\rho}_X]{} & \widehat{\mathrm{Div}}^0(X) & \longrightarrow & 0, \end{array}$$

with vertical arrows denoted by $\hat{\pi}^*$.

LEMMA 11.3. — *All vertical maps $\hat{\pi}*$ in the diagram above are injective.*

Proof. — Follows from the finiteness of torsion in $\mathrm{Div}^0(X)/\mathrm{Div}^0(C)$. □

Every $\nu \in \mathcal{DV}_K$ gives rise to a homomorphism

$$\hat{\nu} : \hat{K}^* \to \mathbb{Z}_\ell$$

and a homomorphism

$$\hat{\mathrm{res}}_\nu \,:\, \mathrm{Ker}(\hat{\nu}) \to \hat{\boldsymbol{K}}^*_\nu,$$

and a symbol

$$\hat{\varrho}_\nu \,:\, \hat{K}^* \times \hat{K}^* \to \hat{\boldsymbol{K}}^*_\nu.$$

On a smooth model X, where $\nu = \nu_D$ for a divisor $D \subset X$, $\hat{\nu}(\hat{f})$ is the ℓ-adic coefficient at D of $\mathrm{div}(\hat{f})$, while $\hat{\varrho}_\nu$ is the natural $\mathbb{Z}_\ell$-bilinear generalization of (10.1).

LEMMA 11.4. — *Let X be a smooth surface or a smooth curve over k and $K = k(X)$. Then*

$$\mathcal{T}_\ell(K) = \cap_{\nu \in \mathcal{DV}_K} \mathrm{Ker}(\hat{\nu}).$$

Proof. — Follows from the definition. □

In particular, we have the map $\hat{\mathrm{res}}_\nu \,:\, \mathcal{T}_\ell(K) \to \hat{\boldsymbol{K}}^*_\nu$.

LEMMA 11.5. — *For all $\nu \in \mathcal{DV}_K$ we have*

$$\hat{\mathrm{res}}_\nu(\mathcal{T}_\ell(K)) \subset \mathcal{T}_\ell(\boldsymbol{K}_\nu).$$

Proof. — Let X be a model of K such that $\nu = \nu_D$, where D is a smooth curve. We may assume (after blowing up) that X contains a divisor D' intersecting D in exactly one point. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(\nu) & \longrightarrow & \mathrm{Div}^0(X \setminus D) & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{res}_\nu} & & \downarrow & & \downarrow{\scriptstyle \delta} & & \\ 0 & \longrightarrow & \boldsymbol{K}^*_\nu & \longrightarrow & \mathrm{Div}^0(D) & \longrightarrow & \mathrm{Pic}^0(D) & \longrightarrow & 0 \end{array}$$

where $\mathrm{Div}^0(X \setminus D)$ is the $\mathbb{Z}$-module spanned by divisors different from D. By the choice of X, the restriction δ induces a surjection $\mathrm{NS}(X) \to \mathrm{NS}(D)$. Tensoring all $\mathbb{Z}$-modules with $\mathbb{Z}/\ell^n$ and passing to the projective limit we obtain a map

$$\mathcal{T}_\ell(K) \to \mathcal{T}_\ell(\boldsymbol{K}_\nu),$$

and the claim. □

12. ℓ-adic analysis: finite support

Our goal is to characterize the ℓ-adic space $K^*/k^* \otimes \mathbb{Z}_\ell \subset \hat{K}^*$. The Galois datum $(\mathcal{G}_K^a, \Sigma_K)$ allows us to distinguish between rational and nonrational irreducible divisors on X (via the corresponding valuations) and to describe intrinsically a subspace $\mathcal{FS}(K) \subset \hat{K}^*$ (of divisors with finite nonrational support, see 12.1 and 12.2). In this section we further shrink $\mathcal{FS}(K)$, giving an intrinsic characterization of those elements which have finite divisorial support on every smooth model X.

By Lemma 2.17, if $\mathcal{T}_\ell(K) \neq 0$ then either X contains only finitely many rational curves, or X, modulo purely inseparable covers, is a rational pencil over a curve C of genus $\mathrm{g}(C) \geq 1$.

NOTATIONS 12.1. — We put

$$\begin{array}{lll} \mathrm{supp}_K(\hat{f}) & := \{ \quad \nu \in \mathcal{DV}_K & | \quad \hat{f} \text{ nontrivial on } \mathcal{I}_\nu^a \ \}; \\ \mathrm{supp}_X(\hat{f}) & := \{ \quad D_m \subset X & | \quad \hat{a}_m \neq 0 \ \}, \end{array}$$

where $\hat{a}_m$ is the coefficient at D_m in the expansion (11.5).

DEFINITION 12.2. — *We say that $\hat{f}$ has* finite nonrational support *if the set of nonrational $\nu \in \mathrm{supp}_K(\hat{f})$ is finite (see Lemma 9.4 for the definition and Galois-theoretic characterization of nonrational valuations). Let*

$$\mathcal{FS}(K) \subset \hat{K}^*$$

be the subgroup of such elements.

Note that for $\hat{f} \in \hat{K}^*$, its nonrational component $\hat{\rho}_{X,\mathrm{nr}}(\hat{f})$ is independent of the model X. More precisely, for any birational morphism $X' \to X$ we can identify $\widehat{\mathrm{Div}_{\mathrm{nr}}}(X') = \widehat{\mathrm{Div}_{\mathrm{nr}}}(X)$ and write $\widehat{\mathrm{Div}_{\mathrm{nr}}}(X) = \widehat{\mathrm{Div}_{\mathrm{nr}}}(K)$. Under this identification

$$\hat{\rho}_{X',\mathrm{nr}}(\hat{f}) = \hat{\rho}_{X,\mathrm{nr}}(\hat{f}).$$

DEFINITION 12.3. — *We say that $\hat{f}$ has* finite support *on the model X if* $\mathrm{supp}_X(\hat{f})$ *is finite. Put*

$$\mathcal{FS}_X(K) \ := \{\hat{f} \in \hat{K}^* \mid \hat{\rho}_X(\hat{f}) \in \mathrm{Div}^0(X)_\ell \subset \widehat{\mathrm{Div}}^0(X)\}.$$

Comparing the exact sequences (11.4) obtain the sequence

$$\mathcal{T}_\ell(K) \hookrightarrow \mathcal{FS}_X(K) \twoheadrightarrow \mathrm{Pic}^0(X)_\ell, \tag{12.1}$$

with cohomology of the central term equal to $K^*/k^* \otimes \mathbb{Z}_\ell$. In particular, if $\mathcal{T}_\ell(K)$ and consequently $\mathrm{Pic}^0(X)_\ell$ vanish, then $\mathcal{FS}_X(K) = K^*/k^* \otimes \mathbb{Z}_\ell$.

LEMMA 12.4. — *The definition of $\mathcal{FS}_X(K)$ does not depend on the choice of a smooth model X.*

Proof. — For any two smooth models X', X'' we can find a smooth model X dominating both. The difference between the sets of irreducible divisors $\mathrm{Div}(X')$, resp. $\mathrm{Div}(X'')$, and $\mathrm{Div}(X)$ is finite and consists of rational curves. □

DEFINITION 12.5. — *We say that $\hat{f}, \hat{g} \in \hat{K}^*$* commute *if $\hat{\varrho}_\nu(\hat{f}, \hat{g}) = 1$, for all divisorial ν. We say that they have* disjoint support *if for all divisorial valuations $\nu \in \mathcal{DV}_K$*

$$\hat{\nu}(\hat{f}) \cdot \hat{\nu}(\hat{g}) = 0.$$

We say that $\hat{f} \in \mathcal{FS}(K)$ has nontrivial commutators if there exists a $\hat{g} \in \mathcal{FS}(K)$ which commutes with $\hat{f}$, has nontrivial nonrational support and whose support is disjoint from the support of $\hat{f}$.

We proceed to give a Galois-theoretic characterization of $\mathcal{FS}_X(K)$. There are three cases to consider:

- I X contains at most finitely many rational curves.
- II A finite inseparable cover of X admits a fibration over a curve C of genus ≥ 1, with generic fiber a rational curve.
- III X is not of type I or II.

The Cases I, II and III have the following Galois-theoretic characterization. Consider the projection

$$\hat{\rho}_{X,\mathrm{nr}} \,:\, \hat{K}^* \to \widehat{\mathrm{Div}_{\mathrm{nr}}}(K).$$

In Case I the kernel of $\hat{\rho}_{X,\mathrm{nr}}$ is topologically finitely generated (it consists of $\mathcal{T}_\ell(K)$ and at most finitely many summands corresponding to the finitely many rational curves on X). In Case II, $\mathcal{T}_\ell(K) \neq 0$ and the kernel of $\hat{\rho}_{X,\mathrm{nr}}$

contains $\widehat{k(C)}^*$ which has infinite $\mathbb{Z}_\ell$-rank. Indeed, this condition means that there is a nontrivial map to the Albanese variety with infinitely many rational curves in the fibers. If we are not in Case I or II, then $\mathrm{Pic}^0(X) = 0$, so that in Case III, $\mathcal{T}_\ell(K) = 0$ and the rank of the kernel of $\hat{\rho}_{X,\mathrm{nr}}$ is infinite, since in this case $\mathrm{Pic}(X)$ is finitely generated and there is an infinite number of linearly independent relations between rational curves on X.

Case I. Let K be the function field of a surface X containing only finitely many rational curves. Then

$$\mathcal{FS}(K) = \mathcal{FS}_X(K).$$

Case II. Assume that, after a purely inseparable extension, X admits a fibration over a curve C of genus ≥ 1, with generic fiber a rational curve.

Let $\mathcal{FS}'(K) \subset \mathcal{FS}(K) \subset \hat{K}^*$ be the $\mathbb{Z}_\ell$-module consisting of finite (multiplicative) $\mathbb{Z}_\ell$-linear combinations of elements $\hat{f}$ having nontrivial commutators. Then, for every model X of K, we have

$$\mathcal{FS}'(K) + \mathcal{T}_\ell(K) = \mathcal{FS}_X(K)$$

(in fact, a direct sum, as will be seen in the proof of Proposition 15.1). Indeed, an infinite rational tail in $\hat{f}$ would contain an infinite number of fibers of $X \to C$. Let $\hat{g}$ be a nontrivial commutator for $\hat{f}$ and $\{C_i\}$ be the finitely many nonrational divisors in the support of $\hat{g}$. One of those irreducible rational fibers in the infinite tail of $\hat{f}$ would have disjoint intersections with the C_i and for the corresponding valuation we would get

$$\hat{\varrho}_\nu(\hat{f}, \hat{g}) \neq 1,$$

contradiction.

