Abelian fibrations and rational points on symmetric products

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Abstract

Given a variety over a number field, are its rational points potentially dense, i.e., does there exist a finite extension over which rational points are Zariski dense? We study the question of potential density for symmetric products of surfaces. Contrary to the situation for curves, rational points are not necessarily potentially dense on a sufficiently high symmetric product. Our main result is that rational points are potentially dense for the Nth symmetric product of a K3 surface, where N is explicitly determined by the geometry of the surface. The basic construction is that for some N, the Nth symmetric power of a K3 surface is birational to an abelian fibration over $\mathbb{P}^N$. It is an interesting geometric problem to find the smallest N with this property.

1 Introduction

Let X be an algebraic variety defined over a number field K and $X(K)$ its set of K-rational points. We are interested in properties of $X(K)$ imposed by the global geometry of X. We say that rational points on X are potentially dense if there exists a finite field extension L/K such that $X(L)$ is Zariski dense. It is expected - at least for surfaces - that if there are no finite étale covers of X dominating a variety of general type then rational points on X are potentially dense. This expectation complements the conjectures of Bombieri, Lang and Vojta predicting that rational points on varieties of general type are always contained in Zariski closed subsets. This dichotomy holds for curves: the nondensity for curves of genus ≥ 2 is a deep theorem of Faltings and the potential density for curves of genus 0 and 1 is classical.

In higher dimensions there are at present no general techniques to prove nondensity. Of course, potential density holds for abelian and unirational varieties. Beyond this, density results rely on the classification and explicit projective geometry of the classes of varieties under consideration. In dimension two potential density is unknown for K3 surfaces with finite automorphisms and without elliptic fibrations (see [6]). In dimension 3 potential density is unknown, for example, for double covers $W_2 \rightarrow \mathbb{P}^3$ ramified in a smooth surface of degree 6, for general conic bundles, as well as for Calabi-Yau varieties (for density results see [12], [5]).

In this paper we study density properties of rational points on symmetric products $X^{(n)} = X^n/\mathbb{S}_n$. If C is a curve of genus g and $n > 2g-2$ the symmetric product admits a bundle structure over the Jacobian $\mathrm{Jac}(C)$, with fibers projective spaces $\mathbb{P}^{n-g}$. We see that in this case rational points on $C^{(n)}$ are potentially dense. Contrary to the situation for curves, we are not guaranteed to find many rational points on sufficiently high symmetric products of arbitrary surfaces. In Section 2 we show that if the Kodaira dimension of a smooth surface X is equal to k then the Kodaira dimension of $X^{(n)}$ is is equal to nk. This leads us to expect the behavior of rational points on $X^{(n)}$ and X to be quite similar. At the same time we observe that symmetric products of K3 surfaces admit (at least birationally) abelian fibrations over projective spaces. In fact, even symmetric squares of certain (nonelliptic) K3 surfaces have the structure of abelian surface fibrations over $\mathbb{P}^2$. This is the starting point for proofs of potential density of rational points.

Let us emphasize that if X is a variety over a number field K then Zariski density of rational points on X defined over degree n field extensions of K is not equivalent to Zariski density of K-rational points on $X^{(n)}$. Of course, the first condition is weaker than the second. Furthermore, if rational points on X are potentially dense then they are potential dense on $X^{(n)}$ as well.

This paper is organized as follows. In Section 2 we recall general properties of symmetric products and Hilbert schemes of surfaces. Section 3 sets up generalities concerning abelian fibrations $\mathcal{A} \rightarrow B$. Potential density for $\mathcal{A}$ follows once one can find a "nondegenerate" multisection for which potential density holds. In Section 4 we prove widely-known results concerning the existence of elliptic curves on K3 surfaces. Then we turn to potential density for symmetric products of K3 surfaces. First, in Sections 5 and 6, we prove potential density for sufficiently high symmetric powers of arbitrary K3 surfaces. This is followed in Section 7 with more precise results for symmetric squares of K3 surfaces of degree $2m^2$.

Throughout this paper, *generic* means 'in a nonempty Zariski open subset' whereas *general* means 'in the complement of a countable union of Zariski closed proper subsets.'

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2 Generalities on symmetric products

Let X be a smooth projective variety over a field K. Denote by $X^n = X \times_K ... \times_K X$ the n-fold product of X. The symmetric group $\mathbb{S}_n$ acts on X^n. The quotient $X^{(n)} = X^n/\mathbb{S}_n$ is a projective variety, called the symmetric product.

If X has dimension one then $X^{(n)}$ is smooth and for $n > 2g-2$ the symmetric product $X^{(n)}$ is a projective bundle over the Jacobian $\mathrm{Jac}(X)$, with fibers projective spaces of dimension $n-g$ (see [22], Ch. 4). In particular, rational points on $X^{(n)}$ are potentially dense for $n > 2g-2$.

