Let X_5 be the del Pezzo surface obtained by blowing up $\mathbb{P}^2$ at the five points with homogeneous coordinates $[0,0,1]$, $[0,1,0]$, $[1,0,0]$, $[1,1,1]$, $[1,\alpha,\beta]$.

Let S be the polynomial ring in the sixteen variables

$$\begin{array}{lll} x_{ij} & , \text{ for } & 1 \le i < j \le 5 \\ x_k & , \text{ for } & 1 \le k \le 5 \\ x & & \end{array}$$

We may sometimes denote S by $k[x_{ij}, x_k, x]$; we order the variables by declaring $x_{ij} < x_k < x$, for all admissible choices of i, j, k, and then we order the x_{ij} (resp. x_k) lexicographically (thus, for instance, $x_{23} < x_{25} < x_{34}$). Let $Q \subset S$ be the ideal generated by the twenty quadrics

$$q_5 := x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} \qquad q_{25} := -x_{12}x_1 + x_{23}x_3 + x_{24}x_4$$

$$q_4 := x_{12}x_{35} - x_{13}x_{25} + x_{15}x_{23} \qquad q_{15} := -x_{12}x_2 + x_{13}x_3 + x_{14}x_4$$

$$q_3 := x_{12}x_{45} - x_{14}x_{25} + x_{15}x_{24} \qquad q_{35} := -x_{13}x_1 + x_{23}x_2 + x_{34}x_4$$

$$q_2 := x_{13}x_{45} - x_{14}x_{35} + x_{15}x_{34} \qquad q_{45} := -x_{14}x_1 + x_{24}x_2 - x_{34}x_3$$

$$q_1 := x_{23}x_{45} - x_{24}x_{35} + x_{25}x_{34} \qquad q_{54} := \beta x_{15}x_1 - \alpha x_{25}x_2 + x_{35}x_3$$

$$q'_5 := \alpha(1-\beta)x_{12}x_{34} + (\beta-\alpha)x_{13}x_{24} + xx_5 \qquad q_{24} := -\beta x_{12}x_1 + x_{23}x_3 + x_{25}x_5$$

$$q'_4 := (1-\beta)x_{12}x_{35} + (\beta-\alpha)x_{13}x_{25} + xx_4 \qquad q_{14} := -\alpha x_{12}x_2 + x_{13}x_3 + x_{15}x_5$$

$$q'_3 := \beta x_{12}x_{45} + (\alpha-\beta)x_{14}x_{25} + xx_3 \qquad q_{34} := -\beta x_{13}x_1 + \alpha x_{23}x_2 + x_{35}x_5$$

$$q'_2 := \beta x_{13}x_{45} + (1-\beta)x_{14}x_{35} + xx_2 \qquad q_{43} := (1-\beta)x_{14}x_1 + (\alpha-1)x_{24}x_2 + x_{45}x_5$$

$$q'_1 := \alpha x_{23}x_{45} + (1-\alpha)x_{24}x_{35} + xx_1 \qquad q_{53} := (1-\beta)x_{15}x_1 + (\alpha-1)x_{25}x_2 + x_{45}x_4$$

Our goal is to show that S/Q is isomorphic to the ring $\mathrm{Cox}(X_5)$ if $\mathrm{char}k = 0$ and $[1,\alpha,\beta]$ is a general point of $\mathbb{P}^2$.

Note that these twenty quadrics are the relations obtained by considering the ten pencils of curves $C \subset X_5$ with the properties $-K \cdot C = 2$ and $C^2 = 0$ (that is the curves mapping isomorphically to conics under che anti-canonical embedding).