Thus $\mathcal{FS}'(K)$ consists of elements with finite support on X and hence $\mathcal{FS}'(K) + \mathcal{T}_\ell(K) \subset \mathcal{FS}_X(K)$. Note that $\mathcal{FS}'(K)$ contains all elements K^*/k^*. Indeed any $f \in K^*/k^*$ with a nontrivial nonrational support is contained in $\mathcal{FS}'(K)$ since it commutes with any element in $k(f)^*/k^* \subset K^*/k^*$. On the other hand any $g \in K^*/k^*$ is a product of at most two elements of K^*/k^* with a nontrivial nonrational support. Similarly any element $h \in \mathrm{Pic}^0(X)$ has finite order and can be represented by a divisor $\sum a_i D_i$ with a nonrational component . Hence $m(\sum a_i D_i) = \mathrm{Div}(f)$

for some $f \in K^*/k^*$ and an integer m. Thus there is an element $\hat{h} \in \mathcal{FS}(K)$ with a divisor $\sum a_i D_i = \mathrm{div}(\hat{h})$ and $m\hat{h} \in k(f)^*/k^*$. It implies that $\hat{h}$ commutes with any element in $k(f)^*/k^* \subset K^*/k^*$ and hence $\hat{h} \in \mathcal{FS}'(K)$ (see more detailed analysis in Ch. 14).

It implies that any divisor in $\mathrm{Div}^0(X)_\ell$ is a divisor on X of an element $\hat{h} \in \mathcal{FS}'(K)$. Note that $\mathrm{Div}^0(X)_\ell$ coincides with the group of divisors of the elements of $\mathcal{FS}_X(K)$ on X.

Thus if $D = \mathrm{div}(\hat{f})$, $\hat{f} \in \mathcal{FS}_X(K)$ then there is $\hat{h} \in \mathcal{FS}'(K)$ with $\mathrm{div}(\hat{h}) = D = \mathrm{div}(\hat{f})$. In particular $\hat{h} - \hat{f} \in \mathcal{T}_\ell(K)$. Thus $\mathcal{FS}'(K) + \mathcal{T}_\ell = \mathcal{FS}_X(K)$. Since Ψ^* induces a canonical isomorphism between $\mathcal{T}_\ell(K)$ and $\mathcal{T}_\ell(L)$, and between $\mathcal{FS}'(K)$ and $\mathcal{FS}'(L)$, it follows that Ψ^* induces a canonical isomorphism $\mathcal{FS}_Y(L) \to \mathcal{FS}_X(K)$, where Y is a model of L and X a model of K.

Case III. By Lemma 2.17, we can now assume that $\mathrm{Pic}^0(X) = 0$ and that we are not in Case I.

Let $\mathcal{F}_X(K)$ be the set of all $f \in k(X)^*/k^*$ such that

(1) $\rho_{X,\mathrm{nr}}(f) \neq 0$ and

(2) for every rational curve $D \subset X$ with $\nu = \nu_D$ either D is in a fiber of π_f (the map defined by f) or $\mathrm{res}_\nu(f) \neq 0 \mod \ell$ in $\boldsymbol{K}_\nu^*/k^*$.

Geometrically, condition (2) means that if a rational curve D is not a component of a fiber of f then there is a point on the normalization of D whose multiplicity (i.e., valuation of the restriction of f) is prime to ℓ.

LEMMA 12.6. — *Let $x \in K^*$ be such that $\rho_{X,\mathrm{nr}}(x) \neq 0$. As $\mathrm{Pic}^0(X) = 0$, the normal closure of $k(x)$ in K is a rational function field $k(y)$. Let $\pi_y : \mathbb{P}^1_y \to \mathbb{P}^1_x$ be the corresponding morphism. Assume that*

(1) $k(y)/k(x)$ is a separable extension of degree > 1;

(2) the preimage under π_y^{-1} of the divisor $0 - \infty \in \mathrm{Div}^0(\mathbb{P}^1_x)$ contains at least 4 points with multiplicities prime to ℓ.

Then the image of x in K^*/k^* is in $\mathcal{F}_X(K)$.

Proof. — Let X be a smooth model of K and

$$\beta_x : X \to \mathbb{P}^1_x, \quad \beta_y : X \to \mathbb{P}^1_y$$

regular maps with $\beta_x = \pi_y \circ \beta_y$. Let R be an irredicible curve in X which surjects geometrically onto $\mathbb{P}^1_x$ (the curve R is not a component of the fiber). We can assume that the map $\beta_x : R \to \mathbb{P}^1_x$ is separable (after a Frobenius twist of the function field of R).

Assume that the multiplicities of all poles and zeroes of the function x on R are divisible by ℓ (this does not change after a Frobenius twist). Thus the map $\beta_y : R \to \mathbb{P}^1_y$ has ramifications over 4-points divisible by ℓ. By the Hurwitz formula, $\mathrm{g}(R) > 0$.

In particular, for any rational curve $D \subset X$ either $\beta_y(D)$ is constant, so that D is contained in the fiber of β_y, or the intersection of D with some component of $\mathrm{Div}(x)$ contains points with multiplicity prime to ℓ. Thus the image of x in K^*/k^* is in $\mathcal{F}_X(K)$. □

COROLLARY 12.7. — *The set $\mathcal{F}_X(K)$ generates K^*/k^*.*

Proof. — The multiplicative group of a normally closed subfield $k(y) \subset K$ is generated by x satisfying Lemma 12.6. Indeed, for x which are not generating the field $k(y)$ and which are not an ℓ-th power, all elements of the form $x(y-a)/(y-b)$, where a, b run through k minus $\mathrm{Div}(x) \subset \mathbb{P}^1_y$, satisfy the lemma. By assumption, $\mathrm{Pic}^0(X) = 0$ so that every closed one-dimensional subfield of K is isomorphic to $k(y)$ for some y. □

LEMMA 12.8. — *For X with $\mathrm{Pic}^0(X) = 0$, and for every pair of nonzero commuting elements $\hat{f}, \hat{g} \in \mathcal{FS}(K)$ with nontrivial nonrational support and disjoint support such that there exists an $f \in \mathcal{F}_X(K)$ with*

$$f = \hat{f} \mod \ell, \text{ in } \hat{K}^* \tag{12.2}$$

one has $\hat{f} \in \mathcal{FS}_X(K)$ and $\hat{g} \in \mathcal{FS}_X(K)$.

Proof. — Note that an f satisfying (12.2) is automatically nonconstant - it has nontrivial nonrational support. Write

$$\begin{aligned}
\hat{\rho}_X(\hat{f}) &= \textstyle\sum_{i\in I} n_i D_i + \ell \sum_{j=1} n_j C_j, \\
\hat{\rho}_X(\hat{g}) &= \textstyle\sum_{i\in I'} n'_i D'_i + \ell \sum_{j=1} n'_j C'_j,
\end{aligned}$$

where I, I' are finite sets and the second sum is a series over distinct rational curves $C_j, C'_j \subset X$ which may be infinite. By assumption, the sets $\{D_i\}_{i\in I}$, $\{C_j\}_{j\in\mathbb{N}}$, $\{D'_i\}_{i\in I'}$, $\{C'_j\}_{j\in\mathbb{N}}$ are disjoint.

Further, $\hat{\varrho}_\nu(\hat{f},\hat{g}) = 1$, for all ν. For $\nu = \nu_D$, where $D \in \mathrm{supp}_X(\hat{g})$, this symbol equals the residue of $\hat{f}$ on D, which equals the corresponding residue of $f \mod \ell$. For rational curves in the support of $\hat{g}$ and not in a fiber of f it is nonzero by assumption on $\hat{f}$ in the lemma. Since the generic fiber of f is nonrational, there are only a finite number of rational curves on X which are mapped to points by f. It follows that every divisor in $\mathrm{supp}_X(\hat{g})$ is nonrational, unless it is in the fiber of f, and that $\hat{g} \in \mathcal{FS}_X(K)$.

Since $\mathcal{T}_\ell(K) = 0$, we have $\mathcal{FS}_X(K) = (K^*/k^*) \otimes \mathbb{Z}_\ell$ and $\hat{g} = \prod_{i\in I''} g_i^{b_i}$ with I'' a finite set, $g_i \in K^*/k^*$ and $b_i \in \mathbb{Z}_\ell$, linearly independent over $\mathbb{Z}$. Note that $\mathrm{div}(\hat{g}) = \sum_{m\in M} a_m D_m$, $a_m \in \mathbb{Z}_\ell$, where M is a finite set and some $D_m \subset X$ are nonrational divisors.

The restriction of g_i to every irreducible component of the divisor of $\hat{f}$ is identically zero. This means that under the map

$$\pi_{g_i} : X \to C$$

all components of $\mathrm{supp}_X(\hat{f})$ map to points (note that $C = \mathbb{P}^1$, since $\mathrm{Pic}^0(X) = 0$). Since some components of the divisor of g_i, for some i, are nonrational, the generic fiber of π_{g_i} is also nonrational. Thus $\mathrm{supp}_X(\hat{f})$ contains only a finite number of rational divisors, so that $\hat{f} \in \mathcal{FS}_X(K)$. □

The following corollary shows that in Case III, we have a more explicit description of elements of $\mathcal{FS}_X(K)$:

COROLLARY 12.9. — *Assume that* $\mathrm{Pic}^0(X) = 0$ *and let* $\Gamma \subset \mathcal{FS}(K)$ *be subgroup. Assume that* $r_\ell : \Gamma \to K^*/\ell$ *is onto and that every element of* Γ *with a nontrivial nonrational support has a nontrivial commutator. Then* $\Gamma \subset \mathcal{FS}_X(K)$.