If X has dimension two then $X^{(n)}$ is no longer smooth. It has Gorenstein singularities since the group action factors through the special linear group (see [31]), i.e., for any point with nontrivial stabilizer, the induced representation of the stabilizer on the tangent space factors through the special linear group. The Hilbert scheme of length n zero-dimensional subschemes is a crepant resolution of $X^{(n)}$

$$\varphi : X^{[n]} \to X^{(n)}$$

(see [2], Section 6 and the references therein). In particular, $\varphi^*\omega_{X^{(n)}} = \omega_{X^{[n]}}$. The same holds for pluricanonical differentials. On the other hand, we have the isomorphism

$$H^0(X^n, \omega_{X^n}^m)^{\mathbb{S}_n} = H^0(X^{(n)}, \omega_{X^{(n)}}^m).$$

We are using the fact that the quotient map $X^n \to X^{(n)}$ is unramified away from a codimension two subset and pluricanonical differentials extend over codimension two subsets. We conclude that pluricanonical differentials on the Hilbert scheme correspond to $\mathbb{S}_n$-invariant differentials on the n-fold product X^n:

$$H^0(X^{[n]}, \omega_{X^{[n]}}^m) \simeq H^0(X^n, \omega_{X^n}^m)^{\mathbb{S}_n}.$$

Since

$$H^0(X^n, \omega_{X^n}^m)^{\mathbb{S}_n} \simeq \mathrm{Sym}^n H^0(X, \omega_X^m)$$

we obtain the following:

Proposition 2.1 *Let X be a smooth surface. If X has Kodaira dimension k then $X^{(n)}$ has Kodaira dimension nk.*

Remark 2.2 Arapura and Archava have recently proved a more general statement [1].

If X is a K3 surface we can be more precise: $X^{[n]}$ is a holomorphic symplectic manifold (see [2], Section 6). In particular, the canonical bundle of $X^{[n]}$ remains trivial.

An important ingredient in the proofs of potential density is the construction of a multisection of the abelian fibration. The following proposition will help us verify that certain subvarieties are multisections:

Proposition 2.3 *Let X be a smooth projective surface and $C_1, ..., C_n$ distinct irreducible curves. Consider the image Z of $C_1 \times ... \times C_n$ under the quotient map $X^n \to X^{(n)}$. The scheme-theoretic preimage $\varphi^{-1}(Z) \subset X^{[n]}$ has a unique irreducible component of dimension $\geq n$, denoted by $C_1 * ... * C_n$. In particular, the homology class of $C_1 * ... * C_n$ is uniquely determined by the homology classes of $C_1, ..., C_n$.*

Proof. Let $(a_1, .., a_k)$ be a partition of n and let $\mathcal{D}_{a_1,...,a_k}$ be the corresponding stratum in $X^{(n)}$. In particular, the $\mathcal{D}_{a_1,...,a_k}$ are disjoint. The intersection of Z with $\mathcal{D}_{a_1,...,a_k}$ has dimension at most $\#\{a_j \mid a_j = 1\}$. Each fiber of φ over $\mathcal{D}_{a_1,\ldots,a_k}$ is irreducible of dimension $\sum_{j=1}^{k}(a_j - 1)$ (see [7]). It follows that the preimage of $Z \cap \mathcal{D}_{a_1,\ldots,a_k}$ has dimension at most

$$\#\{a_j \,|\, a_j = 1\} + \sum_{j=1}^{k}(a_j - 1),$$

which is less than n, provided the a_j are not all equal to 1. □

3 Generalities on abelian fibrations

Let $\mathcal{A}$ be an abelian variety defined over a field K (not necessarily a number field). A point $\sigma \in \mathcal{A}(K)$ is *nondegenerate* if the subgroup generated by σ is Zariski dense in $\mathcal{A}$.

Proposition 3.1 *Let $\mathcal{A}$ be an abelian variety over a number field K. Then there exists a finite field extension L/K such that $\mathcal{A}(L)$ contains a nondegenerate point.*

Proof. We include an argument for completeness, since we could not find a reference.

Lemma 3.2 Let $\mathcal{A}$ be an abelian variety of dimension $\dim(\mathcal{A})$ defined over a number field K. Then there exists a finite field extension L/K such that the rank of the Mordell-Weil group $\mathcal{A}(L)$ is strictly bigger than the rank of $\mathcal{A}(K)$.

Proof. As pointed out to us by M. Jarden, this follows from Theorem 10.1 of [9] and the subsequent remark. We provide an alternate proof suggested by B. Mazur.

We first assume $\dim(\mathcal{A}) > 1$. Let Γ be the *saturation* of $\mathcal{A}(K)$ in $\mathcal{A}(\bar{K})$ (where $\bar{K}$ is the algebraic closure of K). This means that Γ consists of all points p such that a positive multiple of p lies in $\mathcal{A}(K)$; in particular it contains all torsion points. Find a smooth curve C of genus ≥ 2 in $\mathcal{A}$, defined over a finite extension K_1/K. By Raynaud's version of the Manin-Mumford conjecture (see [16], I 6.4 or [26] Theorem 1) we have that $C \cap \Gamma$ is finite. There exists a L/K_1 such that $C(L)$ contains a point q outside $C \cap \Gamma$. It follows that $\mathcal{A}(L)$ has higher rank.

