Homomorphisms of abelian varieties over finite fields

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Abstract. We give a proof of Tate's theorems on homomorphisms of abelian varieties over finite fields and the corresponding \(\ell\)-divisible groups.

The aim of this note is to give a proof of Tate's theorems on homomorphisms of abelian varieties over finite fields and the corresponding \(\ell\)-divisible groups [27,12], using ideas of [32,33]. We give a unified treatment for both \(\ell \neq p\) and \(\ell = p\) cases. In fact, we prove a slightly stronger version of those theorems with "finite coefficients". We use neither the existence (and properties) of the Frobenius endomorphism (for \(\ell \neq p\)) nor Dieudonné modules (for \(\ell = p\)).

The paper is organized as follows. (A rather long) Section 1 contains auxiliary results about finite commutative group schemes and abelian varieties with special reference to isogenies and polarizations. We discuss \(\ell\)-divisible groups (aka Barsotti-Tate groups) in Section 2. Section 3 contains useful results that play a crucial role in the proof of main results that are stated in Section 4.

The next five Sections contain proofs of results that were stated in Section 3. In Section 5 we discuss abelian subvarieties of a given abelian variety. Section 6 deals with the finiteness of the set of abelian varieties of given dimension and "bounded degree" over a finite field. In Section 7 we present a so called quaternon trick. In Section 8 we prove a crucial result about arbitrary finite group subschemes of abelian varieties over finite fields. In Section 9 we try to divide endomorphisms of a given abelian variety modulo \(n\).

The main results of this paper are proven in Section 10. Their variants for Tate modules are discussed in Section 11. An example of non-isomorphic elliptic curves over a finite field with isomorphic \(\ell\)-divisible groups (for all primes \(\ell\)) is discussed in Section 12.

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1. Definitions and statements

Throughout this paper $K$ is a field and $\bar{K}$ its algebraic closure. If $X$ (resp. $W$) is an algebraic variety (resp. group scheme) over $K$ then we write $\bar{X}$ (resp. $\bar{W}$) for the corresponding algebraic variety $X \times_{\text{Spec}(K)} \text{Spec}(\bar{K})$ (resp. group scheme $W \times_{\text{Spec}(K)} \text{Spec}(\bar{K})$) over $\bar{K}$. If $f : X \to Y$ is a a regular map of algebraic varieties over $K$ then we write $f$ for the corresponding map $\bar{X} \to \bar{Y}$.

1.1. Finite commutative group schemes over fields. We refer the reader to the books of Oort [17], Waterhouse [31] and Demazure–Gabriel [3] for basic properties of commutative group schemes; see also [25,21].

Recall that a group scheme $V$ over $K$ is called finite if the structure morphism $V \to \text{Spec}(K)$ is finite. Since $\text{Spec}(K)$ is a one-point set, it follows from the definition of finite morphism [7, Ch. II, Sect. 3] that $V$ is an affine scheme and $\Gamma(V, \mathcal{O}_V)$ is a finite-dimensional commutative $K$-algebra. The $K$-dimension of the $\Gamma(V, \mathcal{O}_V)$ is called the order of $V$ and denoted by $\#(V)$. An analogue of Lagrange theorem [19] asserts that multiplication by $\#(V)$ kills commutative $V$.

Let $V$ and $W$ be finite commutative group schemes over $K$ and let $u : V \to W$ be a morphism of group $K$-schemes. Both $V$ and $W$ are affine schemes, $A = \Gamma(V, \mathcal{O}_V)$ and $B = \Gamma(W, \mathcal{O}_W)$ are finite-dimensional (commutative) $K$-algebras (with 1), $V = \text{Spec}(A), W = \text{Spec}(B)$ and $u$ is induced by a certain $K$-algebra homomorphism

$$u^* : B \to A.$$ 

Since $V$ and $W$ are commutative group schemes, $A$ and $B$ are cocommutative Hopf $K$-algebras. Since $u$ is a morphism of group schemes, $u^*$ is a morphism of Hopf algebras. It follows that $C := u^*(B)$ is a $K$-subalgebra and also a Hopf subalgebra in $A$. It follows that $U := \text{Spec}(C)$ carries the natural structure of a finite group scheme over $K$ such that the natural scheme morphism $U \to V$ induced by $u^* : B \to u^*(B) = C$ is a morphism of group schemes. In addition, the inclusion $C \subset A$ induces the morphism of schemes $V \to U$, which is also a morphism of group schemes. The latter morphism is an epimorphism in the category of finite commutative group schemes over $K$, because the corresponding map

$$C = \Gamma(U, \mathcal{O}_U) \to \Gamma(V, \mathcal{O}_V) = A$$

is nothing else but the inclusion map $C \subset A$ and therefore is injective [18] (see also [5]).

On the other hand, the surjection $B \to C$ provides us with a canonical isomorphism $U \cong \text{Spec}(B/\ker(u^*))$; in addition, we observe that $\text{Spec}(B/\ker(u^*))$ is a (closed) group subscheme of $\text{Spec}(B) = W$. We denote $\text{Spec}(B/\ker(u^*))$ by $u(V)$ and call it the image of $u$ or the image of $V$ with respect to $u$ and denote by $u(V)$. Notice that the set theoretic image of $u$ is closed and our definition of the image of $u$ coincides with the one given in [4, Sect. 5.1.1].

One may easily check that the closed embedding $j : u(V) \hookrightarrow V$ induced by $B \to B/\ker(u^*)$ is an image in the category of (affine) schemes over $K$. This
means that if $\alpha, \beta : W \to S$ are two morphisms of schemes over $K$ such that their restrictions to $u(V)$ do coincide, i.e., $\alpha j = \beta j$ (as morphisms from $u(V)$ to $S$) then $\alpha u = \beta u$ (as morphisms from $U$ to $S$). It follows that $j$ is also an image in the category of finite commutative group schemes. group [21, Sect. 10].

**Theorem 1.2** (Theorem of Gabriel [18,5]). The category of finite commutative group schemes over a field is abelian.

**Remark 1.3.** Let $V$ be a finite commutative group scheme over $K$ and let $W$ be its finite closed group subscheme. If $V \to U$ is a surjective morphism of finite commutative group schemes over $K$ then $[5]$

$$\#(V) = \#(W) \cdot \#(U).$$

Recall that $\Gamma(W, \mathcal{O}_W)$ is the quotient of $\Gamma(V, \mathcal{O}_V)$. In particular, if the orders of $V$ and $W$ do coincide then $V = W$.

**1.4. Abelian varieties over fields.** We refer the reader to the books of Mumford [16], Shimura [26] for basic properties of abelian varieties (see also Lang’s book [8] and papers of Waterhouse [30], Deligne [2], Milne [13] and Oort [20]). If $X$ is an abelian variety over $K$ then we write $\text{End}(X)$ for the ring of all $K$-endomorphisms of $X$. If $m$ is an integer then write $m_X$ for the multiplication by $m$ in $X$; in particular, $1_X$ is the identity map. (Sometimes we will use notation $m$ instead of $m_X$.)

If $Y$ is an abelian variety over $K$ then we write $\text{Hom}(X, Y)$ for the group of all $K$-endomorphisms $X \to Y$.

**Remark 1.5.** Warning: sometimes in the literature, including my own papers, the notation $\text{End}(X)$ is used for the ring of $\bar{K}$-endomorphisms.

It is well known [16, Sect. 19, Theorem 3] that $\text{Hom}(X, Y)$ is a free commutative group of finite rank. We write $X^t$ for the dual of $X$ (See [13, Sect. 9–10] for the definition and basic properties of the dual of an abelian variety.) In particular, $X^t$ is also an abelian variety over $K$ that is isogenous to $X$ (over $K$). If $u \in \text{Hom}(X, Y)$ then we write $u^t$ for its dual in $\text{Hom}(Y, X)$. We have

$$\bar{X}^t = X^t.$$

If $n$ is a positive integer then we write $X_n$ for the kernel of $n_X$; it is a finite commutative (sub)group scheme (of $X$) over $K$ of rank $2\dim(X)$. By definition, $X_n(K)$ is the kernel of multiplication by $n$ in $X(K)$.

If $n$ is not divisible by $\text{char}(K)$ then $X_n$ is an étale group scheme and it is well-known [16, Sect. 4] that $X_n(K)$ is a free $\mathbb{Z}/n\mathbb{Z}$-module of rank $2\dim(X)$ and all $K$-points of $X_n$ are defined over a finite separable extension of $K$. In particular, $X_n(K)$ carries a natural structure of Galois module.

**1.6. Isogenies.** Let $W \subset X$ be a finite group subscheme over $K$. It follows from the analogue of Lagrange theorem that $W \subset X_d$ for $d = \#(W)$. The quotient $Y := X/W$ is an abelian variety over $K$ and the canonical isogeny $\pi : X \to X/W = Y$
has kernel $W$ and degree $\#(W)$ ([16, Sect. 12, Corollary 1 to Theorem 1], [3, Sect. 2, pp. 307-314]). In particular, every homomorphism of abelian varieties $u : X \to Z$ over $K$ with $W \subseteq \ker(u)$ factors through $\pi$, i.e., there exists a unique homomorphism of abelian varieties $v : Y \to Z$ over $K$ such that

$$u = v\pi.$$ 

If $m$ is a positive integer then

$$\pi m_X = m_Y \pi \in \operatorname{Hom}(X, Y).$$

Let us put

$$m^{-1}W := \ker(\pi m_X) = \ker(m_Y \pi) \subset X.$$ 

For every commutative $K$-algebra $R$ the group of $R$-points $m^{-1}W(R)$ is the set of all $x \in X(R)$ with

$$mx \in W(R) \subset X(R).$$

For example, if $W = X_n$ then

$$Y = X, \pi = n_X, m^{-1}X_n = X_{nm}.$$ 

In general, if $W \subseteq X_n$ then $m^{-1}W$ is a closed group subscheme in $X_{nm}$. E.g., $W$ is always a closed group subscheme of $X_{dn}$ and therefore is a finite group subscheme of $X$ over $K$. The order

$$\#(m^{-1}W) = \deg(\pi m_X) = \deg(\pi) \deg(m_X) = \#(W) \cdot m^{2\dim(X)}.$$ 

We have

$$X_m \subset m^{-1}W, m_X(m^{-1}W) \subset W$$

and the kernel of $m_X : m^{-1}W \to W$ coincides with $X_m$.

