ON THE COX RING OF DEL PEZZO SURFACES

ULRICH DERENTHAL

Abstract. Let S_r be the blow-up of $\mathbb{P}^2$ in r general points, i.e., a smooth Del Pezzo surface of degree $9-r$. For $r \leq 7$, we determine the quadratic equations defining its Cox ring explicitly. The ideal of the relations in $\mathrm{Cox}(S_8)$ is calculated up to radical. As conjectured by Batyrev and Popov, all the generating relations are quadratic.

Contents

1. Introduction

A Del Pezzo surface S_r of degree $d = 9-r$ over an algebraically closed field $\mathbb{K}$ is the blow-up of $\mathbb{P}^2$ in r points in general position[1] ($1 \leq r \leq 8$). Its Picard group is a free $\mathbb{Z}$-module of rank $r+1$.

Once we have chosen representatives $\mathcal{L}_0, \dots, \mathcal{L}_r$ for a basis of $\mathrm{Pic}(S_r)$, we can define its *Cox ring*, or *total coordinate ring*, as

$$\mathrm{Cox}(S_r) := \bigoplus_{(\nu_0,\dots,\nu_r)\in\mathbb{Z}^{r+1}} \Gamma(S_r, \mathcal{L}_0^{\otimes\nu_0} \otimes \dots \otimes \mathcal{L}_r^{\otimes\nu_r}).$$

The multiplication of sections induces the multiplication in $\mathrm{Cox}(S_r)$. The Cox ring is graded by $\mathrm{Pic}(S_r)$ and is independent of the choice of the basis.

The *intersection form* is a non-degenerate bilinear form on $\mathrm{Pic}(S_r)$. We will write it as (D_1, D_2) for $D_1, D_2 \in \mathrm{Pic}(S_r)$. (We will often use the same notation for divisors and their class in $\mathrm{Pic}(S_r)$. It will be clear from the context what is meant.) A prime divisor D whose self-intersection number (D, D) is negative is called a *negative curve*. On smooth Del Pezzo surfaces, every negative curve has self-intersection number -1.

For $r \in \{3, \dots, 7\}$, $\mathrm{Cox}(S_r)$ is generated by non-zero sections of the N_r negative curves ([BP04, Theorem 3.2]), see Table 1 for the values of N_r. For $r = 8$, we must add two independent sections of $\Gamma(S_8, -K_{S_8})$. Let R_r be

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[1] I.e., no three points on one line, no six points on a conic, no eight points with one of them a double point on a cubic.

the free polynomial ring whose variables correspond to these generators of $\operatorname{Cox}(S_r)$. We want to determine the relations between these generators.

For $r \leq 3$, the Cox ring is a polynomial ring in $r+3$ generators. This is due to the fact that in these cases, S_r is toric (see [Cox95] for Cox rings of toric varieties).

Definition 1.1. For $n \geq 1$, a divisor class D is called an (n)-*ruling* if $D = D_1 + D_2$ for two negative curves D_1, D_2 whose intersection number (D_1, D_2) is $n \geq 1$. A (1)-ruling is also called a *ruling*.

Each (n)-ruling defines a quadratic relation between generators of $\operatorname{Cox}(S_r)$, see Lemma 2.1. Relations coming from (1)-rulings define an ideal $I_r \subset R_r$. For $r \in \{4,5,6\}$, $\operatorname{Cox}(S_r) = R_r/\operatorname{rad}(I_r)$ by [BP04, Theorem 4.9]. We extend this result to $r \in \{7,8\}$ as follows:

Theorem 1.2. *For $r \in \{4,\dots,8\}$, we have $\operatorname{Cox}(S_r) = R_r/\operatorname{rad}(J_r)$, where*

- *for $r \in \{4,5,6\}$, $J_r := I_r$;*
- *the ideal J_7 is generated by the 504 quadratic relations coming from the 126 rulings, and 25 quadratic relations coming from the (2)-ruling $-K_{S_7}$;*
- *the ideal J_8 is generated by the 10800 quadratic relations coming from the 2160 rulings, 6480 quadratic relations coming from 240 (2)-rulings, and 119 quadratic relations coming from the (3)-ruling $2 \cdot (-K_{S_8})$.*

It is known that the ideal I_4 is radical (see [BP04]). Batyrev proved that the same holds for I_5 (unpublished). Here we prove:

Theorem 1.3. *For $r \in \{4,\dots,7\}$, the ideals J_r are radical, and*

$$\operatorname{Cox}(S_r) = R_r/J_r.$$

It was conjectured by Batyrev and Popov that the ideal of relations defining $\operatorname{Cox}(S_r)$ is generated by quadrics for $r \in \{4,\dots,8\}$, see [BP04, Conjecture 4.3]. To prove this conjecture, it now remains to show that J_8 is radical.

After recalling some general results on Del Pezzo surfaces in Section 2, we will handle the cases $r \in \{6,7,8\}$ separately.

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2. Smooth Del Pezzo surfaces

In this section, we summarize some facts on smooth Del Pezzo surfaces.

- Let $E_1, \dots, E_r$ be the exceptional divisors of the blowup of $\mathbb{P}^2$ in r points $p_1, \dots, p_r$ in general position. A basis of $\operatorname{Pic}(S_r)$ is given by (the classes of) $H, E_1, \dots, E_r$, where H is the pullback of the hyperplane section in $\mathbb{P}^2$.
- In terms of this basis, the intersection form is given by a diagonal matrix of size $r+1$ whose diagonal is $(1,-1,\dots,-1)$. The anticanonical divisor is $-K_{S_r} = 3H - (E_1 + \dots + E_r)$.

- The curves with self-intersection number -1 are described in [BP04, Theorem 2.1]. There are no curves whose self-intersection is ≤ -2.
- The Weyl group W_r acting on $\mathrm{Pic}(S_r)$ depends on r:

r	1	2	3	4	5	6	7	8
W_r	$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{A}_2 + \mathbf{A}_1$	$\mathbf{A}_4$	$\mathbf{D}_5$	$\mathbf{E}_6$	$\mathbf{E}_7$	$\mathbf{E}_8$

For more details, see [BP04, Section 2].

As explained in the introduction, for $r \leq 6$, all relations in the Cox ring are induced by rulings, and these relations also play an important role for $r \in \{7, 8\}$. More precisely, by the discussion following [BP04, Remark 4.7], each ruling is represented in $r-1$ different ways as the sum of two negative curves, giving $r-3$ linearly independent quadratic relation in $\mathrm{Cox}(S_r)$. Therefore, if each of the N_r negative curves intersects n_r negative curves with intersection number 1, we have $N'_r = (N_r \cdot n_r)/2$ pairs, the number of rulings is $N''_r = N'_r/(r-1)$, and the number of quadratic relations coming from rulings is $N''_r \cdot (r-3)$ (see Table 1).

r	3	4	5	6	7	8
N_r	6	10	16	27	56	240
n_r	2	3	5	10	27	126
N''_r	3	5	10	27	126	2160
relations	0	5	20	81	504	10800

Table 1. The number of relations coming from rulings.

Now we describe how to obtain explicit equations for $\mathrm{Cox}(S_r)$ and how to prove Theorems 1.2 and Theorem 1.3. We isolate the steps that must be carried out for each of the degrees 3, 2, and 1 and complete the proofs in the following sections.

Choice of coordinates. Choose coordinates for $p_1, \dots, p_r \in \mathbb{P}^2$. We may assume that the first for points are

$$p_1 = (1:0:0), \quad p_2 = (0:1:0), \quad p_3 = (0:0:1), \quad p_4 = (1:1:1). \tag{2.1}$$

By the general position requirement, the other points must have non-zero coordinates, and we can write $p_j = (1 : \alpha_j : \beta_j)$ for $j \in \{5, \dots, r\}$.

Curves in $\mathbb{P}^2$. As explained in the introduction, $\mathrm{Cox}(S_r)$ is generated by sections of the negative curves for $r \leq 7$. For a negative curve D, we denote the corresponding section by $\xi(D)$, and for a generating section ξ, let $D(\xi)$ be the corresponding divisor. For $r = 8$, we need two further generators: linearly independent sections κ_1, κ_2 of $-K_{S_8}$. Let $K_1 := D(\kappa_1)$, $K_2 := D(\kappa_2)$ be the corresponding divisors in the divisor class $-K_{S_8}$.

