Homomorphisms of abelian varieties over finite fields

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Abstract. We give a proof of Tate's theorems on homomorphisms of abelian varieties over finite fields and the corresponding ℓ-divisible groups.

The aim of this note is to give a proof of Tate's theorems on homomorphisms of abelian varieties over finite fields and the corresponding ℓ-divisible groups [27,12], using ideas of [32,33]. We give a unified treatment for both $\ell \neq p$ and $\ell = p$ cases. In fact, we prove a slightly stronger version of those theorems with "finite coefficients". We use neither the existence (and properties) of the Frobenius endomorphism (for $\ell \neq p$) nor Dieudonne modules (for $\ell = p$).

The paper is organized as follows. (A rather long) Section 1 contains auxiliary results about finite commutative group schemes and abelian varieties with special reference to isogenies and polarizations. We discuss ℓ-divisible groups (aka Barsotti–Tate groups) in Section 2. Section 3 contains useful results that play a crucial role in the proof of main results that are stated in Section 4.

The next five Sections contain proofs of results that were stated in Section 3. In Section 5 we discuss abelian subvarieties of a given abelian variety. Section 6 deals with the finiteness of the set of abelian varieties of given dimension and "bounded degree" over a finite field. In Section 7 we present a so called *quaternion trick*. In Section 8 we prove a crucial result about arbitrary finite group subschemes of abelian varieties over finite fields. In Section 9 we try to divide endomorphisms of a given abelian variety modulo n.

The main results of this paper are proven in Section 10. Their variants for Tate modules are discussed in Section 11. An example of non-isomorphic elliptic curves over a finite field with isomorphic ℓ-divisible groups (for all primes ℓ) is discussed in Section 12.

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1. Definitions and statements

Throughout this paper K is a field and $\bar{K}$ its algebraic closure. If X (resp. W) is an algebraic variety (resp. group scheme) over K then we write $\bar{X}$ (resp. $\bar{W}$) for the corresponding algebraic variety $X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\bar{K})$ (resp. group scheme $W \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\bar{K})$) over $\bar{K}$. If $f : X \to Y$ is a a regular map of algebraic varieties over K then we write $\bar{f}$ for the corresponding map $\bar{X} \to \bar{Y}$.

1.1. Finite commutative group schemes over fields. We refer the reader to the books of Oort [17], Waterhouse [31] and Demazure–Gabriel [3] for basic properties of commutative group schemes; see also [25,21].

Recall that a group scheme V over K is called finite if the structure morphism $V \to \mathrm{Spec}(K)$ is finite. Since $\mathrm{Spec}(K)$ is a one-point set, it follows from the definition of finite morphism [7, Ch. II, Sect. 3] that V is an affine scheme and $\Gamma(V, \mathcal{O}_V)$ is a finite-dimensional commutative K-algebra. The K-dimension of the $\Gamma(V, \mathcal{O}_V)$ is called the *order* of V and denoted by $\#(V)$. An analogue of Lagrange theorem [19] asserts that multiplication by $\#(V)$ kills commutative V.

Let V and W be finite commutative group schemes over K and let $u : V \to W$ be a morphism of group K-schemes. Both V and W are affine schemes, $A = \Gamma(V, \mathcal{O}_V)$ and $B = \Gamma(W, \mathcal{O}_W)$ are finite-dimensional (commutative) K-algebras (with 1), $V = \mathrm{Spec}(A), W = \mathrm{Spec}(B)$ and u is induced by a certain K-algebra homomorphism

$$u^* : B \to A.$$

Since V and W are commutative group schemes, A and B are cocommutative Hopf K-algebras. Since u is a morphism of group schemes, u^* is a morphism of Hopf algebras. It follows that $C := u^*(B)$ is a K-subalgebra and also a Hopf subalgebra in A. It follows that $U := \mathrm{Spec}(C)$ carries the natural structure of a finite group scheme over K such that the natural scheme morphism $U \to V$ induced by $u^* : B \twoheadrightarrow u^*(B) = C$ is a morphism of group schemes. In addition, the inclusion $C \subset A$ induces the morphism of schemes $V \to U$, which is also a morphism of group schemes. The latter morphism is an epimorphism in the category of finite commutative group schemes over K, because the corresponding map

$$C = \Gamma(U, \mathcal{O}_U) \to \Gamma(V, \mathcal{O}_V) = A$$

is nothing else but the inclusion map $C \subset A$ and therefore is injective [18] (see also [5]).

On the other hand, the surjection $B \twoheadrightarrow C$ provides us with a canonical isomorphism $U \cong \mathrm{Spec}(B/\ker(u^*))$; in addition, we observe that $\mathrm{Spec}(B/\ker(u^*))$ is a (closed) group subscheme of $\mathrm{Spec}(B) = W$. We denote $\mathrm{Spec}(B/\ker(u^*))$ by $u(V)$ and call it the image of u or the image of V with respect to u and denote by $u(V)$. Notice that the set theoretic image of u is closed and our definition of the image of u coincides with the one given in [4, Sect. 5.1.1].

One may easily check that the closed embedding $j : u(V) \hookrightarrow V$ induced by $B \twoheadrightarrow B/\ker(u^*)$ is an image in the category of (affine) schemes over K. This means that if $\alpha, \beta : W \to S$ are two morphisms of schemes over K such that their *restrictions* to $u(V)$ do coincide, i.e., $\alpha j = \beta j$ (as morphisms from $u(V)$ to S) then $\alpha u = \beta u$ (as morphisms from U to S). It follows that j is also an image in the category of finite commutative group schemes. group [21, Sect. 10].

Theorem 1.2 (Theorem of Gabriel [18,5]). *The category of finite commutative group schemes over a field is abelian.*

Remark 1.3. Let V be a finite commutative group scheme over K and let W be its finite closed group subscheme. If $V \to U$ is a *surjective* morphism of finite commutative group schemes over K then [5]

$$\#(V) = \#(W) \cdot \#(U).$$

Recall that $\Gamma(W, \mathcal{O}_W)$ is the quotient of $\Gamma(V, \mathcal{O}_V)$. In particular, if the orders of V and W do coincide then $V = W$.

1.4. Abelian varieties over fields. We refer the reader to the books of Mumford [16], Shimura [26] for basic properties of abelian varieties (see also Lang's book [8] and papers of Waterhouse [30], Deligne [2], Milne [13] and Oort [20]). If X is an abelian variety over K then we write $\mathrm{End}(X)$ for the ring of all K-endomorphisms of X. If m is an integer then write m_X for the multiplication by m in X; in particular, 1_X is the identity map. (Sometimes we will use notation m instead of m_X.)

If Y is an abelian variety over K then we write $\mathrm{Hom}(X, Y)$ for the group of all K-endomorphisms $X \to Y$.

Remark 1.5. Warning: sometimes in the literature, including my own papers, the notation $\mathrm{End}(X)$ is used for the ring of $\bar{K}$-endomorphisms.

It is well known [16, Sect. 19, Theorem 3] that $\mathrm{Hom}(X, Y)$ is a free commutative group of finite rank. We write X^t for the dual of X (See [13, Sect. 9–10] for the definition and basic properties of the dual of an abelian variety.) In particular, X^t is also an abelian variety over K that is isogenous to X (over K). If $u \in \mathrm{Hom}(X, Y)$ then we write u^t for its dual in $\mathrm{Hom}(Y, X)$. We have

$$\bar{X}^t = \overline{X^t}.$$

If n is a positive integer then we write X_n for the kernel of n_X; it is a finite commutative (sub)group scheme (of X) over K of rank $2\dim(X)$. By definition, $X_n(\bar{K})$ is the kernel of multiplication by n in $X(\bar{K})$.

If n is not divisible by $\mathrm{char}(K)$ then X_n is an étale group scheme and it is well-known [16, Sect. 4] that $X_n(\bar{K})$ is a free $\mathbb{Z}/n\mathbb{Z}$-module of rank $2\dim(X)$ and all $\bar{K}$-points of X_n are defined over a finite separable extension of K. In particular, $X_n(\bar{K})$ carries a natural structure of Galois module.

1.6. Isogenies. Let $W \subset X$ be a finite group subscheme over K. It follows from the analogue of Lagrange theorem that $W \subset X_d$ for $d = \#(W)$. The quotient $Y := X/W$ is an abelian variety over K and the canonical isogeny $\pi : X \to X/W = Y$

has kernel W and degree $\#(W)$ ([16, Sect. 12, Corollary 1 to Theorem 1], [3, Sect. 2, pp. 307-314]). In particular, every homomorphism of abelian varieties $u: X \to Z$ over K with $W \subset \ker(u)$ factors through π, i.e., there exists a unique homomorphism of abelian varieties $v: Y \to Z$ over K such that

$$u = v\pi.$$

If m is a positive integer then

$$\pi m_X = m_Y \pi \in \mathrm{Hom}(X, Y).$$

Let us put

$$m^{-1}W := \ker(\pi m_X) = \ker(m_Y \pi) \subset X.$$

For every commutative K-algebra R the group of R-points $m^{-1}W(R)$ is the set of all $x \in X(R)$ with

$$mx \in W(R) \subset X(R).$$

For example, if $W = X_n$ then

$$Y = X, \pi = n_X, m^{-1}X_n = X_{nm}.$$

In general, if $W \subset X_n$ then $m^{-1}W$ is a closed group subscheme in X_nm. E.g., W is always a closed group subscheme of X_{dm} and therefore is a finite group subscheme of X over K. The order

$$\#(m^{-1}W) = \deg(\pi m_X) = \deg(\pi)\deg(m_X) = \#(W) \cdot m^{2\dim(X)}.$$

We have

$$X_m \subset m^{-1}W, \ m_X(m^{-1}W) \subset W$$

and the kernel of $m_X : m^{-1}W \to W$ coincides with X_m.

Lemma 1.7. *The image* $m_X(m^{-1}W) = W$.