**Lemma 1.7.** The image $m_X(m^{-1}W) = W$.

**Proof.** Let us denote the image by $G$. By Remark 1.3, $\#(G)$ is the ratio

$$\#(m^{-1}W)/\#(X_m) = \dim(W),$$

i.e., the orders of $G$ and $W$ do coincide. Since $G \subset W$, we have (by the same Remark) $G = W$.  

**Example 1.8.** If $W = X_n$ then $m^{-1}X_n = X_{nm}$ and therefore $m(X_{nm}) = X_n$.

**Lemma 1.9.** If $r$ is a positive integer then $r(X_n) = X_{n_1}$ where $n_1 = n/(n, r)$. 


Proof. We have \( r = (n, r) \cdot r_1 \) where \( r_1 \) is a positive integer such that \( n_1 \) and \( r_1 \) are relatively prime. This implies that \( r_1(X_{n_1}) = X_{n_1} \). By Lemma 1.9, \( (n, r)(X_n) = X_{n_1} \). This implies that

\[
r(X_n) = r_1((n, r)(X_n)) = r_1(X_{n_1}) = X_{n_1}.
\]

\[\square\]

**Lemma 1.10.** Let \( X \) and \( Y \) be abelian varieties over a field \( K \). Let \( u : X \to Y \) be a \( K \)-homomorphism of abelian varieties. Let \( n > 1 \) be an integer and \( u_n : X_n \to Y \) the morphism of commutative group schemes over \( K \) induced by \( u \).

(i) Suppose that \( u \) is an isogeny and \( \deg(u) \) and \( n \) are relatively prime. Then \( u_n : X_n \to Y \) is an isomorphism.

(ii) Suppose that \( u_n : X_n \to Y \) is an isomorphism. Then \( u \) is an isogeny and \( \deg(u) \) and \( n \) are relatively prime.

Proof. Let \( u \) be an isogeny such that \( m := \deg(u) \) and \( n \) are relatively prime. Then \( \ker(u) \subset X_m \). It follows that there exists a \( K \)-isogeny \( v : Y \to X \) such that

\[
v u = m X, u v = m Y.
\]

(i) Since multiplication by \( m \) is an automorphism of both \( X_n \) and \( Y_m \), we conclude that \( u_n : X_n \to Y_n \) and \( v_n : Y_n \to X_n \) are isomorphisms.

(ii) Suppose that \( u_n \) is an isomorphism. This implies that the orders of \( X_n \) and \( Y_n \) coincide and therefore \( \dim(X) = \dim(Y) \). We need to prove that \( u \) is isogeny and \( \deg(u) \) and \( n \) are relatively prime. In order to do that, we may assume that \( K \) is algebraically closed (replacing \( K, X, Y, u \) by \( \bar{K}, \bar{X}, \bar{Y}, \bar{u} \) respectively). Let us put \( Z := u(Y) \subset X \): clearly, \( Z \) is a (closed) abelian subvariety of \( Y \) and therefore \( \dim(Z) \leq \dim(Y) \). It is also clear that \( u : X \to Y \) coincides with the composition of the natural surjection \( X \to u(X) = Z \) and the inclusion map \( j : Z \hookrightarrow X \). This implies that \( u_n(X_n) \) is a (closed) group subscheme of \( j_n(Z_n) \subset Y_n \). It follows that

\[
\#(u_n(X_n)) \leq \#(j_n(Z_n)) \leq \#(Z_n) = n^{2 \dim(Z)}.
\]

Since \( u_n \) is an isomorphism, \( u_n(X_n) = Y_n \) and therefore

\[
\#(u_n(X_n)) = \#(Y_n) = n^{2 \dim(Y)}.
\]

It follows that

\[
n^{2 \dim(Y)} \leq n^{2 \dim(Z)}
\]

and therefore \( \dim(Y) \leq \dim(Z) \). (Here we use that \( n > 1 \).) Since \( Z \) is a closed subvariety in \( Y \), we conclude that \( \dim(Z) = \dim(Y) \) and \( Y = Z \). In other words, \( u \) is surjective. Taking into account that \( \dim(X) = \dim(Y) \), we conclude that \( u \) is an isogeny.
Now let \( m = dr \) where \( d \) is the largest common divisor of \( n \) and \( m \). Then \( r \) and \( n \) are relatively prime; in particular, multiplication by \( r \) is an automorphism of \( X_n \). Let us denote \( \ker(u) \) by \( W \); it is a finite commutative group scheme over \( K \) of order \( m \) and therefore

\[
W \subset X_m.
\]

This implies that for every commutative \( K \)-algebra \( R \) we have

\[
m \cdot W(R) = \{0\}.
\]

On the other hand, since \( u_n \) is an isomorphism, the kernel of \( W(R) \overset{\sim}{\longrightarrow} W(R) \) is \( \{0\} \). Since \( d \mid n \), the kernel of \( W(R) \overset{d}{\longrightarrow} W(R) \) is also \( \{0\} \). This implies that \( r \cdot W(R) = \{0\} \) for all \( R \). Hence \( W \subset X_r \). It follows that \( \deg(u) = \#(W) \) divides \( \#(X_r) = r^{2\dim(X)} \) and therefore is coprime to \( n \).

The next statement will be used only in Section 12.

\textbf{Proposition 1.11.} Let \( X \) and \( Y \) be abelian varieties over a field \( K \). Suppose that for every prime \( \ell \) there exists an isogeny \( X \rightarrow Y \), whose degree is not divisible by \( \ell \). Then for every positive integer \( n \) there exists an isogeny \( X \rightarrow Y \), whose degree is coprime to \( n \). In particular, \( X_n \cong Y_n \).

\textbf{Proof.} Recall that the additive group \( \text{Hom}(X, Y) \) is isomorphic to \( \mathbb{Z}^\rho \) for some nonnegative integer \( \rho \). In our case, \( X \) and \( Y \) are isogenous over \( K \) and therefore \( \rho > 0 \).

Let \( n \) be a positive integer and let \( P(n) \) be the (finite) set of its prime divisors. For each \( \ell \in P(n) \) pick an isogeny \( v(\ell) : X \rightarrow Y \), whose degree is not divisible by \( \ell \). By Lemma 1.10(i), \( v(\ell) \) induces an isomorphism \( X_\ell \cong Y_\ell \). Now, by the Chinese Remainder Theorem, there exists \( u \in \text{Hom}(X, Y) \cong \mathbb{Z}^\rho \) such that

\[
u(\ell) \in \ell \cdot \text{Hom}(X, Y) \quad \forall \ell \in P.
\]

This implies that for each \( \ell \in P \) the homomorphisms \( u \) and \( v(\ell) \) induce the same morphism \( X_\ell \cong Y_\ell \), which, as we know, is an isomorphism. It follows from Lemma By Lemma 1.10(ii) that \( u \) is an isogeny, whose degree is not divisible by \( \ell \). Hence \( \deg(u) \) and \( n \) are coprime. Applying again Lemma 1.10(i), we conclude that \( u \) induces an isomorphism \( X_n \cong Y_n \).

\textbf{1.12. Polarizations.} A homomorphism \( \lambda : X \rightarrow X' \) is a \textit{polarization} if there exists an ample invertible sheaf \( L \) on \( X' \) such that \( \lambda \) coincides with

\[
\Lambda_L : X' \overset{\sim}{\longrightarrow} X, \ z \mapsto \text{cl}(T_z L \otimes L^{-1})
\]

where \( T_z : X \rightarrow X' \) is the translation map

\[
x \mapsto x + z.
\]
and cl stands for the isomorphism class of an invertible sheaf. Recall [16, Sect. 6, Proposition 1; Sect. 8, Theorem 1; Sect. 13, Corollary 5] that a polarization is an isogeny. If \( \lambda \) is an isomorphism, i.e., \( \deg(\lambda) = 1 \), we call \( \lambda \) a principal polarization and the pair \((X, \lambda)\) is called a principally polarized abelian variety (over \( K \)).

If \( n := \deg(\lambda) = \#(\ker(\lambda)) \) then \( \ker(\lambda) \) is killed by multiplication by \( n \), i.e., \( \ker(\lambda) \subset X_n \). For every positive integer \( m \) we write \( \lambda^m \) for the polarization \( X^m \to (X^m)^t = (X^t)^m, (x_1, \ldots, x_m) \mapsto (\lambda(x_1), \ldots, \lambda(x_m)) \) that corresponds to the ample invertible sheaf \( \otimes_{i=1}^m \text{pr}_i^* L \) where \( \text{pr}_i: X^m \to X \) is the \( i \)th projection map. We have

\[
\dim(X^m) = m \cdot \dim(X), \quad \deg(\lambda^m) = \deg(\lambda)^m
\]

and \( \ker(\lambda^m) = \ker(\lambda)^m \subset (X^m)_n \) if \( \ker(\lambda) \subset X_n \).

There exists a Riemann form - a skew-symmetric pairing of group schemes over \( \overline{K} \) [16, Sect. 23]

\[
e_\lambda: \ker(\lambda) \times \ker(\lambda) \to G_m
\]

where \( G_m \) is the multiplicative group scheme over \( K \).