We now do the case of an elliptic curve $\mathcal{E}$. Write $\mathcal{A} = \mathcal{E} \times \mathcal{E}$ with projections π_1 and π_2; we have $\mathcal{A}(K) = \mathcal{E}(K) \times \mathcal{E}(K)$. The argument above gives a point $q \in \mathcal{A}(L)$ not contained in the saturation of $\mathcal{A}(K)$. It follows that either $\pi_1(q)$ or $\pi_2(q)$ is not contained in the saturation of $\mathcal{E}(K)$. $\square$

We prove the proposition. We may replace $\mathcal{A}$ with an isogenous abelian variety, so we may assume that $\mathcal{A}$ is a product of geometrically simple abelian varieties. Our proof proceeds by induction on the number of simple components. Any nontorsion point p of a geometrically simple abelian variety is nondegenerate. Indeed, the Zariski closure of $\mathbb{Z}p$ is a finite union of translates of abelian subvarieties. Hence it suffices to prove the inductive step:

Lemma 3.3 Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be abelian varieties over a number field K. Assume that $\mathcal{A}_2$ is geometrically simple and $\mathcal{A}_1$ and $\mathcal{A}_2$ have nondegenerate K-points p_1 and p_2. Then $\mathcal{A}_1 \times \mathcal{A}_2$ has a nondegenerate point over some finite extension L/K.

Proof. For any pair of abelian varieties $\mathcal{A}_1, \mathcal{A}_2$ the group of homomorphisms $\mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2)$ is finitely generated as a module over $\mathbb{Z}$. After a finite extension, we may assume these are all defined over K. We also consider $\mathrm{Hom}^0(\mathcal{A}_1, \mathcal{A}_2) := \mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2) \otimes \mathbb{Q}$, the group of homomorphisms defined up to isogeny (see [25], p. 172-176).

Assume that (p_1, p_2) is contained in a proper abelian subvariety $\mathcal{B} \subsetneq \mathcal{A}_1 \times \mathcal{A}_2$. Note that the projections $\pi_i|\mathcal{B}$ are surjective. Let $\mathcal{K}_1 \subset \mathcal{B}$ be the kernel of $\pi_1|\mathcal{B}$, which may be regarded as an abelian subvariety of $\mathcal{A}_2$. A dimension count shows that $\mathcal{K}_1 \subsetneq \mathcal{A}_2$, hence $\mathcal{K}_1$ is finite (because $\mathcal{A}_2$ is simple). It follows that $\pi_1|\mathcal{B}$ is an isogeny and we can regard $\mathcal{B}$ as an element $\beta \in \mathrm{Hom}^0(\mathcal{A}_1, \mathcal{A}_2)$. In particular, $(d\beta)(p_1) = dp_2$ for some nonzero integer d. We choose a $\mathbb{Z}$-basis $(Z_1, ..., Z_k)$ for $\mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2)$. There exist integers $b_1, ..., b_k$, such that $(b_1 Z_1 + ... + b_k Z_k)(p_1) = dp_2$ in the Mordell-Weil group. Hence p_2 is contained in the saturation of the subgroup of $\mathcal{A}_2(K)$ generated by the images of p_1 under the Z_i. Conversely, if q is not contained in this subgroup then (p_1, q) is nondegenerate. Applying Lemma 3.2, we obtain a finite field extension L/K and a point $q \in \mathcal{A}_2(L)$ with the desired property. $\square$

Let $\mathcal{T}$ be an $\mathcal{A}$-torsor defined over a field K, i.e., there is an action $\mathcal{A} \times \mathcal{T} \to \mathcal{T}$ so that the induced map

$$\mathcal{A} \times \mathcal{T} \to \mathcal{T} \times \mathcal{T} \quad (a, t) \to (at, t)$$

is an isomorphism (here all morphisms and fiber products are defined over K.) In particular, if M/K is a finite extension and $p \in \mathcal{T}(M)$ then the induced map restricts to an isomorphism $\mathcal{A}(M) \rightarrow \mathcal{T}(M)$.

Consider the Albanese $\mathrm{Alb}(\mathcal{T})$ (see, for example, [15] II. 3). It is an abelian variety defined over K, such that there is a morphism $\mathcal{T} \times \mathcal{T} \rightarrow \mathrm{Alb}(\mathcal{T})$ corresponding to $(t_1, t_2) \rightarrow t_1 - t_2$. For each zero-cycle of $\mathcal{T}$, defined over K and of degree zero, we obtain a point in $\mathrm{Alb}(\mathcal{T})(K)$. Interpret $\mathcal{T}$ as the zero-cycles on $\mathcal{T}$ of degree one, so that the addition map $\mathrm{Alb}(\mathcal{T}) \times \mathcal{T} \rightarrow \mathcal{T}$ makes $\mathcal{T}$ into an $\mathrm{Alb}(\mathcal{T})$-torsor as well. In particular $\mathcal{A}$ and $\mathrm{Alb}(\mathcal{T})$ are both isomorphic over K to the identity component of the automorphism group of $\mathcal{T}$.

