Looking for rational curves on cubic hypersurfaces

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Abstract. We study rational points and rational curves on varieties over finite fields. The main new result is the construction of rational curves passing through a given collection of points on smooth cubic hypersurfaces over finite fields.

1. Introduction

The aim of these lectures is to study rational points and rational curves on varieties, mainly over finite fields $\mathbb{F}_q$. We concentrate on hypersurfaces X^n of degree $\leq n+1$ in $\mathbb{P}^{n+1}$, especially on cubic hypersurfaces.

The theorem of Chevalley–Warning (cf. Esnault's lectures) guarantees rational points on low degree hypersurfaces over finite fields. That is, if $X \subset \mathbb{P}^{n+1}$ is a hypersurface of degree $\leq n+1$, then $X(\mathbb{F}_q) \neq \emptyset$.

In particular, every cubic hypersurface of dimension ≥ 2 defined over a finite field contains a rational point, but we would like to say more.

- Which cubic hypersurfaces contain more than one rational point?
- Which cubic hypersurfaces contain rational curves?
- Which cubic hypersurfaces contain many rational curves?

Note that there can be rational curves on X even if X has a unique $\mathbb{F}_q$-point. Indeed, $f : \mathbb{P}^1 \to X$ could map all $q+1$ points of $\mathbb{P}^1(\mathbb{F}_q)$ to the same point in $X(\mathbb{F}_q)$, even if f is not constant.

So what does it mean for a variety to contain many rational curves? As an example, let us look at $\mathbb{CP}^2$. We know that through any 2 points there is a line, through any 5 points there is a conic, and so on. So we might say that a variety X_K contains many rational curves if through any number of points $p_1, \dots, p_n \in X(K)$ there is a rational curve defined over K.

However, we are in trouble over finite fields. A smooth rational curve over $\mathbb{F}_q$ has only $q+1$ points, so it can never pass through more than $q+1$ points in $X(\mathbb{F}_q)$. Thus, for cubic hypersurfaces, the following result, proved in Section 9, appears to be optimal:

Theorem 1.1. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic hypersurface over $\mathbb{F}_q$. Assume that $n \geq 2$ and $q \geq 8$. Then every map of sets $\phi : \mathbb{P}^1(\mathbb{F}_q) \to X(\mathbb{F}_q)$ can be extended to a map of $\mathbb{F}_q$-varieties $\Phi : \mathbb{P}^1 \to X$.*

In fact, one could think of stronger versions as well. A good way to formulate what it means for X to contain many (rational and nonrational) curves is the following:

Conjecture 1.2. [KS03] $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $\leq n+1$ defined over a finite field $\mathbb{F}_q$. Let C be a smooth projective curve and $Z \subset C$ a zero-dimensional subscheme. Then any morphism $\phi : Z \to X$ can be extended to C. That is, there is a morphism $\Phi : C \to X$ such that $\Phi|_Z = \phi$.

More generally, this should hold for any separably rationally connected variety X, see [KS03]. We define this notion in Section 4.

The aim of these notes is to explore these and related questions, especially for cubic hypersurfaces. The emphasis will be on presenting a variety of methods, and we end up outlining the proof of two special cases of the Conjecture.

Theorem 1.3. *Conjecture (1.2) holds in the following two cases.*

1. *[KS03] For arbitrary X, when q is sufficiently large (depending on $\dim X, g(C)$ and $\deg Z$), and*
2. *for cubic hypersurfaces when $q \geq 8$ and Z contains only odd degree points.*

As a warm-up, let us prove the case when $X = \mathbb{P}^n$. This is essentially due to Lagrange. The case of quadrics is already quite a bit harder, see (4.7).

Example 1.4 (Polynomial interpolation)**.** Over $\mathbb{F}_q$, let C be a smooth projective curve, $Z \subset C$ a zero-dimensional subscheme and $\phi : Z \to \mathbb{P}^n$ a given map.

Fix a line bundle L on C such that $\deg L \geq |Z| + 2g(C) - 1$ and choose an isomorphism $\mathcal{O}_Z \cong L|_Z$. Then ϕ can be given by $n+1$ sections $\phi_i \in H^0(Z, L|_Z)$. From the exact sequence

$$0 \to L(-Z) \to L \to L|_Z \to 0$$

we see that $H^0(C, L) \twoheadrightarrow H^0(Z, L|_Z)$. Thus each ϕ_i lifts to $\Phi_i \in H^0(C, L)$ giving the required extension $\Phi : C \to \mathbb{P}^n$.

1.5 (The plan of the lectures)**.** In Section 2, we study hypersurfaces with a unique point. This is mostly for entertainment, though special examples are frequently useful.

Then we prove that a smooth cubic hypersurface containing a K-point is unirational over K. That is, there is a dominant map $g : \mathbb{P}^n \dashrightarrow X$. This of course gives plenty of rational curves on X as images of rational curves on $\mathbb{P}^n$. Note however, that in general, $g : \mathbb{P}^n(K) \dashrightarrow X(K)$ is not onto. (In fact, one expects the image to be very small, see [Man86, Sec.VI.6].) Thus unirationality does not guarantee that there is a rational curve through every K-point.

As a generalization of unirationality, the notion of separably rationally connected varieties is introduced in Section 4. This is the right class to study the existence of many rational curves. Spaces parametrizing all rational curves on a variety are constructed in Section 5 and their deformation theory is studied in Section 6.

The easy case of Conjecture 1.2 is when Z is a single K-point. Here a complete answer to the analogous question is known over $\mathbb{R}$ or $\mathbb{Q}_p$. Over $\mathbb{F}_q$, the Lang-Weil estimates give a positive answer for q large enough; this is reviewed in Section 7.

The first really hard case of (1.2) is when $C = \mathbb{P}^1$ and $Z = \{0, \infty\}$. The geometric question is: given X with $p, p' \in X(\mathbb{F}_q)$, is there a rational curve defined over $\mathbb{F}_q$ passing through p, p'? We see in Section 8 that this is much harder than the 1-point case since it is related to Lefschetz-type results on the fundamental groups of open subvarieties. We use this connection to settle the case for q large enough and p, p' in general position.

Finally, in Section 9 we use the previous result and the "third intersection point map" to prove Theorem 1.1.

Remark 1.6. The first indication that the 2-point case of (1.2) is harder than the 1-point case is the different behavior over $\mathbb{R}$. Consider the cubic surface S defined by the affine equation $y^2 + z^2 = x^3 - x$. Then $S(\mathbb{R})$ has two components (a compact and an infinite part).

- If p, p' lie in different components, there is no rational curve over $\mathbb{R}$ through p, p', since $\mathbb{R}\mathbb{P}^1$ is connected.
- If p, p' lie in the same component, there is no topological obstruction. In fact, in this case an $\mathbb{R}$-rational curve through p, p' always exists, see [Kol99, 1.7].

2. Hypersurfaces with a unique point

The first question has been answered by Swinnerton-Dyer. We state it in a seemingly much sharpened form.

Proposition 2.1. *Let X be a smooth cubic hypersurface of dimension ≥ 2 defined over a field K with a unique K-point. Then $\dim X = 2$, $K = \mathbb{F}_2$ and X is unique up to projective equivalence.*

Proof. We show in the next section that X is unirational. Hence, if K is infinite, then X has infinitely many K-points. So this is really a question about finite fields.

If $\dim X \geq 3$ then $|X(K)| \geq q + 1$ by (2.3). Let us show next that there is no such surface over $\mathbb{F}_q$ for $q \geq 3$.

Assume to the contrary that S contains exactly one rational point $x \in S(\mathbb{F}_q)$. There are q^3 hyperplanes in $\mathbb{P}^3$ over $\mathbb{F}_q$ not passing through x.

The intersection of each hyperplane with S is a curve of degree 3, which is either an irreducible cubic curve, or the union of a line and a conic, or the union of three lines. In the first case, the cubic curve contains a rational point (if C contains a singular point, this points is defined over $\mathbb{F}_q$; if C is smooth, the Weil conjectures show that $|\#C(\mathbb{F}_q) - (q + 1)| \leq 2\sqrt{q}$, so $\#C(\mathbb{F}_q) = 0$ is impossible);

in the second case, the line is rational and therefore contains rational points; in the third case, at least one line is rational unless all three lines are conjugate.