Proof. Let us denote the image by G. By Remark 1.3, $\#(G)$ is the ratio

$$\#(m^{-1}W)/\#(X_m) = \dim(W),$$

i.e., the orders of G and W do coincide. Since $G \subset W$, we have (by the same Remark) $G = W$. □

Example 1.8. If $W = X_n$ then $m^{-1}X_n = X_{nm}$ and therefore $m(X_{nm}) = X_n$.

Lemma 1.9. *If r is a positive integer then $r(X_n) = X_{n_1}$ where $n_1 = n/(n, r)$.*

Proof. We have $r = (n,r) \cdot r_1$ where r_1 is a positive integer such that n_1 and r_1 are relatively prime. This implies that $r_1(X_{n_1}) = X_{n_1}$. By Lemma 1.9, $(n,r)(X_n) = X_{n_1}$. This implies that

$$r(X_n) = r_1(n,r)(X_n) = r_1((n,r)(X_n)) = r_1(X_{n_1}) = X_{n_1}.$$

□

Lemma 1.10. *Let X and Y be abelian varieties over a field K. Let $u : X \to Y$ be a K-homomorphism of abelian varieties. Let $n > 1$ be an integer and $u_n : X_n \to Y_n$ the morphism of commutative group schemes over K induced by u.*

- *(i) Suppose that u is an isogeny and $\deg(u)$ and n are relatively prime. Then $u_n : X_n \to Y_n$ is an isomorphism.*
- *(ii) Suppose that $u_n : X_n \to Y_n$ is an isomorphism. Then u is an isogeny and $\deg(u)$ and n are relatively prime.*

Proof. Let u be an isogeny such that $m := \deg(u)$ and n are relatively prime. Then $\ker(u) \subset X_m$. It follows that there exists a K-isogeny $v : Y \to X$ such that

$$vu = m_X, uv = m_Y.$$

(i). Since multiplication by m is an automorphism of both X_n and Y_m, we conclude that $u_n : X_n \to Y_n$ and $v_n : Y_n \to X_n$ are isomorphisms.

(ii). Suppose that u_n is an isomorphism. This implies that the orders of X_n and Y_n coincide and therefore $\dim(X) = \dim(Y)$. We need to prove that u is isogeny and $\deg(u)$ and n are relatively prime. In order to do that, we may assume that K is algebraically closed (replacing K, X, Y, u by $\bar{K}, \bar{X}, \bar{Y}, \bar{u}$ respectively). Let us put $Z := u(Y) \subset X$: clearly, Z is a (closed) abelian subvariety of Y and therefore $\dim(Z) \le \dim(Y)$. It is also clear that $u : X \to Y$ coincides with the composition of the natural surjection $X \to u(X) = Z$ and the inclusion map $j : Z \hookrightarrow X$. This implies that $u_n(X_n)$ is a (closed) group subscheme of $j_n(Z_n) \subset Y_n$. It follows that

$$\#(u_n(X_n)) \le \#(j_n(Z_n)) \le \#(Z_n) = n^{2\dim(Z)}.$$

Since u_n is an isomorphism, $u_n(X_n) = Y_n$ and therefore

$$\#(u_n(X_n)) = \#(Y_n) = n^{2\dim(Y)}.$$

It follows that

$$n^{2\dim(Y)} \le n^{2\dim(Z)}$$

and therefore $\dim(Y) \le \dim(Z)$. (Here we use that $n > 1$.) Since Z is a closed subvariety in Y, we conclude that $\dim(Z) = \dim(Y)$ and $Y = Z$. In other words, u is surjective. Taking into account that $\dim(X) = \dim(Y)$, we conclude that u ia an isogeny.

Now let $m = dr$ where d is the largest common divisor of n and m. Then r and n are relatively prime; in particular, multiplication by r is an automorphism of X_n. Let us denote $\ker(u)$ by W: it is a finite commutative group scheme over K of order m and therefore

$$W \subset X_m.$$

This implies that for every commutative K-algebra R we have

$$m \cdot W(R) = \{0\}.$$

On the other hand, since u_n is an isomorphism, the kernel of $W(R) \xrightarrow{n} W(R)$ is $\{0\}$. Since $d \mid n$, the kernel of $W(R) \xrightarrow{d} W(R)$ is also $\{0\}$. This implies that $r \cdot W(R) = \{0\}$ for all R. Hence $W \subset X_r$. It follows that $\deg(u) = \#(W)$ divides $\#(X_r) = r^{2\dim(X)}$ and therefore is coprime to n. □

The next statement will be used only in Section 12.

Proposition 1.11. *Let X and Y be abelian varieties over a field K. Suppose that for every prime ℓ there exists an isogeny $X \to Y$, whose degree is not divisible by ℓ. Then for every positive integer n there exists an isogeny $X \to Y$, whose degree is coprime to n. In particular, $X_n \cong Y_n$.*

Proof. Recall that the additive group $\mathrm{Hom}(X, Y)$ is isomorphic to $\mathbb{Z}^\rho$ for some nonnegative integer ρ. In our case, X and Y are isogenous over K and therefore $\rho > 0$.

Let n be a positive integer and let $P(n)$ be the (finite) set of its prime divisors. For each $\ell \in P(n)$ pick an isogeny $v^{(\ell)} : X \to Y$, whose degree is not divisible by ℓ. By Lemma 1.10(i), $v^{(\ell)}$ induces an isomorphism $X_\ell \cong Y_\ell$. Now, by the Chinese Remainder Theorem, there exists $u \in \mathrm{Hom}(X, Y) \cong \mathbb{Z}^\rho$ such that

$$u - v^{(\ell)} \in \ell \cdot \mathrm{Hom}(X, Y) \ \forall\ \ell \in P.$$

This implies that for each $\ell \in P$ the homomorphisms u and $v^{(\ell)}$ induce the same morphism $X_\ell \cong Y_\ell$, which, as we know, is an isomorphism. It follows from Lemma By Lemma 1.10(ii) that u is an isogeny, whose degree is not divisible by ℓ. Hence $\deg(u)$ and n are coprime. Applying again Lemma 1.10(i), we conclude that u induces an isomorphism $X_n \cong Y_n$. □

1.12. Polarizations. A homomorphism $\lambda : X \to X^t$ is a *polarization* if there exists an ample invertible sheaf $\mathcal{L}$ on $\bar{X}$ such that $\bar{\lambda}$ coincides with

$$\Lambda_{\mathcal{L}} : \bar{X}^t \to \bar{X}^t,\ z \mapsto \mathrm{cl}(T_z^* L \otimes L^{-1})$$

where $T_z : \bar{X} \to \bar{X}$ is the translation map

$$x \mapsto x + z$$

and cl stands for the isomorphism class of an invertible sheaf. Recall [16, Sect, 6, Proposition 1; Sect. 8, Theorem 1; Sect. 13, Corollary 5] that a polarization is an *isogeny.* If λ is an isomorphism, i.e., $\deg(\lambda)=1$, we call λ a *principal polarization* and the pair (X,λ) is called a principally polarized abelian variety (over K).

If $n:=\deg(\lambda)=\#(\ker(\lambda))$ then $\ker(\lambda)$ is killed by multiplication by n, i.e., $\ker(\lambda)\subset X_n$. For every positive integer m we write λ^n for the polarization

$$X^m\to (X^m)^t=(X^t)^m,\ (x_1,\dots,x_m)\mapsto(\lambda(x_1),\dots,\lambda(x_m))$$

that corresponds to the ample invertible sheaf $\otimes_{i=1}^m \mathrm{pr}_i^*\mathcal{L}$ where $\mathrm{pr}_i: X^m\to X$ is the ith projection map. We have

$$\dim(X^m)=m\cdot\dim(X),\ \deg(\lambda^m)=\deg(\lambda)^m$$

and $\ker(\lambda^m)=\ker(\lambda)^m\subset (X^m)_n$ if $\ker(\lambda)\subset X_n$.

There exists a *Riemann form* - a skew-symmetric pairing of group schemes over $\bar{K}$ [16, Sect. 23]

$$e_\lambda:\ker(\bar{\lambda})\times\ker(\bar{\lambda})\to\mathbf{G}_{\mathrm{m}}$$

where $\mathbf{G}_{\mathrm{m}}$ is the multiplicative group scheme over $\bar{K}$.

If

$$e_{\lambda^m}:\ker(\bar{\lambda}^m)\times\ker(\bar{\lambda}^m)\to\mathbf{G}_{\mathrm{m}}$$

is the Riemann form for λ^m then in obvious notation

$$e_{\lambda^m}(x,y)=\prod_{i=1}^m e_\lambda(x_i,y_i)$$

where

$$x=(x_1,\dots,x_m),\ y=(y_1,\dots,y_m)\in\ker(\bar{\lambda})^m=\ker(\bar{\lambda}^m).$$

We have

$$\mathrm{Mat}_m(\mathbb{Z})\subset\mathrm{Mat}_m(\mathrm{End}(\bar{X}))=\mathrm{End}(X^m).$$

One may easily check that every $u\in\mathrm{Mat}_m(\mathbb{Z})$ leaves the group subscheme $\ker(\bar{\lambda}^m)$ invariant and

$$e_{\lambda^m}(ux,y)=e_{\lambda^m}(x,u^*y)$$

where u^* is the transpose of the matrix u. Notice that u^* viewed as an element of

$$\mathrm{Mat}_m(\mathbb{Z})\subset\mathrm{Mat}_m(\mathrm{End}(X^t))=\mathrm{End}((X^t)^m)$$

coincides with $u^t\in\mathrm{End}((X^m)^t)$.

1.13. Polarizations and isogenies. Let $W \subset \ker(\lambda)$ be a finite group subscheme over K. Recall that $Y := X/W$ is an abelian variety over K and the canonical isogeny $\pi : X \to X/W = Y$ has kernel W and degree $\#(W)$.