Proof. — Let $\hat{g}, \hat{f} \in \Gamma$ and $\hat{f} = f \mod \ell$, $f \in \mathcal{F}_X(K)$ and $\hat{g} \in \mathrm{Ker}(r_\ell)$. Assume that $\hat{g}\hat{f}$ has nontrivial nonrational support. By Lemma 12.8, we have $\hat{f}, (\hat{g}\hat{f}) \in \mathcal{FS}_X(K)$. Note that the elements $r_\ell(\hat{f})$, for $\hat{f}$ as above, generate K^*/ℓ, and that $\mathrm{Ker}(r_\ell) \subset \mathcal{FS}_X(K)$, since $\hat{f}\hat{g} = \hat{f} = f \mod \ell$. Thus $\Gamma \subset \mathcal{FS}_X(K)$. □

If we define Γ as a subgroup generated by $\Psi^*(\mathcal{F}_Y(L))$ we obtain that in the case III

$$\Psi^*(\mathcal{F}_Y(L)) \subset \mathcal{F}_X(K),$$

and vice versa. Since $\mathcal{F}_Y(L))$ generates $L^*/l^* \otimes \mathbb{Z}_\ell$ we obtain that $\Psi^*(\mathcal{FS}_Y(L)) = \mathcal{FS}_X(K)$

We now have

PROPOSITION 12.10. — *The isomorphism $\Psi : \mathcal{G}_K^a \to \mathcal{G}_L^a$ induces a canonical isomorphism $\Psi^* : \mathcal{FS}_Y(L) \to \mathcal{FS}_X(K)$, where Y is a model of L and X is a model of K.*

We have shown that $\Psi : \mathcal{G}_K^a \to \mathcal{G}_L^a$ maps inertia subgroups $\mathcal{I}_\nu^a$, for divisorial $\nu \in \mathcal{DV}_K$ isomorphically to inertia subgroups of divisorial valuations in $\mathcal{G}_L^a$. We get the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G}_K^{ram} & \longrightarrow & \mathcal{G}_K^a & \longrightarrow & \mathcal{G}_K^{un} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Psi^{ram}} & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \Psi^{un}} & & \\ 0 & \longrightarrow & \mathcal{G}_L^{ram} & \longrightarrow & \mathcal{G}_L^a & \longrightarrow & \mathcal{G}_L^{un} & \longrightarrow & 0 \end{array}$$

with canonical isomorpisms Ψ^{ram} and Ψ^{un}. This gives canonical dual isomorphisms

$$\Psi^* : \mathcal{T}_\ell(L) \to \mathcal{T}_\ell(K) \text{ and } \Psi^* : \mathrm{Pic}^0(Y)_\ell \to \mathrm{Pic}^0(X)_\ell.$$

Here $\mathcal{T}_\ell(K)$ (resp. $\mathcal{T}_\ell(L)$) is the subgroup of $\hat{K}^*$ (resp. $\hat{L}^*$) of those elements which are identically zero on $\mathcal{G}_K^{ram}$ (resp. $\mathcal{G}_L^{ram}$).

13. ℓ-adic analysis: curves

Here we begin the process of recognition of the lattice

$$K^*/k^* \otimes \mathbb{Z}_{(\ell)} \subset K^*/k^* \otimes \mathbb{Z}_\ell$$

(note that we haven't yet recognized $K^*/k^* \otimes \mathbb{Z}_\ell$ in the case $\mathcal{T}_\ell(X) \neq 0$). We solve an analogous problem for the function field of a rational curve. This result will play an essential role in the analysis of surfaces.

PROPOSITION 13.1. — *Let $\tilde{k}$ be the algebraic closure of a finite field, with* $\mathrm{char}(\tilde{k}) \neq \ell$*, C a curve over $\tilde{k}$ of genus* g *with function field $E = \tilde{k}(C)$ and*

$$\Psi \,:\, \mathcal{G}^a_E \to \mathcal{G}^a_{k(\mathbb{P}^1)}$$

an isomorphism of Galois groups inducing an isomorphism on inertia groups of divisorial valuations, that is, a bijection on the set of such groups and isomorphisms of corresponding groups. Let

$$\Psi^* \,:\, \widehat{k(\mathbb{P}^1)}^* \to \hat{E}^*$$

be the dual isomorphism. Then $E = \tilde{k}(\mathbb{P}^1)$ and there is a constant $a \in \mathbb{Z}_\ell^*$ such that $\Psi^*(k(\mathbb{P}^1)^*/k^*) = a \cdot E^*/\tilde{k}^*$.

Proof. — Recalling the exact sequence (9.2), we have a commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}_\ell(\Delta_{C(\tilde{k})}) & \longrightarrow & \mathcal{M}(C(\tilde{k})) & \longrightarrow & \mathcal{G}^a_E & \longrightarrow & \mathbb{Z}_\ell^{2\mathrm{g}} \longrightarrow 0 \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}_\ell(\Delta_{\mathbb{P}^1(k)}) & \longrightarrow & \mathcal{M}(\mathbb{P}^1(k)) & \longrightarrow & \mathcal{G}^a_{k(\mathbb{P}^1)} & \longrightarrow & 0 \end{array}$$

Since Ψ is an isomorphism on inertia groups $\mathcal{I}^a_w$, for each w, the *sets* $C(\tilde{k})$ and $\mathbb{P}^1(k)$ coincide and we get a *unique* isomorphism of $\mathbb{Z}_\ell$-modules (of maps to $\mathbb{Z}_\ell$)

$$\mathcal{M}(C(\tilde{k})) = \mathcal{M}(\mathbb{P}^1(\tilde{k})).$$

In particular, we find that $\mathrm{g} = 0$ and $E = \tilde{k}(\mathbb{P}^1)$. Further, we have an induced isomorphism

$$\mathbb{Z}_\ell(\sum_{w \in \mathcal{V}_E} \delta_w) = \mathbb{Z}_\ell(\sum_{w' \in \mathcal{V}_{k(\mathbb{P}^1)}} \delta_{w'})$$

so that

$$(\sum_{w \in \mathcal{V}_E} \delta_w) = a(\sum_{w' \in \mathcal{V}_{k(\mathbb{P}^1)}} \delta_{w'})$$

for some $a \in \mathbb{Z}_\ell^*$. This implies that $\delta_w = a\delta_{w'}$, for all $w \in \mathcal{V}_E$ and the corresponding $w' \in \mathcal{V}_{\mathbb{P}^1}$. For the dual groups we obtain

$$E^*/\tilde{k}^* = (K^*/k^*)^a,$$

where $a \in \mathbb{Z}_\ell^*$. □

14. Projections onto curves

In this section we study one-dimensional subfields $E \subset K$, or equivalently, rational surjective maps $\pi : X \to C$, where X is a smooth projective model of K and C a smooth curve, in terms of Galois-structures. We will assume that E is normally closed in K, i.e., that the generic fiber of π is reduced and irreducible, and that $E = k(x)$, i.e., $C = \mathbb{P}^1$. In particular, the embedding $E^*/k^* \to K^*/k^*$ is primitive, i.e., there are no elements $y \in K^*, y \notin E^*$ with $y^{\ell^m} \in E^*$, for some $m \in \mathbb{N}$.

Let $\pi_* : \mathcal{G}^a_K \to \mathcal{G}^a_E$ be the corresponding map of Galois groups. This map is surjective: an abelian finite ℓ-extension of E induces an extension of K with the same Galois group. The dual map $\pi^* : \hat{E}^* \to \hat{K}^*$ is injective, by Lemma 11.3, and we identify $\hat{E}^*$ with its image $\pi^*(\hat{E}^*)$ in $\hat{K}^*$.

LEMMA 14.1. — *The image $\pi^*(\hat{E}^*)$ in $\hat{K}^*$ is primitive.*

Proof. — We have a exact sequence

$$0 \to E^*/k^* \to K^*/k^* \to Q \to 0$$

with Q torsion free. This gives

$$0 \to E^*/k^* \otimes \mathbb{Z}/\ell^n \to K^*/k^* \otimes \mathbb{Z}/\ell^n \to Q \otimes \mathbb{Z}/\ell^n \to 0,$$

for any $n \in \mathbb{N}$. We pass to the limit and observe that the completion of the torsion free module Q is torsion free. Indeed, $Q \otimes \mathbb{Z}/\ell^n$ is flat over $\mathbb{Z}/\ell^n$, hence free, and if $Q \otimes \mathbb{Z}/\ell \sim (\mathbb{Z}/\ell)^{(I)}$, for some index set I, then the completion

$$\hat{Q} \sim \{(a_i)_{i \in I} \,|,\ a_i \to 0\},$$

and is hence torsion free. □

Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E^*/k^* & \xrightarrow{\sim} & \mathrm{Div}^0(C) & \longrightarrow & 0 & & \\ & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} & & & & \\ 0 & \longrightarrow & K^*/k^* & \longrightarrow & \mathrm{Div}^0(X) & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & 0 \end{array}$$

Note that the image of K^*/k^* in $\mathrm{Div}^0(X)$ will fail to be primitive, as soon as $\mathrm{Pic}^0(X) \neq 0$. In this case, the image of $\mathrm{Div}^0(C)$ in $\mathrm{Div}^0(X)$ *may* fail to be primitive. We now discuss the arising issue of primitivization in the geometric setting. Let $\mathrm{Div}_{\mathrm{vert}}(X)$ be the subgroup of divisors supported in the fibers (i.e., spanned by irreducible components of the fibers) and $\mathrm{Div}_{\mathrm{hor}}(X)$ the group spanned by irreducible horizontal divisors, i.e., divisors surjecting onto C. Let $S \subset C$ be a nonempty finite set of points, containing all $q \in C$ such that the fiber $\pi^{-1}(q)$ is special for π (e.g., not reduced irreducible). We have the direct sum decomposition of free $\mathbb{Z}$-modules

$$\mathrm{Div}(X) = \mathrm{Div}_S(X) \oplus \mathrm{Div}_{C\setminus S}(X) \oplus \mathrm{Div}_{\mathrm{hor}}(X),$$

where $\mathrm{Div}_S(X) \subset \mathrm{Div}_{\mathrm{vert}}(X)$, resp. $\mathrm{Div}_{C\setminus S}(X) \subset \mathrm{Div}_{\mathrm{vert}}(X)$, is the subgroup of divisors having support over S, resp. over $C \setminus S$. Let $\mathrm{NS}_{\mathrm{vert}}(X)$ be the image of $\mathrm{Div}_{\mathrm{vert}}(X)$ under the natural projection $\mathrm{Div}(X) \to \mathrm{NS}(X)$. The finitely generated group $\mathrm{Div}_S(X)$ surjects onto $\mathrm{NS}_{\mathrm{vert}}(X)$ and we denote by $\mathrm{Div}_S^0(X)$ the kernel of this homomorphism. Let

$$\mathrm{Div}^0_{\mathrm{hor}}(X) := \mathrm{Ker}(\mathrm{Div}_{\mathrm{hor}}(X) \to \mathrm{NS}(X)/\mathrm{NS}_{\mathrm{vert}}(X)).$$

Note that $\mathrm{Div}^0_{\mathrm{hor}}(X)$ is also a free $\mathbb{Z}$-module, since it is a subgroup of a free $\mathbb{Z}$-module. We have a natural homomorphism

$$\mathrm{Div}_S(X) \oplus \mathrm{Div}_{C\setminus S}(X) \oplus \mathrm{Div}^0_{\mathrm{hor}}(X) \to \mathrm{NS}_{\mathrm{vert}}(X).$$

Fix homomorphisms

– $h_1 : \mathrm{Div}_{C\setminus S}(X) \to \mathrm{Div}_S(X)$ and
– $h_2 : \mathrm{Div}^0_{\mathrm{hor}}(X) \to \mathrm{Div}_S(X)$

such that the projection of the corresponding groups to $\mathrm{NS}_{\mathrm{vert}}(X)$ factors through h_1, resp. h_2. Such homomorphisms exist since both modules are free and hence projective. We will work with a particular h_1: choose $q \in S$ and for $D = \pi^{-1}(q')$ with $q' \notin S$ put

$$h_1(D) := \pi^{-1}(q),$$

and extend to $\mathrm{Div}_{C\setminus S}(X)$ by linearity. This exhibits $\mathrm{Div}^0(X)$ as a direct sum of free $\mathbb{Z}$-modules:

$$\mathrm{Div}^0(X) = \mathrm{Div}^0_S(X) \oplus (i - h_1)\mathrm{Div}_{C\setminus S}(X) \oplus (i - h_2)\mathrm{Div}^0_{\mathrm{hor}}(X), \tag{14.1}$$

where i is the standard map into $\mathrm{Div}(X)$. We have a compatible decomposition

$$\mathrm{Div}^0(C) = \mathrm{Div}^0_S(C) \oplus (i - h_1)\mathrm{Div}_{C\setminus S}(C) \tag{14.2}$$

and an injection $\mathrm{Div}^0_S(C) \hookrightarrow \mathrm{Div}^0_S(X)$.