Using the computer program "Singular" we could compute the Hilbert polynomial $h(t)$ of the graded ring S/Q:

$$h(t) = \frac{1}{7!}\big(34t^7 + 476t^6 + 2884t^5 + 9800t^4 + 20146t^3 + 25004t^2 + 17256t + 5040\big)$$

We could also compute a minimal free resolution of the ideal Q:

$$\begin{array}{ccccccccccccccccc} S & \leftarrow & S[-2]^{20} & \leftarrow & S[-4]^{58} & \leftarrow & S[-6]^{179} & \leftarrow & S[-8]^{280} & \leftarrow & S[-10]^{179} & \leftarrow & S[-12]^{58} & \leftarrow & S[-14]^{20} & \leftarrow & S[-16] \\ 0 & & 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 \end{array}$$

From this computation we deduce that the depth of the $S-$module S/Q equals 8. In the paper [BP] there is a proof that the scheme $\mathrm{Spec}(S/Q)$ is irreducible and that $\mathrm{Cox}X_5 = \mathrm{Spec}\big(S/\mathrm{rad}(Q)\big)$. In particular the dimension of the ring S/Q is 8, and we conclude that the $S-$module S/Q is Cohen-Macaulay. Thus S/Q has no embedded ideals, and we only need to prove that S/Q has a smooth point to check that the ideal Q is radical. Using "Maple" we easily verify that the jacobian matrix of the quadrics at the point

$$p := [1,0,-1,0,1,0,0,1,0,0,0,0,0,0,1,\alpha(\beta-1)] \in \mathrm{Proj}\big(\mathrm{Cox}(X_5)\big)$$

$$
\begin{array}{cccc}
x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} & -\alpha\widetilde{\beta}x_{12}x_{34} + (\beta-\alpha)x_{13}x_{24} + xx_5 & -x_{12}x_1 + x_{23}x_3 + x_{24}x_4 & -\beta x_{12}x_1 + x_{23}x_3 + x_{25}x_5 \\
x_{12}x_{35} - x_{13}x_{25} + x_{15}x_{23} & -\widetilde{\beta}x_{12}x_{35} + (\beta-\alpha)x_{13}x_{25} + xx_4 & -x_{12}x_2 + x_{13}x_3 + x_{14}x_4 & -\alpha x_{12}x_2 + x_{13}x_3 + x_{15}x_5 \\
x_{12}x_{45} - x_{14}x_{25} + x_{15}x_{24} & \beta x_{12}x_{45} + (\alpha-\beta)x_{14}x_{25} + xx_3 & -x_{13}x_1 + x_{23}x_2 + x_{34}x_4 & -\beta x_{13}x_1 + \alpha x_{23}x_2 + x_{35}x_5 \\
x_{13}x_{45} - x_{14}x_{35} + x_{15}x_{34} & \beta x_{13}x_{45} - \widetilde{\beta}x_{14}x_{35} + xx_2 & -x_{14}x_1 + x_{24}x_2 - x_{34}x_3 & -\widetilde{\beta}x_{14}x_1 + \widetilde{\alpha}x_{24}x_2 + x_{45}x_5 \\
x_{23}x_{45} - x_{24}x_{35} + x_{25}x_{34} & \alpha x_{23}x_{45} - \widetilde{\alpha}x_{24}x_{35} + xx_1 & \beta x_{15}x_1 - \alpha x_{25}x_2 + x_{35}x_3 & -\widetilde{\beta}x_{15}x_1 + \widetilde{\alpha}x_{25}x_2 + x_{45}x_4
\end{array}
$$