By assumption, for each hyperplane H not passing through x, the intersection $S \cap H$ does not contain rational points. Therefore, $S \cap H$ must be a union of three lines that are not defined over $\mathbb{F}_q$, but conjugate over $\overline{\mathbb{F}}_q$.

This gives $3q^3$ lines on S. However, over an algebraically closed field, a cubic surface contains exactly 27 lines. For $q \geq 3$, we have arrived at a contradiction.

Finally we construct the surface over $\mathbb{F}_2$, without showing uniqueness. That needs a little more case analysis, see [KSC04, 1.39].

We may assume that the rational point is the origin of an affine space on which S is given by the equation

$$f(x, y, z) = z + Q(x, y, z) + C(x, y, z),$$

with Q (resp. C) homogeneous of degree 2 (resp. 3). If C vanishes in (x, y, z), then S has a rational point $(x : y : z)$ on the hyperplane $\mathbb{P}^2$ at infinity. Since C must not vanish in $(1, 0, 0)$, the cubic form C must contain the term x^3, and similarly y^3, z^3. By considering $(1, 1, 0)$, we see that it must also contain x^2y or xy^2, so without loss of generality, we may assume that it contains x^2y, and for similar reasons, we add the terms y^2z, z^2x. To ensure that C does not vanish at $(1, 1, 1)$, we add the term xyz, giving

$$C(x, y, z) = x^3 + y^3 + z^3 + x^2y + y^2z + z^2x + xyz.$$

Outside the hyperplane at infinity, we distinguish two cases: We see by considering a tangent plane $(z = 0)$ that it must intersect S in three conjugate lines. We conclude that $Q(x, y, z) = z(ax + by + cz)$ for certain $a, b, c \in \mathbb{F}_2$. For $z \neq 0$, we must ensure that f does not vanish at the four points $(0, 0, 1)$, $(1, 0, 1)$, $(0, 1, 1)$, $(1, 1, 1)$, resulting in certain restrictions for a, b, c. These are satisfied by $a = b = 0$ and $c = 1$. Therefore, $f(x, y, z) = z + z^2 + C(x, y, z)$ with C as above defines a cubic surface over $\mathbb{F}_2$ that has exactly one $\mathbb{F}_2$-rational point.

There are various ways to check that the cubic is smooth. The direct computations are messy by hand but easy on a computer. Alternatively, one can note that S does contain $3 \cdot 2^3 + 3 = 27$ lines and singular cubics always have fewer than 27. □

Remark 2.2. A variation of the argument in the proof shows, without using the theorem of Chevalley–Warning, that a cubic surface S defined over $\mathbb{F}_q$ must contain at least one rational point:

If S does not contain a rational point, the intersection of S with any of the $q^3 + q^2 + q + 1$ hyperplanes in $\mathbb{P}^3$ consists of three conjugate lines, giving $3(q^3 + q^2 + q + 1) \geq 45 > 27$ lines, a contradiction.

Exercise 2.3. Using Chevalley–Warning, show that for a hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $n + 1 - r$, the number of $\mathbb{F}_q$-rational points is at least $|\mathbb{P}^r(\mathbb{F}_q)| = q^r + \cdots + q + 1$.

Question 2.4. Find more examples of hypersurfaces $X \subset \mathbb{P}^{n+1}$ of degree at most $n + 1$ with $\#X(\mathbb{F}_q) = 1$.

Example 2.5 (H.-C. Graf v. Bothmer)**.** We construct hypersurfaces $X \subset \mathbb{P}^{n+1}$ over $\mathbb{F}_2$ containing exactly one rational point.

We start by constructing an affine equation. Note that the polynomial $f := x_0 \cdots x_{n+1}$ vanishes in every $\mathbf{x} \in \mathbb{F}_2^{n+2}$ except $(1, \dots, 1)$, while $g := (x_0 - 1) \cdots (x_{n+1} - 1)$ vanishes in every point except $(0, \dots, 0)$. Therefore, the polynomial $h := f + g + 1$ vanishes only in $(0, \dots, 0)$ and $(1, \dots, 1)$.

The only monomial of degree at least $n+2$ occurring in f and g is $x_0 \cdots x_{n+1}$, while the constant term 1 occurs in g but not in f. Therefore, h is a polynomial of degree $n + 1$ without constant term. We construct the homogeneous polynomial H of degree $n + 1$ from h by replacing each monomial $x_{i_1} x_{i_2} \cdots x_{i_r}$ of degree $r \in \{1, \dots, n+1\}$ of h with $i_1 < \cdots < i_r$ by $x_{i_1}^{k+1} x_{i_2} \cdots x_{i_r}$ of degree $n+1$ (where $k = n + 1 - r$). Since $a^k = a$ for any $k \geq 1$ and $a \in \mathbb{F}_2$, we have $h(\mathbf{x}) = H(\mathbf{x})$ for any $\mathbf{x} \in \mathbb{F}_2^{n+2}$; the homogeneous polynomial H vanishes exactly in $(0, \dots, 0)$ and $(1, \dots, 1)$.

Therefore, H defines a degree $n+1$ hypersurface $\mathbb{P}^{n+1}$ containing exactly one $\mathbb{F}_2$-rational point $(1 : \cdots : 1)$.

Using a computer, we can check for $n = 2, 3, 4$ that H defines a smooth hypersurface of dimension n. Note that for $n = 2$, the resulting cubic surface is isomorphic to the one constructed in Proposition 2.1. For $n \geq 5$, it is unknown whether H defines a smooth variety.

Example 2.6. Let α_1 be a generator or $\mathbb{F}_{q^m}/\mathbb{F}_q$ with conjugates α_i. It is easy to see that

$$X(\alpha) := \Bigl(\prod_i (x_1 + \alpha_i x_2 + \cdots + \alpha_i^{m-1} x_m) = 0\Bigr) \subset \mathbb{P}^m$$

has a unique $\mathbb{F}_q$-point at $(1 : 0 : \cdots : 0)$. $X(\alpha)$ has degree m, it is irreducible over $\mathbb{F}_q$ but over $\mathbb{F}_{q^m}$ it is the union of m planes.

Assume now that $q \leq m - 1$. Note that $x_i^q x_j - x_i x_j^q$ is identically zero on $\mathbb{P}^m(\mathbb{F}_q)$. Let H be any homogeneous degree m element of the ideal generated by all the $x_i^q x_j - x_i x_j^q$. Then H is also identically zero on $\mathbb{P}^m(\mathbb{F}_q)$, thus

$$X(\alpha, H) := \Bigl(\prod_i (x_1 + \alpha_i x_2 + \cdots + \alpha_i^{m-1} x_m) = H\Bigr) \subset \mathbb{P}^m$$

also has a unique $\mathbb{F}_q$-point at $(1 : 0 : \cdots : 0)$.

By computer it is again possible to find further examples of smooth hypersurfaces with a unique point, but the computations seem exceedingly lengthy for $m \geq 6$.

Remark 2.7. Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d. Then the primitive middle Betti number is

$$\frac{(d-1)^{n+2} + (-1)^n}{d} + 1 + (-1)^n \leq d^{n+1}.$$

Thus by the Deligne-Weil estimates

$$\left|\#X(\mathbb{F}_q) - \#\mathbb{P}^n(\mathbb{F}_q)\right| \leq d^{n+1}q^{n/2}.$$

Thus we get that for $d = n+1$, there are more then $\#\mathbb{P}^{n-1}(\mathbb{F}_q)$ points in $X(\mathbb{F}_q)$ as soon as $q \geq (n+1)^{2+\frac{2}{n}}$.

3. Unirationality

Definition 3.1. A variety X of dimension n defined over a field K is called *unirational* if there is a dominant map $\phi : \mathbb{P}^n \dashrightarrow X$, also defined over K.

Exercise 3.2. [Kol02, 2.3] Assume that there is a dominant map $\phi : \mathbb{P}^N \dashrightarrow X$ for some N. Show that X is unirational.

The following result was proved by Segre [Seg43] in the case $n = 2$, by Manin [Man86] for arbitrary n and general X when K is not a finite field with "too few" elements, and in full generality by Kollár [Kol02].