Let Ψ_r be the set of sections generating $\mathrm{Cox}(S_r)$, and $\mathcal{D}_r$ the set of corresponding divisors (including κ_1, κ_2 respectively K_1, K_2 if $r = 8$).

We need an explicit description of the image of each generator D of $\mathrm{Cox}(S_r)$ under the projection $\pi : S_r \to \mathbb{P}^2$. According to the seven cases in [BP04, Theorem 2.1], $\pi(D)$ can be a curve, determined by a form f_D of degree $d \in \{1, \dots, 6\}$, or a point (if $D = E_i$). If $\pi(D)$ is a point, the convention to choose f_D as a non-zero constant will be useful later.

For $r = 8$, we have the following situation: The image of K_i is a cubic through the eight points $p_1, \dots, p_8$. The choice of two linearly independent sections κ_1, κ_2 corresponds to the choice of two independent cubic forms f_{K_1}, f_{K_2} vanishing in the eight points. Every cubic through these points has the form $a_1 f_{K_1} + a_2 f_{K_2}$ where $(a_1, a_2) \neq (0, 0)$, and the cubic does not change if we replace (a_1, a_2) be a non-zero multiple. This gives a one-dimensional projective space of cubics through the eight points.

Let $X_1, \dots, X_n$ be the monomials of degree d in three variables x_0, x_1, x_2. For $D \in \mathcal{D}_r$, we can write

$$f_D = \sum_{i=1}^{n} a_i \cdot X_i$$

for suitable coefficients a_i, which we can calculate in the following way: If p_j lies on $\pi(D)$, this gives a linear condition on the coefficients a_i by substituting the coordinates of p_j for x_0, x_1, x_2. If p_j is a double point of $\pi(D)$, all partial derivatives of f_D must vanish at this point, giving three more linear conditions. If p_j is a triple point, we get six more linear conditions from the second derivatives. With $p_1, \dots, p_r$ in general position, we check that these conditions determine f_D uniquely up to a non-zero constant.

Relations corresponding to (n)-rulings. Suppose that an (n)-ruling D can be written as $D_j + D'_j$ for k different pairs $D_j, D'_j \in \mathcal{D}_r$ where $j \in \{1, \dots, k\}$. Then the products

$$f_{D_1} \cdot f_{D'_1}, \dots, f_{D_k} \cdot f_{D'_k}$$

are k homogeneous forms of the same dimension d, and they span a vector space of degree $n + 1$ in the space of homogeneous polynomials of degree d. Therefore, there are $k - (n + 1)$ independent relations between them, which we write as

$$\sum_{j=1}^{k} a_{j,i} \cdot f_{D_j} \cdot f_{D'_j} = 0 \qquad \text{for } i \in \{1, \dots, k - (n + 1)\}.$$

for suitable constants $a_{j,i}$. They give an explicit description of the quadric relations coming from D:

Lemma 2.1. *In this situation, the (n)-ruling D gives the following $k-(n+1)$ quadratic relations in* $\mathrm{Cox}(S_r)$*:*

$$q_i := \sum_{j=1}^{k} a_{j,i} \cdot \xi(D_j) \cdot \xi(D'_j) = 0 \qquad \textit{for } i \in \{1, \dots, k - (n + 1)\}.$$

We will describe the (n)-rulings in more detail in the subsequent sections.

Let J_r be the ideal in R_r which is generated by the (n)-rulings (where $n = 1$ for $r \leq 6$, $n \in \{1, 2\}$ for $r = 7$, and $n \in \{1, 2, 3\}$ for $r = 8$).

The proof of Theorem 1.2. For $r \in \{4, 5, 6\}$, this is [BP04, Theorem 4.9]. For $r \in \{7, 8\}$, we use a refinement of its proof.

Let $Z_r = \mathrm{Spec}(R_r / \mathrm{rad}(J_r)) \subset \mathrm{Spec}(R_r)$. We want to prove that Z_r equals $\mathbb{A}(S_r) \subset \mathrm{Spec}(R_r)$. Obviously, $0 \in \mathrm{Spec}(R_r)$ is contained in both Z_r and $\mathbb{A}(S_r)$. Its complement $\mathrm{Spec}(R_r) \setminus \{0\}$ is covered by the open sets

$$U_D := \{\xi(D) \neq 0\}, \qquad \text{where } D \in \mathcal{D}_r.$$

In the case $r = 8$, we will show that it suffices to consider the sets U_D for $D \in \mathcal{D}_8 \setminus \{K_1, K_2\}$.

We want to show

$$Z_r \cap U_D \cong Z_{r-1} \times (\mathbb{A}^1 \setminus \{0\}).$$

Note that we can identify the negative curves $\mathcal{D}_{r-1}$ of S_{r-1} with the subset $\mathcal{D}'_r$ of $\mathcal{D}_r$ containing the negative curves which do not intersect D. We define

$$\begin{array}{cccc} \psi : & Z_r \cap U_D & \to & Z_{r-1} \times (\mathbb{A}^1 \setminus \{0\}) \\ & (\xi(D') \mid D' \in \mathcal{D}_r) & \mapsto & ((\xi(D') \mid D' \in \mathcal{D}_{r-1}), \xi_D) \end{array}.$$

For $r \in \{7, 8\}$, we will prove:

Lemma 2.2. *Every $\xi(D'')$ for $D'' \in \mathcal{D}_r$ intersecting D is determined by*

$$\xi(D) \qquad \textit{and} \qquad \{\xi(D') \mid D' \in \mathcal{D}_r \quad \textit{with} \quad (D', D) = 0\},$$

provided that $\xi(D) \neq 0$ and using the relations generating J_r.

By the proof of [BP04, Prop. 4.4],

$$\mathbb{A}(S_r) \cap U_D \cong \mathbb{A}(S_{r-1}) \times (\mathbb{A}^1 \setminus \{0\}).$$

By induction, $Z_{r-1} = \mathbb{A}(S_{r-1})$. Therefore, $Z_r \cap U_D = \mathbb{A}(S_r) \cap U_D$ for every negative curve D, which implies $Z_r = \mathbb{A}(S_r)$, completing the proof of Theorem 1.2 once Lemma 2.2 is proved.

Proof of Theorem 1.3. We want to show that the ideal J_r is radical.

Lemma 2.3. *The Hilbert polynomial of S_r/J_r has degree $r + 2$.*

For $r = 5$, this was proved by Batyrev. We will prove it for $r \in \{6, 7\}$.

Remark 2.4. The problem of calculating the Hilbert polynomial of J_8 seems out of reach of the current computer algebra packages. It is the only step missing in the proof of Theorem 1.3 for Del Pezzo surfaces of degree 1.

Under the condition of the proof of Lemma 2.3, the depth of R_r/J_r is $r+3$. As $\operatorname{Spec}(R_r/J_r)$ is irreducible by [BP04], and $\operatorname{Cox}(S_r) = \operatorname{Spec}(R_r/\operatorname{rad}(J_r))$ by Theorem 1.2, the R_r-module R_r/J_r is Cohen-Macaulay. Therefore, we need to check the following claim in order to prove that the ideal J_r is radical:

Lemma 2.5. *R_r/J_r has a smooth point.*

3. Degree 3

We consider the case $r = 6$, i.e. smooth cubic surfaces. The set $\mathcal{D}_6$ of negative curves on S_6 consists of the following 27 divisors:

- exceptional divisors $E_1, \dots, E_6$, preimages of $p_1, \dots, p_6 \in \mathbb{P}^2$,
- transforms $m_{i,j} = H - E_i - E_j$ of the 15 lines $m'_{i,j}$ through the points p_i, p_j $(i \neq j \in \{1, \dots, 6\})$, and
- transforms $Q_k = H - (E_1 + \dots + E_6) + E_k$ of the six conics Q'_k through all of the blown-up points except p_k.

With respect to the anticanonical embedding $S_6 \hookrightarrow \mathbb{P}^3$, the negative curves are the 27 lines.