Let $p \in \mathcal{T}(M)$ where M/K is a finite extension of degree $\deg(M)$. Regarding $\mathrm{Spec}(M) \rightarrow \mathcal{T}$ as a morphism of K-schemes, we obtain a zero-cycle on $\mathcal{T}$ of degree $\deg(M)$, defined over K. This pulls back to a zero-cycle on $\mathcal{T}(M)$ denoted tr_M. The zero-cycle $\tau_M := \deg(M)p - \mathrm{tr}_M$ has degree zero and thus gives an element of $\mathrm{Alb}(\mathcal{T})(M)$. We shall say that $p \in \mathcal{T}(M)$ is *nondegenerate* if τ_M is nondegenerate.

Let $\pi : \mathcal{T} \rightarrow B$ be an abelian fibration, that is: $\mathcal{T}$ and B are normal, B is connected, and the fiber $\mathcal{T}_b$ over the generic point b is a torsor for an abelian variety $\mathcal{A}_b$ over $K(b)$. A multisection $\mathcal{M}$ of π is the closure of an M-valued point of $\mathcal{T}_b$, where M is a finite field extension of $K(b)$ of degree $\deg(M)$. It is nondegenerate if the corresponding M-valued point is nondegenerate.

Proposition 3.4 *Let $\pi : \mathcal{T} \rightarrow B$ be an abelian fibration with nondegenerate multisection $\mathcal{M}$, both defined over a number field K. Assume that K-rational points on $\mathcal{M}$ are Zariski dense. Then K-rational points on $\mathcal{T}$ are Zariski dense.*

Proof. We restrict to an open subset of B over which $\mathcal{T}$ and $\mathcal{M}$ are smooth and the torsor action $\mathcal{A} \times_B \mathcal{T} \rightarrow \mathcal{T}$ is well-defined. Let $p : \mathcal{M} \rightarrow \mathcal{T} \times_B \mathcal{M}$ be the section induced by the multisection. Our assumptions mean that $\mathcal{A} \times_B \mathcal{M}$ has a nondegenerate section $\tau_{\mathcal{M}}$. The translates $(n\tau_{\mathcal{M}})(p(\mathcal{M}))$ are defined over K and are Zariski dense in $\mathcal{T} \times_B \mathcal{M}$. Each translate has Zariski dense K-rational points, so we find that rational points in $\mathcal{T} \times_B \mathcal{M}$ are also Zariski dense. Since $\mathcal{T} \times_B \mathcal{M}$ dominates $\mathcal{T}$, rational points are dense in $\mathcal{T}$ as well. $\square$

Remark 3.5 Our argument does not show that rational points are Zariski dense in *any* fiber $\mathcal{T}_x$, where x is an K-rational point of B. However, when the fibers are of dimension 1 there exists a nonempty open subset $U \subset \mathcal{M}$ such that $(\mathcal{T} \times_B \mathcal{M})_x$ has dense rational points for each $x \in U(K)$ (see [30]). Moreover, by a result of Néron, the rank of the Mordell-Weil group of special fibers of abelian fibrations does not drop outside a thin subset of points on the base of the fibration [28].

4 Elliptic families on K3 surfaces

Throughout this section, we work over an algebraically closed field of characteristic 0. An elliptic fibration is an abelian fibration of relative dimension one. In the sequel an elliptic fibration dominating a K3 surface will be called an elliptic family.

The following theorem is attributed to Bogomolov and Mumford (see [23]). We include a detailed proof because it is crucial for our applications.

Theorem 4.1 *Let S be K3 surface and f a divisor class on S such that $h^0(\mathcal{O}_S(f)) > 1$. Then there exists a smooth curve B and an elliptic fibration $\mathcal{E} \rightarrow B$ with the following properties:*

1. *$\mathcal{E}$ dominates S;*
2. *the generic fiber $\mathcal{E}_b$ is mapped birationally onto its image;*
3. *the class $f - \mathcal{E}_b$ is effective.*

Proof. A genus one curve $C \subset S$ is a curve whose normalization $\tilde{C}$ is a connected curve of genus one. It suffices to prove the result for a singular curve B; we can always pull back to the normalization $\tilde{B}$.

We may restrict to the case where S is not an elliptic K3 surface. We assume that $|f|$ has no fixed components (and thus no base points). Indeed, if this is not the case then we extract the moving part of f. Since S is not elliptic, we have $f^2 > 0$. We may also assume that the class f is primitive; otherwise, take the primitive effective generator f' such that $f \in \mathbb{Z}f'$. We still have $h^0(\mathcal{O}(f')) > 1$ and $|f'|$ basepoint free (again, using the fact that S is not elliptic.) See [29] for basic results concerning linear series on K3 surfaces.

We shall use the following lemma, essentially proved in [23]:

Lemma 4.2 For each $n > 0$, a generic polarized K3 surface (S_1, f) of degree $2n$ contains a one-parameter family of irreducible curves with class f, such that the generic member is nodal of genus one.