Suppose that $\bar{W}$ is isotropic with respect to e_λ, i.e., the restriction of e_λ to $\bar{W} \times \bar{W}$ is trivial. Then there exists an ample invertible sheaf $\mathcal{M}$ on $\bar{Y}$ such that $\mathcal{L} \cong \bar{\pi}^* \bar{\mathcal{M}}$ [16, Sect. 23, Corollary to Theorem 2, p. 231] and the $\bar{K}$-polarization $\Lambda_{\bar{\mathcal{M}}} : \bar{Y} \to \bar{Y}^t$ satisfies

$$\bar{\lambda} = \overline{\pi^t} \Lambda_{\bar{\mathcal{M}}} \bar{\pi}.$$

Since $\bar{\pi^t}$ and $\bar{\pi}$ are isogenies that are defined over K, the polarization $\Lambda_{\bar{\mathcal{M}}}$ is also defined over K, i.e., there exists a K-isogeny $\mu : Y \to Y^t$ such that $\Lambda_{\bar{\mathcal{M}}} = \bar{\mu}$ and

$$\lambda = \pi^t \mu \pi.$$

It follows that

$$\deg(\lambda) = \deg(\pi) \deg(\mu) \deg(\pi^t) = \deg(\pi)^2 \deg(\mu) = (\#(W))^2 \deg(\mu).$$

Therefore μ is a principal polarization (i.e., $\deg(\mu) = 1$) if and only if

$$\deg(\lambda) = (\#(W))^2.$$

2. ℓ-divisible groups, abelian varieties and Tate modules

Let h be a non-negative integer and ℓ a prime. The following notion was introduced by Tate [28,25].

Definition 2.1. An ℓ-divisible group G over K of height h is a sequence $\{G_\nu, i_\nu\}_{\nu=1}^\infty$ in which:

- G_ν is a finite commutative group scheme over K of order $\ell^{h\nu}$.
- i_ν is a closed embedding $G_\nu \hookrightarrow G_{\nu+1}$ that is a morphism of group schemes. In addition, $i_\nu(G_\nu)$ is the kernel of multiplication by ℓ^ν in $G_{\nu+1}$.

Example 2.2. Let X be an abelian variety over K of dimension d. Then it is known [28,25] that the sequence $\{X_{\ell^\nu}\}_{\nu=1}^\infty$ is an ℓ-divisible group over K of height $2d$. Here i_ν is the *inclusion map* $X_{\ell^\nu} \hookrightarrow X_{\ell^{\nu+1}}$. We denote this ℓ-divisible group by $X(\ell)$.

2.3. Homomorphisms of ℓ-divisible groups and abelian varieties. If $H = \{H_\nu, j_\nu\}_{\nu=1}^\infty$ is an ℓ-divisible group over K then a morphism $u : G \to H$ is a sequence $\{u_{(\nu)}\}_{\nu=1}^\infty$ of morphisms of group schemes over K

$$u_{(\nu)} : G_\nu \to H_\nu$$

such that the composition

$$u_{(\nu+1)}i_\nu : G_\nu \hookrightarrow G_{\nu+1} \to H_{\nu+1}$$

coincides with

$$j_\nu u_{(\nu)} : G_\nu \to H_\nu \hookrightarrow H_{\nu+1},$$

i.e., the diagram

$$\begin{array}{ccc} G_\nu & \xrightarrow{u_{(\nu)}} & H_\nu \\ {\scriptstyle i_\nu}\downarrow & & \downarrow{\scriptstyle j_\nu} \\ G_{\nu+1} & \xrightarrow{u_{(\nu+1)}} & H_{\nu+1} \end{array}$$

is commutative.

Remark 2.4. A morphism u is an isomorphism of ℓ-divisible groups if and only if all $u_{(\nu)}$ are isomorphisms of the corresponding finite group schemes.

The group $\mathrm{Hom}(G, H)$ of morphisms from G to H carries a natural structure of $\mathbb{Z}_\ell$-module induced by the natural structures of $\mathbb{Z}/\ell^v = \mathbb{Z}_\ell/\ell^\nu$-module on $\mathrm{Hom}(G_\nu, H_\nu)$. Namely, if $u = \{u_{(\nu)}\}_{\nu=1}^{\infty} \in \mathrm{Hom}(G, H)$ and $a \in \mathbb{Z}_\ell$ then $au = \{(au)_{(\nu)}\}_{\nu=1}^{\infty}$ may be defined as follows. For each ν pick $a_\nu \in \mathbb{Z}$ with $a - a_\nu \in \ell^\nu \mathbb{Z}_\ell$ and put

$$(au)_{(\nu)} := a_\nu u_{(\nu)} : G_\nu \to H_\nu.$$

Since multiplication by ℓ^ν kills G_ν, the definition of $(au)_{(\nu)}$ does not depend on the choice of a_ν.

Let X and Y be abelian varieties over K. There is a natural homomorphism of commutative groups $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(X(\ell), Y(\ell))$. Namely, if $u \in \mathrm{Hom}(X, Y)$ then $u(X_{\ell^\nu})$ lies in the kernel of multiplication by ℓ^ν, i.e. $u(X_{\ell^\nu}) \subset Y_{\ell^\nu}$. In fact, we get the natural homomorphism

$$\mathrm{Hom}(X, Y) \otimes \mathbb{Z}/\ell^\nu \to \mathrm{Hom}(X_{\ell^\nu}, Y_{\ell^\nu}),$$

which is known to be an embedding. (See also Lemma 9.1 below.)

Since $\mathrm{Hom}(X(\ell), Y(\ell))$ is a Z_ℓ-module, we get the natural homomorphism of $\mathbb{Z}_\ell$-modules

$$\mathrm{Hom}(X, Y) \otimes \mathbb{Z}_\ell \to \mathrm{Hom}(X(\ell), Y(\ell)).$$

Explicitly, if $u \in \mathrm{Hom}(X, Y) \otimes \mathbb{Z}_\ell$ then for each ν we may pick

$$w(\nu) \in \mathrm{Hom}(X, Y) = \mathrm{Hom}(X, Y) \otimes 1 \subset \mathrm{Hom}(X, Y) \otimes \mathbb{Z}_\ell$$

such that

$$u - w(\nu) \in \ell^\nu \cdot \{\mathrm{Hom}(X,Y) \otimes \mathbb{Z}_\ell\} = \{\ell^\nu \cdot \mathrm{Hom}(X,Y)\} \otimes \mathbb{Z}_\ell = \mathrm{Hom}(X,Y) \otimes \ell^\nu \mathbb{Z}_\ell.$$

Then the corresponding morphism of group schemes $u_{(\nu)} := w(\nu) : X_{\ell^\nu} \to Y$ does not depend on the choice of $w(\nu)$ and defines the corresponding morphism of ℓ-divisible groups

$$u_{(\nu)} : X_{\ell^\nu} \to Y_{\ell^\nu};\ \nu = 1, 2, \dots .$$

Remark 2.5. Since $\mathrm{Hom}(X,Y)$ is a free commutative group of finite rank, the $\mathbb{Z}_\ell$-module $\mathrm{Hom}(X,Y) \otimes \mathbb{Z}_\ell$ is a free module of finite rank.

The following assertion seems to be well known (at least, when $\ell \neq \mathrm{char}(K)$).

Lemma 2.6. *The natural homomorphism of $\mathbb{Z}_\ell$-modules*

$$\mathrm{Hom}(X,Y) \otimes \mathbb{Z}_\ell \to \mathrm{Hom}(X(\ell), Y(\ell))$$

is injective.

Proof. If it is not injective and u lies in the kernel then $u_{(\nu)} \in \ell^\nu \cdot \mathrm{Hom}(X,Y)$ for all ν. Since $u - u_{(\nu)} \in \ell^\nu \cdot \{\mathrm{Hom}(X,Y) \otimes \mathbb{Z}_\ell\}$, we conclude that $u \in \ell^\nu \cdot \{\mathrm{Hom}(X,Y) \otimes \mathbb{Z}_\ell\}$ for all ν. Since $\mathrm{Hom}(X,Y) \otimes \mathbb{Z}_\ell$ is a free $\mathbb{Z}_\ell$-module of finite rank, it follows that $u = 0$. □

Corollary 2.7. *The following conditions are equivalent:*

- *(i) There exists an isogeny $u : X \to Y$, whose degree is not divisible by ℓ.*
- *(ii) There exists $w \in \mathrm{Hom}(X,Y) \otimes \mathbb{Z}_\ell$ that induces an isomorphism of ℓ-divisible groups $X(\ell) \to Y(\ell)$.*

Proof. Let $u : X \to Y$ be an isogeny, whose degree is not divisible by ℓ. Applying Lemma 1.10(i) to all $n = \ell^\nu$, we conclude that u induces an isomorphism $X(\ell) \cong Y(\ell)$.