LEMMA 14.2. — *The quotient* $\mathrm{Div}^0_S(X)/\mathrm{Div}^0_S(C)$ *is finite.*

Proof. — The group $\mathrm{Div}_S(X)$ is generated by the irreducible components $C_{j,s}$ of the fibers $\pi^{-1}(s)$ for $s \in S$. Consider the kernel of the map

$$\varphi_{S,\mathbb{Q}} : \mathrm{Div}_S(X) \otimes \mathbb{Q} \to \mathrm{Pic}(X) \otimes \mathbb{Q} = \mathrm{NS}(X) \otimes \mathbb{Q}.$$

It is generated by relations $\sum m_j C_{j,s} = \pi^{-1}(s)$, for $s \in S$. Thus $\mathrm{Ker}(\varphi_{S,\mathbb{Q}})$ is generated by divisors $\pi^{-1}(s) - \pi^{-1}(q)$, $s \in S$. The divisors $s - q$ on C also generate $\mathrm{Div}^0_S(C)$. Thus $\mathrm{Div}^0_S(X)$ and $\mathrm{Div}^0_S(C)$ generate commensurable finitely generated subgroups in $\mathrm{Ker}(\varphi_{S,\mathbb{Q}})$. In particular, $\mathrm{Div}^0_S(X)/\mathrm{Div}^0_S(C)$ is finite. □

LEMMA 14.3. — *The image of the natural map*

$$\varphi^0 \ : \ \mathrm{Div}^0_S(X) \oplus (1 - h_1)\mathrm{Div}_{C\setminus S}(X) \to \mathrm{Pic}^0(X).$$

concides with $\mathrm{Div}^0_S(X)/\mathrm{Div}^0_S(C)$.

Proof. — The image is a quotient of this group since

$$\mathrm{Div}^0(C) = \mathrm{Div}^0_S(C) \oplus (1 - h_1)\mathrm{Div}_{C\setminus S}(C)$$

maps to zero. On the other hand, if $\mathrm{div}(f)$, $f \in K^*$ is vertical, then f belongs to E^* since the general fiber of the projection $\pi \ : \ X \to C$ is connected. Thus $\mathrm{div}(f) \in \mathrm{Div}^0(C)$. Hence φ^0 is an imbedding on $\mathrm{Div}^0_S(X)/\mathrm{Div}^0_S(C)$, which coincides with the image of φ^0. □

LEMMA 14.4. — *Consider the diagram*

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & \hat{E}^* & \xrightarrow{\sim} & \widehat{\mathrm{Div}^0}(C) & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} & & \\
0 & \longrightarrow & \mathcal{T}_\ell(K) & \longrightarrow & \hat{K}^* & \longrightarrow & \widehat{\mathrm{Div}^0}(X) & \longrightarrow & 0.
\end{array}$$

Then

$$\mathrm{Prim}(\widehat{\mathrm{Div}^0}(C))/\widehat{\mathrm{Div}^0}(C) = \big(\mathrm{Div}^0_S(X)/\mathrm{Div}^0_S(C)\big) \otimes \mathbb{Z}_\ell,$$

where $\mathrm{Prim}(\widehat{\mathrm{Div}^0}(C))$ *is the primitivisation of* $\widehat{\mathrm{Div}^0}(C)$ *in* $\widehat{\mathrm{Div}^0}(X)$.

Proof. — The direct sum decomposition in free $\mathbb{Z}$-modules (14.1) implies the direct decomposition of the corresponding completions

(14.3) $$\widehat{\mathrm{Div}^0}(X) = \widehat{\mathrm{Div}^0}_S(X) \oplus (1-h_1)\widehat{\mathrm{Div}}_{C\setminus S}(X) \oplus (1-h_2)\widehat{\mathrm{Div}}_{\mathrm{hor}}(X),$$

with natural extensions of homomorphisms h_1, h_2. Note that each component in the decomposition (14.3) is primitive in $\widehat{\mathrm{Div}^0}(X)$.

The group $\widehat{\mathrm{Div}^0}(C)$ is not necessarily primitive in $\widehat{\mathrm{Div}^0}(X)$. Then

$$\mathrm{Prim}(\widehat{\mathrm{Div}^0}(C)) = \widehat{\mathrm{Div}^0}_S(X) \oplus (1-h_1)\widehat{\mathrm{Div}}_{C\setminus S}(X)$$

since $\mathrm{Div}^0_S(X)/\mathrm{Div}^0_S(C))$ is a finite group. Thus

$$\mathrm{Prim}(\widehat{\mathrm{Div}^0}(C))/\mathrm{Div}^0(C)) = \big(\mathrm{Div}^0_S(X)/\mathrm{Div}^0(C)\big) \otimes \mathbb{Z}_\ell.$$

□

COROLLARY 14.5. — *We have natural isomorphisms*

$$(\mathrm{Div}^0_S(X)/\mathrm{Div}^0_S(C)) \otimes \mathbb{Z}_\ell \simeq \mathrm{Tors}(\hat{K}^*/\hat{E}^* \cdot \mathcal{T}_\ell(K)).$$

LEMMA 14.6. — *In the group $\mathcal{FS}_X(K)$ we also have a natural isomorphism*

$$(\mathrm{Div}^0_S(X)/\mathrm{Div}^0_S(C)) \otimes \mathbb{Z}_\ell = \mathrm{Tors}(\mathcal{FS}_X(K)/\mathcal{T}_\ell(K) \cdot (\mathcal{FS}_X(K) \cap \hat{E}^*)).$$

Proof. — Since we have

$$K^*/k^* \otimes \mathbb{Z}_\ell \to \mathcal{FS}_X(K)/\mathcal{T}_\ell(K) = \mathrm{Div}^0(X)_\ell \to \mathrm{Pic}^0(X)_\ell,$$

tensoring the exact sequence (14.1) we obtain

$$\mathrm{Div}^0_S(X)_\ell \oplus (1-h_1)\mathrm{Div}_{C\setminus S}(X)_\ell \oplus (1-h_2)\mathrm{Div}^0_{\mathrm{hor}}(X)_\ell \to \mathrm{Pic}^0(X)_\ell.$$

Note that the image of vertical divisors in $\mathrm{Pic}^0(X)_\ell$ coincides with the image of the free finite rank $\mathbb{Z}_l$-module $\mathrm{Div}^0_S(X)_\ell$ and is equal to

$$\mathrm{Div}^0_S(X)/\mathrm{Div}^0_S(C) \otimes \mathbb{Z}_\ell.$$

□

COROLLARY 14.7. — *We can recover the homomorphism*

$$\bar{\varphi}_{\ell,K} : \mathcal{FS}_X(K)/\mathcal{T}_\ell(K) \to \mathrm{Pic}^0(X)_\ell$$

in Galois-theoretic terms if we can recover the subgroups $E^*/k^* \otimes \mathbb{Z}_\ell$.

Now we give a Galois-theoretic description of this situation. Consider the map $\mathcal{G}^{ram}_K \to \mathcal{G}^a_E$, where $\mathcal{G}^{ram}_K \subset \mathcal{G}^a_K$ is the subgroup spanned by inertial subgroups $\mathcal{I}^a_\nu$ of divisorial valuations of K.

LEMMA 14.8. — *Let ν be a divisorial valuation, and D the corresponding divisor on a smooth model X' of K which dominates X. Let $\pi' : X' \to C$ be the induced projection. If D surjects onto C, i.e., D is a multisection of π' then $\pi_*(\mathcal{I}^a_\nu) = 0$. Otherwise, let $q := \pi'(D)$. Then $\pi_*(\mathcal{I}^a_\nu) = n_D \mathcal{I}^a_q \subset \mathcal{G}^a_E$, where n_D is the multiplicity of D in the fiber X'_q over q.*

Proof. — The first claim follows from the fact that elements from $E^* \subset K^*$ are in $\mathfrak{o}^*_\nu$ and the exact sequence (6.1), which implies that every element of $\mathcal{I}^a_\nu \subset \mathcal{G}^a_E$, considered as a homomorphism $E^*/k^* \to \mathbb{Z}_\ell$, is trivial on $\mathfrak{o}^*_\nu$.

For the second claim, let t be a local parameter around q. We have a corresponding embedding of $k((t))^* \to K^*_\nu$, the valuation ν defines a multiple of the standard valuation of $k((t))^*$, which equals n_D. This implies the claim. □

COROLLARY 14.9. — *The span of the image of intertia groups $\mathcal{I}^a_{\nu_D}$, for $\pi'(D) = q$, is equal to $n_q \mathcal{I}^a_q$, where n_q is the maximal power of ℓ dividing the multiplicity of the fiber over $q \in C$. The map $\mathcal{G}^{ram}_K \to \mathcal{G}^a_E$ is surjective iff at most one fiber of π has multiplicity divisible by ℓ.*

Proof. — The first claim is clear. The second follows from the description of $\mathcal{G}^a_E$ in Section 9, in particular (9.2) for $\mathrm{g}(C) = 0$. □

15. ℓ-adic analysis: surfaces

For any normally closed one-dimensional subfield $E = k(C) \subset K$ the subgroup $\hat{E}^* \subset \hat{K}^*$ consists of commuting elements. In this section we prove the converse, for subgroups of $\mathcal{FS}_X(K)$.

We will need an ℓ-adic version of Lemma 10.1 .