Theorem 3.3. *Let K be an arbitrary field and $X \subset \mathbb{P}^{n+1}$ a smooth cubic hypersurface ($n \geq 2$). Then the following are equivalent:*

1. *X is unirational over K.*
2. *$X(K) \neq \emptyset$.*

Proof. Let us start with the easy direction: (1) $\Rightarrow$ (2). The proof of the other direction will occupy the rest of the section.

If K is infinite, then K-rational points in $\mathbb{P}^n$ are Zariski-dense, so ϕ is defined on most of them, giving K-rational points of X as their image.

If K is a finite field, ϕ might not be defined on any K-rational point. Here, the result is a special case of the following.

Lemma 3.4 (Nishimura). *Given a smooth variety Y defined over K with $Y(K) \neq \emptyset$ and a rational map $\phi : Y \dashrightarrow Z$ with Z proper, we have $Z(K) \neq \emptyset$.*

Proof (after E. Szabó). We proceed by induction on the dimension of Y. If $\dim Y = 0$, the result is clear. If $\dim Y = d > 0$, we extend $\phi : Y \dashrightarrow Z$ to $\phi' : Y' \dashrightarrow Z$, where Y' is the blow-up of Y in $p \in Y(K)$. Since a rational map is defined outside a closed subset of codimension at least 2, we can restrict ϕ' to the exceptional divisor, which is isomorphic to $\mathbb{P}^{d-1}$. This restriction is a map satisfying the induction hypothesis. Therefore, $X(K) \neq \emptyset$. □

3.5 (Third intersection point map). Let $C \subset \mathbb{P}^2$ be a smooth cubic curve. For $p, p' \in C$ the line $\langle p, p' \rangle$ through them intersects C in a unique third point, denote it by $\phi(p, p')$. The resulting morphism $\phi : C \times C \to C$ is, up to a choice of the origin and a sign, the group law on the elliptic curve C.

For an arbitrary cubic hypersurface X defined over a field K, we can construct the analogous rational map $\phi : X \times X \dashrightarrow X$ as follows. If $p \neq p'$ and if the line $\langle p, p' \rangle$ does not lie completely in X, it intersects X in a unique third point $\phi(p, p')$. If $X_{\bar{K}}$ is irreducible, this defines ϕ on an open subset of $X \times X$.

It is very tempting to believe that out of ϕ one can get an (at least birational) group law on X. This is, unfortunately, not at all the case. The book [Man86] gives a detailed exploration of this direction.

We use ϕ to obtain a dominant map from a projective space to X, relying on two basic ideas:

- Assume that $Y_1, Y_2 \subset X$ are rational subvarieties such that $\dim Y_1 + \dim Y_2 \geq \dim X$. Then, if Y_1, Y_2 are in "general position," the restriction of ϕ gives a dominant map $Y_1 \times Y_2 \dashrightarrow X$. Thus X is unirational since $Y_1 \times Y_2$ is birational to a projective space.
- How can we find rational subvarieties of X? Pick a rational point $p \in X(K)$ and let Y_p be the intersection of X with the tangent hyperplane T_p of X in p. Note that Y_p is a cubic hypersurface of dimension $n-1$ with a singularity at p.
  If p is in "general position," then Y_p is irreducible and not a cone. Thus $\pi : Y_p \dashrightarrow \mathbb{P}^{n-1}$, the projection from p, is birational and so Y_p is rational.

From this we conclude that if $X(K)$ has at least 2 points in "general position," then X is unirational. In order to prove unirationality, one needs to understand the precise meaning of the above "general position" restrictions, and then figure out what to do if there are no points in "general position." This is especially a problem over finite fields.

Example 3.6. [Hir81] Check that over $\mathbb{F}_2, \mathbb{F}_4$ and $\mathbb{F}_{16}$ all points of $(x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0)$ lie on a line. In particular, the curves Y_p are reducible whenever p is over $\mathbb{F}_{16}$. Thus there are no points in "general position."

3.7 (End of the proof of (3.3))**.** In order to prove (3.3), we describe 3 constructions, working in increasing generality.

(3.7.1) Pick $p \in X(K)$. If Y_p is irreducible and not a cone, then Y_p is birational over K to $\mathbb{P}^{n-1}$. This gives more K-rational points on Y_p. We pick $p' \in Y_p(K) \subset X(K)$, and if we are lucky again, $Y_{p'}$ is also birational over K to $\mathbb{P}^{n-1}$. This results in

$$\Phi_{1,p,p'} : \mathbb{P}^{2n-2} \overset{bir}{\dashrightarrow} Y_p \times Y_{p'} \dashrightarrow X,$$

where the first map is birational and the second map is dominant.

This construction works when K is infinite and Y_p is irreducible and not a cone.

(3.7.2) Over K, it might be impossible to find $p \in X(K)$ such that Y_p is irreducible. Here we try to give ourselves a little more room by passing to a quadratic field extension and then coming back to K using the third intersection point map ϕ.

Given $p \in X(K)$, a line through p can intersect X in two conjugate points s, s' defined over a quadratic field extension K'/K. If $Y_s, Y_{s'}$ (the intersections of X with the tangent hyperplanes in s resp. s') are birational to $\mathbb{P}^{n-1}$ over K', consider the map

$$\Phi_{1,s,s'} : Y_s \times Y_{s'} \dashrightarrow X.$$

So far $\Phi_{1,s,s'}$ is defined over K'.

Note however that $Y_s, Y_{s'}$ are conjugates of each other by the Galois involution of K'/K. Furthermore, if $z \in Y_s$ and $\bar{z} \in Y_{s'}$ is its conjugate then the line $\phi(z, \overline{z})$ is defined over K. Indeed, the Galois action interchanges $z, \overline{z}$ hence the line $\langle z, \overline{z} \rangle$ is Galois invariant, hence the third intersection point $\Phi_{1,s,s'}(z, \overline{z})$ is defined over K.

That is, the involution $(z_1, z_2) \mapsto (\bar{z}_2, \bar{z}_1)$ makes $Y_s \times Y_{s'}$ into a K-variety and $\Phi_{1,s,s'}$ then becomes a K-morphism. Thus we obtain a dominant map

$$\Phi_{2,p,L} : \mathfrak{R}_{K'/K}\mathbb{P}^{n-1} \dashrightarrow X$$

where $\mathfrak{R}_{K'/K}\mathbb{P}^{n-1}$ is the Weil restriction of $\mathbb{P}^{n-1}$ (cf. Example 3.8).

This construction works when K is infinite, even if Y_p is reducible or a cone. However, over a finite field, it may be impossible to find a suitable line L.

As a last try, if none of the lines work, let's work with all lines together!

(3.7.3) Consider the *universal line* through p instead of choosing a specific line. That is, we are working with all lines at once. To see what this means, choose an affine equation such that p is at the origin:

$$L(x_1, \dots, x_{n+1}) + Q(x_1, \dots, x_{n+1}) + C(x_1, \dots, x_{n+1}) = 0,$$

where L is linear, Q is quadratic and C is cubic. The universal line is given by $(m_1 t, \dots, m_n t, t)$ where the m_i are algebraically independent over K and the quadratic formula gives the points s, s' at

$$t = \frac{-Q(m_1, \dots, m_n, 1) \pm \sqrt{D(m_1, \dots, m_n, 1)}}{2C(m_1, \dots, m_{n-1}, 1)},$$

where $D = Q^2 - 4LC$ is the discriminant.

Instead of working with just one pair $Y_s, Y_{s'}$, we work with the universal family of them defined over the field

$$K\Big(m_1, \dots, m_n, \sqrt{D(m_1, \dots, m_n, 1)}\Big)$$

It does not matter any longer that Y_s may be reducible for every $m_1, \dots, m_n \in K$ since we are working with all the Y_s together and the generic Y_s is irreducible and not a cone.

Thus we get a map

$$\Phi_{3,p} : \mathbb{P}^n \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \overset{bir}{\dashrightarrow} \mathfrak{R}_{K\left(m_1, \dots, m_n, \sqrt{D}\right)/K\left(m_1, \dots, m_n\right)}\mathbb{P}^{n-1} \dashrightarrow X.$$

The last step is the following observation. Unirationality of X_K changes if we extend K. However, once we have a K-map $\mathbb{P}^{3n-2} \dashrightarrow X$, its dominance can be checked after any field extension. Since $\Phi_{3,p}$ incorporates all $\Phi_{2,p,L}$, we see that the K-map $\Phi_{3,p}$ is dominant if $\Phi_{2,p,L}$ is dominant for some $\bar{K}$-line L.