$$
\begin{array}{cccccccccccccccc}
x_{34} & -x_{24} & x_{23} & 0 & x_{14} & -x_{13} & 0 & x_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_{35} & -x_{25} & 0 & x_{23} & x_{15} & 0 & -x_{13} & 0 & x_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_{45} & 0 & -x_{25} & x_{24} & 0 & x_{15} & -x_{14} & 0 & 0 & x_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x_{45} & -x_{35} & x_{34} & 0 & 0 & 0 & x_{15} & -x_{14} & x_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x_{45} & -x_{35} & x_{34} & x_{25} & -x_{24} & x_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\alpha\widetilde{\beta}x_{34} & (\beta-\alpha)x_{24} & 0 & 0 & 0 & (\beta-\alpha)x_{13} & 0 & -\alpha\widetilde{\beta}x_{12} & 0 & 0 & 0 & 0 & 0 & 0 & x & x_5 \\
-\widetilde{\beta}x_{35} & (\beta-\alpha)x_{25} & 0 & 0 & 0 & 0 & (\beta-\alpha)x_{13} & 0 & -\widetilde{\beta}x_{12} & 0 & 0 & 0 & 0 & x & 0 & x_4 \\
\beta x_{45} & 0 & (\alpha-\beta)x_{25} & 0 & 0 & 0 & (\alpha-\beta)x_{14} & 0 & 0 & \beta x_{12} & 0 & 0 & x & 0 & 0 & x_3 \\
0 & \beta x_{45} & -\widetilde{\beta}x_{35} & 0 & 0 & 0 & 0 & 0 & -\widetilde{\beta}x_{14} & \beta x_{13} & 0 & x & 0 & 0 & 0 & x_2 \\
0 & 0 & 0 & 0 & \alpha x_{45} & -\widetilde{\alpha}x_{35} & 0 & 0 & -\widetilde{\alpha}x_{24} & \alpha x_{23} & x & 0 & 0 & 0 & 0 & x_1 \\
-x_1 & 0 & 0 & 0 & x_3 & x_4 & 0 & 0 & 0 & 0 & -x_{12} & 0 & x_{23} & x_{24} & 0 & 0 \\
-x_2 & x_3 & x_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_{12} & x_{13} & x_{14} & 0 & 0 \\
0 & -x_1 & 0 & 0 & x_2 & 0 & 0 & x_4 & 0 & 0 & -x_{13} & x_{23} & 0 & x_{34} & 0 & 0 \\
0 & 0 & -x_1 & 0 & 0 & x_2 & 0 & -x_3 & 0 & 0 & -x_{14} & x_{24} & -x_{34} & 0 & 0 & 0 \\
0 & 0 & 0 & \beta x_1 & 0 & 0 & -\alpha x_2 & 0 & x_3 & 0 & \beta x_{15} & -\alpha x_{25} & x_{35} & 0 & 0 & 0 \\
-\beta x_1 & 0 & 0 & 0 & x_3 & 0 & x_5 & 0 & 0 & 0 & -\beta x_{12} & 0 & x_{23} & 0 & x_{25} & 0 \\
-\alpha x_2 & x_3 & 0 & x_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha x_{12} & x_{13} & 0 & x_{15} & 0 \\
0 & -\beta x_1 & 0 & 0 & \alpha x_2 & 0 & 0 & 0 & x_5 & 0 & -\beta x_{13} & \alpha x_{23} & 0 & 0 & x_{35} & 0 \\
0 & 0 & -\widetilde{\beta}x_1 & 0 & 0 & \widetilde{\alpha}x_2 & 0 & 0 & 0 & x_5 & -\widetilde{\beta}x_{14} & \widetilde{\alpha}x_{24} & 0 & 0 & x_{45} & 0 \\
0 & 0 & 0 & -\widetilde{\beta}x_1 & 0 & 0 & \widetilde{\alpha}x_2 & 0 & 0 & x_4 & -\widetilde{\beta}x_{15} & \widetilde{\alpha}x_{25} & 0 & x_{45} & 0 & 0
\end{array}
$$

$$\widetilde{\alpha} := \alpha - 1 \qquad \widetilde{\beta} := \beta - 1$$

Jacobian matrix of the quadratic equations has rank eight, and thus the Zariski tangent space to $\mathrm{Spec}(S/Q)$ at p has codimension equal to seven, which is exactly the codimension of $\mathrm{Spec}(S/Q)$ in $\mathbb{A}^{16}$.

Thus p is a smooth of $\mathrm{Spec}(S/Q)$ and we conclude that $Q = \mathrm{rad}(Q)$.