Now suppose that $w \in \mathrm{Hom}(X,Y) \otimes \mathbb{Z}_\ell$ that induces an isomorphism of ℓ-divisible groups $X(\ell) \to Y(\ell)$. In particular, w induces an isomorphism of finite group schemes $w_{(1)} : X_\ell \cong Y_\ell$. On the other hand, there exists $u \in \mathrm{Hom}(X,Y)$ such that

$$w - u \in \ell \cdot \{\mathrm{Hom}(X,Y) \otimes \mathbb{Z}_\ell\} = \mathrm{Hom}(X,Y) \otimes \ell\mathbb{Z}_\ell.$$

This implies that u and w induce the same morphism of finite group schemes $X_\ell \to Y_\ell$. It follows that the morphism

$$u_\ell = u_{(1)} : X_\ell \to Y_\ell$$

induced by u coincides with $w_{(1)}$ and therefore is an isomorphism. Now Lemma 1.10(ii) implies that u is an isogeny, whose degree is not divisible by ℓ. □

2.8. Tate modules. In this subsection we assume that ℓ is a prime different from $\mathrm{char}(K)$. If $n = \ell^\nu$ then X_n is an étale finite group scheme of order $n^{2\dim(X)}$ and we will identify its with the Galois module of its $\bar{K}$-points. (Actually, all points of X_n are defined over a separable algebraic extension of K). The Tate ℓ-module $T_\ell(X)$ is defined as the projective limit of Galois modules X_{ℓ^ν} where the transition map $X_{\ell^{\nu+1}} \to X_{\ell^\nu}$ is multiplication by ℓ. The Tate module carries a natural structure of free $\mathbb{Z}_\ell$-module of rank $2\dim(X)$; it is also provided with a natural structure of Galois module in such a way that natural homomorphisms $T_\ell(X) \to X_{\ell^\nu}$ induce isomorphisms of Galois modules

$$T_\ell(X) \otimes \mathbb{Z}/\ell^\nu \cong X_{\ell^\nu}.$$

Explicitly, $T_\ell(X)$ is the set of all collections $x = \{x_\nu\}_{\nu=1}^\infty$ with

$$x_\nu \in X_{\ell^\nu}, \quad x_{\nu+1} = \ell x_\nu \ \forall \nu.$$

The map $x \mapsto x_\nu$ defines the surjective homomorphism of Galois modules $T_\ell(X) \to X_{\ell^\nu}$, whose kernel coincides with $\ell^\nu \cdot T_\ell(X)$ and therefore induces the isomorphism of Galois modules $T_\ell(X)/\ell^\nu \cong X_{\ell^\nu}$ mentioned above.

If Y is an abelian variety over K then we write $\mathrm{Hom}_{\mathrm{Gal}}(T_\ell(X), T_\ell(Y))$ for the $\mathbb{Z}_\ell$-module of all homomorphisms of $\mathbb{Z}_\ell$-modules $T_\ell(X) \to T_\ell(Y)$ that commute with the Galois action(s), i.e., are also homomorphisms of Galois modules.

The $\mathbb{Z}_\ell$-module $\mathrm{Hom}_{\mathrm{Gal}}(T_\ell(X), T_\ell(Y))$ is the set of collections $w = \{w_\nu\}_{\nu=1}^\infty$ of homomorphisms of Galois modules

$$w_\nu : T_\ell(X)/\ell^\nu = X_{\ell^\nu} \to Y_{\ell^\nu} = T_\ell(Y)/\ell^\nu$$

such that

$$w_\nu(x_\nu) = \ell \cdot u w_{\nu+1}(x_{\nu+1}) \quad \forall x = \{x_\nu\}_{\nu=1}^\infty \in T_\ell(X).$$

Now if $z \in X_{\ell^\nu}$ then there exists $x \in T_\ell(X)$ with $x_\nu = z$. We have $\ell x_{\nu+1} = x_\nu = z$ and

$$w_\nu(z) = w_\nu(x_\nu) = \ell \cdot w_{\nu+1}(x_{\nu+1}) = w_{\nu+1}(\ell x_{\nu+1}) = w_{\nu+1}(x_\nu) = w_{\nu+1}(z),$$

i.e., the restriction of $w_{\nu+1}$ to X_{ℓ^ν} coincides with w_ν. This means that the collection $\{w_\nu\}_{\nu=1}^\infty$ defines a morphism of ℓ-divisible groups over K

$$X(\ell) \to Y(\ell).$$

Conversely, if $u = \{u_{(\nu)}\}_{\nu=1}^\infty$ is a morphism $X(\ell) \to Y(\ell)$ over K then

$$u_{(\nu)} : X_{\ell^\nu} \to Y_{\ell^\nu}$$

is a homomorphism of Galois modules; in addition, the restriction of $u_{(\nu+1)}$ to X_{ℓ^ν} coincides with $u_{(\nu)}$. This implies that for each $\{x_\nu\}_{\nu=1}^\infty \in T_\ell(X)$

$$u_{(\nu)}(x_\nu) = u_{(\nu+1)}(x_\nu) = u_{(\nu+1)}(\ell x_{\nu+1}) = \ell u_{(\nu+1)}(x_{\nu+1})$$

for all ν. This means that the collection $\{u_{(\nu)}\}_{\nu=1}^{\infty}$ defines a homomorphism of Galois modules $T_\ell(X) \to T_\ell(Y)$. Those observations give us the natural isomorphism of $\mathbb{Z}_\ell$-modules

$$\mathrm{Hom}(X(\ell), Y(\ell)) = \mathrm{Hom}_{\mathrm{Gal}}(T_\ell(X), T_\ell(Y)).$$

3. Useful results

Theorem 3.1 ([32,34,14]). *Let X be an abelian variety of positive dimension over a field K and X^t its dual. Then $(X \times X^t)^4$ admits a principal K-polarization.*

We prove Theorem 3.1 in Section 7.

Theorem 3.2 ([11]). *Let X be an abelian variety over K. The set of abelian K-subvarieties of X is finite, up to the action of the group $\mathrm{Aut}(X)$ of K-automorphisms of X.*

We sketch the proof of Theorem 3.2 in Section 5.

Lemma 3.3 (Tate ([27], Sect. 2, p. 136)). *Let K be a finite field, and let g and d be positive integers. The set of K-isomorphism classes of g-dimensional abelian varieties over K that admit a K-polarization of degree d is finite.*

Lemma 3.3 will be proven in Section 6.

Theorem 3.4 ([32], Th. 4.1). *Let K be a finite field, g a positive integer. Then the set of K-isomorphism classes of g-dimensional abelian varieties over K is finite.*

Proof of Theorem 3.4 (modulo Theorem 3.1 and Lemma 3.3). Suppose that X is a g-dimensional abelian variety over K. By Lemma 3.3, the set of $4g$-dimensional abelian varieties over K of the form $(X \times X^t)^4$ is finite, up to K-isomorphism. The abelian variety X is isomorphic over K to an abelian subvariety of $(X \times X^t)^4$. In order to finish the proof, one has only to recall that thanks to Theorem 3.2, the set of abelian subvarieties of a given abelian variety is finite, up to a K-isomorphism. □

We need Theorem 1.2 in order to state the following assertion.

Corollary 3.5 (Corollary to Theorem 3.4). *Let X be an abelian variety of positive dimension over a finite field K. There exists a positive integer $r = r(X, K)$ that enjoys the following properties:*

(i) If Y is an abelian variety over K that is K-isogenous to X then there exists a K-isogeny $\beta : X \to Y$ such that $\ker(\beta) \subset X_r$.
(ii) If n is a positive integer and $W \subset X_n$ is a group subscheme over K then there exists an endomorphism $u \in \mathrm{End}(X)$ such that

$$rW \subset uX_n \subset W.$$

Remark 3.6. The assertion 3.5(i) follows readily from Theorem 3.4.

We prove Corollary 3.5(ii) in Section 8.

4. Main results

Theorem 4.1. *Let X be an abelian variety of positive dimension over a finite field K. There exists a positive integer $r_1 = r_1(X, K)$ that enjoys the following properties:*

Let n be a positive integer and $u_n \in \operatorname{End}(X_n)$. Let us put $m = n/(n, r_1)$. Then there exists $u \in \operatorname{End}(X)$ such that the images of u and u_n in $\operatorname{End}(X_m)$ do coincide.

We prove Theorem 4.1 in Section 10.

Applying Theorem 4.1 to a product $X = A \times B$ of abelian varieties A and B, we obtain the following statement.

Theorem 4.2. *Let A, B be abelian varieties of positive dimension over a finite field K. There exists a positive integer $r_2 = r_2(A, B)$ that enjoys the following properties:*

Suppose that n is a positive integer and $u_n : A_n \to B_n$ is a morphism of group schemes over K. Let us put $m = n/(n, r_2)$. Then there exists a homomorphism $u : A \to B$ of abelian varieties over K such that the images of u and u_n in $\operatorname{Hom}(A_m, B_m)$ do coincide.

The following assertions follow readily from Theorem 4.2.

Corollary 4.3 (First Corollary to Theorem 4.2)**.** *If n and r_2 are relatively prime (e.g., n is a prime that does not divide r_2) then the natural injection*

$$\operatorname{Hom}(A, B) \otimes \mathbb{Z}/n \hookrightarrow \operatorname{Hom}(A_n, B_n)$$

is bijective.

Corollary 4.4 (Second Corollary to Theorem 4.2)**.** *Let ℓ be a prime and $\ell^{r(\ell)}$ is the exact power of ℓ dividing r_2. Then for each positive integer i the image of*

$$\operatorname{Hom}(A_{\ell^{i+r(\ell)}}, B_{\ell^{i+r(\ell)}}) \to \operatorname{Hom}(A_{\ell^i}, B_{\ell^i})$$

coincides with the image of

$$\operatorname{Hom}(A, B) \otimes \mathbb{Z}/\ell^i \hookrightarrow \operatorname{Hom}(A_{\ell^i}, B_{\ell^i}).$$

5. Abelian subvarieties

We follow the exposition in [11].

The next statement is a corollary of a finiteness result of Borel and Harish-Chandra [1, Theorem 6.9]; it may also be deduced from the Jordan–Zassenhaus theorem [23, Theorem 26.4].

Proposition 5.1 ([11], p. 514). *Let F be a finite-dimensional semisimple $\mathbb{Q}$-algebra, M a finitely generated right F-module, L a $\mathbb{Z}$-lattice in M. Let G be the group of those automorphisms σ of the F-module M for which $\sigma(L) = L$. Then the number of G-orbits of the set of F-submodules of M is finite.*

Now let X be an abelian variety over K. We are going to apply Proposition 5.1 to

$$F = \mathrm{End}(X) \otimes \mathbb{Q}, \ M = \mathrm{End}(X) \otimes \mathbb{Q}, \ L = \mathrm{End}(X).$$

One may identify G with the group $\mathrm{Aut}(X) = \mathrm{End}(X)^*$ of automorphisms of X: here elements of $\mathrm{End}(X)^*$ act as left multiplications on $\mathrm{End}(X) \otimes \mathbb{Q} = M$.