Together with information from Section 2, it is straightforward to derive:

Lemma 3.1. *The extended Dynkin diagram of negative curves has the following structure:*

(1) *It has 27 vertices corresponding to the 27 lines $E_i, m_{i,j}, Q_i$. Each of them has self-intersection number -1.*
(2) *Every line intersects exactly 10 other lines: E_i intersects $m_{i,j}$ and Q_j (for $j \neq i$); $m_{i,j}$ intersects E_i, E_j, Q_i, Q_j and $m_{k,l}$ (for $\{i,j\} \cap \{k,l\} = \emptyset$); Q_i intersects $m_{i,j}$ and E_j (for $j \neq i$). Correspondingly, there are 135 edges in the Dynkin diagram.*
(3) *There are 45* triangles*, i.e., triples of lines which intersect pairwise: 30 triples $E_i, m_{i,j}, Q_j$ and 15 triples $m_{i_1,j_1}, m_{i_2,j_2}, m_{i_3,j_3}$ where $\{i_1, j_1, i_2, j_2, i_3, j_3\} = \{1, \dots, 6\}$. This corresponds to 45 triangles in the Dynkin diagram, where each edge is contained in exactly one of the triangles, and each vertex belongs to exactly five triangles.*

Lemma 3.2. *The 27 rulings of S_6 are given by $-K_{S_6} - D$ for $D \in \mathcal{D}_6$. Two negative curves D', D'' fulfill $D' + D'' = -K_{S_6} - D$ if and only if D, D', D'' form a triangle in the sense of Lemma 3.1(3). There are five such pairs for any given D.*

Proof. We can check directly that $D + D' + D'' = -K_{S_6}$ if D, D', D'' form a triangle. Therefore, $-K_{S_6} - D$ is a ruling, and as any D is contained in exactly five triangles, it can be expressed in five corresponding ways as $D' + D''$.

On the other hand, by Table 1, the total number of rulings is 27, and each ruling can be expressed in exactly five ways as the sum of two negative curves. □

Let D be one of the 27 lines of S_6, and consider the projection $\psi_D : S_6 \dashrightarrow \mathbb{P}^2$ from D. Then

$$\psi_D^*(\mathcal{O}_{\mathbb{P}^2}(1)) = -K_{S_6} - D = \begin{cases} H - E_i, & \ell = Q_i, \\ 2H - (E_1 + \dots + E_6) + E_i + E_j, & \ell = m_{i,j}, \\ 3H - (E_1 + \dots + E_6) - E_i, & \ell = E_i. \end{cases}$$

These are exactly the rulings.

A generating set Ψ of $\mathrm{Cox}(S_6)$ is given by section $\eta_i, \mu_{i,j}, \lambda_i$ corresponding to the 27 lines $E_i, m_{i,j}, Q_i$, respectively. Let

$$R_6 := \mathbb{K}[\eta_i, \mu_{i,j}, \lambda_i].$$

The quadratic monomials in $\Gamma(S_6, -K_{S_6} - D)$ corresponding to the five ways to express $-K_{S_6} - D$ as the sum of the negative curves are

- $\mu_{i,j}\eta_j$ if $D = Q_i$
- $\eta_i\lambda_j, \eta_j\lambda_i, \mu_{k_1,k_2}\mu_{k_3,k_4}$ if $D = \mu_{i,j}$ (with $\{i, j, k_1, \dots, k_4\} = \{1, \dots, 6\}$)
- $\mu_{i,j}\lambda_j$ if $D = E_i$

In order to calculate the 81 relations in J_6 explicitly as described in Lemma 2.1, we use the coordinates of (2.1) for $p_1, \dots, p_4$, and

$$p_5 = (1 : a : b), \qquad p_6 = (1 : c : d).$$

We write

$$E := (b-1)(c-1) - (a-1)(d-1) \qquad \text{and} \qquad F := bc - ad$$

for simplicity. The three relations corresponding to the lines ℓ will be denoted by $q_\ell, q'_\ell, q''_\ell$.

$$
\begin{aligned}
q_{Q_1} &= -\eta_2\mu_{1,2} - \eta_3\mu_{1,3} + \eta_4\mu_{1,4}\\
q'_{Q_1} &= -a\eta_2\mu_{1,2} - b\eta_3\mu_{1,3} + \eta_5\mu_{1,5}\\
q'_{Q_1} &= -c\eta_2\mu_{1,2} - d\eta_3\mu_{1,3} + \eta_6\mu_{1,6}
\end{aligned}
$$

$$
\begin{aligned}
q_{Q_2} &= \eta_1\mu_{1,2} - \eta_3\mu_{2,3} + \eta_4\mu_{2,4}\\
q'_{Q_2} &= \eta_1\mu_{1,2} - b\eta_3\mu_{2,3} + \eta_5\mu_{2,5}\\
q''_{Q_2} &= \eta_1\mu_{1,2} - d\eta_3\mu_{2,3} + \eta_6\mu_{2,6}
\end{aligned}
$$

$$
\begin{aligned}
q_{Q_3} &= \eta_1\mu_{1,3} + \eta_2\mu_{2,3} + \eta_4\mu_{3,4}\\
q'_{Q_3} &= \eta_1\mu_{1,3} + a\eta_2\mu_{2,3} + \eta_5\mu_{3,5}\\
q''_{Q_3} &= \eta_1\mu_{1,3} + c\eta_2\mu_{2,3} + \eta_6\mu_{3,6}
\end{aligned}
$$

$$
\begin{aligned}
q_{Q_4} &= \eta_1\mu_{1,4} + \eta_2\mu_{2,4} + \eta_3\mu_{3,4}\\
q'_{Q_4} &= (1-b)\eta_1\mu_{1,4} + (a-b)\eta_2\mu_{2,4} + \eta_5\mu_{4,5}\\
q''_{Q_4} &= (1-d)\eta_1\mu_{1,4} + (c-d)\eta_2\mu_{2,4} + \eta_6\mu_{4,6}
\end{aligned}
$$

$$
\begin{aligned}
q_{Q_5} &= 1/b\eta_1\mu_{1,5} + a/b\eta_2\mu_{2,5} + \eta_3\mu_{3,5}\\
q'_{Q_5} &= (1-b)/b\eta_1\mu_{1,5} + (a-b)/b\eta_2\mu_{2,5} + \eta_4\mu_{4,5}\\
q''_{Q_5} &= (b-d)/b\eta_1\mu_{1,5} + F/b\eta_2\mu_{2,5} + \eta_6\mu_{5,6}
\end{aligned}
$$

$$
\begin{aligned}
q_{Q_6} &= 1/d\eta_1\mu_{1,6} + c/d\eta_2\mu_{2,6} + \eta_3\mu_{3,6}\\
q'_{Q_6} &= (1-d)/d\eta_1\mu_{1,6} + (c-d)/d\eta_2\mu_{2,6} + \eta_4\mu_{4,6}\\
q''_{Q_6} &= (b-d)/d\eta_1\mu_{1,6} + F/d\eta_2\mu_{2,6} + \eta_5\mu_{5,6}
\end{aligned}
$$

$$
\begin{aligned}
q_{m_{1,2}} &= \mu_{4,5}\mu_{3,6} - \mu_{3,5}\mu_{4,6} + \mu_{3,4}\mu_{5,6}\\
q'_{m_{1,2}} &= (b-d)\mu_{3,5}\mu_{4,6} + (d-1)\mu_{3,4}\mu_{5,6} + \eta_2\lambda_1\\
q''_{m_{1,2}} &= F\mu_{3,5}\mu_{4,6} + a(d-c)\mu_{3,4}\mu_{5,6} + \eta_1\lambda_2
\end{aligned}
$$

$$
\begin{aligned}
q_{m_{1,3}} &= \mu_{4,5}\mu_{2,6} - \mu_{2,5}\mu_{4,6} + \mu_{2,4}\mu_{5,6}\\
q'_{m_{1,3}} &= (c-a)\mu_{2,5}\mu_{4,6} + (1-c)\mu_{2,4}\mu_{5,6} + \eta_3\lambda_1\\
q''_{m_{1,3}} &= -F\mu_{2,5}\mu_{4,6} + b(c-d)\mu_{2,4}\mu_{5,6} + \eta_1\lambda_3
\end{aligned}
$$