PROPOSITION 15.1. — *Assume that $\hat{f}, \hat{g} \in \mathcal{FS}_X(K)$ have nontrivial support, commute and their support is disjoint. That means*

- $\varrho_\nu(\hat{f}, \hat{g}) = 1$ *for every* $\nu \in \mathcal{DV}_K$*;*
- $\mathrm{supp}_K(\hat{f}) \cap \mathrm{supp}_K(\hat{g}) = \emptyset$,

Then there exist a curve C with function field $E = k(C)$, a map $X \to C$, with generically irreducible fibers, finite families

$$\{f_i\}_{i\in I}, \{g_j\}_{j\in J}$$

of nontrivial elements of $E^*/k^* \subset K^*/k^*$ and $\mathbb{Q}$-rationally independent families $\{a_i\}_{i\in I}, \{b_j\}_{j\in J} \subset \mathbb{Q}_\ell$ such that

$$\hat{f} = t_f \cdot f, \quad \textit{where} \quad f := \prod_{i\in I} f_i^{a_i}, \ \textit{resp.}\ \hat{g} = t_g \cdot g, \quad \textit{where} \quad g := \prod_{j\in J} g_j^{b_j},$$

with $t_f, t_g \in \mathcal{T}_\ell(K) \otimes \mathbb{Q}_\ell$. Moreover, some integral powers of t_f, t_g are contained in the image of $\hat{E}^*$ in $\hat{K}^*$.

Proof. — We have an exact sequence

$$0 \to K^*/k^* \otimes \mathbb{Z}_\ell \to \mathcal{FS}_X(K)/\mathcal{T}_\ell(K) \to \mathrm{Pic}^0(X)_\ell \to 0.$$

Since $\mathrm{Pic}^0(X)$ is torsion, some ℓ-power of any element in $\mathcal{FS}_X(K)/\mathcal{T}_\ell(K)$ is in $K^*/k^* \otimes \mathbb{Z}_\ell$. After choosing a smooth model X we can apply Lemma 11.1 to conclude that

$$\hat{f} = t_f \cdot f, \quad \text{where} \quad f := \prod_{i\in I} f_i^{a_i}, \ \text{resp.}\ \hat{g} = t_g \cdot g, \quad \text{where} \quad g := \prod_{j\in J} g_j^{b_j},$$

where

- $t_f, t_g \in \mathcal{T}_\ell(K) \otimes \mathbb{Q}_\ell$;
- I, J are finite sets;
- $f_i, g_j \in K^*/k^*$ are not equal to 1 for all i, j;
- $\mathrm{supp}_X(f_i) \subset \mathrm{supp}_X(f)$ and $\mathrm{supp}_X(g_j) \subset \mathrm{supp}_X(g)$, for all i, j;
- $a_i \in \mathbb{Q}_\ell$ (resp. $b_j \in \mathbb{Q}_\ell$) are linearly independent over $\mathbb{Z}$.

Fix a valuation $\nu = \nu_D$, where D is in the support of $\hat{g}$ on X. After changing the model, we can assume that $\mathrm{supp}_X(\hat{f})$ and $\mathrm{supp}_X(\hat{g})$ satisfy the assumptions of Lemma 10.2, i.e., the union of the irreducible components in these supports is a divisor with normal crossings. The same holds for f_i and g_j. By assumption,

$$\hat{\mathrm{res}}_\nu(t_f \cdot \prod_{i\in I} f_i^{a_i}) = 1 \in \hat{\boldsymbol{K}}_\nu^*.$$

By Lemma 11.5, $\hat{\mathrm{res}}_\nu(t_f) \in \mathcal{T}_\ell(\boldsymbol{K}_\nu) \otimes \mathbb{Q}_\ell$ so that t_f has trivial support on D. We claim that for all $i \in I$, $\mathrm{res}_\nu(f_i) = 1 \in \boldsymbol{K}_\nu^*/k^*$. The divisor of the restriction of f_i to D is $\sum_{i'} r_{ii'}q_{ii'}$, where $q_{ii'}$ are points on D and $r_{ii'} \in \mathbb{Z}$. This gives a relation

$$\sum_{i\in I} a_i(\sum_{i'} r_{ii'}q_{ii'}) = 0.$$

Since a_i are linearly independent over $\mathbb{Z}$ we have $r_{ii'} = 0$, for all i, i'. In particular, $\mathrm{res}_\nu(f_i) = 1$. The same argument for g shows that g and f commute and that all pairs f_i, g_j commute as well. By Lemma 10.2, all $f_i, g_j \in E^*/k^*$ where $E = k(C) \subset K$ for some curve C, which is independent of i, j (the subfield E is the closure of any $k(\tilde{f}_i)$ or $k(\tilde{g}_j)$ in K, where $\tilde{f}_i, \tilde{g}_j$ are lifts of f_i, g_j from K^*/k^* to K^*).

We have a diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{T}_\ell(E) & \longrightarrow & \hat{E}^* & \longrightarrow & \widehat{\mathrm{Div}}(C) & \longrightarrow & \widehat{\mathrm{Pic}}(C) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{T}_\ell(K) & \longrightarrow & \hat{K}^* & \longrightarrow & \widehat{\mathrm{Div}}(X) & \longrightarrow & \widehat{\mathrm{Pic}}(X) & \longrightarrow & 0
\end{array}$$

Note that $\mathcal{T}_\ell(E) = \hat{E}^* \cap \mathcal{T}_\ell(K)$. Thus we need to show that some integral powers of t_f (resp. t_g) are in the image of $\mathcal{T}_\ell(E)$. Let D be an irreducible component in the divisor of g (resp. f). Changing the model, we may assume that D is smooth. We have a diagram:

$$\begin{array}{ccccc}
D & \longrightarrow & X & & \\
\downarrow & & \downarrow & & \\
\mathrm{Jac}(D) & \xrightarrow[\iota]{} & \mathrm{Alb}(X) & \xrightarrow[\alpha]{} & A = \mathrm{Alb}(X)/B
\end{array}$$

where α is a surjection with irreducible fibers and $B = B_D$ is the minimal abelian subvariety of $A^0(X)$ so that the image of D in $\mathrm{Alb}(X)/B$ is a point, a_D (note that D is irreducible). We have

$$\iota(\mathrm{Jac}(D)) = \alpha^{-1}(a_D) \simeq B.$$

Applying Lemma 11.2 (4) we conclude that the induced sequence

$$\mathcal{T}_\ell(A) \xrightarrow{\alpha_\ell^*} \mathcal{T}_\ell(\mathrm{Alb}(X)) \xrightarrow{\iota_\ell^*} \mathcal{T}_\ell(B)$$

of free finite rank $\mathbb{Z}_\ell$-modules is exact in the middle term. We have shown that $\mathrm{res}_\nu(f_i) \in k^*$, for all $i \in I$. It follows that $\hat{\mathrm{res}}_\nu(t_f) = 1 \in \hat{\boldsymbol{K}}^*_\nu$, where

$\nu = \nu_D$ is the corresponding valuation. In particular,

$$t_f = 1 \in \mathcal{T}_\ell(B) \hookrightarrow \mathcal{T}_\ell(\boldsymbol{K}_\nu) = \mathcal{T}_\ell(D).$$

It follows that there is an $a \in \mathcal{T}_\ell(A)$ such that $\alpha_\ell^*(a) = t_f$.

We apply this argument to every component D_j of the divisor of g and find that t_f is induced from quotients $\mathrm{Alb}(X)/B_j$, where $B_j := B_{D_j}$, for $j \in J$. Let B be the abelian subvariety of $A^0(X)$ generated by B_j. By Lemma 2.18, $\mathrm{Alb}(A)/B \simeq \mathrm{Jac}(C)$, and X maps to C with irreducible fibers. We have the diagrams

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Alb}(X) \\ \downarrow & & \downarrow \\ C & \longrightarrow & \mathrm{Jac}(C) \end{array} \qquad \begin{array}{ccc} \mathcal{T}_\ell(X) & \longleftarrow & \mathcal{T}_\ell(\mathrm{Alb}(X)) \\ \uparrow & & \uparrow \\ \mathcal{T}_\ell(C) & \xleftarrow{\sim} & \mathcal{T}_\ell(\mathrm{Jac}(C)) \end{array}$$

It follows that t_f, and similarly t_g, is in the image of $\mathcal{T}_\ell(E)$ in $\mathcal{T}_\ell(K)$. □

COROLLARY 15.2. — *Let $E = k(x) \subset K$ be a normally closed subfield and $\pi : X \to \mathbb{P}^1$ the corresponding projection from a smooth model X. Let $b \in \mathbb{P}^1$ and $\pi^{-1}(b) = \sum n_i D_i$, where each D_i is an irreducible component of the fiber. Then the homomorphism $\mathcal{T}_\ell(K) \to \oplus \mathcal{T}_\ell(D_i)$ is injective.*

Proof. — Indeed, in this case the variety B from above coincides with $\mathrm{Alb}(X)$, which implies triviality of the kernel of the map above. □

REMARK 15.3. — For every $f \in K^*$ and $g = (f+a)/(f+b)$, where $a, b \in k^*$, $a \neq b$, their images in K^*/k^* satisfy the conditions of Proposition 15.1.

REMARK 15.4. — Proposition 15.1 shows that normally closed subfields $E = k(x) = k(C)$ such that $\widehat{\mathrm{Div}}^0(C) \subset \mathcal{FS}_X(K)/\mathcal{T}_\ell(K)$ is primitive have an intrinsic Galois-theoretic description. Note that $\widehat{\mathrm{Div}}^0(C) \subset \mathcal{FS}_X(K)/\mathcal{T}_\ell(K)$ coincides with the image of $\hat{E}^* \bigcap \mathcal{FS}_X(K)$ in $\mathcal{FS}_X(K)/\mathcal{T}_\ell(K)$.

By Corollary 15.2, if $E = k(x)$ is normally closed then any element $f \in E^*/k^* \bigcap \mathcal{FS}_X(K)$, with nonzero support, does not commute with any nontrivial $t \in \mathcal{T}_\ell(K)$. Thus such fields E are characterized by the fact that the map

$$E^*/k^* \bigcap \mathcal{FS}_X(K) \to \mathcal{FS}_X(K)/\mathcal{T}_\ell(K)$$

is an embedding. If the image of $E^*/k^* \bigcap \mathcal{FS}_X(K)$ under this projection is primitive in $\mathcal{FS}_X(K)/\mathcal{T}_\ell(K)$ then the image of $E^*/k^* \bigcap \mathcal{FS}_X(K)$ in $\mathrm{Pic}^0(X)_\ell$

is trivial, by Lemma 14.6. It follows that the homomorphism Ψ^* induces a bijection between the sets of such subgroups in $\mathcal{FS}_Y(L)$ and $\mathcal{FS}_X(K)$.