On the other hand, to each abelian K-subvariety $Y \subset X$ corresponds the right ideal

$$I(Y) = \{u \in \mathrm{End}(X) \mid u(X) \subset Y\}$$

and the F-submodule

$$I(Y)_{\mathbb{Q}} = I(Y) \otimes \mathbb{Q} \subset \mathrm{End}(X) \otimes \mathbb{Q} = M.$$

Using the theorem of Poincaré–Weil [13, Proposition 12.1], one may prove ([11, p. 515] that $I(Y)_{\mathbb{Q}}$ uniquely determines Y. Even better, if Y' is an abelian K-subvariety of X and

$$uI(Y)_{\mathbb{Q}} = I(Y')_{\mathbb{Q}}$$

for $u \in \mathrm{Aut}(X) = \mathrm{End}(X)^*$ then $Y' = u(Y)$. Now Proposition 5.1 implies the finiteness of the number of orbits of the set of abelian K-subvarieties of X under the natural action of $\mathrm{Aut}(X)$. This proves Theorem 3.2. (See [10] for variants and complements.)

6. Polarized abelian varieties

Lemma 6.1 (Mumford's lemma [15]). *Let X be an abelian variety of positive dimension over a field K. If $\lambda : X \to X^t$ is a polarization then there exists an ample invertible sheaf $\mathcal{L}$ on X such that*

$$\Lambda_{\bar{\mathcal{L}}} = 2\bar{\lambda}$$

where $\bar{\mathcal{L}}$ is the invertible sheaf on $\bar{X}$ induced by $\mathcal{L}$.

Proof. See [15, Ch. 6, Sect. 2, pp. 120–121] where a much more general case of abelian schemes is considered. (In notation of [15], S is the spectrum of K.) Let me just recall an explicit construction of $\mathcal{L}$. Let $\mathbb{P}$ be the universal Poincaré invertible sheaf on $X \times X^t$ [13, Sect. 9]. Then $\mathcal{L} := (1_X, \lambda)^*\mathbb{P}$ where $(1_X, \lambda) : X \to X \times X^t$ is defined by the formula

$$x \mapsto (x, \lambda(x)).$$

□

Proof of Lemma 3.3. So, let X be a g-dimensional abelian variety over a finite field K and let $\lambda : X \to X^t$ be a polarization of degree d. We follow the exposition in [22, p. 243]. By Lemma 6.1, there exists an invertible ample sheaf $\mathcal{L}$ on X such that the self-intersection index of $\bar{\mathcal{L}}$ equals $2^g dg!$ [16, Sect. 16]. The invertible sheaf $\bar{\mathcal{L}}^3$ is very ample, its space of global section has dimension $6^g d$; the self-intersection index of $\mathcal{L}$ equals $6^g dg!$ [16, Sect. 16]. This implies that $\mathcal{L}^3$ is also very ample and gives us an embedding (over K) of X into the $6^g d-1$-dimensional projective space as a closed K-subvariety of degree $6^g dg!$. All those subvarieties are uniquely determined by their Chow forms ([29, Ch. 1, Sect. 6.5], [6, Lecture 21, pp. 268–273]), whose coefficients are elements of K. Since K is finite and the number of coefficients depends only on the degree and dimension, we get the desired finiteness result. □

7. Quaternion trick

Let X be an abelian variety of positive dimension over a field K and $\lambda : X \to X^t$ a K-polarization. Pick a positive integer n such that

$$\ker(\lambda) \subset X_n.$$

Lemma 7.1. *Suppose that there exists an integer a such that a^2+1 is divisible by n. Then $X \times X^t$ admits a principal polarization that is defined over K.*

Proof. Let

$$V \subset \ker(\lambda) \times \ker(\lambda) \subset X_n \times X_n \subset X \times X$$

be the graph of multiplication by a in $\ker(\lambda)$. Clearly, V is a finite group subscheme over K that is isomorphic to $\ker(\lambda)$ and therefore its order is equal to $\deg(\lambda)$. Notice that $\deg(\lambda)$ is the square root of $\deg(\lambda^2)$.

For each commutative $\bar{K}$-algebra R the group $\bar{V}(R)$ of R-points coincides with the set of all the pairs (x, ax) with $x \in \ker(\bar{\lambda}) \subset \bar{X}_n$. This implies that for all $(x, ax), (y, ay) \in \bar{V}(R)$ we have

$$e_{\lambda^2}((x, ax), (y, ay)) = e_\lambda(x, y) \cdot e_\lambda(ax, ay) = e_\lambda(x, y) \cdot e_\lambda(a^2 x, y) =$$

$$e_\lambda(x, y) \cdot e_\lambda(-x, y) = e_\lambda(x, y)/e_\lambda(x, y) = 1.$$

In other words, $\bar{V}$ is isotropic with respect to e_{λ^2}; in addition,

$$\#(\bar{V})^2 = \deg(\lambda)^2 = \deg(\lambda^2).$$

This implies that X^2/V is a principally polarized abelian variety over K. On the other hand, we have an isomorphism of abelian varieties over K

$$f : X \times X \to X \times X = X^2,\ (x, y) \mapsto (x, ax) + (0, y) = (x, ax + y)$$

and

$$V = f(\ker \lambda \times \{0\}) \subset f(X \times \{0\}).$$

Thus, we obtain K-isomorphisms

$$X^2/V \cong X/\ker(\lambda) \times X = X^t \times X = X \times X^t.$$

In particular, $X \times X^t$ admits a principal K-polarization and we are done. □

Proof of Theorem 3.1. Choose a quadruple of integers a, b, c, d such that

$$0 \neq s := a^2 + b^2 + c^2 + d^2$$

is congruent to -1 modulo n. We denote by $\mathcal{I}$ the "quaternion"

$$\mathcal{I} = \begin{pmatrix} a & -b & -c & -d \\ b & a & d & c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} \in \mathrm{Mat}_4(\mathbb{Z}) \subset \mathrm{Mat}_4(\mathrm{End}(X) = \mathrm{End}(X*^4)).$$

We have

$$\mathcal{I}^*\mathcal{I} = a^2 + b^2 + c^2 + d^2 = s \in \mathbb{Z} \subset \mathrm{Mat}_4(\mathbb{Z}) \subset \mathrm{Mat}_4(\mathrm{End}(X) = \mathrm{End}(X^4).$$

Let

$$V \subset \ker(\lambda^4) \times \ker(\lambda^4) \subset (X^4)_n \times (X^4)_n \subset X^4 \times X^4 = X^8$$

be the graph of

$$\mathcal{I} : \ker(\lambda^4) \to \ker(\lambda^4).$$

Clearly, V is a finite group subscheme over K and its order is equal to $\deg(\lambda^4)$. Notice that $\deg(\lambda^4)$ is the square root of $\deg(\lambda^8)$.

For each commutative $\bar{K}$-algebra R the group $\bar{V}(R)$ of R-points consists of all the pairs $(x, \mathcal{I}x)$ with $x \in \ker(\bar{\lambda}^4) \subset (\bar{X^4})_n$. This implies that for all $(x, \mathcal{I}x), (y, \mathcal{I}y) \in \bar{V}(R)$ we have

$$e_{\lambda^4}((x, \mathcal{I}x), (y, \mathcal{I}y)) = e_{\lambda^4}(x, y) \cdot e_{\lambda^4}(\mathcal{I}x, \mathcal{I}y) = e_{\lambda^4}(x, y) \cdot e_{\lambda}(x, \mathcal{I}^t\mathcal{I}y) =$$

$$e_{\lambda}(x, y) \cdot e_{\lambda}(x, sy) = e_{\lambda}(x, y) \cdot e_{\lambda}(x, -y) = e_{\lambda}(x, y)/e_{\lambda}(x, y) = 1.$$

In other words, $\bar{V}$ is isotropic with respect to e_{λ^4}; in addition,

$$\#(\bar{V})^2 = \deg(\lambda^4)^2 = \deg(\lambda^8).$$

This implies that X^8/V is a principally polarized abelian variety over K. On the other hand, we have an isomorphism of abelian varieties over K

$$f : X^4 \times X^4 \to X^4 \times X^4 = X^8,\ (x,y) \mapsto (x, \mathcal{I}x) + (0,y) = (x, \mathcal{I}x + y)$$

and

$$V = f(\ker(\lambda^4) \times \{0\}) \subset f(X^4 \times \{0\}).$$

Thus, we obtain K-isomorphisms

$$X^4/V \cong X^4/\ker \lambda^4 \times X^4 = (X^4)^t \times X^4 = (X \times X^t)^4.$$

In particular, $(X \times X^t)^4$ admits a principal K-polarization and we are done. □

Remark 7.2. We followed the exposition in [32, Lemma 2.5], [34, Sect. 5]. See [14, Ch. IX, Sect. 1] where Deligne's proof is given.