If \( e_{\lambda^m} : \ker(\lambda^m) \times \ker(\lambda^m) \to G_m \) is the Riemann form for \( \lambda^m \) then in obvious notation

\[
e_{\lambda^m}(x, y) = \prod_{i=1}^m e_\lambda(x_i, y_i)
\]

where

\[
x = (x_1, \ldots, x_m), \quad y = (y_1, \ldots, y_m) \in \ker(\lambda)^m = \ker(\bar{\lambda}^m).
\]

We have

\[
\text{Mat}_m(\mathbb{Z}) \subset \text{Mat}_m(\text{End}(X)) = \text{End}(X^m).
\]

One may easily check that every \( u \in \text{Mat}_m(\mathbb{Z}) \) leaves the group subscheme \( \ker(\lambda^m) \) invariant and

\[
e_{\lambda^m}(ux, y) = e_{\lambda^m}(x, u^*y)
\]

where \( u^* \) is the transpose of the matrix \( u \). Notice that \( u^* \) viewed as an element of

\[
\text{Mat}_m(\mathbb{Z}) \subset \text{Mat}_m(\text{End}(X^t)) = \text{End}((X^t)^m)
\]

coincides with \( u^t \in \text{End}((X^m)^t) \).
1.13. Polarizations and isogenies. Let $W \subset \ker(\lambda)$ be a finite group subscheme over $K$. Recall that $Y := X/W$ is an abelian variety over $K$ and the canonical isogeny $\pi : X \to X/W = Y$ has kernel $W$ and degree $#W$.

Suppose that $W$ is isotropic with respect to $e_{\lambda}$, i.e., the restriction of $e_{\lambda}$ to $W \times \bar{W}$ is trivial. Then there exists an ample invertible sheaf $\mathcal{M}$ on $\bar{Y}$ such that $L \cong \pi^* \mathcal{M}$ [16, Sect. 23, Corollary to Theorem 2, p. 231] and the $K$-polarization $\Lambda_{\mathcal{M}} : \bar{Y} \to \bar{Y}$ satisfies

$$\bar{\lambda} = \pi^* \Lambda_{\mathcal{M}} \bar{\pi}.$$ 

Since $\bar{\pi}$ and $\bar{\pi}$ are isogenies that are defined over $K$, the polarization $\Lambda_{\mathcal{M}}$ is also defined over $K$, i.e., there exists a $K$-isogeny $\mu : Y \to Y$ such that $\Lambda_{\mathcal{M}} = \bar{\mu}$ and

$$\lambda = \pi^t \mu \pi.$$ 

It follows that

$$\deg(\lambda) = \deg(\pi) \deg(\mu) \deg(\pi^t) = \deg(\pi)(\deg(\mu))^2 = (\#W)^2 \deg(\mu).$$ 

Therefore $\mu$ is a principal polarization (i.e., $\deg(\mu) = 1$) if and only if

$$\deg(\lambda) = (\#W)^2.$$ 

2. $\ell$-divisible groups, abelian varieties and Tate modules

Let $h$ be a non-negative integer and $\ell$ a prime. The following notion was introduced by Tate [28, 25].

**Definition 2.1.** An $\ell$-divisible group $G$ over $K$ of height $h$ is a sequence $\{G_\nu, i_\nu\}_{\nu=1}^\infty$ in which:

- $G_\nu$ is a finite commutative group scheme over $K$ of order $\ell^{h\nu}$.
- $i_\nu$ is a closed embedding $G_\nu \hookrightarrow G_{\nu+1}$ that is a morphism of group schemes.

In addition, $i_\nu(G_\nu)$ is the kernel of multiplication by $\ell^\nu$ in $G_{\nu+1}$.

**Example 2.2.** Let $X$ be an abelian variety over $K$ of dimension $d$. Then it is known [28, 25] that the sequence $\{X_\nu\}_{\nu=1}^\infty$ is an $\ell$-divisible group over $K$ of height $2d$. Here $i_\nu$ is the inclusion map $X_\nu \hookrightarrow X_{\nu+1}$. We denote this $\ell$-divisible group by $X(\ell)$.

**2.3. Homomorphisms of $\ell$-divisible groups and abelian varieties.** If $H = \{H_\nu, j_\nu\}_{\nu=1}^\infty$ is an $\ell$-divisible group over $K$ then a morphism $u : G \to H$ is a sequence $\{u_\nu\}_{\nu=1}^\infty$ of morphisms of group schemes over $K$

$$u_\nu : G_\nu \to H_\nu$$

such that the composition
\[ u_{(\nu+1)i_\nu} : G_\nu \leftrightarrow G_{\nu+1} \rightarrow H_{\nu+1} \]
coinsides with
\[ j_\nu u_{(\nu)} : G_\nu \rightarrow H_\nu \leftrightarrow H_{\nu+1}, \]
i.e., the diagram
\[
\begin{array}{ccc}
G_\nu & \xrightarrow{u_{(\nu)}} & H_\nu \\
\downarrow i_\nu & & \downarrow j_\nu \\
G_{\nu+1} & \xrightarrow{u_{(\nu+1)}} & H_{\nu+1}
\end{array}
\]
is commutative.

**Remark 2.4.** A morphism \( u \) is an isomorphism of \( \ell \)-divisible groups if and only if all \( u_{(\nu)} \) are isomorphisms of the corresponding finite group schemes.

The group \( \text{Hom}(G, H) \) of morphisms from \( G \) to \( H \) carries a natural structure of \( \mathbb{Z}_\ell \)-module induced by the natural structures of \( \mathbb{Z}/\ell^\nu = \mathbb{Z}_\ell/\ell^\nu \)-module on \( \text{Hom}(G_{\nu}, H_{\nu}) \). Namely, if \( u = \{u_{(\nu)}\}_{\nu=1}^\infty \in \text{Hom}(G, H) \) and \( a \in \mathbb{Z}_\ell \) then
\[ au = \{(au)_{(\nu)}\}_{\nu=1}^\infty \]
may be defined as follows. For each \( \nu \) pick \( a_\nu \in \mathbb{Z} \) with
\[ a = a_\nu \in \ell^\nu \mathbb{Z}_\ell \]
and put
\[ (au)_{(\nu)} := a_\nu u_{(\nu)} : G_\nu \rightarrow H_\nu. \]
Since multiplication by \( \ell^\nu \) kills \( G_\nu \), the definition of \( (au)_{(\nu)} \) does not depend on the choice of \( a_\nu \).

Let \( X \) and \( Y \) be abelian varieties over \( K \). There is a natural homomorphism of commutative groups \( \text{Hom}(X, Y) \rightarrow \text{Hom}(X(\ell), Y(\ell)) \). Namely, if \( u \in \text{Hom}(X, Y) \) then \( u(X_{\ell^\nu}) \) lies in the kernel of multiplication by \( \ell^\nu \), i.e. \( u(X_{\ell^\nu}) \subset Y_{\ell^\nu} \). In fact, we get the natural homomorphism
\[ \text{Hom}(X, Y) \otimes \mathbb{Z}/\ell^\nu \rightarrow \text{Hom}(X_{\ell^\nu}, Y_{\ell^\nu}), \]
which is known to be an embedding. (See also Lemma 9.1 below.)

Since \( \text{Hom}(X(\ell), Y(\ell)) \) is a \( \mathbb{Z}_\ell \)-module, we get the natural homomorphism of \( \mathbb{Z}_\ell \)-modules
\[ \text{Hom}(X, Y) \otimes \mathbb{Z}_\ell \rightarrow \text{Hom}(X(\ell), Y(\ell)). \]
Explicitly, if \( u \in \text{Hom}(X, Y) \otimes \mathbb{Z}_\ell \) then for each \( \nu \) we may pick
\[ w(\nu) \in \text{Hom}(X, Y) = \text{Hom}(X, Y) \otimes 1 \subset \text{Hom}(X, Y) \otimes \mathbb{Z}_\ell \]
such that
Then the corresponding morphism of group schemes \( u_\nu := w_\nu : X_\ell^\nu \to Y \) does not depend on the choice of \( w_\nu \) and defines the corresponding morphism of \( \ell \)-divisible groups

\[
u - w(\nu) \in \ell^\nu \cdot \{\text{Hom}(X, Y) \otimes Z_\ell\} = \{\ell^\nu \cdot \text{Hom}(X, Y)\} \otimes Z_\ell = \text{Hom}(X, Y) \otimes \ell^\nu Z_\ell.
\]

Remark 2.5. Since \( \text{Hom}(X, Y) \) is a free commutative group of finite rank, the \( Z_\ell \)-module \( \text{Hom}(X, Y) \otimes Z_\ell \) is a free module of finite rank.

The following assertion seems to be well known (at least, when \( \ell \neq \text{char}(K) \)).

**Lemma 2.6.** The natural homomorphism of \( Z_\ell \)-modules

\[
\text{Hom}(X, Y) \otimes Z_\ell \to \text{Hom}(X(\ell), Y(\ell))
\]

is injective.

**Proof.** If it is not injective and \( u \) lies in the kernel then \( u_\nu \in \ell^\nu \cdot \text{Hom}(X, Y) \) for all \( \nu \). Since \( u - u_\nu \in \ell^\nu \cdot \{\text{Hom}(X, Y) \otimes Z_\ell\} \), we conclude that \( u \in \ell^\nu \cdot \{\text{Hom}(X, Y) \otimes Z_\ell\} \) for all \( \nu \). Since \( \text{Hom}(X, Y) \otimes Z_\ell \) is a free \( Z_\ell \)-module of finite rank, it follows that \( u = 0 \).

**Corollary 2.7.** The following conditions are equivalent:

(i) There exists an isogeny \( u : X \to Y \), whose degree is not divisible by \( \ell \).

(ii) There exists \( w \in \text{Hom}(X, Y) \otimes Z_\ell \) that induces an isomorphism of \( \ell \)-divisible groups \( X(\ell) \to Y(\ell) \).

**Proof.** Let \( u : X \to Y \) be an isogeny, whose degree is not divisible by \( \ell \). Applying Lemma 1.10(i) to all \( n = \ell^\nu \), we conclude that \( u \) induces an isomorphism \( X(\ell) \cong Y(\ell) \).