$$
\begin{aligned}
q_{m_{2,3}} &= \mu_{4,5}\mu_{1,6} - \mu_{1,5}\mu_{4,6} + \mu_{1,4}\mu_{5,6}\\
q'_{m_{2,3}} &= (a-c)\mu_{1,5}\mu_{4,6} + a(c-1)\mu_{1,4}\mu_{5,6} + \eta_3\lambda_2\\
q''_{m_{2,3}} &= (b-d)\mu_{1,5}\mu_{4,6} + b(d-1)\mu_{1,4}\mu_{5,6} + \eta_2\lambda_3
\end{aligned}
$$

$$
\begin{aligned}
q_{m_{1,4}} &= \mu_{3,5}\mu_{2,6} - \mu_{2,5}\mu_{3,6} + \mu_{2,3}\mu_{5,6}\\
q'_{m_{1,4}} &= -E\mu_{2,5}\mu_{3,6} + (b-1)(c-1)\mu_{2,3}\mu_{5,6} + \eta_4\lambda_1\\
q''_{m_{1,4}} &= -F\mu_{2,5}\mu_{3,6} + bc\mu_{2,3}\mu_{5,6} + \eta_1\lambda_4
\end{aligned}
$$

$$
\begin{aligned}
q_{m_{2,4}} &= \mu_{3,5}\mu_{1,6} - \mu_{1,5}\mu_{3,6} + \mu_{1,3}\mu_{5,6}\\
q'_{m_{2,4}} &= E\mu_{1,5}\mu_{3,6} + (a-b)(c-1)\mu_{1,3}\mu_{5,6} + \eta_4\lambda_2\\
q''_{m_{2,4}} &= (b-d)\mu_{1,5}\mu_{3,6} - b\mu_{1,3}\mu_{5,6} + \eta_2\lambda_4
\end{aligned}
$$

$$
\begin{aligned}
q_{m_{3,4}} &= \mu_{2,5}\mu_{1,6} - \mu_{1,5}\mu_{2,6} + \mu_{1,2}\mu_{5,6}\\
q'_{m_{3,4}} &= -E\mu_{1,5}\mu_{2,6} + (a-b)(1-d)\mu_{1,2}\mu_{5,6} + \eta_4\lambda_3\\
q''_{m_{3,4}} &= (c-a)\mu_{1,5}\mu_{2,6} + a\mu_{1,2}\mu_{5,6} + \eta_3\lambda_4
\end{aligned}
$$

$$
\begin{aligned}
q_{m_{1,5}} &= \mu_{3,4}\mu_{2,6} - \mu_{2,4}\mu_{3,6} + \mu_{2,3}\mu_{4,6}\\
q'_{m_{1,5}} &= -E\mu_{2,4}\mu_{3,6} + (a-c)(1-b)\mu_{2,3}\mu_{4,6} + \eta_5\lambda_1\\
q''_{m_{1,5}} &= (d-c)\mu_{2,4}\mu_{3,6} + c\mu_{2,3}\mu_{4,6} + \eta_1\lambda_5
\end{aligned}
$$

$$
\begin{aligned}
q_{m_{2,5}} &= \mu_{3,4}\mu_{1,6} - \mu_{1,4}\mu_{3,6} + \mu_{1,3}\mu_{4,6}\\
q'_{m_{2,5}} &= aE\mu_{1,4}\mu_{3,6} + (a-b)(c-a)\mu_{1,3}\mu_{4,6} + \eta_5\lambda_2\\
q''_{m_{2,5}} &= (1-d)\mu_{1,4}\mu_{3,6} - \mu_{1,3}\mu_{4,6} + \eta_2\lambda_5
\end{aligned}
$$

$$
\begin{aligned}
q_{m_{3,5}} &= \mu_{2,4}\mu_{1,6} - \mu_{1,4}\mu_{2,6} + \mu_{1,2}\mu_{4,6}\\
q'_{m_{3,5}} &= -bE\mu_{1,4}\mu_{2,6} + (a-b)(b-d)\mu_{1,2}\mu_{4,6} + \eta_5\lambda_3\\
q''_{m_{3,5}} &= (c-1)\mu_{1,4}\mu_{2,6} + \mu_{1,2}\mu_{4,6} + \eta_3\lambda_5
\end{aligned}
$$

$$
\begin{aligned}
q_{m_{4,5}} &= \mu_{2,3}\mu_{1,6} - \mu_{1,3}\mu_{2,6} + \mu_{1,2}\mu_{3,6}\\
q'_{m_{4,5}} &= b(c-a)\mu_{1,3}\mu_{2,6} + a(b-d)\mu_{1,2}\mu_{3,6} + \eta_5\lambda_4\\
q''_{m_{4,5}} &= (c-1)\mu_{1,3}\mu_{2,6} + (1-d)\mu_{1,2}\mu_{3,6} + \eta_4\lambda_5
\end{aligned}
$$

$$
\begin{aligned}
q_{m_{1,6}} &= \mu_{3,4}\mu_{2,5} - \mu_{2,4}\mu_{3,5} + \mu_{2,3}\mu_{4,5}\\
q'_{m_{1,6}} &= -E\mu_{2,4}\mu_{3,5} + (a-c)(1-d)\mu_{2,3}\mu_{4,5} + \eta_6\lambda_1\\
q''_{m_{1,6}} &= (b-a)\mu_{2,4}\mu_{3,5} + a\mu_{2,3}\mu_{4,5} + \eta_1\lambda_6
\end{aligned}
$$

$$
\begin{aligned}
q_{m_{2,6}} &= \mu_{3,4}\mu_{1,5} - \mu_{1,4}\mu_{3,5} + \mu_{1,3}\mu_{4,5}\\
q'_{m_{2,6}} &= cE\mu_{1,4}\mu_{3,5} + (a-c)(d-c)\mu_{1,3}\mu_{4,5} + \eta_6\lambda_2\\
q''_{m_{2,6}} &= (1-b)\mu_{1,4}\mu_{3,5} - \mu_{1,3}\mu_{4,5} + \eta_2\lambda_6
\end{aligned}
$$

$$q_{m_{3,6}} = \mu_{2,4}\mu_{1,5} - \mu_{1,4}\mu_{2,5} + \mu_{1,2}\mu_{4,5}$$
$$q'_{m_{3,6}} = -dE\mu_{1,4}\mu_{2,5} + (d-b)(d-c)\mu_{1,2}\mu_{4,5} + \eta_6\lambda_3$$
$$q''_{m_{3,6}} = (a-1)\mu_{1,4}\mu_{2,5} + \mu_{1,2}\mu_{4,5} + \eta_3\lambda_6$$

$$q_{m_{4,6}} = \mu_{2,3}\mu_{1,5} - \mu_{1,3}\mu_{2,5} + \mu_{1,2}\mu_{3,5}$$
$$q'_{m_{4,6}} = d(c-a)\mu_{1,3}\mu_{2,5} + c(b-d)\mu_{1,2}\mu_{3,5} + \eta_6\lambda_4$$
$$q''_{m_{4,6}} = (a-1)\mu_{1,3}\mu_{2,5} + (1-b)\mu_{1,2}\mu_{3,5} + \eta_4\lambda_6$$

$$q_{m_{5,6}} = \mu_{2,3}\mu_{1,4} - \mu_{1,3}\mu_{2,4} + \mu_{1,2}\mu_{3,4}$$
$$q'_{m_{5,6}} = d(c-1)\mu_{1,3}\mu_{2,4} + c(1-d)\mu_{1,2}\mu_{3,4} + \eta_6\lambda_5$$
$$q''_{m_{5,6}} = b(a-1)\mu_{1,3}\mu_{2,4} + a(1-b)\mu_{1,2}\mu_{3,4} + \eta_5\lambda_6$$

$$q_{E_1} = (d-b)/E\mu_{1,2}\lambda_2 + (c-a)/E\mu_{1,3}\lambda_3 + \mu_{1,4}\lambda_4$$
$$q'_{E_1} = (d-1)/E\mu_{1,2}\lambda_2 + (c-1)/E\mu_{1,3}\lambda_3 + \mu_{1,5}\lambda_5$$
$$q''_{E_1} = (b-1)/E\mu_{1,2}\lambda_2 + (a-1)/E\mu_{1,3}\lambda_3 + \mu_{1,6}\lambda_6$$