8. Finite group subschemes of abelian varieties

Proof of Corollary 3.5(ii). Let r be as in 3.5(i). Let us consider the abelian variety $Y := X/W$ and the canonical K-isogeny $\pi : X \to X/W = Y$. Clearly,

$$W = \ker(\pi).$$

Since $W \subset X_n$, there exists a K-isogeny $v : Y \to X/X_n = X$ such that the composition $v\pi$ coincides with multiplication by n in X; in addition,

$$\pi n_X = n_Y \pi : X \to Y$$

is a K-isogeny, whose degree is $\#(W) \times n^{2\dim(X)}$. Here n_X (resp. n_Y) stands for multiplication by n in X (resp. in Y). Let us put

$$U = \ker(\pi n_X) = \ker(n_Y \pi) \subset X;$$

it is a finite commutative group K-(sub)scheme and

$$\#(U) = \#(W) \times n^{2\dim(X)}.$$

Then

$$X_n \subset U,\ W \subset U;\ \pi(U) \subset Y_n,\ n_X(U) \subset W.$$

The order arguments imply that the natural morphisms of group K-schemes

$$\pi : U \to Y_n,\ n_X : U \to W$$

are surjective, i.e.,

$$\pi(U) = Y_n,\ nU = W.$$

We have

$$v(Y_n) = v(\pi(U)) = v\pi(U) = nU = W,$$

i.e.,

$$v(Y_n) = W.$$

By 3.5(i), there exists a K-isogeny $\beta : X \to Y$ with $\ker(\beta) \subset X_r$. Then there exists a K-isogeny $\gamma : Y \to X$ such that $\gamma\beta = r_X$. This implies that

$$\gamma r_Y = r_X\gamma = \gamma\beta\gamma = \gamma(\beta\gamma),$$

i.e.,

$$\gamma r_Y = \gamma(\beta\gamma).$$

It follows that $r_Y = \beta\gamma$, because $\ker(\gamma)$ is finite while $(r_Y - \beta\gamma)Y$ is an abelian subvariety. This implies that

$$\beta(X_n) \supset \beta(\gamma(Y_n)) = \beta\gamma(Y_n) = rY_n.$$

Let us put

$$u = v\beta \in \mathrm{End}(X).$$

We have

$$Y_n \supset \beta(X_n) \supset rY_n.$$

This implies that

$$W = v(Y_n) \supset v(\beta)(X_n) = u(X_n),$$

$$u(X_n) = v(\beta(X_n)) \supset v(rY_n) = r(W)$$

and therefore

$$W \supset u(X_n) \supset r(W).$$

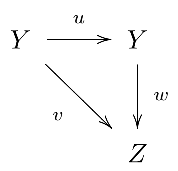
□

9. Dividing homomorphisms of abelian varieties

Results of this Section will be used in the proof of Theorem 4.1 in Section 10.

Throughout this Section, Y is an abelian variety over a field K. The following statement is well known.

Lemma 9.1. *let $u : Y \to Y$ be a K-isogeny. Suppose that Z is an abelian variety over K. Let $v \in \mathrm{Hom}(Y, Z)$ and $\ker(u) \subset \ker(v)$ (as a group subscheme in Y). Then there exists exactly one $w \in \mathrm{Hom}(Y, T)$ such that $v = wu$, i.e., the diagram*



is commutative. In addition, w is an isogeny if and only if v is an isogeny.

Proof. We have $Y \cong Y/\ker(u)$. Now the result follows from the universality property of quotient maps.

□

Let n be a positive integer and u an endomorphism of Y. Let us consider the homomorphism of abelian varieties over K

$$(n_Y, u) : Y \to Y \times Y, \quad y \mapsto (ny, uy).$$

Then

$$\ker((n_Y, u)) = \ker(Y_n \xrightarrow{u} Y_n) \subset Y_n \subset Y.$$

Slightly abusing notation, we denote the finite commutative group K-(sub)scheme $\ker((n_Y, u))$ by $\{\ker(u) \bigcap Y_n\}$.

Lemma 9.2. *Let Y be an abelian variety of positive dimension over a field K. Then there exists a positive integer $h = h(Y, K)$ that enjoys the following properties:*

If n is a positive integer, $u, v \in \mathrm{End}(Y)$ are endomorphisms such that

$$\{\ker(u) \bigcap Y_n\} \subset \{\ker(v) \bigcap Y_n\}$$

then there exists a K-isogeny $w : Y \to Y$ such that

$$hv - wu \in n \cdot \mathrm{End}(Y).$$

In particular, the images of hv and wu in $\mathrm{End}(Y_n)$ do coincide.

Proof. Since $\mathcal{O} := \mathrm{End}(Y)$ is an order in the semisimple finite-dimensional $\mathbb{Q}$-algebra $\mathrm{End}(Y) \otimes \mathbb{Q}$, the Jordan–Zassenhaus theorem [23, Th. 26.4] implies that there exists a positive integer M that enjoys the following properties:

if I is a left ideal in $\mathcal{O}$ that is also a subgroup of finite index then there exists $a_I \in \mathcal{O}$ such that the principal left ideal $a \cdot \mathcal{O}$ is a subgroup in I of finite index dividing M; in particular,

$$M \cdot I \subset a_I \cdot \mathcal{O} \subset I.$$

Clearly, such a_I is invertible in $\mathrm{End}(Y) \otimes \mathbb{Q}$ and therefore is an isogeny. Let us put

$$h := M^3.$$

Let us consider the left ideals

$$I = n\mathcal{O} + u\mathcal{O}, \ J = n\mathcal{O} + v\mathcal{O}$$

in $\mathcal{O}$. Then both I and J are subgroups of finite index in $\mathcal{O}$. So, there exist K-isogenies

$$a_I : Y \to Y, \ a_J : Y \to Y$$

such that

$$M \cdot I \subset a_I \cdot \mathcal{O} \subset I, \ M \cdot I \subset a_J \cdot \mathcal{O} \subset J.$$

In particular, there exist $b, c \in \mathcal{O}$ such that

$$Ma_I - bu \in n \cdot \mathcal{O}, \ Mv = ca_J.$$

In obvious notation

$$\{\ker(v) \bigcap Y_n\} \subset \ker(a_J) \subset \{\ker(Mv) \bigcap Y_{Mn}\} = M^{-1}\{\ker(v) \bigcap Y_n\} \subset Y,$$

$$\{\ker(u) \bigcap Y_n\} \subset \ker(a_I) \subset \{\ker(Mu) \bigcap Y_{Mn}\} = M^{-1}\{\ker(u) \bigcap Y_n\} \subset Y.$$

This implies that

$$\ker(a_I) \subset M^{-1}\{\ker(u) \bigcap Y_n\} \subset M^{-1}\{\ker(v) \bigcap Y_n\} \subset M^{-1} \ker(a_J) = \ker(Ma_J)$$

and therefore

$$\ker(a_I) \subset \ker(Ma_J).$$

By Lemma 9.1, there exists a K-isogeny $z : Y \to Y$ such that $Ma_J = za_I$ and therefore $M^2 a_J = Mza_I$. This implies that

$$M^3v = M^2ca_J = Mc(Ma_J) = Mc(za_I) = cz(Ma_I) =$$

$$cz[bu + (Ma_I - bu)] = (czb)u + cz(Ma_I - bu).$$

Since $h = M^3$ and $bu - Ma_I$ is divisible by n in $\mathcal{O} = \mathrm{End}(Y)$,

$$hv - (czb)u \in n \cdot \mathrm{End}(Y).$$

So, we may put $w = czb$. $\square$

10. Endomorphisms of group schemes

Proof of Theorem 4.1. Let X be an abelian variety of positive dimension over a finite field K. Let us put $Y := X \times X$. Let $h = h(Y)$ be as in Lemma 9.2 and $r = r(Y, K)$ be as in Corollary 3.5. Let us put

$$r_1 = r_1(X, K) := r(Y, K)h(Y, K).$$

Let n be a positive integer and $u_n \in \mathrm{End}(X_n)$. Let W be the graph of u_n in $X_n \times X_n = (X \times X)_n = Y_n$, i.e., the image of

$$(\mathbf{1}_n, u_n) : X_n \hookrightarrow X_n \times X_n = (X \times X)_n = Y_n.$$

Here $\mathbf{1}_n$ is the identity automorphism of X_n.

By Corollary 3.5, there exists $v \in \mathrm{End}(Y)$ such that

$$rW \subset u(Y_n) \subset W.$$

Let $\mathrm{pr}_1, \mathrm{pr}_2 : Y = X \times X \to X$ be the projection maps and

$$q_1 : X = X \times \{0\} \subset X \times X = Y,\ q_2 : X = \{0\} \times X \subset X \times X = Y$$

be the inclusion maps. Let us consider the homomorphisms

$$\mathrm{pr}_1 v, \mathrm{pr}_2 v : Y \to X$$

and the endomorphisms

$$v_1 = q_1 \mathrm{pr}_1 v,\ v_2 = q_1 \mathrm{pr}_2 v \in \mathrm{End}(X \times X) = \mathrm{End}(Y).$$

Clearly,

$$v : Y \to Y = X \times X$$

is "defined" by pair

$$(\mathrm{pr}_1 v, \mathrm{pr}_2 v) : Y \to X \times X = Y.$$

Since W is a graph,

$$\mathrm{pr}_1(W) = X_n,\ v(Y_n) \subset W$$

and

$$\{\ker(\mathrm{pr}_1 v) \bigcap Y_n\} \subset \{\ker(\mathrm{pr}_2 v) \bigcap Y_n\}.$$

Since q_1 and q_2 are embeddings,

$$\{\ker(v_1) \bigcap Y_n\} \subset \{\ker(v_2) \bigcap Y_n\}.$$

By Lemma 9.2, there exists a K-isogeny $w : Y \to Y$ such that the restrictions of hv_2 and wv_1 to Y_n do coincide. Taking into account that

$$v_1(X \times X) \subset X \times \{0\},\ v_2(X \times X) \subset \{0\} \times X,$$

we conclude that if we put

$$w_{12} = \mathrm{pr}_2 w q_1 \in \mathrm{End}(X)$$

then the images of $h\ \mathrm{pr}_2 v$ and $w_{12}\mathrm{pr}_1 v$ in $\mathrm{Hom}(Y_n, X_n) = \mathrm{Hom}(X_n \times X_n, X_n)$ do coincide.