LEMMA 15.5. — *The homomorphism* $\Psi : \mathcal{G}_K^a \to \mathcal{G}_L^a$ *induces a canonical isomorphism*

$$\Psi^* : L^*/l^* \otimes \mathbb{Z}_\ell \to K^*/k^* \otimes \mathbb{Z}_\ell.$$

Proof. — Proposition 12.10 establishes canonical isomorphisms

$$\Psi^* : \mathcal{FS}_Y(L) \to \mathcal{FS}_X(K) \quad \text{and} \quad \Psi^* : \mathcal{T}_\ell(L) \to \mathcal{T}_\ell(K).$$

Let $\mathcal{S}(K)$ (resp. $\mathcal{S}(L)$) be the set of one-dimensional subfields $E = k(C)$ of K (resp. L) such that

- $\mathcal{T}_\ell(K) \cap \hat{E}^* = 0$;
- $\widehat{\mathrm{Div}^0}(C) \cap (\mathcal{FS}_X(K)/\mathcal{T}_\ell(K))$ is primitive in $\mathcal{FS}_X(K)/\mathcal{T}_\ell(K)$.

Then $E = k(x)$ for some $x \in K^*$ and $\widehat{\mathrm{Div}^0}(C) \cap \mathcal{FS}_X(K)/\mathcal{T}_\ell(K)$ is in the kernel of the map $\bar{\varphi}_{\ell,K} : \mathcal{FS}_X(K)/\mathcal{T}_\ell(K) \to \mathrm{Pic}^0(X)$. It suffices to show that the groups $\widehat{\mathrm{Div}^0}(C) \cap (\mathcal{FS}_X(K)/\mathcal{T}_\ell(K))$, for E as above, generate the whole kernel of $\bar{\varphi}_{\ell,K}$ Note that condition 3 in the Definition 2.20 of an ℓ-Lefschetz pencil implies that none of the fibers of the map $\pi : X \to C$ has multiplicity divisible by ℓ. It follows that $\mathrm{Div}_S^0(X)/\mathrm{Div}_S^0(C) \otimes \mathbb{Z}_\ell = 0$ and, by Lemma 14.6, that the image of the subgroup $\widehat{\mathrm{Div}^0}(C)$ in $\mathcal{FS}_X(K)/\mathcal{T}_\ell(K)$ is primitive. Thus the subfields corresponding to ℓ-Lefschetz pencils belong to $\mathcal{S}(K)$. The multiplicative groups $E^*/k^* \otimes \mathbb{Z}_{(\ell)}$ of fields corresponding to ℓ-Lefschetz pencils generate $K^*/k^* \otimes \mathbb{Z}_{(\ell)}$, by Proposition 2.22.

Propositions 15.1 and Lemma 14.6 imply that Ψ^* induces a bijection $\mathcal{S}(L) \to \mathcal{S}(K)$. The obtained Galois-theoretic characterization of subgroups in $\mathcal{S}(K)$, resp. $\mathcal{S}(L)$, gives a canonical bijection $\Psi^* : \mathrm{Ker}(\varphi_{\ell,L}) \to \mathrm{Ker}(\varphi_{\ell,K})$ and shows that there is a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{T}_\ell(L) & \longrightarrow & \mathcal{FS}_Y(L) & \xrightarrow{\varphi_{\ell,L}} & \mathrm{Pic}^0(Y)_\ell & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Psi^*} & & \downarrow{\scriptstyle \Psi^*} & & \downarrow{\scriptstyle \Psi^*} & & \\
0 & \longrightarrow & \mathcal{T}_\ell(K) & \longrightarrow & \mathcal{FS}_X(K) & \xrightarrow[\varphi_{\ell,K}]{} & \mathrm{Pic}^0(X)_\ell & \longrightarrow & 0
\end{array}$$

with canonical vertical isomorphisms.

It follows that the isomorphism

$$\Psi^* : \mathcal{FS}_Y(L) \to \mathcal{FS}_X(K)$$

obtained in Proposition 12.10 induces a canonical isomorphism

$$\Psi^* : L^*/l^* \otimes \mathbb{Z}_\ell \to K^*/k^* \otimes \mathbb{Z}_\ell,$$

of the middle cohomology groups in the horizontal sequences of the above commutative diagram. □

COROLLARY 15.6. — *For any normally closed one-dimensional subfield $E = k(x) \subset K$ there exists a one-dimensional field $F = l(y) \subset L$ such that Ψ^* induces an isomorphism*

$$F^*/l^* \otimes \mathbb{Z}_\ell \to E^*/k^* \otimes \mathbb{Z}_\ell.$$

PROPOSITION 15.7. — *Let $\mathfrak{M}^* \subset K^*/k^* \otimes \mathbb{Z}_\ell$ be a subgroup with the following properties:*

(1) *$\mathfrak{M}^* \cap E^*/k^* \otimes \mathbb{Z}_\ell = a_E \cdot E^*/k^*$, for every one-dimensional normally closed subfield $E = k(x) \subset K$, with $a_E \in \mathbb{Z}_\ell^*$;*
(2) *there exists a $\nu_0 \in \mathcal{DV}_K$ such that*

$$\{[\delta_0, \hat{f}] \,|\, \hat{f} \in \mathfrak{M}^*\} = \mathbb{Z}$$

for a topological generator δ_0 of $\mathcal{I}^a_{\nu_0}$. (Here $[\cdot,\cdot]$ is the value of $\hat{f}$ on the element of the Galois group δ_0, see Theorem 5.5.)

Then

$$\mathfrak{M}^* \subset K^*/k^* \otimes \mathbb{Z}_{(\ell)} \quad \textit{and} \quad \mathfrak{M}^* \otimes \mathbb{Z}_{(\ell)} = K^*/k^* \otimes \mathbb{Z}_{(\ell)}.$$

Proof. — For $x \in K \setminus k$ let $E = k(x)$ be the corresponding one-dimensional field, assumed to be normally closed in K. By assumption, there exists an $a_E \in \mathbb{Z}_\ell^*$ such that

$$\mathfrak{M}^* \cap E^*/k^* \otimes \mathbb{Z}_\ell = a_E \cdot E^*/k^*.$$

If some (any) topological generator δ_0 of $\mathcal{I}^a_{\nu_0}$ is not identically zero on the group $E^*/k^* \otimes \mathbb{Z}_\ell$ then there exists a (smooth) model X, where ν_0 is realized by a divisor D_0, together with a morphism

$$X \to \mathbb{P}^1 = \mathbb{P}^1_E$$

such that D_0 dominates $\mathbb{P}^1$. It follows that

$$a_E \in \mathbb{Q} \cap \mathbb{Z}_\ell^* = \mathbb{Z}_{(\ell)}.$$

It remains to observe that every $x \in K^*$ can be written as a product

$$x = x' \cdot x''$$

such that δ_0 is nontrivial for both fields $E' = k(x')$ and $E'' = k(x'')$.

Finally, every group $k(x)^*/k^* \otimes \mathbb{Z}_{(\ell)}$ is generated over $\mathbb{Z}_{(\ell)}$ by elements from $\mathfrak{M}^*$. □

COROLLARY 15.8. — *There exists a constant $c \in \mathbb{Z}_\ell^*$ such that*

$$c\Psi^* \,:\, L^*/l^* \otimes \mathbb{Z}_{(\ell)} \to K^*/k^* \otimes \mathbb{Z}_{(\ell)}.$$

is an isomorphism.

Proof. — Note that $\Psi^*(L^*/l^*)$ satisfies all conditions of Proposition 15.7, except possibly (3). Multiplication of the lattice $\Psi(L^*/l^*)$ by a constant $c \in \mathbb{Z}_\ell^*$ gives (3). □

Thus every subgroup $\mathcal{I}_\nu^a$ contains an element δ_ν such that $[f, \delta_\nu] = \mathbb{Z}_{(\ell)}$. Moreover, this δ_ν is unique modulo multiplication by $\mathbb{Z}_{(\ell)}^*$. Since $\mathbb{Z}_{(\ell)}^* \subset \mathbb{Q}^*$, we can define a monoid

$$\mathfrak{M}_{\delta_\nu}^{>0} := \{f \mid [f, \delta_\nu] > 0\} \subset K^*/k^* \otimes \mathbb{Z}_{(\ell)}.$$

Note that $\mathfrak{M}_{a\delta_\nu}^{>0} = \mathfrak{M}_{\delta_\nu}^{>0}$ if $a \in \mathbb{Z}_{(\ell)}^*, a > 0$ and $\mathfrak{M}_{a\delta_\nu}^{>0} = (\mathfrak{M}_{\delta_\nu}^{>0})^{-1}$, if $a < 0$.

Geometrically, if ν corresponds to a divisor D_ν on some model X then $\mathfrak{M}_{\delta_\nu}^{>0}$ consists either of functions $f \in K^*$ which have nontrivial zero along D_ν or a nontrivial pole.

We would like to make a compatible choice of such orientations for all ν. More precisely, we want it to resemble the standard orientation coming from the field structure, when $\mathfrak{M}_\nu^{>0}$ is the multiplicative monoid of functions having a nontrivial zero along D_ν.

Let $E = k(x)$ and let w, w' be two different valuations on on $k(x)$. Consider $\delta_w \in \mathcal{I}_w^a, \delta_{w'} \in \mathcal{I}_{w'}^a$ with $[k(x)^* \otimes \mathbb{Z}_{(\ell)}, \delta_w] \subset \mathbb{Z}_{(\ell)}$ and similarly for $\delta_{w'}$. We say that δ_w, δ_w' are compatible if for any generating $y \in k(x)^*$ with support in w and w', $y \in \mathfrak{M}_{\delta_w}^{>0}$ implies $y \in \mathfrak{M}_{-\delta_{w'}}^{>0}$.

For $K = k(X)$ let $\nu, \nu' \in \mathcal{DV}_K$. We say that δ_ν, δ_ν' are compatible if for every normally closed one-dimensional subfield $E = k(x) \subset K$ such that the restrictions ν, ν' to E define distinct valuations, the corresponding restrictions of δ_ν and $\delta_{\nu'}$ are compatible, or if the restrictions are dependent. We say that the system $\{\delta_\nu\}_{\nu \in \mathcal{DV}_K}$ is compatible, if any two of these are compatible.