In fact, we may be more precise: $\mathrm{ProjCox}(X_5) = \mathrm{Proj}(S/Q)$ has exactly sixteen singular points, corresponding to the sixteen points with all coordinates equal to zero except for one and at these points, the Zariski tangent space has dimension equal to 10. To prove this, observe that the group of automorphisms of the Picard group, preserving the intersection form and the canonical class, acts transitively on the classes of the $(-1)-$curves and also on pairs of $(-1)-$curves with a given intersection product; therefore, it is enough to check that the rank of the jacobian matrix of the quadrics has rank 5 at the point

$$[1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0] \in \mathrm{Proj}\big(\mathrm{Cox}(X_5)\big)$$

(which is immediate using "Maple") and that at any other point the rank of the jacobian matrix is at least eight. This follows easily by considering the jacobian matrix: the $5 \times 5-$minor M obtained from the rows 6 through 10 and the columns 11 through 15 is clearly not zero if x is not zero.

Removing the rows and columns corresponding to the minor M (and the last column which consists then of only zeros) we are left with the matrix

$$\begin{pmatrix}
x_{34} & -x_{24} & x_{23} & 0 & x_{14} & -x_{13} & 0 & x_{12} & 0 & 0 \\
x_{35} & -x_{25} & 0 & x_{23} & x_{15} & 0 & -x_{13} & 0 & x_{12} & 0 \\
x_{34} & 0 & -x_{25} & x_{24} & 0 & x_{15} & -x_{14} & 0 & 0 & x_{12} \\
0 & x_{34} & -x_{35} & x_{34} & 0 & 0 & 0 & x_{15} & -x_{14} & x_{13} \\
0 & 0 & 0 & 0 & x_{34} & -x_{35} & x_{34} & x_{25} & -x_{24} & x_{23} \\
-x_1 & 0 & 0 & 0 & x_3 & x_4 & 0 & 0 & 0 & 0 \\
-\beta x_1 & 0 & 0 & 0 & x_3 & 0 & x_5 & 0 & 0 & 0 \\
-x_2 & x_3 & x_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\alpha x_2 & x_3 & 0 & x_5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -x_1 & 0 & 0 & x_2 & 0 & 0 & x_4 & 0 & 0 \\
0 & -\beta x_1 & 0 & 0 & \alpha x_2 & 0 & 0 & 0 & x_5 & 0 \\
0 & 0 & -x_1 & 0 & 0 & x_2 & 0 & -x_3 & 0 & 0 \\
0 & 0 & (1-\beta)x_1 & 0 & 0 & (\alpha-1)x_2 & 0 & 0 & 0 & x_5 \\
0 & 0 & 0 & \beta x_1 & 0 & 0 & -\alpha x_2 & 0 & x_3 & 0 \\
0 & 0 & 0 & (1-\beta)x_1 & 0 & 0 & (\alpha-1)x_2 & 0 & 0 & x_4
\end{pmatrix}$$

Up to the action of the automorphism group of the lattice $\mathrm{Pic}(X_5)$ preserving the intersection form and the canonical class, we only need to find a $3\times 3-$minor with non-zero determinant whenever x_{12} (resp. x_5) is non-zero. Clearly the minor in the top right corner (resp. the minor corresponding to the 7th, 11th and 13th rows and the 7th, 9th and 10th columns) works.

By intersecting with successive hyperplane sections we obtain the following Hilbert polynomials:

$$
\begin{aligned}
h_6(t) &= \frac{1}{6!}\left(34t^6+306t^5+1210t^4+2670t^3+3436t^2+2424t+720\right)\\
h_5(t) &= \frac{1}{5!}\left(34t^5+170t^4+410t^3+550t^2+396t+120\right)\\
h_4(t) &= \frac{1}{4!}\left(34t^4+68t^3+110t^2+76t+24\right)\\
h_3(t) &= \frac{1}{3!}\left(34t^3+38t^2\right)\\
h_2(t) &= \frac{1}{2!}\left(34t^2-34t+24\right)\\
h_1(t) &= \frac{1}{1!}\left(34t-34\right)
\end{aligned}
$$

References

[BP] V. Batyrev, O. Popov, *The Cox ring of a del Pezzo surface*, Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002), 85–103, Progr. Math., 226, Birkhäuser.