Now suppose that \( w \in \text{Hom}(X, Y) \otimes Z_\ell \) that induces an isomorphism of \( \ell \)-divisible groups \( X(\ell) \to Y(\ell) \). In particular, \( w \) induces an isomorphism of finite group schemes \( w(1) : X_\ell \cong Y_\ell \). On the other hand, there exists \( u \in \text{Hom}(X, Y) \) such that

\[
w - u \in \ell \cdot \{\text{Hom}(X, Y) \otimes Z_\ell\} = \text{Hom}(X, Y) \otimes \ell Z_\ell.
\]

This implies that \( u \) and \( w \) induce the same morphism of finite group schemes \( X_\ell \to Y_\ell \). It follows that the morphism

\[
u_\ell = u(1) : X_\ell \to Y_\ell
\]

induced by \( u \) coincides with \( w(1) \) and therefore is an isomorphism. Now Lemma 1.10(ii) implies that \( u \) is an isogeny, whose degree is not divisible by \( \ell \).
2.8. Tate modules. In this subsection we assume that \( \ell \) is a prime different from \( \text{char}(K) \). If \( n = \ell^k \) then \( X_n \) is an étale finite group scheme of order \( n^{2\dim(X)} \) and we will identify it with the Galois module of its \( K \)-points. (Actually, all points of \( X_n \) are defined over a separable algebraic extension of \( K \).) The Tate \( \ell \)-module \( T_\ell(X) \) is defined as the projective limit of Galois modules \( X_{\ell^k} \) where the transition map \( X_{\ell^{k+1}} \to X_{\ell^k} \) is multiplication by \( \ell \). The Tate module carries a natural structure of free \( \mathbb{Z}_\ell \)-module of rank \( 2\dim(X) \); it is also provided with a natural structure of Galois module in such a way that natural homomorphisms \( T_\ell(X) \to X_{\ell^k} \) induce isomorphisms of Galois modules

\[
T_\ell(X) \otimes \mathbb{Z}/\ell^k \cong X_{\ell^k}.
\]

Explicitly, \( T_\ell(X) \) is the set of all collections \( x = \{x_\nu\}_{\nu=1}^\infty \) with

\[
x_\nu \in X_{\ell^k}, \quad x_{\nu+1} = \ell x_\nu \forall \nu.
\]

The map \( x \mapsto x_\nu \) defines the surjective homomorphism of Galois modules \( T_\ell(X) \to X_{\ell^k} \), whose kernel coincides with \( \ell^k \cdot T_\ell(X) \) and therefore induces the isomorphism of Galois modules \( T_\ell(X)/\ell^k \cong X_{\ell^k} \) mentioned above.

If \( Y \) is an abelian variety over \( K \) then we write \( \text{Hom}_{\text{Gal}}(T_\ell(X), T_\ell(Y)) \) for the \( \mathbb{Z}_\ell \)-module of all homomorphisms of \( \mathbb{Z}_\ell \)-modules \( T_\ell(X) \to T_\ell(Y) \) that commute with the Galois action(s), i.e., are also homomorphisms of Galois modules.

The \( \mathbb{Z}_\ell \)-module \( \text{Hom}_{\text{Gal}}(T_\ell(X), T_\ell(Y)) \) is the set of collections \( w = \{w_\nu\}_{\nu=1}^\infty \) of homomorphisms of Galois modules

\[
w_\nu : T_\ell(X)/\ell^k \to X_{\ell^k} = T_\ell(Y)/\ell^k
\]

such that

\[
w_\nu(x_\nu) = \ell \cdot w_{\nu+1}(x_{\nu+1}) \quad \forall x = \{x_\nu\}_{\nu=1}^\infty \in T_\ell(X).
\]

Now if \( z \in X_{\ell^k} \) then there exists \( x \in T_\ell(X) \) with \( x_\nu = z \). We have \( \ell x_{\nu+1} = x_\nu = z \) and

\[
w_\nu(z) = w_\nu(x_\nu) = \ell \cdot w_{\nu+1}(x_{\nu+1}) = w_{\nu+1}(\ell x_{\nu+1}) = w_{\nu+1}(x_{\nu+1}) = w_{\nu+1}(z),
\]

i.e., the restriction of \( w_{\nu+1} \) to \( X_{\ell^k} \) coincides with \( w_\nu \). This means that the collection \( \{w_\nu\}_{\nu=1}^\infty \) defines a morphism of \( \ell \)-divisible groups over \( K \)

\[
X(\ell) \to Y(\ell).
\]

Conversely, if \( u = \{u_\nu(x)\}_{\nu=1}^\infty \) is a morphism \( X(\ell) \to Y(\ell) \) over \( K \) then

\[
u_\nu : X_{\ell^k} \to Y_{\ell^k}
\]

is a homomorphism of Galois modules; in addition, the restriction of \( u_{\nu+1} \) to \( X_{\ell^k} \) coincides with \( u_\nu \). This implies that for each \( \{x_\nu\}_{\nu=1}^\infty \in T_\ell(X) \)}
for all $\nu$. This means that the collection \(\{u(\nu)\}_\nu\) defines a homomorphism of Galois modules $T_\ell(X) \to T_\ell(Y)$. These observations give us the natural isomorphism of $\mathbb{Z}_\ell$-modules $\text{Hom}(X(\ell), Y(\ell)) = \text{Hom}_{\text{Gal}}(T_\ell(X), T_\ell(Y))$.

### 3. Useful results

**Theorem 3.1** ([32,34,14]). Let $X$ be an abelian variety of positive dimension over a field $K$ and $X^t$ its dual. Then $(X \times X^t)^4$ admits a principal $K$-polarization.

We prove Theorem 3.1 in Section 7.

**Theorem 3.2** ([11]). Let $X$ be an abelian variety over $K$. The set of abelian $K$-subvarieties of $X$ is finite, up to the action of the group $\text{Aut}(X)$ of $K$-automorphisms of $X$.

We sketch the proof of Theorem 3.2 in Section 5.

**Lemma 3.3** (Tate ([27], Sect. 2, p. 136)). Let $K$ be a finite field, and let $g$ and $d$ be positive integers. The set of $K$-isomorphism classes of $g$-dimensional abelian varieties over $K$ that admit a $K$-polarization of degree $d$ is finite.

Lemma 3.3 will be proven in Section 6.

**Theorem 3.4** ([32], Th. 4.1). Let $K$ be a finite field, $g$ a positive integer. Then the set of $K$-isomorphism classes of $g$-dimensional abelian varieties over $K$ is finite.

**Proof of Theorem 3.4** (modulo Theorem 3.1 and Lemma 3.3). Suppose that $X$ is a $g$-dimensional abelian variety over $K$. By Lemma 3.3, the set of $4g$-dimensional abelian varieties over $K$ of the form $(X \times X^t)^4$ is finite, up to $K$-isomorphism. The abelian variety $X$ is isomorphic over $K$ to an abelian subvariety of $(X \times X^t)^4$. In order to finish the proof, one has only to recall that thanks to Theorem 3.2, the set of abelian subvarieties of a given abelian variety is finite, up to a $K$-isomorphism.

We need Theorem 1.2 in order to state the following assertion.

**Corollary 3.5** (Corollary to Theorem 3.4). Let $X$ be an abelian variety of positive dimension over a finite field $K$. There exists a positive integer $r = r(X, K)$ that enjoys the following properties:

(i) If $Y$ is an abelian variety over $K$ that is $K$-isogenous to $X$ then there exists a $K$-isogeny $\beta : X \to Y$ such that $\ker(\beta) \subset X_r$.

(ii) If $n$ is a positive integer and $W \subset X_n$ is a group subscheme over $K$ then there exists an endomorphism $u \in \text{End}(X)$ such that $rW \subset uX_n \subset W$. 

\[u(\nu)(x_\nu) = u(\nu+1)(x_\nu) = u(\nu+1)(\ell x_{\nu+1}) = \ell u(\nu+1)(x_{\nu+1})\]
Remark 3.6. The assertion 3.5(i) follows readily from Theorem 3.4.

We prove Corollary 3.5(ii) in Section 8.

4. Main results

Theorem 4.1. Let $X$ be an abelian variety of positive dimension over a finite field $K$. There exists a positive integer $r_1 = r_1(X, K)$ that enjoys the following properties:

Let $n$ be a positive integer and $u_n \in \text{End}(X_n)$. Let us put $m = n/(n, r_1)$. Then there exists $u \in \text{End}(X)$ such that the images of $u$ and $u_n$ in $\text{End}(X_m)$ do coincide.

We prove Theorem 4.1 in Section 10.

Applying Theorem 4.1 to a product $X = A \times B$ of abelian varieties $A$ and $B$, we obtain the following statement.

Theorem 4.2. Let $A, B$ be abelian varieties of positive dimension over a finite field $K$. There exists a positive integer $r_2 = r_2(A, B)$ that enjoys the following properties:

Suppose that $n$ is a positive integer and $u_n : A_n \to B_n$ is a morphism of group schemes over $K$. Let us put $m = n/(n, r_2)$. Then there exists a homomorphism $u : A \to B$ of abelian varieties over $K$ such that the images of $u$ and $u_n$ in $\text{Hom}(A_m, B_m)$ do coincide.

The following assertions follow readily from Theorem 4.2.

Corollary 4.3 (First Corollary to Theorem 4.2). If $n$ and $r_2$ are relatively prime (e.g., $n$ is a prime that does not divide $r_2$) then the natural injection

$$\text{Hom}(A, B) \otimes \mathbb{Z}/n \hookrightarrow \text{Hom}(A_n, B_n)$$

is bijective.