$$q_{E_2} = F/E\mu_{1,2}\lambda_1 + (c-a)/E\mu_{2,3}\lambda_3 + \mu_{2,4}\lambda_4$$
$$q'_{E_2} = (c-d)/E\mu_{1,2}\lambda_1 + (c-1)/E\mu_{2,3}\lambda_3 + \mu_{2,5}\lambda_5$$
$$q''_{E_2} = (a-b)/E\mu_{1,2}\lambda_1 + (a-1)/E\mu_{2,3}\lambda_3 + \mu_{2,6}\lambda_6$$

$$q_{E_3} = F/E\mu_{1,3}\lambda_1 + (b-d)/E\mu_{2,3}\lambda_2 + \mu_{3,4}\lambda_4$$
$$q'_{E_3} = (c-d)/E\mu_{1,3}\lambda_1 + (1-d)/E\mu_{2,3}\lambda_2 + \mu_{3,5}\lambda_5$$
$$q''_{E_3} = (a-b)/E\mu_{1,3}\lambda_1 + (1-b)/E\mu_{2,3}\lambda_2 + \mu_{3,6}\lambda_6$$

$$q_{E_4} = F/(a-c)\mu_{1,4}\lambda_1 + (b-d)/(a-c)\mu_{2,4}\lambda_2 + \mu_{3,4}\lambda_3$$
$$q'_{E_4} = c/(a-c)\mu_{1,4}\lambda_1 + 1/(a-c)\mu_{2,4}\lambda_2 + \mu_{4,5}\lambda_5$$
$$q''_{E_4} = a/(a-c)\mu_{1,4}\lambda_1 + 1/(a-c)\mu_{2,4}\lambda_2 + \mu_{4,6}\lambda_6$$

$$q_{E_5} = (d-c)/(c-1)\mu_{1,5}\lambda_1 + (d-1)/(c-1)\mu_{2,5}\lambda_2 + \mu_{3,5}\lambda_3$$
$$q'_{E_5} = -c/(c-1)\mu_{1,5}\lambda_1 - 1/(c-1)\mu_{2,5}\lambda_2 + \mu_{4,5}\lambda_4$$
$$q''_{E_5} = -1/(c-1)\mu_{1,5}\lambda_1 - 1/(c-1)\mu_{2,5}\lambda_2 + \mu_{5,6}\lambda_6$$

$$q_{E_6} = (b-a)/(a-1)\mu_{1,6}\lambda_1 + (b-1)/(a-1)\mu_{2,6}\lambda_2 + \mu_{3,6}\lambda_3$$
$$q'_{E_6} = -a/(a-1)\mu_{1,6}\lambda_1 - 1/(a-1)\mu_{2,6}\lambda_2 + \mu_{4,6}\lambda_4$$
$$q''_{E_6} = -1/(a-1)\mu_{1,6}\lambda_1 - 1/(a-1)\mu_{2,6}\lambda_2 + \mu_{5,6}\lambda_5$$

Proof of Lemma 2.3. We calculate the Hilbert polynomial of R_6/J_6 over the field of fractions of the polynomial ring $\mathbb{Q}[a,b,c,d]$ using `Magma`:

$$h(t) = \frac{1}{8!}(372t^8 + 4464t^7 + 25200t^6 + 86184t^5 + 193788t^4 + 291816t^3 + 284640t^2 + 161856t + 40320).$$

Its degree is $r + 2 = 8$ as required.

Proof of Lemma 2.5. The point p with coordinates

$$(\eta_5, \eta_6, \mu_{1,2}, \mu_{1,4}, \mu_{2,3}, \mu_{3,4}, \lambda_5, \lambda_6) = (c(d-1), a(b-1), 1, -1, 1, 1, 1, 1),$$

and all other coordinates zero is a smooth point of R_6/J_6. Indeed, we check that p fulfills all the relations generating J_r (which is obvious for all of them except $q_{m_{5,6}}, q'_{m_{5,6}}, q''_{m_{5,6}}$), and we calculate directly that the 81×27 Jacobian matrix has full rank 18 at this point.

4. Degree 2

Let S_7 be a smooth Del Pezzo surface of degree $d = 2$, i.e. the blow-up of $\mathbb{P}^2$ in $r = 7$ points. The set $\mathcal{D}_7$ contains 56 negatives curves which are the transforms of the following curves in $\mathbb{P}^2$:

- blow-ups $E_1, \dots, E_7$ of $p_1, \dots, p_7$;
- 21 lines $m'_{i,j}$ through p_i, p_j, where
$$m_{i,j} = H - E_i - E_j;$$
- 21 conics $Q'_{i,j}$ through five of the seven points, missing p_i, p_j, where
$$Q_{i,j} = 2H - (E_1 + \dots + E_7) + E_i + E_j;$$
- 7 singular cubics C'_i through all seven points, where p_i is a double point, and
$$C_i = 3H - (E_1 + \dots + E_7) - E_i.$$

The Cox ring $\mathrm{Cox}(S_7)$ is generated by the sections $\eta_i, \mu_{i,j}, \nu_{i,j}, \lambda_i$ corresponding to the 56 negative curves $E_i, m_{i,j}, Q_{i,j}, C_i$, respectively. Let

$$R_7 := \mathbb{K}[\eta_i, \mu_{i,j}, \nu_{i,j}, \lambda_i]$$

be the polynomial ring in 56 generators.

Consider the ideal $I_7 \subset R_7$ generated by the quadratic relations corresponding to rulings. In view of Lemma 2.1, we need to know the six different ways to write each of the 126 rulings as a sum of two negative curves in order to describe I_7 explicitly. Here, we do not write the resulting 504 relations down because of the length of this list.

Lemma 4.1. *Each of the 126 rulings can be written in six ways as a sum of two negative curves:*

(1) *For the seven rulings $H - E_i$:*

$$\{E_j + m_{i,j} \mid j \neq i\}.$$

(2) *For the 35 rulings $2H - (E_1 + \dots + E_7) + E_i + E_j + E_k$:*

$$\{E_i + Q_{j,k}, E_j + Q_{i,k}, E_k + Q_{i,j}, m_{l_1,l_2} + m_{l_3,l_4} \mid \{i,j,k,l_1,l_2,l_3,l_4\} = \{1,\dots,7\}\}$$

(3) *For the 42 rulings* $3H-(E_1+\cdots+E_7)+E_i-E_j$*:*

$$\{E_i+C_j, Q_{i,k}+m_{j,k} \mid k\neq i,j\}.$$

(4) *For the 35 rulings* $4H-(E_1+\cdots+E_7)-E_i-E_j-E_k$*:*

$$\{C_i+m_{j,k}, C_j+m_{i,k}, C_k+m_{i,j}, Q_{l_1,l_2}+Q_{l_3,l_4} \mid \{i,j,k,l_1,l_2,l_3,l_4\}=\{1,\dots,7\}\}$$

(5) *For the seven rulings* $5H-2(E_1+\cdots+E_7)+E_i$*:*

$$\{C_j+Q_{i,j} \mid j\neq i\}.$$

However, we have more quadratic relations in $\mathrm{Cox}(S_7)$: Note that the point q, with $\eta_1=\lambda_1=1$ and other coordinates zero, satisfies the 504 relations. Indeed, $(E_1,C_1)=2$, but all quadratic monomials which occur in the relations correspond to pairs of divisors whose intersection number is 1. Therefore, all these monomials and all the relations vanish in q. On the other hand, we check that the 504×56 Jacobian matrix has rank 54 in this point, which means that q is contained in a component of the variety defined by I_7 which has dimension 2. As $\mathbb{A}(S_7)$ is irreducible of dimension 10, we must find other relations to exclude such components.