Thus we can check dominance over the algebraic closure of K, where the techniques of the previous cases work.

There are a few remaining points to settle (mainly that Y_p is irreducible and not a cone for general $p \in X(\bar{K})$ and that $\Phi_{1,p,p'}$ is dominant for general $p, p' \in X(\bar{K})$). These are left to the reader. For more details, see [Kol02, Section 2].

Example 3.8. We give an explicit example of the construction of the Weil restriction. The aim of Weil restriction is to start with a finite field extension L/K and an L-variety X and construct in a natural way a K-variety $\mathfrak{R}_{L/K}X$ such that $X(L) = \big(\mathfrak{R}_{L/K}X\big)(K)$.

As a good example, assume that the characteristic is $\neq 2$ and let $L = K(\sqrt{a})$ be a quadratic field extension with $G := \mathrm{Gal}(L/K) = \{\mathrm{id}, \sigma\}$. Let X be an L-variety and X^σ its conjugate over K.

Then $X \times X^\sigma$ is an L-variety. We can define a G-action on it by

$$\sigma : (x_1, x_2^\sigma) \mapsto (x_2, x_1^\sigma).$$

This makes $X \times X^\sigma$ into the K-variety $\mathfrak{R}_{L/K}X$.

We explicitly construct $\mathfrak{R}_{L/K}\mathbb{P}^1$, which is all one needs for the surface case of (3.3).

Take the product of two copies of $\mathbb{P}^1$ with the G-action

$$((s_1 : t_1), (s_2^\sigma : t_2^\sigma)) \mapsto ((s_2 : t_2), (s_1^\sigma : t_1^\sigma)).$$

Sections of $\mathcal{O}(1,1)$ invariant under G are

$$u_1 := s_1 s_2^\sigma, \quad u_2 := t_1 t_2^\sigma, \quad u_3 := s_1 t_2^\sigma + s_2^\sigma t_1, \quad u_4 := \frac{1}{\sqrt{a}}(s_1 t_2^\sigma - s_2^\sigma t_1).$$

These sections satisfy $u_3^2 - a u_4^2 = 4u_1u_2$, and in fact, this equation defines $\mathfrak{R}_{L/K}\mathbb{P}^1$ as a subvariety of $\mathbb{P}^3$ over K.

Thus $\mathfrak{R}_{L/K}\mathbb{P}^1$ is a quadric surface with K-points (e.g., $(1:0:0:0)$), hence rational over K.

Let us check that $(\mathfrak{R}_{L/K}\mathbb{P}^1)(K) = \mathbb{P}^1(L)$. Explicitly, one direction of this correspondence is as follows. Given $(x_1 + \sqrt{a}x_2 : y_1 + \sqrt{a}y_2) \in \mathbb{P}^1(L)$ with $x_1, x_2, y_1, y_2 \in K$, we get the G-invariant point

$$\big((x_1 + \sqrt{a}x_2 : y_1 + \sqrt{a}y_2), (x_1 - \sqrt{a}x_2 : y_1 - \sqrt{a}y_2)\big) \in ((X \times X^\sigma)(L))^G.$$

From this, we compute

$$u_1 = x_1^2 - a x_2^2, \quad u_2 = y_1^2 - a y_2^2, \quad u_3 = 2(x_1y_1 - a x_2 y_2), \quad u_4 = 2(x_2y_1 - x_1y_2).$$

Then $(u_1 : u_2 : u_3 : u_4) \in \mathfrak{R}_{L/K}\mathbb{P}^1(K)$.

For the precise definitions and for more information, see [BLR90, Section 7.6] or [Kol02, Definition 2.1].

In order to illustrate the level of our ignorance about unirationality, let me mention the following problem.

Question 3.9. Over any field K, find an example of a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ with $\deg X \le n+1$ and $X(K) \neq \emptyset$ that is not unirational. So far, no such X is known.

The following are 2 further incarnations of the third intersection point map.

Exercise 3.10. Let X^n be an irreducible cubic hypersurface. Show that S^2X is birational to $X \times \mathbb{P}^n$, where S^2X denotes the symmetric square of X, that is, $X \times X$ modulo the involution $(x, x') \mapsto (x', x)$.

Exercise 3.11. Let X^n be an irreducible cubic hypersurface defined over K and L/K any quadratic extension. Show that there is a map $\mathfrak{R}_{L/K}X \dashrightarrow X$.

4. Separably rationally connected varieties

Before we start looking for rational curves on varieties over finite fields, we should contemplate which varieties contain plenty of rational curves over an algebraically closed field. There are various possible ways of defining what we mean by lots of rational curves, here are some of them.

4.1. Let X be a smooth projective variety over an algebraically closed field $\overline{K}$. Consider the following conditions:

1. For any given $x, x' \in X$, there is $f : \mathbb{P}^1 \to X$ such that $f(0) = x, f(\infty) = x'$.
2. For any given $x_1, \dots, x_m \in X$, there is $f : \mathbb{P}^1 \to X$ such that $\{x_1, \dots, x_m\} \subset f(\mathbb{P}^1)$.
3. Let $Z \subset \mathbb{P}^1$ be a zero-dimensional subscheme and $f_Z : Z \to X$ a morphism. Then f_Z can be extended to $f : \mathbb{P}^1 \to X$.
4. Conditions (3) holds, and furthermore $f^*T_X(-Z)$ is ample. (That is, f^*T_X is a sum of line bundles each of degree at least $|Z| + 1$.)
5. There is $f : \mathbb{P}^1 \to X$ such that f^*T_X is ample.

Theorem 4.2. *[KMM92b], [Kol96, Sec.4.3] Notation as above.*

1. *If $\overline{K}$ is an uncountable field of characteristic* 0 *then the conditions 4.1.1–4.1.5 are equivalent.*
2. *For any $\overline{K}$, condition 4.1.5 implies the others.*

Definition 4.3. Let X be a smooth projective variety over a field K. We say that X is *separably rationally connected* or *SRC* if the conditions 4.1.1–4.1.5 hold for $X_{\overline{K}}$.

Remark 4.4. There are 2 reasons why the conditions 4.1.1–4.1.5 are not always equivalent.

First, in positive characteristic, there are inseparably unirational varieties. These also satisfy the conditions 4.1.1–4.1.2, but usually not 4.1.5. For instance, if X is an inseparably unirational surface of general type, then 4.1.5 fails. Such examples are given by (resolutions of) a hypersurface of the form $z^p = f(x, y)$ for $\deg f \gg 1$.

Second, over countable fields, it could happen that (4.1.1) holds but X has only countably many rational curves. In particular, the degree of the required f depends on x, x'. These examples are not easy to find, see [BT05] for some over $\overline{\mathbb{F}}_q$. It is not known if this can happen over $\overline{\mathbb{Q}}$ or not.

Over countable fields of characteristic 0, we must require the existence of $f : \mathbb{P}^1 \to X$ of *bounded degree* in these conditions in order to obtain equivalence with 4.1.5.

Example 4.5. Let $S \subset \mathbb{P}^3$ be a cubic surface. Over the algebraic closure, S is the blow-up of $\mathbb{P}^2$ in six points. Considering f mapping $\mathbb{P}^1$ to a line in $\mathbb{P}^2$ not passing through any of the six points, we see that S is separably rationally connected.

More generally, any rational surface is separably rationally connected.

It is not quite trivial to see that for any normal cubic surface S that is not a cone, there is a morphism to the smooth locus $f : \mathbb{P}^1 \to S^{ns}$ such that f^*T_S is ample.

Any normal cubic hypersurface is also separably rationally connected, except cones over cubic curves. To see this, take repeated general hyperplane sections until we get a normal cubic surface $S \subset X$ which is not a cone. The normal bundle of S in X is ample, hence the $f : \mathbb{P}^1 \to S^{ns}$ found earlier also works for X.

In characteristic 0, any smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $\leq n+1$ is SRC; see [Kol96, Sec.V.2] for references and various stronger versions. Probably every normal hypersurface is also SRC, except for cones.