The compatibility condition can be expressed in terms of the Galois group. Namely, we can select δ_ν, δ'_ν modulo multiplication by $\mathbb{Z}^*_{(\ell)}$. We have also defined subgroups $E^*/k^* \times \mathbb{Z}_{(\ell)}$ for a normally closed subfield $E = k(x) \subset k(X)$. Any δ_ν defines a homomorphism

$$h_\nu = [f, \delta_\nu] \,:\, E^*/k^* \times \mathbb{Z}_{(\ell)} \to \mathbb{Z}_{(\ell)},$$

and we consider valuations defining different homomorphisms, modulo scalar multiplication. The powers y of generating elements in $E^*/k^* \times \mathbb{Z}_{(\ell)}$ are defined as the elements of $E^*/k{*}\times\mathbb{Z}_{(\ell)}$ for which there are exactly two nontrivial homomorphisms h_ν. This correspond to the condition that they have support in exactly two points on $\mathbb{P}^1$. Thus the compatibility condition translates to

$$y \in \mathfrak{M}^{>0}_{\delta_\nu} \Rightarrow y \in \mathfrak{M}^{>0}_{-\delta_{\nu'}}.$$

COROLLARY 15.9. — *Choose $\nu_0 \in \mathcal{DV}_K$ and $\delta_0 \in \mathcal{I}^a_{\nu_0}$ as in Proposition 15.7. Then there exists only one compatible system $\{\delta_\nu\}_{\nu\in\mathcal{DV}_K}$, modulo multiplication by positive numbers in $\mathbb{Z}_{(\ell)}$.*

Proof. — Consider a smooth irreducible divisor D_0 on a smooth model X and another smooth divisor D_1 which intersects it transversally at a point q. Then there is an element $f \in K = k(X)$ with $\mathrm{div}(f) = D_0 - D_1 + D'$, where D' does not contain a component which passes through q and f is a generators of the normally closed subfield $k(f) \subset K = k(X)$. The restrictions of the valuations ν_0, ν_1 to $k(f)$ are independent. Hence there is a unique δ_1 compatible with δ_0. If ν_2 is another divisorial valuation we can choose a model X' and D_1 such that a proper preimage of D_1 on X' and D_2 intersect transversally at one point. Thus for δ_1 there is only one compatible δ_2. Hence there is at most one compatible set of δ_ν for a given δ_0 and it corresponds to the standard field structure on K or its inverse. □

Recall that we have a natural embedding

$$K^*/k^* \otimes \mathbb{Z}_\ell \to \mathcal{FS}_X(K) \subset \hat{K}^*, \tag{15.1}$$

from Diagram (11.4). Note that under this embedding $K^*/k^*\otimes\mathbb{Z}_\ell$ is contained in the kernel of $\varphi_{\ell,K} \,:\, \mathcal{FS}_X(K) \to \mathrm{Pic}^0(X)_\ell$ and is canonically isomorphic to the cohomology of the sequence

$$O \to \mathcal{T}_\ell(K) \to \mathcal{FS}_X(K) \to \mathrm{Pic}^0(X)_\ell \to 0$$

under the natural projection. The following lemma shows that $K^*/k^* \otimes \mathbb{Z}_\ell$ is equivariant with respect to certain Galois automorphisms.

LEMMA 15.10. — *Let $\Delta : \mathcal{G}_K^a \to \mathcal{G}_K^a$ be an isomorphism preserving Σ_K. Let*

$$\Delta^* : \hat{K}^* \to \hat{K}^*$$

be the dual isomorphism, inducing an isomorphism

$$\Delta^* : \mathcal{FS}_X(K) \to \mathcal{FS}_X(K).$$

Consider the sequence

$$0 \to \mathcal{T}_\ell(K) \to \mathcal{FS}_X(K) \to \mathrm{Pic}^0(X)_\ell \to 0.$$

Assume that Δ^* induces the identity on the cohomology $K^*/k^* \otimes \mathbb{Z}_\ell$ of this sequence. Then Δ is the identity.

Proof. — Note that Δ^* is the identity on $\mathcal{FS}_X(K)/\mathcal{T}_\ell(K)$, since this is a torsion free module, whose quotient by $K^*/k^* \otimes \mathbb{Z}_\ell$ is the torsion module $\mathrm{Pic}^0(X)_\ell$.

We claim that Δ^* is the identity on the subgroups $E^*/k^* \otimes \mathbb{Z}_\ell \subset \mathcal{FS}_X(K)$, for any one-dimensional normally closed subfield $E = k(x) \subset K$. Note that such E^*/k^* generate K^*/k^*, by Proposition 2.22. We have

$$\mathcal{T}_\ell(K) \cap E^*/k^* \otimes \mathbb{Z}_\ell = 0.$$

(since E is a rational function field, see Proposition 15.1). Moreover, for every nonzero element $t \in \mathcal{T}_\ell(K)$ and every nonconstant element $f \in E^*/k^*$, the commutator $\varrho(t, f)$ is nontrivial, by Corollary 15.2. More generally, for any nontrivial $f, f' \in E^*/k^*$ as above, and any $t, t' \in \mathcal{T}_\ell(K)$ such that at least one of these is nonzero,

$$\varrho(tf, t'f') = \varrho(t, f') - \varrho(t', f) \neq 0,$$

provided there is at least one general fiber X_η of π which is contained in the support of f and is not contained in the support of f' (or vice versa). Thus, $E^*/k^* \otimes \mathbb{Z}_\ell$ is the unique subgroup of commuting elements which surjects isomorphically onto its image in the quotient $\mathcal{FS}_X(K)/\mathcal{T}_\ell(K)$.

Hence Δ^* acts as the identity on the section $K^*/k^* \otimes \mathbb{Z}_\ell \subset \mathcal{FS}_X(K)$. Since $K^*/k^* \otimes \mathbb{Z}_\ell$ is dense in $\hat{K}^*$, the map Δ^* is the identity. □

16. Projective structure

In Section 15 we have proved that

$$c\Psi^*(L^*/l^*) \subset K^*/k^* \otimes \mathbb{Z}_{(\ell)} \supset K^*/k^*$$

for some $c \in \mathbb{Z}_\ell^*$. Let $\mathfrak{M}^* := c\Psi^*(L^*/l^*) \cap (K^*/k^*)$ be the intersection. Then $\mathfrak{M}^* \subset K^*/k^*$ and $(c\Psi^*)^{-1}(\mathfrak{M}^*) \subset L^*/l^*$ satisfy all conditions of Proposition 15.7. Moreover, the full preimages of these groups to K^*, resp. L^*, satisfy the conditions of Proposition 2.13. Therefore, there exist subfields $K_1 \subset K$ and $L_1 \subset L$ so that K/K_1 and L/L_1 are finite purely inseparable extensions and

$$c\Psi^*(L_1/l^*) = \mathfrak{M}^* = K_1^*/k^*.$$

The sets $c\Psi^*(L_1/l^*)$ and K_1^*/k^* carry canonical projective structures coming from field structures of L_1 and K_1. A priori, this induces two projective structures on $\mathfrak{M}^*$. The last essential step is to show that these structures on $\mathfrak{M}^*$ coincide. It suffices to show that primary lines in both structures are the same on $\mathfrak{M}^*$ (see Definition 2.4 and Definition 3.7).

LEMMA 16.1. — *Let $x \in K^*$ be a generating element, $E := k(x)$ and $r = r(x) \in \mathbb{N}$ the smallest positive integer such that x^r, modulo k^* is in $\mathfrak{M}^*$. Then*

- *$r = p^m$ for some $m \in \mathbb{N}$ (with $p = \mathrm{char}(k)$);*
- *$(E^*/k^*) \cap \mathfrak{M}^* = (E^{p^m})^*/k^*$;*
- *(pointwise) p^m-th powers of primary lines in E^*/k^* coincide with primary lines in $(E^{p^m})^*/k^*$.*

Proof. — The first property follows since K/K_1 is a finite purely inseparable extension, by Propositions 2.13 and 15.7. Next, we claim that a generating element $y \in K_1$ (see 2.4) is a p^m-th power of a generating element of K (for some m depending on y). Indeed, $E := \overline{k(y)}^K \subset K$ is a finite and purely inseparable extension of $k(y)$, $E := k(x)$ (for some $x \in K$). Thus

$$y = (ax^{p^m} + b)/(cx^{p^m} + d) = ((a'x + b')/(c'x + d'))^{p^m}$$

for some $m \in \mathbb{Z}$, $a, b, c, d \in k$ and their p^m-th roots $a', b', c', d' \in k$ (since k is algebraically closed).

In particular, a generating element $y \in K_1$ is in $E^*/k^* \cap \mathfrak{M}^*$ (and is the minimal positive power of a generator in E contained in $E^*/k^* \cap \mathfrak{M}^*$). This implies the third property: the generating elements of E^{p^m} are p^m-th powers of generators of E. □

LEMMA 16.2. — *The isomorphism* $c\Psi^* : L_1^*/l^* \to K_1^*/k^*$ *induces isomorphisms of multiplicative groups*

$$c\Psi^* : l(t)^*/l^* \to k(x)^*/k^*,$$

where $l(t)$*, resp.* $k(x)$ *are normally closed one-dimensional subfields in* L_1*, resp.* K_1*, inducing a bijection on (the images of) generating elements of the corresponding fields.*

Proof. — For elements of $l(t)^*/l^*$, resp. $k(x)^*/k^*$, we have a Galois-theoretic notion of divisorial "support". This characterizes elements of minimal, by inclusion, divisorial support. These elements have also minimal support on $\mathbb{P}^1_x$ and hence their support on $\mathbb{P}^1_x$ consists of two points. Thus they are powers of the images of generating elements in $k(x)$. Among all elements with fixed minimal divisorial support we distinguished the primitive elements (with respect to multiplication). These primitive elements are generating elements of L_1, resp. K_1, and $c\Psi^*$ establishes a bijection on (images in L_1^*/l^*, resp. K_1^*/k^*, of) generating elements. □

COROLLARY 16.3. — *The isomorphism* $c\Psi^* : L_1^*/l^* \to K_1^*/k^*$ *identifies primary lines of the corresponding projective structures.*

Proof. — By Corollary 15.9 we can Galois-theoretically distinguish zeroes and poles of elements in L_1^*/l^* and K_1^*/k^*. By Lemma 16.2, if $l(t)$, resp. $k(x)$, is a normally closed one-dimensional subfield in L_1, resp. K_1, then the restriction

$$c\Psi^* : l(t)^*/l^* \to k(x)^*/k^*$$

induces a bijection on (the images of) generating elements which have the same poles. The set of elements of $l(t)$, resp. $k(x)$, with the same pole is a primary line in $\mathbb{P}_l(L_1)$, resp. $\mathbb{P}_k(K_1)$. In particular, $c\Psi^*$ identifies the primary lines in the projective structures on $\mathcal{M}^*$. □

By Corollary 3.10, we conclude:

COROLLARY 16.4. — *The isomorphism* $c\Psi^*$ *induces an isomorphism of fields*

$$L \supset L_1 \simeq K_1 \subset K,$$

and of perfect closures of L *and* K*.*

This isomorphism will be denoted by $\Psi_1^* : L_1 \to K_1$. It induces a homomorphism $\Psi_1 : \mathcal{G}_K^a = \mathcal{G}_{K_1}^a \to \mathcal{G}_{L_1}^a = \mathcal{G}_L^a$, inducing a bijection $\Sigma_K \to \Sigma_L$. Consider

$$\Delta := \Psi_1^{-1} \circ \Psi.$$

LEMMA 16.5. — *The isomorphism Δ satisfies the conditions of Lemma 15.10.*

Proof. — Indeed, if we start our reconstruction procedure with Ψ_1^*, which is induced by an isomorphism of fields $L_1 \to K_1$, we will arrive, after passage to the Galois group, then its dual $\hat{L}^*$, then $\mathcal{FS}_Y(L)$, at the same isomorphism

$$\Psi_1^* : L/l \otimes \mathbb{Z}_\ell = L_1^*/l^* \otimes \mathbb{Z}_\ell \to K^*/k^* \otimes \mathbb{Z}_\ell.$$

□

17. Proof

In this section we prove our main theorem: if

$$\Psi : (\mathcal{G}_K^a, \Sigma_K) \xrightarrow{\sim} (\mathcal{G}_L^a, \Sigma_L),$$

is an isomorphism, where L is a function field over an algebraic closure of a finite field of characteristic $\neq \ell$, then K is a purely inseparable extension of L. Moreover, for some $c \in \mathbb{Z}_\ell^*$, $c\Psi$ is induced by an isomorphism

$$\bar{\Psi}^* : L^{\mathrm{perf}} \to K^{\mathrm{perf}}$$

of the perfect closure of L with the perfect closure of K.