Corollary 4.4 (Second Corollary to Theorem 4.2). Let $\ell$ be a prime and $\ell^{r_2}$ is the exact power of $\ell$ dividing $r_2$. Then for each positive integer $i$ the image of

$$\text{Hom}(A_{\ell^{i+r_2}}, B_{\ell^{i+r_2}}) \to \text{Hom}(A_{\ell^i}, B_{\ell^i})$$

coincides with the image of

$$\text{Hom}(A, B) \otimes \mathbb{Z}/\ell^i \hookrightarrow \text{Hom}(A_{\ell^i}, B_{\ell^i}).$$

5. Abelian subvarieties

We follow the exposition in [11].

The next statement is a corollary of a finiteness result of Borel and Harish-Chandra [1, Theorem 6.9]; it may also be deduced from the Jordan–Zassenhaus theorem [23, Theorem 26.4].
Proposition 5.1 ([11], p. 514). Let $F$ be a finite-dimensional semisimple $\mathbb{Q}$-algebra, $M$ a finitely generated right $F$-module, $L$ a $\mathbb{Z}$-lattice in $M$. Let $G$ be the group of those automorphisms $\sigma$ of the $F$-module $M$ for which $\sigma(L) = L$. Then the number of $G$-orbits of the set of $F$-submodules of $M$ is finite.

Now let $X$ be an abelian variety over $K$. We are going to apply Proposition 5.1 to $F = \text{End}(X) \otimes \mathbb{Q}$, $M = \text{End}(X) \otimes \mathbb{Q}$, $L = \text{End}(X)$.

One may identify $G$ with the group $\text{Aut}(X) = \text{End}(X)^*$ of automorphisms of $X$: here elements of $\text{End}(X)^*$ act as left multiplications on $\text{End}(X) \otimes \mathbb{Q} = M$.

On the other hand, to each abelian $K$-subvariety $Y \subset X$ corresponds the right ideal

$$I(Y) = \{ u \in \text{End}(X) \mid u(X) \subset Y \}$$

and the $F$-submodule

$$I(Y)_\mathbb{Q} = I(Y) \otimes \mathbb{Q} \subset \text{End}(X) \otimes \mathbb{Q} = M.$$ 

Using the theorem of Poincaré-Weil [13, Proposition 12.1], one may prove ([11, p. 515] that $I(Y)_\mathbb{Q}$ uniquely determines $Y$. Even better, if $Y'$ is an abelian $K$-subvariety of $X$ and

$$uI(Y)_\mathbb{Q} = I(Y')_\mathbb{Q}$$

for $u \in \text{Aut}(X) = \text{End}(X)^*$ then $Y' = u(Y)$. Now Proposition 5.1 implies the finiteness of the number of orbits of the set of abelian $K$-subvarieties of $X$ under the natural action of $\text{Aut}(X)$. This proves Theorem 3.2. (See [10] for variants and complements.)

6. Polarized abelian varieties

Lemma 6.1 (Mumford’s lemma [15]). Let $X$ be an abelian variety of positive dimension over a field $K$. If $\lambda : X \to X^t$ is a polarization then there exists an ample invertible sheaf $\mathcal{L}$ on $X$ such that

$$\Lambda_\mathcal{L} = 2\lambda$$

where $\mathcal{L}$ is the invertible sheaf on $X$ induced by $\mathcal{L}$.

Proof. See [15, Ch. 6, Sect. 2, pp. 120–121] where a much more general case of abelian schemes is considered. (In notation of [15], $S$ is the spectrum of $K$.) Let me just recall an explicit construction of $\mathcal{L}$. Let $\mathbb{P}$ be the universal Poincaré invertible sheaf on $X \times X^t$ [13, Sect. 9]. Then $\mathcal{L} := (1_X, \lambda)^*\mathbb{P}$ where $(1_X, \lambda) : X \to X \times X^t$ is defined by the formula
Proof of Lemma 3.3. So, let $X$ be a $g$-dimensional abelian variety over a finite field $K$ and let $\lambda : X \to X^t$ be a polarization of degree $d$. We follow the exposition in [22, p. 243]. By Lemma 6.1, there exists an invertible ample sheaf $L$ on $X$ such that the self-intersection index of $\bar{L}$ equals $2^g d!$ [16, Sect. 16]. The invertible sheaf $\bar{L}$ is very ample, its space of global section has dimension $6^g d!$ [16, Sect. 16]. This implies that $L^3$ is also very ample and gives us an embedding (over $K$) of $X$ into the $6^g d!-1$-dimensional projective space as a closed $K$-subvariety of degree $6^g d!$. All those subvarieties are uniquely determined by their Chow forms ([29, Ch. 1, Sect. 6.5], [6, Lecture 21, pp. 268–273]), whose coefficients are elements of $K$. Since $K$ is finite and the number of coefficients depends only on the degree and dimension, we get the desired finiteness result.

7. Quaternion trick

Let $X$ be an abelian variety of positive dimension over a field $K$ and $\lambda : X \to X^t$ a $K$-polarization. Pick a positive integer $n$ such that

$$\ker(\lambda) \subset X_n.$$ 

Lemma 7.1. Suppose that there exists an integer $a$ such that $a^2 + 1$ is divisible by $n$. Then $X \times X^t$ admits a principal polarization that is defined over $K$.

Proof. Let

$$V \subset \ker(\lambda) \times \ker(\lambda) \subset X_n \times X_n \subset X \times X$$

be the graph of multiplication by $a$ in $\ker(\lambda)$. Clearly, $V$ is a finite group subscheme over $K$ that is isomorphic to $\ker(\lambda)$ and therefore its order is equal to $\deg(\lambda)$. Notice that $\deg(\lambda)$ is the square root of $\deg(\lambda^2)$.

For each commutative $K$-algebra $R$ the group $V(R)$ of $R$-points coincides with the set of all the pairs $(x, ax)$ with $x \in \ker(\lambda) \subset X_n$. This implies that for all $(x, ax), (y, ay) \in V(R)$ we have

$$e_{\lambda^2}((x, ax), (y, ay)) = e_{\lambda}(x, y) \cdot e_{\lambda}(ax, ay) = e_{\lambda}(x, y) \cdot e_{\lambda}(a^2 x, y) =$$

$$e_{\lambda}(x, y) \cdot e_{\lambda}(-x, y) = e_{\lambda}(x, y) / e_{\lambda}(x, y) = 1.$$ 

In other words, $V$ is isotropic with respect to $e_{\lambda^2}$; in addition,

$$\#(V)^2 = \deg(\lambda)^2 = \deg(\lambda^2).$$
This implies that $X^2/V$ is a principally polarized abelian variety over $K$. On the other hand, we have an isomorphism of abelian varieties over $K$

$$f : X \times X \to X \times X = X^2, \quad (x, y) \mapsto (x, ax + y)$$

and

$$V = f(\ker \lambda \times \{0\}) \subset f(X \times \{0\}).$$

Thus, we obtain $K$-isomorphisms

$$X^2/V \cong X/\ker(\lambda) \times X = X^t \times X = X \times X^t.$$

In particular, $X \times X^t$ admits a principal $K$-polarization and we are done.

*Proof of Theorem 3.1.* Choose a quadruple of integers $a, b, c, d$ such that

$$0 \neq s := a^2 + b^2 + c^2 + d^2$$

is congruent to $-1$ modulo $n$. We denote by $I$ the "quaternion"

$$I = \begin{pmatrix}
    a & -b & -c & -d \\
    b & a & d & c \\
    c & -d & a & b \\
    d & c & -b & a
\end{pmatrix} \in \Mat_4(\mathbb{Z}) \subset \Mat_4(\End(X) = \End(X^8)).$$

We have

$$I^*I = a^2 + b^2 + c^2 + d^2 = s \in \mathbb{Z} \subset \Mat_4(\mathbb{Z}) \subset \Mat_4(\End(X) = \End(X^4)).$$

Let

$$V \subset \ker(\lambda^4) \times \ker(\lambda^4) \subset (X^4)_n \times (X^4)_n \subset X^4 \times X^4 = X^8$$

be the graph of

$$I : \ker(\lambda^4) \to \ker(\lambda^4).$$

Clearly, $V$ is a finite group subscheme over $K$ and its order is equal to $\deg(\lambda^4)$. Notice that $\deg(\lambda^4)$ is the square root of $\deg(\lambda^8)$.

For each commutative $K$-algebra $R$ the group $\tilde{V}(R)$ of $R$-points consists of all the pairs $(x, Ix)$ with $x \in \ker(\lambda^4) \subset (X^4)_n$. This implies that for all $(x, Ix), (y, Iy) \in \tilde{V}(R)$ we have

$$e_{\lambda^4}((x, Ix), (y, Iy)) = e_{\lambda^4}(x, y) \cdot e_{\lambda^4}(Ix, Iy) = e_{\lambda^4}(x, y) \cdot e_{\lambda}(x, I^t Iy) =$$

$$e_{\lambda}(x, y) \cdot e_{\lambda}(x, sy) = e_{\lambda}(x, y) \cdot e_{\lambda}(x, -y) = e_{\lambda}(x, y) - e_{\lambda}(x, y) = 1.$$
In other words, $\bar{V}$ is isotropic with respect to $e_{4\lambda}$; in addition,

$$\#(\bar{V})^2 = \deg(\lambda^4)^2 = \deg(\lambda^8).$$

This implies that $X^8/V$ is a principally polarized abelian variety over $K$. On the other hand, we have an isomorphism of abelian varieties over $K$

$$f : X^4 \times X^4 \to X^4 \times X^4 = X^8, \quad (x, y) \mapsto (x, \lambda x + y)$$

and

$$V = f(\ker(\lambda^4) \times \{0\}) \subset f(X^4 \times \{0\}).$$

Thus, we obtain $K$-isomorphisms

$$X^4/V \cong X^4/\ker \lambda^4 \times X^4 = (X^4)^t \times X^4 = (X \times X^t)^4.$$

In particular, $(X \times X^t)^4$ admits a principal $K$-polarization and we are done. \(\square\)

**Remark 7.2.** We followed the exposition in [32, Lemma 2.5], [34, Sect. 5]. See [14, Ch. IX, Sect. 1] where Deligne’s proof is given.