In positive characteristic the situation is more complicated. A general hypersurface of degree $\leq n+1$ is SRC, but it is not known that every smooth hypersurface of degree $\leq n+1$ is SRC. There are some mildly singular hypersurfaces which are not SRC, see [Kol96, Sec.V.5].

4.6 (Effective bounds for hypersurfaces)**.** Let $X \subset \mathbb{P}^{n+1}$ be a smooth SRC hypersurface over $\bar{K}$. Then (4.2) implies that there are rational curves through any point or any 2 points. Here we consider effective bounds for the degrees of such curves.

First, if $\deg X < n+1$ then through every point there are lines. For a general point, the general line is also free (cf. (5.2)).

If $\deg X = n+1$ then there are no lines through a general point, but usually there are conics. However, on a cubic surface there are no irreducible conics through an Eckart point p. (See (7.5) for the definition and details.)

My guess is that in all cases there are free twisted cubics through any point, but this may be difficult to check. I don't know any reasonable effective upper bound.

For 2 general points $x, x' \in X$, there is an irreducible rational curve of degree $\leq n(n+1)/(n+2-\deg X)$ by [KMM92a]. The optimal result should be closer to $n+1$, but this is not known. Very little is known about non-general points.

Next we show that (1.2) depends only on the birational class of X. The proof also shows that (4.1.3) is also a birational property.

Proposition 4.7. *Let K be a field and X, X' smooth projective K-varieties which are birational to each other. Then, if (1.2) holds for X, it also holds for X'.*

Proof. Assume for notational simplicity that K is perfect. Fix embeddings $X \subset \mathbb{P}^N$ and $X' \subset \mathbb{P}^M$ and represent the birational maps $\phi : X \dashrightarrow X'$ and $\phi^{-1} : X' \dashrightarrow X$ with polynomial coordinate functions.

Given $f'_Z : Z \to X'$, we construct a thickening $Z \subset Z_t \subset C$ and $f_{Z_t} : Z_t \to X$ such that if $f : C \to X$ extends f_{Z_t} then $f' := \phi \circ f$ extends f'_Z.

Pick a point $p \in Z$ and let $\hat{\mathcal{O}}_p \cong L[[t]]$ be its complete local ring where $L = K(p)$. Then the corresponding component of Z is $\operatorname{Spec}_K L[[t]]/(t^m)$ for some $m \geq 1$.

By choosing suitable local coordinates at $f'_Z(p) \in X'$, we can define its completion by equations

$$y_{n+i} = G_i(y_1, \dots, y_n) \quad \text{where } G_i \in L[[y_1, \dots, y_n]].$$

Thus f'_Z is given by its coordinate functions

$$\bar{y}_1(t), \dots, \bar{y}_n(t), \dots \in L[[t]]/(t^m).$$

The polynomials $\bar{y}_i$ for $i = 1, \dots, n$ can be lifted to $y_i(t) \in L[[t]]$ arbitrarily. These then determine liftings $y_{n+i}(t) = G_i(y_1(t), \dots, y_n(t))$ giving a map $F' : \operatorname{Spec}_K L[[t]] \to X'$. In particular, we can choose a lifting such that ϕ^{-1} is a local isomorphism at the image of the generic point of $\operatorname{Spec}_K L[[t]]$. Thus $\phi^{-1} \circ F'$ and $\phi \circ \phi^{-1} \circ F'$ are both defined and $\phi \circ \phi^{-1} \circ F' = F'$.

Using the polynomial representations for ϕ, ϕ^{-1}, write

$$\begin{aligned} \phi^{-1} \circ (1, y_1(t), \dots, y_n(t), \dots) &= (x_0(t), \dots, x_n(t), \dots), \quad \text{and} \\ \phi \circ (x_0(t), \dots, x_n(t), \dots) &= (z_0(t), \dots, z_n(t), \dots). \end{aligned}$$

Note that $(z_0(t), \dots, z_n(t), \dots)$ and $(1, y_1(t), \dots, y_n(t), \dots)$ give the same map $F' : \operatorname{Spec}_K L[[t]] \to X'$, but we map to projective space. Thus all we can say is that

$$y_i(t) = z_i(t)/z_0(t) \quad \forall\, i \geq 1.$$

Assume now that we have $(x_0^*(t), \dots, x_n^*(t), \dots)$ and the corresponding

$$\phi \circ (x_0^*(t), \dots, x_n^*(t), \dots) = (z_0^*(t), \dots, z_n^*(t), \dots).$$

such that

$$x_i(t) \equiv x_i^*(t) \mod t^s \quad \forall\, i.$$

Then also

$$z_i(t) \equiv z_i^*(t) \mod t^s \quad \forall\, i.$$

In particular, if $s > r := \operatorname{mult}_0 z_0(t)$, then $\operatorname{mult}_0 z_0^*(t) = \operatorname{mult}_0 z_0(t)$ and so

$$y_i^*(t) := \frac{z_i^*(t)}{z_0^*(t)} \equiv y_i(t) \mod \left(t^{s-r}\right).$$

That is, if $F^* : \operatorname{Spec}_K L[[t]] \to X$ agrees with $\phi^{-1} \circ F'$ up to order $s = r + m$ then $\phi \circ F^*$ agrees with F' up to order $m = s - r$.

We apply this to every point in Z to obtain the thickening $Z \subset Z_t \subset C$ and $f_{Z_t} : Z_t \to X$ as required. □

5. Spaces of rational curves

Assume that X is defined over a non-closed field K and is separably rationally connected. Then X contains lots of rational curves over $\overline{K}$, but what about rational curves over K? We are particularly interested in the cases when K is one of $\mathbb{F}_q$, $\mathbb{Q}_p$ or $\mathbb{R}$.

5.1 (Spaces of rational curves)**.** Let X be any variety. Subvarieties or subschemes of X come in families, parametrized by the Chow variety or the Hilbert scheme. For rational curves in X, the easiest to describe is the space of maps $\operatorname{Hom}(\mathbb{P}^1, X)$.

Pick an embedding $X \subset \mathbb{P}^N$ and let F_i be homogeneous equations of X.

Any map $f : \mathbb{P}^1 \to \mathbb{P}^N$ of fixed degree d is given by $N + 1$ homogeneous polynomials $(f_0(s,t), \dots, f_N(s,t))$ of degree d in two variables s, t (up to scaling of these polynomials). Using the coefficients of $f_0, \dots, f_N$, we can regard f as a point in $\mathbb{P}^{(N+1)(d+1)-1}$.

We have $f(\mathbb{P}^1) \subset X$ if and only if the polynomials $F_i(f_0(s,t), \dots, f_N(s,t))$ are identically zero. Each F_i gives $d \cdot \deg F_i + 1$ equations of degree $\deg F_i$ in the coefficients of $f_0, \dots, f_N$.

If $f_0, \dots, f_N$ have a common zero, then we get only a lower degree map. We do not count these in $\operatorname{Hom}_d(\mathbb{P}^1, X)$. By contrast we allow the possibility that $f \in \operatorname{Hom}_d(\mathbb{P}^1, X)$ is not an embedding but a degree e map onto a degree d/e rational curve in X. These maps clearly cause some trouble but, as it turns out, it would be technically very inconvenient to exclude them from the beginning.

Thus $\operatorname{Hom}_d(\mathbb{P}^1, X)$ is an open subset of a subvariety of $\mathbb{P}^{(N+1)(d+1)-1}$ defined by equations of degree $\leq \max_i\{\deg F_i\}$.

$\operatorname{Hom}(\mathbb{P}^1, X)$ is the disjoint union of the $\operatorname{Hom}_d(\mathbb{P}^1, X)$ for $d = 1, 2, \dots$.

Therefore, finding a rational curve $f : \mathbb{P}^1 \to X$ defined over K is equivalent to finding K-points on $\operatorname{Hom}(\mathbb{P}^1, X)$.

In a similar manner one can treat the space $\operatorname{Hom}(\mathbb{P}^1, X, 0 \mapsto x)$ of those maps $f : \mathbb{P}^1 \to X$ that satisfy $f(0) = x$ or $\operatorname{Hom}(\mathbb{P}^1, X, 0 \mapsto x, \infty \mapsto x')$, those maps $f : \mathbb{P}^1 \to X$ that satisfy $f(0) = x$ and $f(\infty) = x'$.