Since W is the graph of u_n and $u(Y_n) \subset W$,

$$\mathrm{pr}_2 v = u_n \mathrm{pr}_1 v \in \mathrm{Hom}(Y_n, X_n);$$

here both sides are viewed as morphisms of group schemes $Y_n \to X_n$. This implies that in $\mathrm{Hom}(Y_n, X_n)$ we have

$$w_{12}\mathrm{pr}_1 v = h\ \mathrm{pr}_2 v = h\ u_n \mathrm{pr}_1 v.$$

This implies that $w_{12} = h\ u_n$ on

$$\mathrm{pr}_1 v(Y_n) \subset X_n.$$

We have

$$\mathrm{pr}_1 v(Y_n) \supset r\ \mathrm{pr}_1(r(W)) = r(X_n)$$

and therefore $w_{12} = h\ u_n$ on $r(X_n)$. By Lemma 1.8,

$$r(X_n) = X_{n_1},$$

where $n_1 = n/(n,r)$. So, $w_{12} = h\ u_n$ on X_{n_1}. Let us put $d := (n_1, h)$. Clearly, $X_d \subset X_{n_1}$ and $w_{12} = hu_n$ kills X_d, because d divides h. This implies that there exists $u \in \mathrm{End}(X)$ such that $w_{12} = d\ u$. If we put $m = n_1/d$ then h/d is a positive integer relatively prime to m and $(h/d)\ u\ d = (h/d)\ u_n\ d$ on X_{n_1} and therefore $(h/d)\ u = (h/d)\ u_n$ on $d(X_{n_1}) = X_m$. Since multiplication by (h/d) is an automorphism of X_m, we conclude that $u = u_n$ on X_m.

□

Corollary 10.1. *Let K be a finite field, X and Y abelian varieties over K. Let S be the set of positive integers n such that the finite commutative group K-schemes X_n and Y_n are isomorphic. If S is infinite then X and Y are isogenous over K. In addition, if S is the set of powers of a prime ℓ then there exists a K-isogeny $X \to Y$, whose degree is not divisible by ℓ.*

Proof. Pick $n \in S$ such that $n > r_2 := r_2(X, Y)$ where r_2 is as in Theorem 4.2. Then $m := n/(n, r_2)$ is strictly greater than 1. (In addition, if n is a power of ℓ then m is also a power of ℓ.(Fix an isomorphism $w_n : X_n \cong Y_n$. By Theorem 4.2, there exists $u \in \mathrm{Hom}(X, Y)$ such that the induced morphism $u_m : X_m \to Y_m$ coincides with the restriction (image) of w_n to (in) $\mathrm{Hom}(X_m, Y_m)$. But this restriction is an isomorphism, since w_n is an isomorphism. It follows that u_m is an isomorphism. Now the desired result follows from Lemma 1.10(ii).

□

Theorem 10.2 (Tate's theorem on homomorphisms)**.** *Let K be a finite field, ℓ an arbitrary prime, X and Y abelian varieties over K of positive dimension. Let $X(\ell)$ and $Y(\ell)$ be the ℓ-divisible groups attached to X and Y respectively. Then the natural embedding*

$$\mathrm{Hom}(X, Y) \otimes \mathbb{Z}_\ell \hookrightarrow \mathrm{Hom}(X(\ell), Y(\ell))$$

is bijective.

Remark 10.3. Our proof will work for both cases $\ell \neq \mathrm{char}(K)$ and $\ell = \mathrm{char}(K)$.

Proof of Theorem 10.2. Any element of $\mathrm{Hom}(X(\ell), Y(\ell))$ is a collection

$$\{w_{(\nu)} \in \mathrm{Hom}(X_{\ell^\nu}, Y_{\ell^\nu})\}_{\nu=1}^{\infty}$$

such that every $w_{(\nu)}$ coincides with the "restriction" of $w_{(\nu+1)}$ to X_{ℓ^ν}. It follows from Corollary 4.4 that there exists $u_\nu \in \mathrm{Hom}(X, Y) \otimes \mathbb{Z}/\ell^\nu$ such that $w_{(\nu)} = u_\nu$. This implies that the image of $u_{\nu+1}$ in $\mathrm{Hom}(X, Y) \otimes \mathbb{Z}/\ell^\nu$ coincides with u_ν for all ν. This means that if u is the projective limit of u_ν in $\mathrm{Hom}(X, Y) \otimes \mathbb{Z}_\ell$ then u induces (for all ν) the morphism from X_{ℓ^ν} to Y_{ℓ^ν} that coincides with u_ν and therefore with $w_{(\nu)}$. □

Corollary 10.4. *Let K be a finite field, ℓ an arbitrary prime, X and Y abelian varieties over K of positive dimension. Then the following conditions are equivalent:*

- *There exists a K-isogeny $X \to Y$, whose degree is not divisible by ℓ.*
- *The ℓ-divisible groups $X(\ell)$ and $Y(\ell)$ are isomorphic.*

Proof. It follows readily from Theorem 10.2 and Corollary 2.7. □

11. Homomorphisms of Tate modules and isogenies

Throughout this Section, K is a finite field and ℓ is a prime $\neq \mathrm{char}(K)$.

Combining Theorem 10.2 with results of Section 2.8, we obtain the following statement.

Theorem 11.1 (Tate [27]). *Let X and Y be abelian varieties over K. Then*

$$\mathrm{Hom}(X,Y)\otimes \mathbb{Z}_\ell = \mathrm{Hom}_{\mathrm{Gal}}(T_\ell(X), T_\ell(Y)).$$

Let X be an abelian variety over K. Let us consider the $\mathbb{Q}_\ell$-vector space

$$V_\ell(X) = T_\ell(X)\otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

provided with the natural structure of Galois module. We have

$$\dim_{\mathbb{Q}_\ell}(V_\ell(X)) = 2\dim(X)$$

and the map

$$T_\ell(X) \hookrightarrow V_\ell(X),\ z \mapsto z\otimes 1$$

identifies $T_\ell(X)$ with a Galois-invariant $\mathbb{Z}_\ell$-lattice. This implies that the natural map

$$\mathrm{Hom}_{\mathrm{Gal}}(T_\ell(X), T_\ell(Y))\otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \to \mathrm{Hom}_{\mathrm{Gal}}(V_\ell(X), V_\ell(Y))$$

is bijective. Here $\mathrm{Hom}_{\mathrm{Gal}}(V_\ell(X), V_\ell(Y))$ is the $\mathbb{Q}_\ell$-vector space of $\mathbb{Q}_\ell$-linear homomorphisms of Galois modules $V_\ell(X)\to V_\ell(Y)$.

Applying Theorem 11.1, we obtain the following statement.

Theorem 11.2 (Tate [27]). *Let X and Y be abelian varieties over K. Then the natural map*

$$\mathrm{Hom}(X,Y)\otimes \mathbb{Q}_\ell = \mathrm{Hom}_{\mathrm{Gal}}(V_\ell(X), V_\ell(Y))$$

is bijective.

The following assertion is very useful.

Corollary 11.3 (Tate's isogeny theorem [27]). *Let X and Y be abelian varieties over K. Then X and Y are isogenous over K if and only if the Galois modules $V_\ell(X)$ and $V_\ell(Y)$ are isomorphic.*

Proof. If X and Y are isogenous over K then there exist a positive integer N and isogenies

$$\alpha : X\to Y,\ \beta : Y\to X$$

such that

$$\beta\alpha = N_X, \ \alpha\beta = N_Y.$$

By functoriality, α and β induce homomorphisms of Galois modules

$$\alpha(\ell): V_\ell(X) \to V_\ell(Y), \ \beta(\ell): V_\ell(Y) \to V_\ell(X)$$

such that the compositions $\beta(\ell)\alpha(\ell)$ and $\alpha(\ell)\beta(\ell)$ coincide with multiplication by N in $V_\ell(X)$ and $V_\ell(Y)$ respectively. It follows that $\alpha(\ell)$ is an isomorphism of Galois modules $V_\ell(X)$ and $V_\ell(Y)$.

Suppose now that the Galois modules $V_\ell(X)$ and $V_\ell(Y)$ are isomorphic. Then their $\mathbb{Q}_\ell$-dimensions coincide and therefore

$$\dim(X) = \dim(Y).$$

Choose an isomorphism

$$w: V_\ell(X) \cong V_\ell(Y)$$

of Galois modules. Replacing (if necessary) w by $\ell^M w$ for sufficiently large positive integer M, we may and will assume that

$$w(T_\ell(X)) \subset T_\ell(Y).$$

The image $w(T_\ell(X))$ is a $\mathbb{Z}_\ell$-lattice in $V_\ell(Y)$. This implies that $w(T_\ell(X))$ is a subgroup of finite index in $T_\ell(Y)$. So, we may view w as an *injective* homomorphism $T_\ell(X) \to T_\ell(Y)$ of Galois modules. There exists a positive integer M such that if

$$w' \in \mathrm{Hom}_{\mathrm{Gal}}(T_\ell(X), T_\ell(Y)), \ w' - w \in \ell^M \cdot \mathrm{Hom}_{\mathrm{Gal}}(T_\ell(X), T_\ell(Y))$$

then

$$w': T_\ell(X) \to T_\ell(Y)$$

is also injective. Since $\mathrm{Hom}(X, Y)$ is everywhere dense with respect to ℓ-adic topology in

$$\mathrm{Hom}(X, Y) \otimes \mathbb{Z}_\ell = \mathrm{Hom}_{\mathrm{Gal}}(T_\ell(X), T_\ell(Y)),$$

there exists $u \in \mathrm{Hom}(X, Y)$ such that the induced (by u) homomorphism of Galois modules

$$u(\ell): T_\ell(X) \to T_\ell(Y)$$

is injective. This implies that

$$\mathrm{rk}_{\mathbb{Z}_\ell}(u(\ell)(T_\ell(X))) = \mathrm{rk}_{\mathbb{Z}_\ell}(T_\ell(X)) = 2\dim(X) = 2\dim(Y).$$

I claim that u is an isogeny. Indeed, let us put $Z := u(X)$: it is a (closed) abelian subvariety of Y that is defined over K. The homomorphism $u: X \to Y$ coincides with the composition of the natural surjection $X \to Z$ and the inclusion map $j: Z \hookrightarrow X$. This implies that $u(\ell)(T_\ell(X))$ is contained in $j(\ell)(T_\ell(Z))$ where

$$j(\ell): T_\ell(Z) \to T_\ell(Y)$$

is the homomorphism of Tate modules induced by j. It follows that

$$2\dim(Z) = \mathrm{rk}(T_\ell(Z)) \geq \mathrm{rk}(j(\ell)(T_\ell(Z))) \geq$$

$$\mathrm{rk}(u(\ell)(T_\ell(X))) = 2\dim(X) = 2\dim(Y)$$

and therefore $\dim(Z) \geq \dim(Y)$. (Hereafter rk stands for the rank of a free $\mathbb{Z}_\ell$-module.)