As $E_1+C_1=-K_{S_7}$, we look for more relations in degree $-K_{S_7}$ of $\mathrm{Cox}(S_7)$: We check that in this degree, we have exactly 28 monomials:

$$\{\eta_i\lambda_i \mid 1\leq i\leq 7\}\cup\{\mu_{j,k}\nu_{j,k} \mid 1\leq j<k\leq 7\},$$

corresponding to $-K_{S_7}=E_i+C_i=m_{j,k}+Q_{j,k}$. As $\dim\Gamma(S_7,-K_{S_7})=3$, and as none of the relations coming from rulings induces a relation in this degree, we obtain 25 independent relations. Note that $-K_{S_7}$ is the unique (2)-ruling of S_7.

We can calculate the relations explicitly as they correspond to the relations between the polynomials $f_{E_i}\cdot f_{C_i}$ and $f_{m_{i,j}}\cdot f_{Q_{i,j}}$, which are homogeneous of degree 3, as described in Lemma 2.1.

Let J_7 be the ideal generated by these 529 relations.

Proof of Lemma 2.2. In order to prove that $\mathrm{Cox}(S_7)$ is described by $\mathrm{rad}(J_7)$, we must prove Lemma 2.2 in the case $r=7$.

For any $D\in\mathcal{D}_7$, consider a coordinate $\xi(D')$ where $(D,D')=1$. This is determined by the ruling $D+D'$. Indeed, this ruling induces a relation of the form

$$\xi(D)\xi(D')=\sum a_i\xi(D_i)\xi(D'_i),$$

where $D_i+D'_i=D+D'$. Therefore,

$$(D,D_i+D'_i)=(D,D+D')=(D,D)+(D,D')=-1+1=0,$$

which implies $(D,D_i)=(D,D'_i)=0$ since the only divisor with a negative intersection number with D is D itself. Since $\xi(D)\neq 0$, the only unknown variable $\xi(D')$ is determined by this relation.

Furthermore, there is exactly one coordinate $\xi(D'')$ where $(D,D'')=2$. The unique (2)-ruling $D+D''=-K_{S_7}$ induces a relation of the form

$$\xi(D)\xi(D'')=\sum a_i\xi(D_i)\xi(D'_i),$$

where $\xi(D'')$ is the only unknown variable.

Proof of Lemma 2.3. In a special case, we can calculate the Hilbert polynomial:

Example 4.2. Over the field $\mathbb{F}_{101}$ with $p_1, \dots, p_4$ as in (2.1) and

$$p_5 = (1 : 2 : 3), \quad p_6 = (1 : 5 : 7), \quad p_7 = (1 : 13 : 17)$$

in general position, we can use Macaulay to calculate the Hilbert polynomial of J_7 as

$$h(t) = \frac{1}{9!} \cdot (9504t^9 + 85536t^8 + 412992t^7 + 1294272t^6 + 2860704t^5 + 4554144t^4 + 5125248t^3 + 3863808t^2 + 1752192t + 362880). \tag{4.1}$$

The Hilbert polynomial does not depend on the choice of the field or the points. Therefore, $h(t)$ is the Hilbert polynomial of R_7/J_7. Its degree is $r + 2 = 9$.

5. Degree 1

In this section, we consider blow-ups of $\mathbb{P}^2$ in $r = 8$ points in general position, i.e., Del Pezzo surfaces S_8 of degree 1.

The set $\mathcal{D}_8$ contains the transforms of the following 242 curves:

- Blow-ups $E_1, \dots, E_8$ of $p_1, \dots, p_8$;
- 28 lines $m'_{i,j}$ through p_i, p_j:
$$m_{i,j} = H - E_i - E_j;$$
- 56 conics $Q'_{i,j,k}$ through 5 points, missing p_i, p_j, p_k:
$$Q_{i,j,k} = 2H - (E_1 + \dots + E_8) + E_i + E_j + E_k;$$
- 56 cubics $C'_{i,j}$ through 7 points missing p_j, where p_i is a double point:
$$C_{i,j} = 3H - (E_1 + \dots + E_8) - E_i + E_j;$$
- 56 quartics $V'_{i,j,k}$ through all points, where p_i, p_j, p_k are double points:
$$V_{i,j,k} = 4H - (E_1 + \dots + E_8) - (E_i + E_j + E_k);$$
- 28 quintics $F'_{i,j}$ through all points, where p_i, p_j are simple points and the other six are double points:
$$F_{i,j} = 5H - 2(E_1 + \dots + E_8) + E_i + E_j;$$
- 8 sextics T'_i, where p_i a triple point and the other seven points are double points:
$$T_i = 6H - 2(E_1 + \dots + E_8) - E_i;$$
- two independent cubics K'_1, K'_2 through the eight points:
$$[K_1] = [K_2] = -K_{S_8} = 3 - (E_1 + \dots + E_8).$$

The Cox ring of S_8 is generated by the 242 sections

$$\eta_i,\ \mu_{i,j},\ \nu_{i,j,k},\ \lambda_{i,j},\ \phi_{i,j,k},\ \psi_{i,j},\ \sigma_i,\ \kappa_i$$

of E_i, $m_{i,j}$, $Q_{i,j,k}$, $C_{i,j}$, $V_{i,j,k}$, $F_{i,j}$, T_i, K_i, respectively.

Lemma 5.1. *Each of the 2160 rulings can be expressed in the following seven ways as a sum of two negative curves:*

- 8 *rulings of the form* $H - E_i$*:*
$$\{E_j + m_{i,j} \mid j \neq i\}.$$
- $\binom{8}{4} = 70$ *rulings of the form* $2H - (E_i + E_j + E_k + E_l)$*:*
$$\left\{\begin{matrix} m_{i,j} + m_{k,l}, m_{i,k} + m_{j,l}, \\ m_{i,l} + m_{j,k}, E_a + Q_{b,c,d} \end{matrix}\middle| \{a, b, c, d, i, j, k, l\} = \{1, \ldots, 8\}\right\}.$$
- $8 \cdot \binom{7}{2} = 168$ *rulings of the form* $3H - (E_1 + \cdots + E_8) - E_i + E_j + E_k$*:*
$$\{E_j + C_{i,k}, E_k + C_{i,j}, m_{i,l} + Q_{j,k,l} \mid l \notin \{i, j, k\}\}.$$
- $8 \cdot \binom{7}{3} = 280$ *rulings* $4H - (E_1 + \cdots + E_8) + E_i - (E_j + E_k + E_l)$*:*
$$\left\{\begin{matrix} E_i + V_{j,k,l}, Q_{i,a,b} + Q_{i,c,d}, \\ C_{j,i} + m_{k,l}, C_{k,i} + m_{j,l}, C_{l,i} + m_{j,k} \end{matrix}\middle| \{a, b, c, d, i, j, k, l\} = \{1, \ldots, 8\}\right\},$$
and 8 *rulings of the form* $4H - (E_1 + \cdots + E_8) - 2E_i$*:*
$$\{m_{i,j} + C_{i,j} \mid j \neq i\}.$$
- $8 \cdot 7 = 56$ *rulings of the form* $5H - 2(E_1 + \cdots + E_8) + 2E_i + E_j$*:*
$$\{E_i + F_{i,j}, C_{k,i} + Q_{i,j,k} \mid k \notin \{i, j\}\},$$
and $8 \cdot \binom{7}{3} = 280$ *rulings* $5H - (E_1 + \cdots + E_8) - 2E_i - (E_j + E_k + E_l)$*:*
$$\left\{\begin{matrix} m_{i,j} + V_{i,k,l}, m_{i,k} + V_{i,j,l}, \\ m_{i,l} + V_{i,j,k}, C_{i,a} + Q_{b,c,d} \end{matrix}\middle| \{a, b, c, d, i, j, k, l\} = \{1, \ldots, 8\}\right\}.$$
- $\binom{8}{2} \cdot \binom{6}{2} = 420$ *rulings* $6H - 2(E_1 + \cdots + E_8) - (E_i + E_j) + E_k + E_l$*:*
$$\{m_{i,j} + F_{k,l}, V_{i,j,m} + Q_{k,l,m}, C_{i,k} + C_{j,l}, C_{i,l} + C_{j,k} \mid m \notin \{i, j, k, l\}\}.$$
- $8 \cdot 7 = 56$ *rulings of the form* $7H - 2(E_1 + \cdots + E_8) - 2E_i - E_j$*:*
$$\{m_{i,j} + T_i, C_{i,k} + V_{i,j,k} \mid k \notin \{i, j\}\},$$
and $8 \cdot \binom{7}{3} = 280$ *rulings of the form* $7H - 3(E_1 + \cdots + E_8) + 2E_i + E_j + E_k + E_l$*:*
$$\left\{\begin{matrix} F_{i,j} + Q_{i,k,l}, F_{i,k} + Q_{i,j,l}, \\ F_{i,l} + Q_{i,j,k}, C_{a,i} + V_{b,c,d} \end{matrix}\middle| \{a, b, c, d, i, j, k, l\} = \{1, \ldots, 8\}\right\}.$$
- $8 \cdot \binom{7}{3} = 280$ *rulings* $8H - 3(E_1 + \cdots + E_8) - E_i + E_j + E_k + E_l$*:*
$$\left\{\begin{matrix} C_{i,j} + F_{k,l}, C_{i,k} + F_{j,l}, C_{i,l} + F_{j,k}, \\ T_i + Q_{j,k,l}, V_{i,a,b} + V_{i,c,d} \end{matrix}\middle| \{a, b, c, d, i, j, k, l\} = \{1, \ldots, 8\}\right\},$$
and 8 *rulings of the form* $8H - 3(E_1 + \cdots + E_8) + 2E_i$*:*
$$\{F_{i,j} + C_{j,i} \mid j \neq i\}.$$
- $8 \cdot \binom{7}{2} = 168$ *rulings of the form* $9H - 3(E_1 + \cdots + E_8) + E_i - (E_j + E_k)$*:*
$$\{S_j + C_{k,i}, S_k + C_{j,i}, F_{i,l} + V_{j,k,l} \mid l \notin \{i, j, k\}\}.$$
- $\binom{8}{4} = 70$ *rulings of the form* $10H - 4(E_1 + \cdots + E_8) + E_i + E_j + E_k + E_l$*:*
$$\{F_{i,j} + F_{k,l}, F_{i,k} + F_{j,l}, F_{i,l} + F_{j,k}, S_a + V_{b,c,d} \mid \{a, b, c, d, i, j, k, l\} = \{1, \ldots, 8\}\}.$$