5.2 (Free and very free maps)**.** In general, the local structure of the spaces $\operatorname{Hom}(\mathbb{P}^1, X)$ can be very complicated, but everything works nicely in certain important cases.

We say that $f : \mathbb{P}^1 \to X$ is *free* if f^*T_X is semi-positive, that is a direct sum of line bundles of degree ≥ 0. We see in (6.4) that if f is free then $\operatorname{Hom}(\mathbb{P}^1, X)$ and $\operatorname{Hom}(\mathbb{P}^1, X, 0 \mapsto f(0))$ are both smooth at $[f]$.

We say that $f : \mathbb{P}^1 \to X$ is *very free* if f^*T_X is positive or ample, that is, a direct sum of line bundles of degree ≥ 1. This implies that $\operatorname{Hom}(\mathbb{P}^1, X, 0 \mapsto f(0), \infty \mapsto f(\infty))$ is also smooth at $[f]$.

Remark 5.3. Over a nonclosed field K there can be smooth projective curves C such that $C_{\bar{K}} \cong \mathbb{P}^1_{\bar{K}}$ but $C(K) = \emptyset$, thus C is not birational to $\mathbb{P}^1_K$. When we work with $\mathrm{Hom}(\mathbb{P}^1, X)$, we definitely miss these curves. There are various ways to remedy this problem, but for us this is not important.

Over a finite field K, every rational curve is in fact birational to $\mathbb{P}^1_K$, thus we do not miss anything.

To get a feeling for these spaces, let us see what we can say about the irreducible components of $\mathrm{Hom}_d(\mathbb{P}^1, X)$ for cubic surfaces.

Example 5.4. Let $S \subset \mathbb{P}^3$ be a cubic surface defined over a non-closed field K. Consider $\mathrm{Hom}_d(\mathbb{P}^1, S)$ for low values of d.

- For $d = 1$, over $\overline{K}$, there are 27 lines on S, so $\mathrm{Hom}_1(\mathbb{P}^1, S)$ has 27 components which may be permuted by the action of the Galois group $G = \mathrm{Gal}(\overline{K}/K)$.
- For $d = 2$, over $\overline{K}$, there are 27 one-dimensional families of conics, each obtained by intersecting S with the pencil of planes containing a line on S. These 27 families again may have a non-trivial action of G.
- For $d = 3$, over $\overline{K}$, there are 72 two-dimensional families of twisted cubics on S (corresponding to the 72 ways to map S to $\mathbb{P}^2$ by contracting 6 skew lines; the twisted cubics are preimages of lines in $\mathbb{P}^2$ not going through any of the six blown-up points). Again there is no reason to assume that any of these 72 families is fixed by G.
  However, there is exactly one two-dimensional family of plane rational cubic curves on S, obtained by intersecting S with planes tangent to the points on S outside the 27 lines. This family is defined over K and is geometrically irreducible.

All this is not very surprising. A curve C on S determines a line bundle $\mathcal{O}_S(C) \in \mathrm{Pic}(S) \cong \mathbb{Z}^7$, hence we see many different families in a given degree because there are many different line bundles of a given degree. It turns out that, for cubic surfaces, once we fix not just the degree but also the line bundle $L = \mathcal{O}_S(C)$, the resulting spaces $\mathrm{Hom}_L(\mathbb{P}^1, X)$ are irreducible.

This, however, is a very special property of cubic surfaces and even for smooth hypersurfaces X it is very difficult to understand the irreducible components of $\mathrm{Hom}(\mathbb{P}^1, X)$. See [HRS04,HS05,HRS05,dJS04] for several examples.

Thus, in principle, we reduced the question of finding rational curves defined over K to finding K-points of the scheme $\mathrm{Hom}(\mathbb{P}^1, X)$. The problem is that $\mathrm{Hom}(\mathbb{P}^1, X)$ is usually much more complicated than X.

5.5 (Plan to find rational curves)**.** We try to find rational curves defined over a field K in 2 steps.

1. For any field K, we will be able to write down reducible curves C and morphisms $f : C \to X$ defined over K and show that $f : C \to X$ can be naturally viewed as a smooth point $[f]$ in a suitable compactification of $\mathrm{Hom}(\mathbb{P}^1, X)$ or $\mathrm{Hom}(\mathbb{P}^1, X, 0 \mapsto x)$.

2. Then we argue that for certain fields K, a smooth K-point in a compactification of a variety U leads to a K-point inside U.
   There are 2 main cases where this works.
   (a) (Fields with an analytic inverse function theorem)
   These include $\mathbb{R}, \mathbb{Q}_p$ or the quotient field of any local, complete Dedekind domain, see [GR71]. For such fields, any smooth point in $\bar{U}(K)$ has an analytic neighborhood biholomorphic to $0 \in K^n$. This neighborhood has nontrivial intersection with any nonempty Zariski open set, hence with U.
   (b) (Sufficiently large finite fields)
   This method relies on the Lang-Weil estimates. Roughly speaking these say that a variety U over $\mathbb{F}_q$ has points if $q \gg 1$, where the bound on q depends on U. We want to apply this to $U = \mathrm{Hom}_d(\mathbb{P}^1, X)$. We know bounds on its embedding dimension and on the degrees of the defining equations, but very little else. Thus we need a form of the Lang-Weil estimates where the bound for q depends only on these invariants.

We put more detail on these steps in the next sections, but first let us see an example.

Example 5.6. Let us see what we get in a first computation trying to find a degree 3 rational curve through a point p on a cubic surface S over $\mathbb{F}_q$.

The intersection of S with the tangent plane at p usually gives a rational curve C_p which is singular at p. If we normalize to get $n : \mathbb{P}^1 \to C$, about half the time, $n^{-1}(p)$ is a conjugate pair of points in $\mathbb{F}_{q^2}$. This is not what we want.

So we have to look for planes $H \subset \mathbb{P}^3$ that pass through p and are tangent to S at some other point. How to count these?

Projecting S from p maps to $\mathbb{P}^2$ and the branch curve $B \subset \mathbb{P}^2$ has degree 4. Moreover, B is smooth if p is not on any line. The planes we are looking for correspond to the tangent lines of B.

By the Weil estimates, a degree 4 smooth plane curve has at least $q+1-6\sqrt{q}$ points. For $q > 33$ this guarantees a point in $B(\mathbb{F}_q)$ and so we get a plane H through p which is tangent to S at some point.

However, this is not always enough. First, we do not want the tangency to be at p. Second, for any line $L \subset S$, the plane spanned by p and L intersects S in L and a residual conic. These correspond to double tangents of B. The 28 double tangents correspond to 56 points on B. Thus we can guarantee an irreducible degree 3 rational curve only if we find an $\mathbb{F}_q$-point on B which different from these 56 points. This needs $q > 121$ for the Weil estimates to work. This is getting quite large!

Of course, a line is a problem only if it is defined over $\mathbb{F}_q$ and then the corresponding residual conic is a rational curve over $\mathbb{F}_q$ passing through p, unless the residual conic is a pair of lines. In fact, if we look for rational curves of degree ≤ 3, then $q > 33$ works.

I do not know what the best bound for q is. In any case, we see that even this simple case leads either to large bounds or to case analysis.

The current methods work reasonably well when $q \gg 1$, but, even for cubic hypersurfaces, the bounds are usually so huge that I do not even write them down.

Then we see by another method that for cubics we can handle small values of q. The price we pay is that the degrees of the rational curves found end up very large.

It would be nice to figure out a reasonably sharp answer at least for cubics, Just to start the problem, let me say that I do not know the answer to the following.

Question 5.7. Let X be a smooth cubic hypersurface over $\mathbb{F}_q$ and $p, p' \in X(\mathbb{F}_q)$ two points. Is there a degree ≤ 9 rational curve defined over $\mathbb{F}_q$ passing through p and p'?

Exercise 5.8. Let $S \subset \mathbb{P}^3$ be the smooth cubic surface constructed in (2.1). Show that S does not contain any rational curve of degree ≤ 8 defined over $\mathbb{F}_2$.

Hints. First prove that the Picard group of S is generated by the hyperplane sections. Thus any curve on S has degree divisible by 3.

A degree 3 rational curve would be a plane cubic, these all have at least 2 points over $\mathbb{F}_2$.