Since Z is a closed subvariety of Y, we conclude that $\dim(Z) = \dim(Y)$ and therefore $Z = Y$. This implies that $u: X \to Y$ is surjective. Since $\dim(X) = \dim(Y)$, we conclude that u is an isogeny. □

Corollary 11.3 admits the following "refinement".

Corollary 11.4. *Let X and Y be abelian varieties over K. The following assertions are equivalent.*

- *There exists an isogeny $X \to Y$, whose degree is not divisible by ℓ.*
- *The Galois modules $T_\ell(X)$ and $T_\ell(Y)$ are isomorphic.*

Proof. It follows readily from Corollary 10.4 and the last displayed formula in Subsection 2.8. □

12. An example

Corollaries 10.1 and Corollary 10.4 suggest the following question: if X and Y are abelian varieties over a finite field K such that $X_n \cong Y_n$ for all n and $X(\ell) \cong Y(\ell)$ for all ℓ then is it true that X and Y are isomorphic? The aim of this Section is to give a negative answer to this question. Our construction is based on the theory of elliptic curves with complex multiplication [24,9].

We start to work over the field $\mathbb{C}$ of complex numbers. Let $F \subset \mathbb{C}$ be an imaginary quadratic field with the ring of integers $\mathcal{O}_F$. For every non-zero ideal $\mathfrak{b} \subset \mathcal{O}_F$ there exists an elliptic curve $E^{(\mathfrak{b})}$ over $\mathbb{C}$ such that that its group of complex points $E^{(\mathfrak{b})}(\mathbb{C})$ (viewed as a complex Lie group) is $\mathbb{C}/\mathfrak{b}$. There is a natural ring isomorphism $\mathcal{O}_F \cong \mathrm{End}(E^{(\mathfrak{b})})$ where any $a \in \mathcal{O}_F$ acts on $E^{(\mathfrak{b})}(\mathbb{C})$ as

$$z + \mathfrak{b} \mapsto az + \mathfrak{b}.$$

In particular, $E^{(\mathfrak{b})}$ is an elliptic curve with complex multiplication and $\mathrm{j}(E^{(\mathfrak{b})}) \in \mathbb{C}$ is an *algebraic integer*.

Let us put $E := E^{(\mathcal{O}_F)}$. There is a natural bijection of groups

$$\mathfrak{b} \cong \mathrm{Hom}(E, E^{(\mathfrak{b})}),\ c \mapsto u(c),$$

where homomorphism $u(c)$ acts on complex points as

$$u(c) : \mathbb{C}/\mathcal{O}_F \to \mathbb{C}/\mathfrak{b},\ z + \mathcal{O}_F \mapsto cz + \mathfrak{b}.$$

In addition, for every non-zero c the homomorphism $u(c) : E \to E^{(\mathfrak{b})}$ is an isogeny, whose degree is the order of the (finite) quotient $\mathfrak{b}/c\mathcal{O}_F$. In particular, E and $E^{(\mathfrak{b})}$ are isomorphic if and only if $\mathfrak{b}$ is a principal ideal. This implies that if $\mathfrak{b}$ is not principal then

$$\mathrm{j}(E^{(\mathfrak{b})}) \neq \mathrm{j}(E).$$

Lemma 12.1. *For every prime ℓ there exists a non-zero $c \in \mathfrak{b}$ such that the order of $\mathfrak{b}/c\mathcal{O}_F$ is not divisible by ℓ.*

Proof. We may assume that $\mathfrak{b}$ is not principal. If $\ell\mathcal{O}_F$ is a prime ideal in $\mathcal{O}_F$, pick any $c \in \mathfrak{b} \setminus \ell\mathfrak{b}$. If $\ell\mathcal{O}_F$ is a square $\mathfrak{L}^2$ of a prime ideal $\mathfrak{L}$, pick any $c \in \mathfrak{b} \setminus \mathfrak{L} \cdot \mathfrak{b}$. If $\ell\mathcal{O}_F$ is a product $\mathfrak{L}_1\mathfrak{L}_2$ of two distinct prime ideals $\mathfrak{L}_1, \mathfrak{L}_2 \subset \mathcal{O}_F$, pick

$$c_1 \in \mathfrak{L}_1 \cdot b \setminus \mathfrak{L}_2 \cdot \mathfrak{b},\ c_2 \in \mathfrak{L}_2 \cdot \mathfrak{b} \setminus \mathfrak{L}_1 \cdot \mathfrak{b}$$

and put $c = c_1 + c_2$; clearly,

$$c \notin \mathfrak{L}_1 \cdot \mathfrak{b},\ c \notin \mathfrak{L}_2 \cdot \mathfrak{b}.$$

In all three cases

$$c\mathcal{O}_F = \mathfrak{M} \cdot \mathfrak{b}$$

where the ideal $\mathfrak{M} = \prod_{\mathfrak{P}} \mathfrak{P}^{m_{\mathfrak{P}}}$ is a (finite) product of powers of (non-zero) prime ideals $\mathfrak{P}$, none of which divides ℓ. It follows that $\mathfrak{b}/c\mathcal{O}_F$ is a (finite) $\mathcal{O}_F/\mathfrak{M}$-module. By the Chinese Remainder Theorem,

$$\mathcal{O}_F/\mathfrak{M} = \oplus_{\mathfrak{P}}\mathcal{O}_F/\mathfrak{P}^{m_{\mathfrak{P}}}.$$

Therefore $\mathfrak{b}/c\mathcal{O}_F$ is a product of finite $\mathcal{O}_F/\mathfrak{P}^{m_{\mathfrak{P}}}$-modules. Since the multiplication by the residual characteristic of $\mathfrak{P}$ kills $\mathcal{O}_F/\mathfrak{P}$, it follows that the $m_{\mathfrak{P}}$th power of this characteristic kills every $\mathcal{O}_F/\mathfrak{P}^{m_{\mathfrak{P}}}$-module. This implies that the order of $\mathfrak{b}/c\mathcal{O}_F$ is a product of powers of residual characteristics of $\mathfrak{P}$'s and therefore is not divisible by ℓ. □

Corollary 12.2. *For every prime ℓ there exists an isogeny $E \to E^{(\mathfrak{b})}$, whose degree is not divisible by ℓ.*

12.3. The construction. Choose an imaginary quadratic field F with class number > 1 and pick a *non*-principal ideal $\mathfrak{b} \subset \mathcal{O}_F$. We have

$$\mathrm{j}(E^{(\mathfrak{b})}) \neq \mathrm{j}(E).$$

There exists an algebraic number field $L \subset \mathbb{C}$ such that:

- L contains F, $\mathrm{j}(E)$ and $\mathrm{j}(E^{(\mathfrak{b})})$.
- The elliptic curves E and $E^{(\mathfrak{b})}$ are defined over L.
- All homomorphisms between E and $E^{(\mathfrak{b})}$ are defined over L.

Let us choose a maximal ideal $\mathfrak{q} \subset \mathcal{O}_F$ such that both E and $E^{(\mathfrak{b})}$ have good reduction at $\mathfrak{q}$ and $j(E) - j(E^{(\mathfrak{b})})$ does *not* lie in $\mathfrak{q}$. (Those conditions are satisfied by all but finitely many $\mathfrak{q}$.) Let K be the (finite) residue field at $\mathfrak{q}$, let $\mathbf{E}$ and $\mathbf{E}^{(b)}$ be the reductions at $\mathfrak{q}$ of E and $E^{(\mathfrak{b})}$ respectively: they are elliptic curves over K. Then $\mathrm{j}(\mathbf{E})$ and $\mathrm{j}(\mathbf{E}^{(b)})$ are the reductions modulo $\mathfrak{q}$ of $\mathrm{j}(E)$ and $\mathrm{j}(E^{(\mathfrak{b})})$ respectively. Our assumptions on $\mathfrak{q}$ imply that

$$\mathrm{j}(\mathbf{E}) \neq \mathrm{j}(\mathbf{E}^{(b)}).$$

Therefore $\mathbf{E}$ and $\mathbf{E}^{(b)}$ are not isomorphic over K and even over $\bar{K}$!

On the other hand, it is known [9, Ch. 9, Sect. 3] that there is a natural embedding

$$\mathrm{Hom}(E, E^{(\mathfrak{b})}) \hookrightarrow \mathrm{Hom}(\mathbf{E}, \mathbf{E}^{(\mathfrak{b})})$$

that respects the degrees of isogenies. It follows from Corollary 12.2 that for every prime ℓ there exists an isogeny $\mathbf{E} \to \mathbf{E}^{(\mathfrak{b})}$, whose degree is not divisible by ℓ. Now Proposition 1.11 implies that $\mathbf{E}_n \cong \mathbf{E}^{(\mathfrak{b})}{}_n$ for all positive integers n. It follows from Corollary 10.4 that the ℓ-divisible groups $\mathbf{E}(\ell)$ and $\mathbf{E}^{(\mathfrak{b})}(\ell)$ are isomorphic for all ℓ, including $\ell = \mathrm{char}(K)$. Since both $\mathbf{E}(\bar{K})$ and $\mathbf{E}^{(\mathfrak{b})}(\bar{K})$ are torsion groups, they are isomorphic as Galois modules. This implies that their subgroups of all Galois invariants are isomorphic, i.e., the finite groups $\mathbf{E}(K)$ and $\mathbf{E}^{(\mathfrak{b})}(K)$ are isomorphic.

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