8. Finite group subschemes of abelian varieties

**Proof of Corollary 3.5(ii).** Let $r$ be as in 3.5(i). Let us consider the abelian variety $Y := X/W$ and the canonical $K$-isogeny $\pi : X \to X/W = Y$. Clearly,

$$W = \ker(\pi).$$

Since $W \subset X_n$, there exists a $K$-isogeny $v : Y \to X/X_n = X$ such that the composition $v\pi$ coincides with multiplication by $n$ in $X$; in addition,

$$\pi n_X = n_Y \pi : X \to Y$$

is a $K$-isogeny, whose degree is $\#(W) \times n^{2\dim(X)}$. Here $n_X$ (resp. $n_Y$) stands for multiplication by $n$ in $X$ (resp. in $Y$). Let us put

$$U = \ker(\pi n_X) = \ker(n_Y \pi) \subset X;$$

it is a finite commutative group $K$-(sub)scheme and

$$\#(U) = \#(W) \times n^{2\dim(X)}.$$

Then

$$X_n \subset U, \quad W \subset U; \quad \pi(U) \subset Y_n, \quad n_X(U) \subset W.$$
The order arguments imply that the natural morphisms of group $K$-schemes

$$
\pi : U \to Y_n, \quad n_X : U \to W
$$

are surjective, i.e.,

$$
\pi(U) = Y_n, \quad nU = W.
$$

We have

$$
v(Y_n) = v(\pi(U)) = v\pi(U) = nU = W,
$$

i.e.,

$$
v(Y_n) = W.
$$

By 3.5(i), there exists a $K$-isogeny $\beta : X \to Y$ with $\ker(\beta) \subset X_r$. Then there exists a $K$-isogeny $\gamma : Y \to X$ such that $\gamma\beta = r_X$. This implies that

$$
\gamma r_Y = r_X \gamma = \gamma\beta \gamma = \gamma(\beta\gamma),
$$

i.e.,

$$
\gamma r_Y = \gamma(\beta\gamma).
$$

It follows that $r_Y = \beta\gamma$, because $\ker(\gamma)$ is finite while $(r_Y - \beta\gamma)Y$ is an abelian subvariety. This implies that

$$
\beta(X_n) \supset \beta(\gamma(Y_n)) = \beta\gamma(Y_n) = rY_n.
$$

Let us put

$$
u = v\beta \in \text{End}(X).
$$

We have

$$
Y_n \supset \beta(X_n) \supset rY_n.
$$

This implies that

$$
W = v(Y_n) \supset v(\beta(X_n)) = u(X_n),
$$

$$
u(X_n) = v(\beta(X_n)) \supset v(rY_n) = r(W)
$$

and therefore

$$
W \supset u(X_n) \supset r(W).
$$
9. Dividing homomorphisms of abelian varieties

Results of this Section will be used in the proof of Theorem 4.1 in Section 10.

Throughout this Section, \( Y \) is an abelian variety over a field \( K \). The following statement is well known.

**Lemma 9.1.** Let \( u : Y \rightarrow Y \) be a \( K \)-isogeny. Suppose that \( Z \) is an abelian variety over \( K \). Let \( v \in \text{Hom}(Y, Z) \) and \( \ker(u) \subset \ker(v) \) (as a group subscheme in \( Y \)). Then there exists exactly one \( w \in \text{Hom}(Y, Z) \) such that \( v = wu \), i.e., the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{u} & Y \\
\downarrow{v} & & \downarrow{w} \\
Z & & 
\end{array}
\]

is commutative. In addition, \( w \) is an isogeny if and only if \( v \) is an isogeny.

**Proof.** We have \( Y \cong Y/\ker(u) \). Now the result follows from the universality property of quotient maps. \( \square \)

Let \( n \) be a positive integer and \( u \) an endomorphism of \( Y \). Let us consider the homomorphism of abelian varieties over \( K \)

\[
(n_Y, u) : Y \rightarrow Y \times Y, \quad y \mapsto (ny, uy).
\]

Then

\[
\ker((n_Y, u)) = \ker(Y_n \xrightarrow{u} Y_n) \subset Y_n \subset Y.
\]

Slightly abusing notation, we denote the finite commutative group \( K \)-(sub)scheme \( \ker((n_Y, u)) \) by \( \{\ker(u) \cap Y_n\} \).

**Lemma 9.2.** Let \( Y \) be an abelian variety of positive dimension over a field \( K \). Then there exists a positive integer \( h = h(Y, K) \) that enjoys the following properties:

If \( n \) is a positive integer, \( u, v \in \text{End}(Y) \) are endomorphisms such that

\[
\{\ker(u) \cap Y_n\} \subset \{\ker(v) \cap Y_n\}
\]

then there exists a \( K \)-isogeny \( w : Y \rightarrow Y \) such that

\[
hv - wu \in n \cdot \text{End}(Y).
\]

In particular, the images of \( hv \) and \( wu \) in \( \text{End}(Y_n) \) do coincide.
Proof. Since $O := \text{End}(Y)$ is an order in the semisimple finite-dimensional $\mathbb{Q}$-algebra $\text{End}(Y) \otimes \mathbb{Q}$, the Jordan-Zassenhaus theorem [23, Th. 26.4] implies that there exists a positive integer $M$ that enjoys the following properties:

- if $I$ is a left ideal in $O$ that is also a subgroup of finite index then there exists $a_I \in O$ such that the principal left ideal $a \cdot O$ is a subgroup in $I$ of finite index dividing $M$; in particular,

$$M \cdot I \subset a_I \cdot O \subset I.$$  

Clearly, such $a_I$ is invertible in $\text{End}(Y) \otimes \mathbb{Q}$ and therefore is an isogeny. Let us put

$$h := M^3.$$  

Let us consider the left ideals

$$I = nO + uO, \quad J = nO + vO$$  

in $O$. Then both $I$ and $J$ are subgroups of finite index in $O$. So, there exist $K$-isogenies

$$a_I : Y \to Y, \quad a_J : Y \to Y$$  

such that

$$M \cdot I \subset a_I \cdot O \subset I, \quad M \cdot I \subset a_J \cdot O \subset J.$$  

In particular, there exist $b, c \in O$ such that

$$Ma_I - bu \in n \cdot O, \quad Mv = ca_J.$$  

In obvious notation

$$\{\ker(v) \cap Y_n\} \subset \ker(a_J) \subset \{\ker(Mv) \cap M^nY_n\} = M^{-1}\{\ker(v) \cap Y_n\} \subset Y,$$

$$\{\ker(u) \cap Y_n\} \subset \ker(a_I) \subset \{\ker(Mu) \cap M^nY_n\} = M^{-1}\{\ker(u) \cap Y_n\} \subset Y.$$  

This implies that

$$\ker(a_I) \subset M^{-1}\{\ker(u) \cap Y_n\} \subset M^{-1}\{\ker(v) \cap Y_n\} \subset M^{-1}\ker(a_J) = \ker(Ma_J)$$  

and therefore

$$\ker(a_I) \subset \ker(Ma_J).$$  

By Lemma 9.1, there exists a $K$-isogeny $z : Y \to Y$ such that $Ma_J = za_I$ and therefore $M^2a_J = Mza_I$. This implies that
\[ M^3v = M^2caJ = Mc(MaJ) = Mc(zaI) = cz(MaI) = \]
\[ cz[bu + (MaI - bu)] = (czb)u + cz(MaI - bu). \]

Since \( h = M^3 \) and \( bu - MaI \) is divisible by \( n \) in \( \mathcal{O} = \text{End}(Y) \),

\[ hv - (czb)u \in n \cdot \text{End}(Y). \]

So, we may put \( w = czb. \) \( \square \)

10. Endomorphisms of group schemes

**Proof of Theorem 4.1.** Let \( X \) be an abelian variety of positive dimension over a finite field \( K \). Let us put \( Y := X \times X \). Let \( h = h(Y) \) be as in Lemma 9.2 and \( r = r(Y, K) \) be as in Corollary 3.5. Let us put

\[ r_1 = r_1(X, K) := r(Y, K)h(Y, K). \]

Let \( n \) be a positive integer and \( u_n \in \text{End}(X_n) \). Let \( W \) be the graph of \( u_n \) in \( X_n \times X_n = (X \times X)_n = Y_n \), i.e., the image of

\[ (1_n, u_n) : X_n \hookrightarrow X_n \times X_n = (X \times X)_n = Y_n. \]

Here \( 1_n \) is the identity automorphism of \( X_n \).

By Corollary 3.5, there exists \( v \in \text{End}(Y) \) such that

\[ rW \subset u(Y_n) \subset W. \]

Let \( \text{pr}_1, \text{pr}_2 : Y = X \times X \rightarrow X \) be the projection maps and

\[ q_1 : X = X \times \{0\} \subset X \times X = Y, \quad q_2 : X = \{0\} \times X \subset X \times X = Y \]

be the inclusion maps. Let us consider the homomorphisms

\[ \text{pr}_1v, \text{pr}_2v : Y \rightarrow X \]

and the endomorphisms

\[ v_1 = q_1\text{pr}_1v, \quad v_2 = q_1\text{pr}_2v \in \text{End}(X \times X) = \text{End}(Y). \]

Clearly,

\[ v : Y \rightarrow Y = X \times X \]

is “defined” by pair
(pr\_1v, pr\_2v) : Y \rightarrow X \times X = Y.

Since W is a graph,

pr\_1(W) = X_n, v(Y_n) \subset W

and

\{\ker(pr\_1v) \cap Y_n\} \subset \{\ker(pr\_2v) \cap Y_n\}.

Since q\_1 and q\_2 are embeddings,

\{\ker(v\_1) \cap Y_n\} \subset \{\ker(v\_2) \cap Y_n\}.