Next show that any rational curve defined over $\mathbb{F}_2$ must have multiplicity 3 or more at the unique $p \in S(\mathbb{F}_2)$. A degree 6 rational curve would be a complete intersection of S with a quadric Q. Show that S and Q have a common tangent plane at p and then prove that $S \cap Q$ has only a double point if Q is irreducible.

6. Deformation of combs

Example 6.1. Let S be a smooth cubic surface over $\mathbb{R}$ and p a real point of S. Our aim is to find a rational curve defined over $\mathbb{R}$ passing through p. It is easy to find such a rational curve C defined over $\mathbb{C}$. Its conjugate $\bar{C}$ then also passes through p. Together, they define a curve $C + \bar{C} \subset S$ which is defined over $\mathbb{R}$. So far this is not very useful.

We can view $C + \bar{C}$ as the image of a map $\phi_0 : Q_0 \to S$ where $Q_0 \subset \mathbb{P}^2$ is defined by $x^2+y^2 = 0$. Next we would like to construct a perturbation $\phi_\varepsilon : Q_\varepsilon \to S$ of this curve and of this map. It is easy to perturb Q_0 to get "honest" rational curves over $\mathbb{R}$, for instance $Q_\varepsilon := (x^2 + y^2 = \varepsilon z^2)$.

The key question is, can we extend ϕ_0 to ϕ_ε? Such questions are handled by deformation theory, originated by Kodaira and Spencer. A complete treatment of the case we need is in [Kol96] and [AK03] is a good introduction.

The final answer is that if $H^1(Q_0, \phi_0^* T_S) = 0$, then ϕ_ε exists for $|\varepsilon| \ll 1$. This allows us to obtain a real rational curve on S, and with a little care we can arrange for it to pass through p.

In general, the above method gives the following result:

Corollary 6.2. *[Kol99] Given $X_{\mathbb{R}}$ such that $X_{\mathbb{C}}$ is rationally connected, there is a real rational curve through any $p \in X(\mathbb{R})$.*

We would like to apply a similar strategy to X_K such that $X_{\overline{K}}$ is separably rationally connected. For a given $x \in X(K)$, we find a curve $g_1 : \mathbb{P}^1 \to X$ defined over $\overline{K}$ such that $g_1(0) = x$, with conjugates $g_2, \ldots, g_m$. Then $C :=$

$g_1(\mathbb{P}^1)+\cdots+g_m(\mathbb{P}^1)$ is defined over K. Because of the singularity of C in x, it is harder to find a smooth deformation of C. It turns out that there is a very simple way to overcome this problem: we need to add a whole new $\mathbb{P}^1$ at the point x and look at maps of curves to X which may not be finite.

Definition 6.3. Let X be a variety over a field K. An *m-pointed stable curve of genus 0 over X* is an object $(C, p_1, \dots, p_m, f)$ where

1. C is a proper connected curve with $p_a(C) = 0$ defined over K having only nodes,
2. $p_1, \dots, p_m$ are distinct smooth points in $C(K)$,
3. $f : C \to X$ is a K-morphism, and
4. C has only finitely many automorphisms that commute with f and fix $p_1, \dots, p_m$. Equivalently, there is no irreducible component $C_i \subset C_{\bar{K}}$ such that f maps C_i to a point and C_i contains at most 2 special points (that is, nodes of C or $p_1, \dots, p_m$).

Note that if $f : C \to X$ is finite, then $(C, p_1, \dots, p_m, f)$ is a stable curve of genus 0 over X, even if $(C, p_1, \dots, p_m)$ is not a stable m-pointed genus 0 curve in the usual sense [FP97].

We have shown how to parametrize all maps $\mathbb{P}^1 \to X$ by the points of a scheme $\mathrm{Hom}(\mathbb{P}^1, X)$. Similarly, the methods of [KM94] and [Ale96] show that one can parametrize all m-pointed genus 0 stable curves of degree d with a single scheme $\overline{M}_{0,m}(X, d)$. For a map $(C, p_1, \dots, p_m, f)$, the corresponding point in $\overline{M}_{0,m}(X, d)$ is denoted by $[C, p_1, \dots, p_m, f]$.

Given K-points $x_1, \dots, x_m \in X(K)$, the family of those maps $f : C \to X$ that satisfy $f(p_i) = x_i$ for $i = 1, \dots, m$ forms a closed scheme

$$\overline{M}_{0,m}(X, p_i \mapsto x_i) \subset \overline{M}_{0,m}(X).$$

See [AK03, sec.8] for more detailed proofs.

The deformation theory that we need can be conveniently compacted into one statement. The result basically says that the deformations used in (6.1) exist for any reducible rational curve.

Theorem 6.4. *(cf. [Kol96, Sec.II.7] or [AK03]) Let $f : (p_1, \dots, p_m \in C) \to X$ be an m-pointed genus 0 stable curve. Assume that X is smooth and*

$$H^1(C, f^*T_X(-p_1 - \cdots - p_m)) = 0.$$

Then:

1. *There is a unique irreducible component*

$$\mathrm{Comp}(C, p_1, \dots, p_m, f) \subset \overline{M}_{0,m}(X, p_i \mapsto f(p_i))$$

   which contains $[C, p_1, \dots, p_m, f]$.
2. *$[C, p_1, \dots, p_m, f]$ is a smooth point of $\mathrm{Comp}(C, p_1, \dots, p_m, f)$. In particular, if $f : (p_1, \dots, p_m \in C) \to X$ is defined over K then $\mathrm{Comp}(C, p_1, \dots, p_m, f)$ is geometrically irreducible.*

3. *There is a dense open subset*

$$\text{Smoothing}(C, p_1, \ldots, p_m, f) \subset \text{Comp}(C, p_1, \ldots, p_m, f)$$

which parametrizes free maps of smooth rational curves, that is

$$\text{Smoothing}(C, p_1, \ldots, p_m, f) \subset \text{Hom}^{\text{free}}(\mathbb{P}^1, X, p_i \mapsto f(p_i))$$

(We cheat a little in (6.4.2). In general $[C, p_1, \ldots, p_m, f]$ is smooth only in the stack sense; this is all one needs. Moreover, in all our applications $[C, p, f]$ will be a smooth point.)

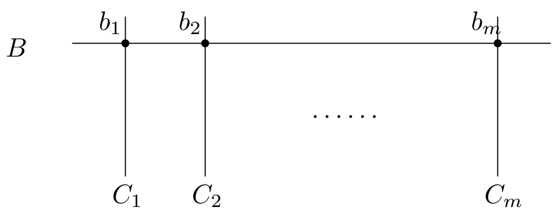
The required vanishing is usually easy to check using the following.

Exercise 6.5. Let $C = C_1 + \cdots + C_m$ be a reduced, proper curve with arithmetic genus 0 and $p \in C$ a smooth point. Let $C_1, \cdots, C_m$ be its irreducible components over $\bar{K}$. Let E be a vector bundle on C and assume that $H^1(C_i, E|_{C_i}(-1)) = 0$ for every i. Then $H^1(C, E(-p)) = 0$.

In particular, if $f : C \to X$ is a morphism to a smooth variety and if each $f|_{C_i}$ is free then $H^1(C, f^*T_X(-p)) = 0$.

Definition 6.6 (Combs)**.** A *comb* assembled from a curve B (the handle) and m curves C_i (the teeth) attached at the distinct points $b_1, \ldots, b_m \in B$ and $c_i \in C_i$ is a curve obtained from the disjoint union of B and of the C_i by identifying the points $b_i \in B$ and $c_i \in C_i$. In these notes we only deal with the case when B and the C_i are smooth, rational.

A comb can be pictured as below:



Comb with m-teeth

Assume now that we have a Galois extension L/K and $g_i : (0 \in \mathbb{P}^1) \to (x \in X)$, a conjugation invariant set of maps defined over L.

We can view this collection as just one map as follows. The maps $[g_i] \in \text{Hom}(\mathbb{P}^1, X)$ form a 0-dimensional reduced K-scheme Z. Then the g_i glue together to a single map

$$G : Z \times (0 \in \mathbb{P}^1) = (Z \subset \mathbb{P}^1_Z) \to (x \in X).$$

Let $j : Z \hookrightarrow \mathbb{P}^1_K$ be an embedding. We can then assemble a comb with handle $\mathbb{P}^1_K$ and teeth $\mathbb{P}^1_Z$. Let us denote it by

$$\text{Comb}(g_1, \ldots, g_m).$$

(The role of j is suppressed, it will not be important for us.)