Proof. We first claim there exists a K3 surface S_0 containing two smooth rational curves D_1 and D_2 meeting transversally at $n+2$ points. Let S_0 be the Kummer surface associated to the product of elliptic curves E_1 and E_2, such that there exists an isogeny $E_1 \rightarrow E_2$ of degree $2n+5$. Let Γ be the graph of this isogeny and $p \in E_2$ a 2-torsion point. Now Γ intersects $E_1 \times p$ transversally in $2n+5$ points, one of which is 2-torsion in $E_1 \times E_2$. We take D_1 to be the image of Γ and D_2 to be the image of $E_1 \times p$; D_1 and D_2 are smooth, rational, and intersect transversally in $n+2$ points. The line bundle $\mathcal{O}(f) := \mathcal{O}_{S_0}(D_1 + D_2)$ is big and nef and thus has no higher cohomology (by Kawamata-Viehweg vanishing).

Let Δ be the spectrum of a discrete valuation ring with closed point 0 and generic point η. Let $\mathcal{S} \rightarrow \Delta$ be a deformation of S_0 such that f remains algebraic. We assume further that the class f is ample and indecomposible in the monoid of effective curves in a (geometric) generic fiber S_1. These conditions are satisfied away from a finite union of irreducible divisors. Since f has no higher cohomology, $D_1 \cup D_2$ is a specialization of curves in the generic fiber and the deformation space $\mathrm{Def}(D_1 \cup D_2)$ is smooth of dimension $n+2$. Consider the locus in $\mathrm{Def}(D_1 \cup D_2)$ parametrizing curves with at least ν nodes; this has dimension $\geq n+2-\nu$. When $\nu = n+1$ the corresponding curves are necessarily rational. Each fiber of $\mathcal{S} \rightarrow \Delta$ is not uniruled, and thus contains a finite number of these curves. In each fiber, the rational curves with $n+1$ nodes deform to positive-dimensional families of curves with n nodes. Hence S_1 contains a family of nodal curves of genus one with the desired properties. $\square$

To complete the proof, we use a proposition suggested by Joe Harris:

Proposition 4.3 *Let $\mathcal{S} \rightarrow D$ be a projective morphism. Then there exists a scheme $\mathcal{K}_g(\mathcal{S}/D)$ such that each connected component is projective over D and the fiber over each $d \in D$ is isomorphic to the corresponding moduli space of stable maps $\mathcal{K}_g(\mathcal{S}_d)$.*

Proof. We refer to Kontsevich's moduli space of stable maps constructed in [14],[10]. We first consider the special case when $\mathcal{S} = \mathbb{P}^n_D$. Then

$$\mathcal{K}_g(\mathbb{P}^n_D/D) = \mathcal{K}_g(\mathbb{P}^n) \times D$$

More generally, given an embedding $\mathcal{S}\to\mathbb{P}^n_D$ over D, we define $\mathcal{K}_g(\mathcal{S}/D)$ as those elements of $\mathcal{K}_g(\mathbb{P}^n_D/D)$ which factor through $\mathcal{S}$. Since it is a closed subscheme it is projective over D. □

We finish the proof of Theorem 4.1. There exists a projective family of K3 surfaces $\mathcal{S}\to\Delta$ equipped with a divisor class f, such that the (geometric) generic fiber satisfies the conditions of Lemma 4.2 and the special fiber is (S, f). Consider the component $\mathcal{K}_1(\mathcal{S}/\Delta, f)$ of $\mathcal{K}_1(\mathcal{S}/\Delta)$ consisting of maps with image in the class f. After a finite base change $\Delta'\to\Delta$, there exists a geometrically irreducible curve $\mathcal{C}_\eta \subset \mathcal{K}_1(\mathcal{S}/\Delta, f)$ corresponding to an elliptic fibration dominating the generic fiber $\mathcal{S}_\eta$. Let $\mathcal{C} \subset \mathcal{K}_1(\mathcal{S}/\Delta, f)$ be the flat extension over Δ and $\mathcal{C}_0$ the corresponding flat limit.

There may not be a 'universal stable map' defined over $\mathcal{C}_0 \subset \mathcal{K}_1(S, f)$. However, for each irreducible reduced component $C_i \subset \mathcal{C}_0$, a universal stable map exists after a finite cover $B_i\to C_i$. (This follows from the existence of a universal stable map over the associated moduli stack.) For some such B_i, the resulting family of stable maps $\mathcal{E}'_i \to B_i$ dominates S. The image of the generic fiber contains a component of genus one because no K3 surface is uniruled. □

5 Density of rational points

In this section S denotes a K3 surface defined over a number field K. Potential density holds for elliptic K3 surfaces and for all but finitely many families of K3 surfaces with Picard group of rank ≥ 3, and consequently for their symmetric products (see [6]). However, a general K3 surface has Picard group of rank 1. In the following sections we will prove density results for symmetric products of general K3 surfaces.

By Theorem 4.1, there is a family of elliptic curves $\mathcal{E}$ dominating S. Let $E_1, \ldots, E_n$ be generic curves in the fibration and assume that $g = [E_i]$ is big; in particular, $\mathcal{E}$ is not an elliptic fibration on S. It follows that the generic member of g is an irreducible curve of genus > 1. Note that we have a well defined class $g * ... * g$ in the homology of $S^{[n]}$, equal to the homology class of $C_1 * ... * C_n$, where the C_i are irreducible curves in g (see Proposition 2.3).