By Lemma 9.2, there exists a K-isogeny w : Y \rightarrow Y such that the restrictions of hv\_2 and wv\_1 to Y\_n do coincide. Taking into account that

v\_1(X \times X) \subset X \times \{0\}, v\_2(X \times X) \subset \{0\} \times X,

we conclude that if we put

w\_12 = pr\_2wq\_1 \in \text{End}(X)

then the images of h pr\_2v and w\_12pr\_1v in Hom(Y\_n, X\_n) = Hom(X\_n \times X\_n, X\_n) do coincide.

Since W is the graph of u\_n and u(Y\_n) \subset W,

pr\_2v = u\_npr\_1v \in \text{Hom}(Y\_n, X\_n);

here both sides are viewed as morphisms of group schemes Y\_n \rightarrow X\_n. This implies that in Hom(Y\_n, X\_n) we have

w\_12pr\_1v = h pr\_2v = h u\_npr\_1v.

This implies that w\_12 = h u\_n on

pr\_1v(Y\_n) \subset X\_n.

We have

pr\_1v(Y\_n) \supset r pr\_1(r(W)) = r(X\_n)

and therefore w\_12 = h u\_n on r(X\_n). By Lemma 1.8,

r(X\_n) = X\_{n1},

where n\_1 = n/(n, r). So, w\_12 = h u\_n on X\_{n1}. Let us put d := (n\_1, h). Clearly, X\_d \subset X\_{n1} and w\_12 = hu\_n kills X\_d, because d divides h. This implies that there
exists $u \in \text{End}(X)$ such that $w_{12} = d \cdot u$. If we put $m = u_1/d$ then $h/d$ is a positive integer relatively prime to $m$ and $(h/d) \cdot u \cdot d = (h/d) \cdot u_n \cdot d$ on $X_n$, and therefore $(h/d) \cdot u = (h/d) \cdot u_n$ on $d(X_n) = X_m$. Since multiplication by $(h/d)$ is an automorphism of $X_m$, we conclude that $u = u_n$ on $X_m$.

**Corollary 10.1.** Let $K$ be a finite field, $X$ and $Y$ abelian varieties over $K$. Let $S$ be the set of positive integers $n$ such that the finite commutative group $K$-schemes $X_n$ and $Y_n$ are isomorphic. If $S$ is infinite then $X$ and $Y$ are isogeneous over $K$. In addition, if $S$ is the set of powers of a prime $\ell$ then there exists a $K$-isogeny $X \to Y$, whose degree is not divisible by $\ell$.

**Proof.** Pick $n \in S$ such that $n > r_2 := r_2(X, Y)$ where $r_2$ is as in Theorem 4.2. Then $m := n/(n, r_2)$ is strictly greater than 1. (In addition, if $n$ is a power of $\ell$ then $m$ is also a power of $\ell$.) Fix an isomorphism $w_n : X_n \cong Y_n$. By Theorem 4.2, there exists $u \in \text{Hom}(X, Y)$ such that the induced morphism $u_m : X_m \to Y_m$ coincides with the restriction (image) of $w_n$ to (in) $\text{Hom}(X_m, Y_m)$. But this restriction is an isomorphism, since $w_n$ is an isomorphism. It follows that $u_m$ is an isomorphism. Now the desired result follows from Lemma 1.10(ii).

**Theorem 10.2** (Tate’s theorem on homomorphisms). Let $K$ be a finite field, $\ell$ an arbitrary prime, $X$ and $Y$ abelian varieties over $K$ of positive dimension. Let $X(\ell)$ and $Y(\ell)$ be the $\ell$-divisible groups attached to $X$ and $Y$ respectively. Then the natural embedding

$$\text{Hom}(X, Y) \otimes \mathbb{Z}_\ell \hookrightarrow \text{Hom}(X(\ell), Y(\ell))$$

is bijective.

**Remark 10.3.** Our proof will work for both cases $\ell \neq \text{char}(K)$ and $\ell = \text{char}(K)$.

**Proof of Theorem 10.2.** Any element of $\text{Hom}(X(\ell), Y(\ell))$ is a collection

$$\{w_{(\nu)} \in \text{Hom}(X_{\ell^v}, Y_{\ell^v})\}_{v=1}^{\infty}$$

such that every $w_{(\nu)}$ coincides with the “restriction” of $w_{(\nu+1)}$ to $X_{\ell^v}$. It follows from Corollary 4.4 that there exists $u_\nu \in \text{Hom}(X, Y) \otimes \mathbb{Z}/\ell^\nu$ such that $w_{(\nu)} = u_\nu$. This implies that the image of $u_{\nu+1}$ in $\text{Hom}(X, Y) \otimes \mathbb{Z}/\ell^{\nu+1}$ coincides with $u_\nu$ for all $\nu$. This means that if $u$ is the projective limit of $u_\nu$ in $\text{Hom}(X, Y) \otimes \mathbb{Z}_\ell$ then $u$ induces (for all $\nu$) the morphism from $X_{\ell^\nu}$ to $Y_{\ell^\nu}$ that coincides with $u_\nu$ and therefore with $w_{(\nu)}$.

**Corollary 10.4.** Let $K$ be a finite field, $\ell$ an arbitrary prime, $X$ and $Y$ abelian varieties over $K$ of positive dimension. Then the following conditions are equivalent:

- There exists a $K$-isogeny $X \to Y$, whose degree is not divisible by $\ell$.
- The $\ell$-divisible groups $X(\ell)$ and $Y(\ell)$ are isomorphic.

**Proof.** It follows readily from Theorem 10.2 and Corollary 2.7.
11. Homomorphisms of Tate modules and isogenies

Throughout this Section, $K$ is a finite field and $\ell$ is a prime $\neq \text{char}(K)$.

Combining Theorem 10.2 with results of Section 2.8, we obtain the following statement.

**Theorem 11.1** (Tate [27]). Let $X$ and $Y$ be abelian varieties over $K$. Then

$$\text{Hom}(X,Y) \otimes \mathbb{Z}_\ell = \text{Hom}_{\text{Gal}}(T_\ell(X), T_\ell(Y)).$$

Let $X$ be an abelian variety over $K$. Let us consider the $\mathbb{Q}_\ell$-vector space

$$V_\ell(X) = T_\ell(X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

provided with the natural structure of Galois module. We have

$$\dim_{\mathbb{Q}_\ell}(V_\ell(X)) = 2\dim(X)$$

and the map

$$T_\ell(X) \hookrightarrow V_\ell(X), \quad z \mapsto z \otimes 1$$

identifies $T_\ell(X)$ with a Galois-invariant $\mathbb{Z}_\ell$-lattice. This implies that the natural map

$$\text{Hom}_{\text{Gal}}(T_\ell(X), T_\ell(Y)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \to \text{Hom}_{\text{Gal}}(V_\ell(X), V_\ell(Y))$$

is bijective. Here $\text{Hom}_{\text{Gal}}(V_\ell(X), V_\ell(Y))$ is the $\mathbb{Q}_\ell$-vector space of $\mathbb{Q}_\ell$-linear homomorphisms of Galois modules $V_\ell(X) \to V_\ell(Y)$.

Applying Theorem 11.1, we obtain the following statement.

**Theorem 11.2** (Tate [27]). Let $X$ and $Y$ be abelian varieties over $K$. Then the natural map

$$\text{Hom}(X,Y) \otimes \mathbb{Q}_\ell = \text{Hom}_{\text{Gal}}(V_\ell(X), V_\ell(Y))$$

is bijective.

The following assertion is very useful.

**Corollary 11.3** (Tate's isogeny theorem [27]). Let $X$ and $Y$ be abelian varieties over $K$. Then $X$ and $Y$ are isogenous over $K$ if and only if the Galois modules $V_\ell(X)$ and $V_\ell(Y)$ are isomorphic.

**Proof.** If $X$ and $Y$ are isogenous over $K$ then there exist a positive integer $N$ and isogenies

$$\alpha : X \to Y, \beta : Y \to X$$
such that
\[ \beta \alpha = N_X, \quad \alpha \beta = N_Y. \]

By functoriality, \( \alpha \) and \( \beta \) induce homomorphisms of Galois modules
\[ \alpha(\ell) : V_\ell(X) \to V_\ell(Y), \quad \beta(\ell) : V_\ell(Y) \to V_\ell(X) \]
such that the compositions \( \beta(\ell)\alpha(\ell) \) and \( \alpha(\ell)\beta(\ell) \) coincide with multiplication by \( N \) in \( V_\ell(X) \) and \( V_\ell(Y) \) respectively. It follows that \( \alpha(\ell) \) is an isomorphism of Galois modules \( V_\ell(X) \) and \( V_\ell(Y) \).