- 8 *rulings of the form* $11H - 4(E_1 + \dots + E_8) + E_i$*:*
$$\{S_j + F_{i,j} \mid j \neq i\}.$$

There is no way to write a ruling as the sum of $-K_{S_8}$ *and negative curves.*

Proof. Because of the Weyl group symmetry, we only need to prove the last statement in one case, say $H - E_1$. In this case, it is obvious.

By Table 1, there can be no other rulings, and each ruling can be expressed in no further ways as the sum of two negative curves. □

With this information, Lemma 2.1 allows us to determine the 10800 relations coming from rulings explicitly.

We can find more quadratic relations in the degrees corresponding to (2)-rulings: Because of the Weyl group symmetry, it is enough to consider the (2)-ruling $D := E_2 + C_{2,1}$. This can also be written as $E_j + C_{j,1}$ for any $j \neq 1$ and as $m_{i,j} + Q_{1,i,j}$ for any $i, j \neq 1$, giving 28 section in $\Gamma(S_8, D)$. As $D = -K_{S_8} + E_1$, we get two further section $\eta_1\kappa_1, \eta_1\kappa_2$. As the previous quadratic relations do not induce relations in this degree of $\mathrm{Cox}(S_8)$, and because we calculate $\dim \Gamma(S_8, D) = 3$ for this nef degree, we obtain 27 relations, which can be calculated explicitly as before.

Every negative curve has intersection number 2 with exactly 56 other curves (e.g. $(E_1, D) = 2$ if and only if $D \in \{C_{1,i}, V_{1,i,j}, F_{i,j}, T_i\}$ for $i, j \neq 1$), so it occurs in exactly 56 (2)-rulings. On the other hand, as every (2)-ruling can be written in 28 ways as the sum of two negative curves, the total number of (2)-rulings is $\frac{240\cdot 56}{2\cdot 28} = 240$. Therefore, we obtain another 6480 relations from the (2)-rulings. To determine them explicitly, we need the following more detailed information:

Lemma 5.2. *Each of the 240* (2)*-rulings can be written as a sum of two negative curves in the following 28 ways:*

- 8 (2)*-rulings of the form*
$$-K_{S_8} + E_i = 3H - (E_1 + \dots + E_8) + E_i :$$
$$\{E_j + C_{j,i}, m_{j,k} + Q_{i,j,k} \mid j, k \neq i\}.$$
- $\binom{8}{2} = 28$ (2)*-rulings of the form*
$$-K_{S_8} + m_{i,j} = 4H - (E_1 + \dots + E_8) - (E_i + E_j) :$$
$$\left\{ \begin{matrix} E_k + V_{i,j,k}, m_{i,k} + C_{j,k}, \\ m_{j,k} + C_{i,k}, Q_{a,b,c} + Q_{d,e,f} \end{matrix} \middle| \begin{matrix} k \notin \{i,j\}, \\ \{i,j,a,b,c,d,e,f\} = \{1,\dots,8\} \end{matrix} \right\}.$$
- $\binom{8}{3} = 56$ (2)*-rulings of the form*
$$-K_{S_8} + Q_{i,j,k} = 5H - 2(E_1 + \dots + E_8) + E_i + E_j + E_k :$$
$$\left\{ \begin{matrix} E_i + F_{j,k}, E_j + F_{i,k}, E_k + F_{i,j}, m_{a,b} + V_{c,d,e}, \\ Q_{i,j,l} + C_{l,k}, Q_{i,k,l} + C_{l,j}, Q_{j,k,l} + C_{l,i} \end{matrix} \middle| \begin{matrix} \{i,j,k,a,b,c,d,e\} \\ = \{1,\dots,8\}, l \notin \{i,j,k\} \end{matrix} \right\}.$$
- $8 \cdot 7 = 56$ (2)*-rulings of the form*
$$-K_{S_8} + C_{i,j} = 6H - 2(E_1 + \dots + E_8) - E_i + E_j :$$
$$\{E_j + T_i, m_{i,k} + F_{j,k}, Q_{j,k,l} + V_{i,k,l}, C_{i,k} + C_{k,j} \mid k, l \notin \{i,j\}\}.$$

- $\binom{8}{3} = 56$ (2)-*rulings of the form*

$$-K_{S_8} + V_{i,j,k} = 7H - 2(E_1 + \cdots + E_8) - (E_i + E_j + E_k) :$$

$$\left\{ \begin{matrix} T_i + m_{j,k}, S_j + m_{i,k}, S_k + m_{i,j}, F_{a,b} + Q_{c,d,e}, \\ V_{i,j,l} + C_{k,l}, V_{i,k,l} + C_{j,l}, V_{j,k,l} + C_{i,l} \end{matrix} \middle| \begin{matrix} \{i,j,k,a,b,c,d,e\} \\ = \{1,\dots,8\}, l \notin \{i,j,k\} \end{matrix} \right\}.$$

- $\binom{8}{2} = 28$ (2)-*rulings of the form*

$$-K_{S_8} + F_{i,j} = 8H - 3(E_1 + \cdots + E_8) + E_i + E_j :$$

$$\left\{ \begin{matrix} S_k + Q_{i,j,k}, F_{i,k} + C_{k,j}, \\ F_{j,k} + C_{k,i}, V_{a,b,c} + V_{d,e,f} \end{matrix} \middle| \begin{matrix} k \notin \{i,j\}, \\ \{i,j,a,b,c,d,e,f\} = \{1,\dots,8\} \end{matrix} \right\}.$$

- 8 (2)-*rulings of the form*

$$-K_{S_8} + T_i = 9H - 3(E_1 + \cdots + E_8) - E_i :$$

$$\{S_j + C_{i,j}, F_{j,k} + V_{i,j,k} \mid j,k \neq i\}.$$

Furthermore, the 242 generators give the 123 quadratic monomials

$$\eta_i\sigma_i, \quad \mu_{i,j}\psi_{i,j}, \quad \nu_{i,j,k}\phi_{i,j,k}, \quad \lambda_{i,j}\lambda_{j,i}, \quad \kappa_1^2, \kappa_1\kappa_2, \kappa_2^2$$

in the 4-dimensional subspace $\Gamma(S_8, 2 \cdot (-K_{S_8}))$ of $\mathrm{Cox}(S_8)$. Note that $2 \cdot (-K_{S_8})$ is the unique (3)-ruling. As the relations coming from rulings and (2)-rulings do induce relations, we obtain another 119 relations. Their equations can be calculated in the same way as before.