Theorem 5.1 *Let S be a K3 surface satisfying the conditions of the previous paragraph. Assume that either*

1. *$S^{[n]}$ admits an abelian fibration $\mathcal{T} \to B$ and $g * ... * g$ intersects the proper transform of the generic fiber positively, or*

2. *$S^{[n]}$ is birational to an abelian fibration, and $E_1 * ... * E_n$ is a multisection.*

Then rational points on $S^{[n]}$ are potentially dense.

Proof. Throughout the proof, L/K is some finite field extension, which we will enlarge as necessary. We want to show that L-rational points are Zariski dense on $S^{[n]}$.

Under the first assumption, for any irreducible curves $C_1, ..., C_n$ in g, $C_1 * ... * C_n \subset S^{[n]}$ gives a multisection of $\mathcal{T} \to B$. In particular, $E_1 * ... * E_n$ is a multisection.

Choose a point $x \in B(L)$ corresponding to a smooth fiber $\mathcal{T}_x$ of $\mathcal{T}$. Choose a nondegenerate cycle in $\mathcal{T}_x$ of length n, represented by $s_1 + ... + s_n \in S^{[n]}$ (see Proposition 3.1). We may assume the s_i are distinct, that each s_i lies in a smooth fiber E_i of our elliptic family, that s_i and E_i are defined over L, and that L-rational points of E_i are Zariski dense. Then we have a multisection $\mathcal{M}$ for $\mathcal{T}$ given as (the proper transform of) $E_1 * ... * E_n$. Note that L-rational points on $\mathcal{M}$ are Zariski dense.

It follows that $\mathcal{M}$ satisfies the nondegeneracy assumptions of Proposition 3.4. Therefore, L-rational points are Zariski dense in $S^{[n]}$. □

We employed two parallel sets of hypotheses because in some applications the abelian fibration is only described over the generic point of B, which makes intersection computations difficult. In other applications, the abelian fibration is given by an explicit linear series, but the multisection is difficult to control.

Remark 5.2 Matsushita has proved a structure theorem for irreducible holomorphic symplectic manifolds of dimension $2n$ admitting a fibration structure. In particular, he proved that the base has dimension n, is Fano, has Picard group of rank 1, and log-terminal singularities. Furthermore, the fibers admit finite étale covers which are abelian varieties (see [19]).

Remark 5.3 We do not know how to produce abelian fibrations on symmetric products of Calabi-Yau varieties of dimension ≥ 3. For example, do they exist for quintic threefolds?

6 Potential density on $S^{[n]}$

In this section we exhibit K3 surfaces S defined over a number field K and satisfying the assumptions of Theorem 5.1.

Theorem 6.1 *Let S be a K3 surface defined over a number field K. Let g be a big line bundle on S of degree $2(n-1)$. Assume that $|g|$ contains the class of an irreducible elliptic curve. Then there exists a finite extension L/K such that L-rational points on $S^{[n]}$ are Zariski dense.*

Proof. Under our hypothesis g is basepoint free; the base locus of any linear series on a K3 surface has pure dimension one (see [29]). We obtain a morphism $S \rightarrow \mathbb{P}^n$ which is generically finite onto its image. Furthermore, the generic member of $|g|$ is smooth of genus n.

There is an abelian fibration over $B \subset \mathbb{P}^n$, where B corresponds to the locus of smooth curves in $|g|$. Indeed, $\mathcal{T} \rightarrow B$ is the degree n component of the relative Picard fibration. We claim that $S^{[n]}$ is birational to $\mathcal{T}$. Given generic points $s_1, ..., s_n$ on S there is a smooth curve $C \in |g|$ passing through those points. The line bundle $\mathcal{O}_C(s_1 + ... + s_n)$ is a generic point of $\mathrm{Pic}_n(C)$, and such a line bundle has a unique representation as an effective divisor. We are using the fact that $C^{[n]}$ is birational to $\mathrm{Pic}_n(C)$. (This idea can also be found in the work of Yau-Zaslow [32] and Beauville [4].)

To apply Theorem 5.1 we must verify that (the proper transform of) $E_1 * ... * E_n$ is a multisection for $\mathcal{T}$. A generic curve $C \in |g|$ intersects the union of the E_i transversally in $n(2n-2)$ points. Under these assumptions, every subscheme parametrized by $C^{[n]} \cap (E_1 * ... * E_n)$ is reduced and there are finitely many such subschemes. It particular, $C^{[n]}$ intersects $E_1 * ... * E_n$ in finitely many points. □

Theorem 6.2 *Let S be a K3 surface defined over a number field K and admitting a polarization f of degree $2(N-1)$. Then there exist a positive integer $n \leq N$ and a finite extension L/K such that the L-rational points of $S^{[n]}$ are Zariski dense.*

Proof. By Theorem 4.1 S is dominated by an elliptic fibration $\mathcal{E} \rightarrow B$, with $\langle \mathcal{E}_b, \mathcal{E}_b \rangle \leq \langle f, f \rangle = 2(N-1)$. Theorem 6.1 gives the result when $g = [\mathcal{E}_b]$ is big. If the class of the fiber is not big it has self-intersection zero, which implies that S is an elliptic K3 surface. In this case, the main theorem of [6] proves our claim with $n = 1$. □

Example 6.3 Let S be a K3 surface of degree 2. Then rational points on $S^{[2]}$ are potentially dense.