Suppose now that the Galois modules \( V_\ell(X) \) and \( V_\ell(Y) \) are isomorphic. Then their \( \mathbb{Q}_\ell \)-dimensions coincide and therefore
\[ \dim(X) = \dim(Y). \]

Choose an isomorphism
\[ w : V_\ell(X) \cong V_\ell(Y) \]
of Galois modules. Replacing (if necessary) \( w \) by \( \ell^M w \) for sufficiently large positive integer \( M \), we may and will assume that
\[ w(T_\ell(X)) \subset T_\ell(Y). \]
The image \( w(T_\ell(X)) \) is a \( \mathbb{Z}_\ell \)-lattice in \( V_\ell(Y) \). This implies that \( w(T_\ell(X)) \) is a subgroup of finite index in \( T_\ell(Y) \). So, we may view \( w \) as an injective homomorphism \( T_\ell(X) \to T_\ell(Y) \) of Galois modules. There exists a positive integer \( M \) such that if
\[ w' \in \text{Hom}_{\text{Gal}}(T_\ell(X), T_\ell(Y)), \quad w' - w \in \ell^M \cdot \text{Hom}_{\text{Gal}}(T_\ell(X), T_\ell(Y)) \]
then
\[ w' : T_\ell(X) \to T_\ell(Y) \]
is also injective. Since \( \text{Hom}(X, Y) \) is everywhere dense with respect to \( \ell \)-adic topology in
\[ \text{Hom}(X, Y) \otimes \mathbb{Z}_\ell = \text{Hom}_{\text{Gal}}(T_\ell(X), T_\ell(Y)), \]
there exists \( u \in \text{Hom}(X, Y) \) such that the induced (by \( u \)) homomorphism of Galois modules
\[ u(\ell) : T_\ell(X) \to T_\ell(Y) \]
is injective. This implies that
\[ \text{rk}_{\mathbb{Z}_\ell}(u(\ell)(T_\ell(X))) = \text{rk}_{\mathbb{Z}_\ell}(T_\ell(X)) = 2\dim(X) = 2\dim(Y). \]
I claim that \( u \) is an isogeny. Indeed, let us put \( Z := u(X) \); it is a (closed) abelian subvariety of \( Y \) that is defined over \( K \). The homomorphism \( u : X \to Y \) coincides with the composition of the natural surjection \( X \twoheadrightarrow Z \) and the inclusion map \( j : Z \hookrightarrow X \). This implies that \( u(\ell)(T_\ell(X)) \) is contained in \( j(\ell)(T_\ell(Z)) \) where

\[
j(\ell) : T_\ell(Z) \to T_\ell(Y)
\]

is the homomorphism of Tate modules induced by \( j \). It follows that

\[
2\dim(Z) = \rk(T_\ell(Z)) \geq \rk(j(\ell)(T_\ell(Z))) \geq \rk(u(\ell)(T_\ell(X))) = 2\dim(X) = 2\dim(Y)
\]

and therefore \( \dim(Z) \geq \dim(Y) \). (Hereafter \( \rk \) stands for the rank of a free \( \mathbb{Z}_\ell \)-module.)

Since \( Z \) is a closed subvariety of \( Y \), we conclude that \( \dim(Z) = \dim(Y) \) and therefore \( Z = Y \). This implies that \( u : X \to Y \) is surjective. Since \( \dim(X) = \dim(Y) \), we conclude that \( u \) is an isogeny. \( \square \)

Corollary 11.3 admits the following “refinement”.

**Corollary 11.4.** Let \( X \) and \( Y \) be abelian varieties over \( K \). The following assertions are equivalent.

- There exists an isogeny \( X \to Y \), whose degree is not divisible by \( \ell \).
- The Galois modules \( T_\ell(X) \) and \( T_\ell(Y) \) are isomorphic.

**Proof.** It follows readily from Corollary 10.4 and the last displayed formula in Subsection 2.8. \( \square \)

**12. An example**

Corollaries 10.1 and Corollary 10.4 suggest the following question: if \( X \) and \( Y \) are abelian varieties over a finite field \( K \) such that \( X_n \cong Y_n \) for all \( n \) and \( X(\ell) \cong Y(\ell) \) for all \( \ell \) then is it true that \( X \) and \( Y \) are isomorphic? The aim of this Section is to give a negative answer to this question. Our construction is based on the theory of elliptic curves with complex multiplication [24,9].

We start to work over the field \( \mathbb{C} \) of complex numbers. Let \( F \subseteq \mathbb{C} \) be an imaginary quadratic field with the ring of integers \( \mathcal{O}_F \). For every non-zero ideal \( b \subset \mathcal{O}_F \) there exists an elliptic curve \( E(b) \) over \( \mathbb{C} \) such that its group of complex points \( E(b)(\mathbb{C}) \) (viewed as a complex Lie group) is \( b \). There is a natural ring isomorphism \( \mathcal{O}_F \cong \text{End}(E(b)) \) where any \( a \in \mathcal{O}_F \) acts on \( E(b)(\mathbb{C}) \) as

\[
z + b \mapsto az + b.
\]

In particular, \( E(b) \) is an elliptic curve with complex multiplication and \( j(E(b)) \in \mathbb{C} \) is an algebraic integer.
Let us put \( E := E^{(\mathcal{O}_F)} \). There is a natural bijection of groups

\[
\begin{align*}
b & \cong \text{Hom}(E, E^{(b)}), \quad c \mapsto u(c),
\end{align*}
\]

where homomorphism \( u(c) \) acts on complex points as

\[
\begin{align*}
u(c): \mathbb{C}/\mathcal{O}_F & \rightarrow \mathbb{C}/b, \quad z + \mathcal{O}_F \mapsto cz + b.
\end{align*}
\]

In addition, for every non-zero \( c \) the homomorphism \( u(c): \mathcal{O}_F \rightarrow \mathcal{O}_F^{(b)} \) is an isogeny, whose degree is the order of the (finite) quotient \( b/c\mathcal{O}_F \). In particular, \( E \) and \( E^{(b)} \) are isomorphic if and only if \( b \) is a principal ideal. This implies that if \( b \) is not principal then

\[
\begin{align*}
j(E^{(b)}) & \neq j(E).
\end{align*}
\]

**Lemma 12.1.** For every prime \( \ell \) there exists a non-zero \( c \in b \) such that the order of \( b/c\mathcal{O}_F \) is not divisible by \( \ell \).

**Proof.** We may assume that \( b \) is not principal. If \( \ell \mathcal{O}_F \) is a prime ideal in \( \mathcal{O}_F \), pick any \( c \in b \setminus \ell b \). If \( \ell \mathcal{O}_F \) is a square \( \mathcal{L}^2 \) of a prime ideal \( \mathcal{L} \), pick any \( c \in b \setminus \mathcal{L} \cdot b \). If \( \ell \mathcal{O}_F \) is a product \( \mathcal{L}_1 \cdot \mathcal{L}_2 \) of two distinct prime ideals \( \mathcal{L}_1, \mathcal{L}_2 \subset \mathcal{O}_F \), pick

\[
\begin{align*}
c_1 & \in \mathcal{L}_1 \cdot b \setminus \mathcal{L}_2 \cdot b, \quad c_2 \in \mathcal{L}_2 \cdot b \setminus \mathcal{L}_1 \cdot b
\end{align*}
\]

and put \( c = c_1 + c_2 \); clearly,

\[
\begin{align*}
c \not\in \mathcal{L}_1 \cdot b, \quad c \not\in \mathcal{L}_2 \cdot b.
\end{align*}
\]

In all three cases

\[
c\mathcal{O}_F = \mathcal{M} \cdot b
\]

where the ideal \( \mathcal{M} = \prod \mathfrak{P}^{m_{\mathfrak{P}}} \) is a (finite) product of powers of (non-zero) prime ideals \( \mathfrak{P} \), none of which divides \( \ell \). It follows that \( b/c\mathcal{O}_F \) is a (finite) \( \mathcal{O}_F/\mathcal{M} \)-module. By the Chinese Remainder Theorem,

\[
\mathcal{O}_F/\mathcal{M} = \bigoplus \mathfrak{P} \mathcal{O}_F/\mathfrak{P}^{m_{\mathfrak{P}}}.
\]

Therefore \( b/c\mathcal{O}_F \) is a product of finite \( \mathcal{O}_F/\mathfrak{P}^{m_{\mathfrak{P}}}-\)modules. Since the multiplication by the residual characteristic of \( \mathfrak{P} \) kills \( \mathcal{O}_F/\mathfrak{P} \), it follows that the \( m_{\mathfrak{P}} \)th power of this characteristic kills every \( \mathcal{O}_F/\mathfrak{P}^{m_{\mathfrak{P}}}-\)module. This implies that the order of \( b/c\mathcal{O}_F \) is a product of powers of residual characteristics of \( \mathfrak{P} \)'s and therefore is not divisible by \( \ell \).

**Corollary 12.2.** For every prime \( \ell \) there exists an isogeny \( E \rightarrow E^{(b)} \), whose degree is not divisible by \( \ell \).
12.3. The construction. Choose an imaginary quadratic field $F$ with class number $> 1$ and pick a non-principal ideal $b \subset O_F$. We have

$$j(E^{(b)}) \neq j(E).$$

There exists an algebraic number field $L \subset \mathbb{C}$ such that:

- $L$ contains $F$, $j(E)$ and $j(E^{(b)})$.
- The elliptic curves $E$ and $E^{(b)}$ are defined over $L$.
- All homomorphisms between $E$ and $E^{(b)}$ are defined over $L$.

Let us choose a maximal ideal $q \subset O_F$ such that both $E$ and $E^{(b)}$ have good reduction at $q$ and $j(E) - j(E^{(b)})$ does not lie in $q$. (Those conditions are satisfied by all but finitely many $q$.) Let $K$ be the (finite) residue field at $q$, let $E$ and $E^{(b)}$ be the reductions at $q$ of $E$ and $E^{(b)}$ respectively; they are elliptic curves over $K$. Then $j(E)$ and $j(E^{(b)})$ are the reductions modulo $q$ of $j(E)$ and $j(E^{(b)})$ respectively. Our assumptions on $q$ imply that

$$j(E) \neq j(E^{(b)}).$$

Therefore $E$ and $E^{(b)}$ are not isomorphic over $K$ and even over $\overline{K}$!

On the other hand, it is known [9, Ch. 9, Sect. 3] that there is a natural embedding

$$\text{Hom}(E, E^{(b)}) \hookrightarrow \text{Hom}(E, E^{(b)})$$

that respects the degrees of isogenies. It follows from Corollary 12.2 that for every prime $\ell$ there exists an isogeny $E \rightarrow E^{(b)}$, whose degree is not divisible by $\ell$. Now Proposition 1.11 implies that $E_n \cong E^{(b)}_n$ for all positive integers $n$. It follows from Corollary 10.4 that the $\ell$-divisible groups $E(\ell)$ and $E^{(b)}(\ell)$ are isomorphic for all $\ell$, including $\ell = \text{char}(K)$. Since both $E(K)$ and $E^{(b)}(K)$ are torsion groups, they are isomorphic as Galois modules. This implies that their subgroups of all Galois invariants are isomorphic, i.e., the finite groups $E(K)$ and $E^{(b)}(K)$ are isomorphic.

References