Step 1. By Theorem 5.5 we have pairing

$$\mathcal{G}_K^a \times \hat{K}^* \to \mathbb{Z}_\ell(1),$$

exhibiting $\hat{K}^*$ as the dual of $\mathcal{G}_K^a$. This induces the dual isomorphism

$$\Psi^* : \hat{L}^* \to \hat{K}^*.$$

Step 2. In Sections 3-8 we characterize intrinsically the inertia and decomposition groups of divisorial valuations:

$$\mathcal{I}_\nu^a \subset \mathcal{D}_\nu^a \subset \mathcal{G}_K^a :$$

every liftable subgroup $\sigma \in \Sigma_K^{\mathrm{div}} \subset \Sigma_K$ contains an inertia element of a divisorial valuation (which is also contained in at least one other $\sigma' \in \Sigma_K$). The corresponding decomposition group is the "centralizer" of the (topologically) cyclic inertia group (the set of all elements which "commute" with inertia).

By assumption, the isomorphism Ψ of Galois groups induces a bijection on the sets of maximal topologically noncyclic liftable subgroups. This gives a bijection of sets of divisorial valuations of the corresponding fields

$$\Psi \,:\, \mathcal{DV}_K \to \mathcal{DV}_L,$$

and induces a canonical isomorphism of Galois groups of the residue fields

$$\Psi_\nu \,:\, \mathcal{D}_\nu^a/\mathcal{I}_\nu^a = \mathcal{G}_{\boldsymbol{K}_\nu}^a \to \mathcal{G}_{\boldsymbol{L}_{\Psi(\nu)}}^a,$$

for all $\nu \in \mathcal{DV}_K$.

Step 3. For every $\nu \in \mathcal{DV}_K$ the isomorphism Ψ_ν defines a canonical isomorphism ψ_v from $\mathcal{G}^a_{K_\nu}$ to $\mathcal{G}^a_{L_{\psi(\nu)}}$. This gives a bijection on points on smooth models of these one-dimensional fields (see Proposition 9.3). In practical terms, this establishes a bijection on the sets of all curves, and all points on these curves, on all models of K, resp. L. This bijection does not change when Ψ is multiplied by a constant $c \in \mathbb{Z}_\ell^*$ and under purely inseparable extensions of K or L.

Step 4. We distinguish divisorial valuations with nonrational centers (see Lemma 9.4 and Remark 9.5).

Step 5. For $\hat{f} \in \hat{K}^*$ we have two notions of support: $\mathrm{supp}_K(\hat{f})$ (intrinsic) and $\mathrm{supp}_X(\hat{f})$ (depending on a model X) and two notions of finiteness: $\hat{f}$ is nontrivial on at most finitely many nonrational divisorial valuations ν, resp. $\hat{f}$ has finite divisorial support on a model. We defined $\mathcal{FS}(K) \subset \hat{K}^*$ as the subgroup of elements satisfying the first notion of finiteness, and $\mathcal{FS}_X(K) \subset \hat{K}^*$ as the subgroup of elements satisfying the second notion (this subgroup does not depend on the choice of a model X of K). By Step 4, the characterization of $\mathcal{FS}(K)$ is Galois-theoretic and Ψ^* induces an isomorphism

$$\Psi^* \,:\, \mathcal{FS}(L) \to \mathcal{FS}(K).$$

Step 6. If some (any) model X of K contains only finitely many rational curves then $\mathcal{FS}(K) = \mathcal{FS}_X(K)$. A priori, the Ψ^*-image of some $g \in L^*/l^*$ could have an "infinite rational tail" on some (every) model X of K:

$$\hat{\rho}_X(\Psi^*(g)) = \hat{\rho}_{X,\mathrm{nr}}(\Psi^*(g)) + \sum_{j\geq 1} n_j C_j,$$

where C_j are irreducible rational curves on X. We exclude this possibility - in Proposition 12.10 we obtain a canonical isomorphism

$$\Psi^* : \mathcal{FS}_Y(L) \to \mathcal{FS}_X(K),$$

where Y is a model of L and X a model of K, as well as the induced canonical isomorphism

$$\Psi^* : \mathcal{T}_\ell(L) \to \mathcal{T}_\ell(K).$$

The diagram after Proposition 12.10 shows that in order to prove that Ψ^* induces a canonical isomorphism

$$L^*/l^* \otimes \mathbb{Z}_\ell \to K^*/k^* \otimes \mathbb{Z}_\ell$$

(of the quotients by $\mathcal{T}_\ell$ inside $\mathcal{FS}_X(K)$, see the diagram at the end of Section 12), it suffices to show that Ψ^* induces an isomorphism between the kernels of $\varphi_{\ell,L} : \mathcal{FS}_Y(L) \to \mathrm{Pic}^0(Y)_\ell$ and of $\varphi_{\ell,K} : \mathcal{FS}_X(K) \to \mathrm{Pic}^0(X)_\ell$.

Step 7. By Proposition 15.1, for every pair of elements $\hat{f}, \hat{g} \in \mathcal{FS}_X(K)$ satisfying

- $\mathrm{supp}_K(\hat{f}) \cap \mathrm{supp}_K(\hat{g}) = \emptyset$;
- $\varrho_\nu(\hat{f}, \hat{g}) = 1$ for all $\nu \in \mathcal{DV}_K$

there exists a subfield $E = k(C) \subset K$ such that $\hat{f}, \hat{g}$ (modulo elements with trivial support, i.e., elements in $\mathcal{T}_\ell(K)$, defined in Section 11) lie in the lattice $E^*/k^* \otimes \mathbb{Z}_\ell \subset K^*$. The isomorphism Ψ^* identifies subgroups $\hat{E}^* \cap \mathcal{FS}_X(K)$ for normally closed subfields $E = k(x) = k(C) \subset K$ with primitive $\widehat{\mathrm{Div}}^0(C)$ in $\widehat{\mathrm{Div}}^0(X)$ with similar subgroups in $\mathcal{FS}_Y(L)$. The one-dimensional fields $E \subset K$ and the corresponding subfields of L have similar properties. The corresponding subgroups generate the kernel $K^*/k^* \otimes \mathbb{Z}_\ell$ of the map

$$\bar{\varphi}_{\ell,K} : \mathcal{FS}_X(K)/\mathcal{T}_\ell(K) \to \mathrm{Pic}^0(K)_\ell.$$

Thus Ψ^* induces a canonical isomorphism of the kernels and defines a canonical isomorphism

$$L^*/l^* \otimes \mathbb{Z}_\ell \to K^*/k^* \otimes \mathbb{Z}_\ell.$$

Step 8. We apply Proposition 13.1 to a normally closed subfield $E \subset K$ of the form $E = k(x)$. This identifies E^*/k^* inside $E^*/k^* \otimes \mathbb{Z}_\ell \subset K^*/k^* \otimes \mathbb{Z}_\ell$, up to conformal equivalence with respect to multiplication by elements in $\mathbb{Z}_\ell^*$. More precisely, there exist an $c \in \mathbb{Z}_\ell^*$, and a $y \in L^* \backslash l^*$, so that $l(y)$ is normally closed in L and

$$c \cdot \Psi^* \,:\, l(y)^*/l^* \to k(x)^*/k^*$$

is an isomorphism of multiplicative groups.

Step 9. Let $\mathfrak{M}^* = c\Psi^*(L^*/l^*) \cap K^*/k^*$. Then $\mathfrak{M}^*$ satisfies the conditions of Proposition 15.7. In particular, we obtain the isomorphism

$$c \cdot \Psi^* \,:\, L^*/l^* \otimes \mathbb{Z}_{(\ell)} \to K^*/k^* \otimes \mathbb{Z}_{(\ell)}.$$

By Proposition 2.13, we have finite purely inseparable extensions K/K_1 and L/L_1 such that $\mathfrak{M}^* = K_1^*/k^*$ and $\mathfrak{M}^* = c\Psi^*(L_1^*/l^*)$, as multiplicative groups. Thus, $\mathfrak{M}^*$ carries two structures of an abstract projective space compatible with the multiplicative structure (see Example 3.5), induced from the additive structure on K_1, resp. L_1.

Step 10. By Theorem 3.6 the field is uniquely determined by the partial projective structure on $\mathfrak{M}^*$ consisting of primary lines (see Lemma 3.8 and Lemma 3.9).

Step 11. Corollary 16.3 shows that the map $c\Psi^*$ identifies primary lines of these two structures. This defines a unique projective structure on $\mathfrak{M}^*$, compatible with multiplication. It follows that $c\Psi^*$ induces an isomorphism of fields

$$L \supset L_1 \simeq K_1 \subset K,$$

denoted by Ψ_1^*, and of perfect closures of L and K. We claim that Ψ_1^* is dual to Ψ (up to multiplication by a constant in $\mathbb{Z}_\ell^*$). Since L_1^* generates $L_1^*/l^* \otimes \mathbb{Z}_\ell = L^*/l^* \otimes \mathbb{Z}_\ell$ the map Ψ_1^* coincides with the map

$$c\Psi^* \,:\, L^*/l^* \otimes \mathbb{Z}_\ell \to K^*/k^* \otimes \mathbb{Z}_\ell.$$

By Kummer theory, L^*/l^* (and K^*/k^*) is a free module and

$$\mathcal{G}^a_{L_1} = \mathrm{Hom}(L_1^*, \mathbb{Z}_\ell) = \mathrm{Hom}(L_1 \otimes \mathbb{Z}_\ell, \mathbb{Z}_\ell) = \mathcal{G}^a_L.$$

Thus $\Psi_1^* = c\Psi^*$ and defines a unique

$$\Psi_1 : \mathcal{G}^a_K \to \mathcal{G}^a_L.$$

Step 12. It remains to show that $\Psi_1 = \Psi$ (modulo multiplication by a constant). This follows from Lemma 15.10 and Lemma 16.5.

This concludes the proof of Theorem 1.

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