7 Potential density on $S^{[2]}$

Given a fixed K3 surface it is a natural problem to determine the smallest possible n for which the theorem holds. (Of course, we expect that we can always take $n = 1$!) As we have seen, the key to proving potential density is the existence of abelian fibrations on $S^{[n]}$.

The intersection form on the Picard group of S is an integer-valued nondegenerate quadratic form, denoted $\langle , \rangle$. We recall that the Picard group of $S^{[n]}$ is also equipped with a natural integer-valued nondegenerate quadratic form $(,)$, the Beauville form [2]. With respect to this form, we have an orthogonal direct sum decomposition

$$\mathrm{Pic}(S^{[n]}) = \mathrm{Pic}(S) \oplus_{\perp} \mathbb{Z}e,$$

where $(e, e) = -2(n-1)$ and $2e$ is the class of the diagonal (more precisely, the nonreduced subchemes in $S^{[n]}$.)

On the K3 surface S, the Picard group together with the quadratic form control much of the geometry of S. For example, if the quadratic form represents zero, then S admits an elliptic fibration over $\mathbb{P}^1$. A naive question would be whether the analog holds for $S^{[n]}$ with $n \geq 2$. More precisely, if the Beauville form represents zero, is $S^{[n]}$ birational to an abelian fibration over $\mathbb{P}^n$ (see [13])? Note that the Beauville form of $S^{[2]}$ represents zero if and only if the intersection form on $\mathrm{Pic}(S)$ represents $2m^2$ for some $m \in \mathbb{Z}$.

Proposition 7.1 *Let S be a generic K3 surface of degree $2m^2$ with $m > 1$. Then $S^{[2]}$ is isomorphic to an abelian surface fibration over $\mathbb{P}^2$.*

Proof. We first consider the case $m = 2$. We asssume that the polarization on S is very ample and that S does not contain a line or a cubic plane curve. Then S can be represented as a complete intersection of a three-dimensional space $\mathcal{I}_S(2)$ of quadrics in $\mathbb{P}^5$. An element of $S^{[2]}$ spans a line $\ell \in \mathbb{P}^5$ and a two dimensional subspace of $\mathcal{I}_S(2)$ contains ℓ. In this way, we obtain a morphism

$$a : S^{[2]} \rightarrow \mathbb{P}^2 \simeq \mathbb{P}(\mathcal{I}_S(2)^*).$$

The generic fiber of a is an abelian surface; the variety of lines on a smooth complete intersection of two quadrics in $\mathbb{P}^5$ is a principally polarized abelian surface (see [11], p. 779). Notice that a is induced by the sections of $f_8 - 2e$, where f_8 is the polarization of degree 8.

When $m > 2$ the proof consists of three steps:

1. construct special K3 surfaces S so that $S^{[2]}$ admits a natural involution;
2. show directly that some of these special K3 surfaces admit an abelian surface fibration and a polarization of degree $2m^2$;
3. verify that this abelian surface fibration deforms to the Hilbert scheme of a generic K3 surface of degree $2m^2$.

We begin with a construction of Beauville and Debarre [8]. Let $S \subset \mathbb{P}^3$ be a smooth quartic hypersurface; in particular, S is a K3 surface and the corresponding polarization is denoted f_4. Then there is a birational involution

$$j : S^{[2]} \dashrightarrow S^{[2]}$$

defined on an open subset of $S^{[2]}$ by the rule $j(p_1 + p_2) = p_3 + p_4$, where p_1, p_2, p_3, and p_4 are collinear points on S. This is a morphism provided that S does not contain a line. The action of j on the Picard group of $S^{[2]}$ is given by

$$j^*x = -x + (f_4 - e, x)\,(f_4 - e).$$

Next, we consider some special quartic K3 surfaces. Let S be a K3 surface with Picard group generated by the ample class f_4 and a second class f_8 satisfying

	f_4	f_8
f_4	4	k
f_8	k	8

where $k > 7$. Such K3 surfaces are parametrized by a nonempty analytic open subset of an irreducible variety of dimension 18. This follows from the Torelli theorem, surjectivity of Torelli, and the structure of the cohomology lattice of K3 surfaces (see [17] Theorem 2.4 and [3]). Note that f_4 is very ample and that the image is a smooth quartic surface not containing a line [29]; here we are using the fact that $k \neq 6$. Furthermore, the same reasoning shows that f_8 is very ample and the image does not contain a line, provided that f_8 is ample. (Here we are using the fact that $k \neq 7$.) If f_8 were not ample then $\langle f_8, C\rangle \leq 0$ for some (-2)-curve C (see [17] 1.6). Clearly $\langle f_8, C\rangle \neq 0$ and if $\langle f_8, C\rangle < 0$ then the Picard-Lefschetz reflection $\rho(f_8) = f_8 + \langle f_8, C\rangle C$

and f_4 generate a sublattice with discriminant greater than $32 - k^2$, which is impossible. Our argument in the $m = 2$ case shows that the $S^{[2]}$ admits an abelian surface fibration, induced by the line bundle $f_8 - 2e$. Composing with the involution j, we obtain a second abelian fibration, induced by

$$j^*(f_8 - 2e) = 2e - f_8 + (f_8 - 2e, f_4 - e)\,(f_4 - e) = (k-4)f_4 - f_8 - (k-6)e.$$