Lemma 5.3. *There are exactly* 17399 *quadratic relations in* $\mathrm{Cox}(S_8)$.

Proof. The relations in $\mathrm{Cox}(S_8)$ are generated by relations which are homogeneous with respect to the $\mathrm{Pic}(S_8)$-grading. A quadratic relation involving a term $\delta_1 \cdot \delta_2$ whose variables correspond to the negative curves D_1, D_2 has degree $D = D_1 + D_2$. The relations of degree $D_1 + D_2$ depend on the intersection number $n = (D_1, D_2)$:

- If $n = 1$, then D is a ruling. As described above, we have exactly 10800 corresponding relations.
- If $n = 2$, then D is a (2)-ruling. We have described the 6480 resulting relations.
- If $n = 3$, then $D = 2 \cdot (-K_{S_8})$, which results in exactly 119 quadratic relations.
- If $n = 0$, then $D = D_1 + D_2$ is not nef since $(D, D_1) = -1$. However, by results of [HT04, Section 3], the relations in $\mathrm{Cox}(S_8)$ are generated by relations in nef degrees.
- If $n = -1$, then $D_1 = D_2$, and $(D, D_1) = -2$, so D is not nef, giving no generating relations as before.

There are no other quadratic relations involving κ_i because the 240 degrees $-K_{S_8} + D_1$ for some negative curve D_1 are exactly the (2)-rulings, and the degree $2 \cdot (-K_{S_8})$ has also been considered. □

Let J_8 be the ideal generated by these 17399 quadratic relations in

$$R_8 = \mathbb{K}[\eta_i, \mu_{i,j}, \nu_{i,j,k}, \lambda_{i,j}, \phi_{i,j,k}, \psi_{i,j}, \sigma_i, \kappa_i].$$

Proof of Lemma 2.2. Let $D \in \mathcal{D}_8 \setminus \{K_1, K_2\}$ be any negative curve. We call a variable $\xi(D')$ for a negative curve $D' \in \mathcal{D}_8$ an (n)*-variable* if $(D, D') = n$.

As for $r = 7$ in the previous section, we show that the rulings determine the (1)-variables in terms of the (0)-variables and $\xi(D) \neq 0$.

For the two variables $\kappa_i = \xi(K_i)$ corresponding to $-K_{S_8}$, we use the (2)-ruling $-K_{S_8}+D$: As $(D, -K_{S_8}+D) = 0$, we have $(D, D_i) = (D, D_i') = 0$ for any other possibility to write $-K_{S_8} + D$ as the sum of two negative curves D_i, D_i'. Since $(-K_{S_8} + D, -K_{S_8}) = 2$, by [BP04, Prop. 3.4], the quadratic monomials $\xi(D_i)\xi(D_i')$ span the 3-dimensional space $\Gamma(S_8, -K_{S_8} + D)$, so this (2)-ruling induces relations of the form

$$\kappa_i \xi(D) = \sum a_i \xi(D_i)\xi(D_i').$$

Therefore, κ_1, κ_2 are determined by $\xi(D)$ and the (0)-variables.

Any (2)-coordinates $\xi(D')$ is determined by the (2)-ruling $D + D'$: As $(D, D + D') = 1$, we have $(D, D_i) = 0$ and $(D, D_i') = 1$ for every other possibility to write $D + D'$ as the sum of two negative curves D_i, D_i'. Furthermore, if $D + D' = -K_{S_8} + D''$, then $(D, D'') = 0$. Therefore, the relations corresponding to this (2)-ruling determine $\xi(D')$ in terms of the (0)- and (1)-variables and $\kappa_1, \kappa_2, \xi(D)$.

Finally, there is a unique (3)-coordinate D', where $D + D' = 2 \cdot (-K_{S_2})$ is the (3)-ruling. As all the other variables are known at this point, the relations corresponding to $2 \cdot (-K_{S_8})$ containing the term $\xi(D)\xi(D')$ determine $\xi(D')$.

Consider a point in U_{K_j}, i.e., with $\kappa_j \neq 0$. As above, by [BP04, Prop. 3.4], $\Gamma(S_8, 2\cdot(-K_{S_8}))$ is spanned by the monomials $\xi(D_i)\xi(D_i')$ for (3)-rulings D_i, D_i'. Therefore, we have relations of the form

$$\kappa_j^2 = \sum a_i \xi(D_i)\xi(D_i'),$$

which shows that $\xi(D_i) \neq 0$ for some i. This proves that $Z_8 \setminus \{0\}$ is covered by the sets U_D for $D \in \mathcal{D}_8 \setminus \{K_1, K_2\}$.

Proof of Lemma 2.5. Let $p \in \operatorname{Spec} R_8$ be the point whose coordinates are zero, except that $\eta_8, \mu_{1,3}, \mu_{2,3}, \mu_{3,4}, \mu_{3,5}, \mu_{3,6}, \mu_{3,8}$ are 1 and

$$(\eta_1, \eta_2, \eta_4, \eta_5, \eta_6) = \left(\frac{\alpha_3\alpha_4}{\alpha}, \frac{\alpha_4}{\alpha}, \frac{(1-\alpha_3)\alpha_4}{\alpha}, \frac{(\alpha_1-\alpha_3)\alpha_4}{\alpha_1\alpha}, \frac{(\alpha_2-\alpha_3)\alpha_4}{\alpha_2\alpha}\right),$$

where $\alpha := \alpha_4 - \alpha_3$. This point satisfies the five equations corresponding to the ruling $H - E_3$:

$$\eta_1\mu_{1,3} - \frac{\alpha_3\alpha_4}{\alpha_3-\alpha_4}\eta_7\mu_{3,7} + \frac{\alpha_3\alpha_4}{\alpha_3-\alpha_4}\eta_8\mu_{3,8},$$
$$\eta_2\mu_{2,3} - \frac{\alpha_3}{\alpha_3-\alpha_4}\eta_7\mu_{3,7} + \frac{\alpha_4}{\alpha_3-\alpha_4}\eta_8\mu_{3,8},$$
$$\eta_4\mu_{3,4} + \frac{\alpha_3\alpha_4-\alpha_3}{\alpha_3-\alpha_4}\eta_7\mu_{3,7} + \frac{-\alpha_3\alpha_4+\alpha_4}{\alpha_3-\alpha_4}\eta_8\mu_{3,8},$$
$$\eta_5\mu_{3,5} + \frac{-\alpha_1\alpha_3+\alpha_3\alpha_4}{\alpha_1\alpha_3-\alpha_1\alpha_4}\eta_7\mu_{3,7} + \frac{\alpha_1\alpha_4-\alpha_3\alpha_4}{\alpha_1\alpha_3-\alpha_1\alpha_4}\eta_8\mu_{3,8},$$
$$\eta_6\mu_{3,6} + \frac{-\alpha_2\alpha_3+\alpha_3\alpha_4}{\alpha_2\alpha_3-\alpha_2\alpha_4}\eta_7\mu_{3,7} + \frac{\alpha_2\alpha_4-\alpha_3\alpha_4}{\alpha_2\alpha_3-\alpha_2\alpha_4}\eta_8\mu_{3,8}.$$

Consider intersection numbers between the negative curves corresponding to the twelve non-zero coordinates. They are zero except for the 6 pairs corresponding to the ruling $H - E_3$. Therefore, no pair of non-zero coordinates occurs in relations corresponding to other (n)-rulings, which shows that $p \in \mathbb{A}(S_8)$. We check directly that the Jacobian in p has full rank 231.

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Mathematisches Institut, Universität Göttingen, Bunsenstr. 3-5, 37075 Göttingen, Germany

E-mail address: derentha@math.uni-goettingen.de