Let $g = (k-4)f_4 - f_8$ and $m = k - 6$ so that $\langle g, g\rangle = 2(k-6)^2 = 2m^2$ and $j^*(f_8 - 2e) = g - me$. Note that g is effective on S.

We turn to the last step. Let $\mathcal{S}\to\Delta$ be a general deformation of S for which g remains algebraic. The class g restricts to a polarization on the generic fiber, since it has Picard group of rank one. The class $g - me$ is algebraic (and nef) on the generic fiber of $\mathcal{S}^{[2]}\to\Delta$. Using deformation theory (see [13] and [27] Cor. 3.4), we find that the generic fiber also admits an abelian fibration with base $\mathbb{P}^2$, induced by the sections of the line bundle $g - me$. We are using the fact that the abelian surface fibration on $S^{[2]}$ is *Lagrangian*; see [13] for the fourfold case and [20] more generally. $\square$

Remark 7.2 Unfortunately, our argument gives little information about how the abelian fibration degenerates for nongeneric K3 surfaces of degree $2m^2$ with $m > 2$. A more precise description would follow from the conjectures of [13].

Remark 7.3 Proposition 7.1 gives a counterexample to the theorem in Section 2, p. 463 of [18]. There it is claimed that $S^{[2]}$ of a K3 surface S admits a (Lagrangian) abelian surface fibration if and only if S is elliptic.

Remark 7.4 We expect the arguments of Proposition 7.1 to generalize to higher symmetric products. More precisely, if S is generic of degree $2nm^2$ then $S^{[n+1]}$ should be birational to an abelian fibration over $\mathbb{P}^{n+1}$. As K. O'Grady pointed out, this is best understood as an application of the Fourier-Mukai transform (see [24]). Essentially, S is isogenous to a K3 surface $\hat{S}$ of degree $2n$, i.e., $\hat{S}$ may be interpretted as a moduli space of vector bundles on S. Applying the Fourier-Mukai transform to ideal sheaves of length $(n+1)$ subschemes of S, one should obtain sheaves supported on hyperplane sections of $\hat{S}$ which are invertible along their support. We have already seen that the relative Jacobian of $\hat{S}$ is birational to an abelian fibration over $\mathbb{P}^{n+1}$.

Theorem 7.5 *Let S_8 be a K3 surface of degree 8, defined over a number field K, embedded in projective space $\mathbb{P}^5$ as a complete intersection of 3 quadrics and not containing a line. Then rational points on $S_8^{[2]}$ are potentially dense. The same result holds for a generic K3 surface of degree $2m^2$.*

Proof. We apply Theorem 5.1, using the first set of assumptions. We use the abelian fibrations constructed in Proposition 7.1.

Let g be the homology class of an irreducible elliptic curve (see Theorem 4.1). We verify that $g * g$ intersects the class of a fiber positively.

We need to compute the intersection on $S^{[2]}$ of $(f-me)\cdot(f-me)\cdot(g*g)$, where f and g are divisor classes on S. Let Σ be the class of subschemes containing a fixed point $p \in S$; note that these subschemes are parametrized by the blow-up of S at p. In particular, $(f-me)\cdot(f-me)\cdot\Sigma = \langle f, f\rangle - m^2$ (because e restricts to the exceptional divisor of the blown-up K3 surface). We also have

$$\begin{aligned} g * g &= g \cdot g - \langle g, g\rangle \Sigma, \\ f \cdot f \cdot g \cdot g &= \langle f, f\rangle\langle g, g\rangle + 2\langle f, g\rangle^2, \\ f \cdot e \cdot g \cdot g &= 0, \\ e \cdot e \cdot g \cdot g &= -2\langle g, g\rangle. \end{aligned}$$

Finally, we obtain

$$(f - me)\cdot(f - me)\cdot(g * g) = 2\langle f, g\rangle^2 - m^2\langle g, g\rangle.$$

In our case, $f = f_{2m^2}$, g is the class of the elliptic curve. To verify the hypothesis of the Theorem 5.1, we need $2\langle f_{2m^2}, g\rangle^2 > m^2\langle g, g\rangle$. Since $\langle g, g\rangle > 0$ we are done by the Hodge index theorem, which implies that the determinant of the matrix

$$\begin{pmatrix} 2m^2 & \langle f_{2m^2}, g\rangle \\ \langle f_{2m^2}, g\rangle & \langle g, g\rangle \end{pmatrix}$$

is negative. $\square$

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