If K is infinite, an embedding $j : Z \hookrightarrow \mathbb{P}^1_K$ always exists. If K is finite, then Z may have too many points, but an embedding exists whenever Z is irreducible over K.

Indeed, in this case $Z = \text{Spec}_K K(a)$ for some $a \in \bar{K}$. Thus $K[t] \to K(a)$ gives an embedding $Z \hookrightarrow \mathbb{A}^1_K$.

Everything is now ready to obtain rational curves through 1 point.

Corollary 6.7. *[Kol99] Given a separably rationally connected variety X defined over a local field $K = \mathbb{Q}_p$ or $K = \mathbb{F}_q((t))$, there is a rational curve defined over K through any $x \in X(K)$.*

Proof. Given $x \in X(K)$, pick a free curve $g_1 : (0 \in \mathbb{P}^1) \to (x \in X)$ over $\bar{K}$ with conjugates $g_2, \ldots, g_m$. As in (6.6), assemble a K-comb

$$f : (0 \in \text{Comb}(g_1, \ldots, g_m)) \to (x \in X).$$

Using (6.4), we obtain

$$\text{Smoothing}(C, 0, f) \subset \text{Hom}^{\text{free}}(\mathbb{P}^1, X, 0 \mapsto f(p))$$

and by (6.4.2) we see that (5.5.2.a) applies. Hence we get K-points in $\text{Smoothing}(p \in C, f)$, as required. □

The finite field case, corresponding to (5.5.2.b), is treated in the next section.

7. The Lang-Weil estimates

Theorem 7.1. *[LW54] Over $\mathbb{F}_q$, let $U \subset \mathbb{P}^N$ be the difference of two subvarieties defined by several equations of degree at most D. If $U_0 \subset U$ is a geometrically irreducible component, then*

$$\left| \#U_0(\mathbb{F}_q) - q^{\dim U_0} \right| \le C(N, D) \cdot q^{\dim U_0 - \frac{1}{2}},$$

where the constant $C(N, D)$ depends only on N and D.

Notes on the proof. The original form of the estimate in [LW54] assumes that U_0 is projective and it uses $\deg U_0$ instead of D. These are, however, minor changes.

First, if $V \subset \mathbb{P}^N$ is an irreducible component of W which is defined by equations of degree at most D, then it is also an irreducible component of some $W' \supset W$ which is defined by $N - \dim V$ equations of degree at most D. Thus, by Bézout's theorem, $\deg V \le D^{N - \dim V}$.

Thus we have a bound required for $\#\bar{U}_0(\mathbb{F}_q)$ and we need an upper bound for the complement $\#(\bar{U}_0 \setminus U_0)(\mathbb{F}_q)$. We assumed that $\bar{U}_0 \setminus U_0$ is also defined by equations of degree at most D. A slight problem is, however, that it may have components which are geometrically reducible. Fortunately, an upper bound for $\#V(\mathbb{F}_q)$ is easy to get. □

Exercise 7.2. Let $V \subset \mathbb{P}^N$ be a closed, reduced subscheme of pure dimension r and degree d. Show that if $q \geq d$ then V does not contain $\mathbb{P}^N(\mathbb{F}_q)$. Use this to show that there is a projection $\pi : V \to \mathbb{P}^r$ defined over $\mathbb{F}_q$ which is finite of degree d. Conclude from this that

$$\#V(\mathbb{F}_q) \leq d \cdot \#\mathbb{P}^r(\mathbb{F}_q) = d \cdot (q^r + \cdots + q + 1).$$

7.3 (Application to $\mathrm{Hom}_d(\mathbb{P}^1, X)$)**.** We are looking for rational curves of degree d on a hypersurface $X \subset \mathbb{P}^{n+1}$ of degree m. We saw in (5.1) that $\mathrm{Hom}_d(\mathbb{P}^1, X)$ lies in $\mathbb{P}^{(n+2)(d+1)-1}$ (hence we can take $N = (n+2)(d+1) - 1$) and its closure is defined by equations of degree m.

The complement of $\mathrm{Hom}_d(\mathbb{P}^1, X)$ in its closure consists of those $(f_0, \dots, f_N)$ with a common zero. One can get explicit equations for this locus as follows. Pick indeterminates λ_i, μ_j. Then $f_0, \dots, f_N$ have a common zero iff the resultant

$$\mathrm{Res}(\textstyle\sum_i \lambda_i f_i, \sum_j \mu_j f_j)$$

is identically zero as a polynomial in the λ_i, μ_j. This gives equations of degree $2d$ in the coefficients of the f_i. Thus we can choose $D = \max\{m, 2d\}$.

Finally, where do we find a geometrically irreducible component of the space $\mathrm{Hom}_d(\mathbb{P}^1, X)$? Here again a smooth point $[f]$ in a suitable compactification of $\mathrm{Hom}(\mathbb{P}^1, X)$ gives the answer by (7.4). Similar considerations show that our methods also apply to $\mathrm{Hom}_d(\mathbb{P}^1, X, 0 \mapsto p)$.

Exercise 7.4. Let W be a K-variety and $p \in W$ a smooth point. Then there is a unique K-irreducible component $W_p \subset K$ which contains p and W_p is also geometrically irreducible if either p is K-point or K is algebraically closed in $K(p)$.

As a first application, let us consider cubic surfaces.

Example 7.5 (Cubic surfaces)**.** Consider a cubic surface $S \subset \mathbb{P}^3$, defined over $K = \mathbb{F}_q$. We would like to use these results to get a rational curve through any $p \in S(\mathbb{F}_q)$.

We need to start with some free rational curves over $\bar{K}$.

The first such possibility is to use conics. If $L \subset S$ is a line, then the plane spanned by p and L intersects S in L plus a residual conic C_L. C is a smooth and free conic, unless p lies on a line.

In general, we get 27 conics and we conclude that if q is large enough, then through every point $p \in S(\mathbb{F}_q)$ which is not on a line, there is rational curve of degree $2 \cdot 27 = 54$, defined over $\mathbb{F}_q$.

If p lies on 1 (resp. 2) lines, then we get only 16 (resp. 8) smooth conics, and so we get even lower degree rational curves.

However, when p lies on 3 lines (these are called Eckart points) then there is no smooth conic through p.

Let us next try twisted cubics. As we saw in (5.4), we get twisted cubics from a morphism $S \to \mathbb{P}^2$ as the birational transforms of lines not passing through any of the 6 blown up points. Thus we get a 2-dimensional family of twisted cubics whenever p is not on one the 6 lines contracted by $S \to \mathbb{P}^2$.

If p lies on 0 (resp. 1, 2, 3) lines, we get 72 (resp. $72-16$, $72-2\cdot 16$, $72-3\cdot 16$) such families.

Hence we obtain that for every $p \in S(\mathbb{F}_q)$, the space $\mathrm{Hom}_d(\mathbb{P}^1, X, 0 \mapsto p)$ has a geometrically irreducible component for some $d \le 3 \cdot 72 = 216$.

As in (7.3), we conclude that if q is large enough, then through every point $p \in S(\mathbb{F}_q)$, there is rational curve of degree at most 216, defined over $\mathbb{F}_q$.

Example 7.6 (Cubic hypersurfaces)**.** Consider a smooth cubic hypersurface $X^n \subset \mathbb{P}^{n+1}$, defined over $K = \mathbb{F}_q$ and let $p \in X(\mathbb{F}_q)$ be a point.

If p lies on a smooth cubic surface section $S \subset X$, then we can assemble a K-comb of degree ≤ 216 and, as before, we can use it to get rational curves through p.

Over a finite field, however, there is no guarantee that X has any smooth cubic surface sections. What can we do then?

We can use a generic cubic surface section through p. This is then defined over a field extension $L = K(y_1, \dots, y_s)$ where the y_i are algebraically independent over K. By the previous considerations we can assemble an L-comb and conclude that $\mathrm{Hom}_d(\mathbb{P}^1, X, 0 \mapsto p)$ has a smooth L-point for some $d \le 3 \cdot 72 = 216$.

By (7.4), this implies that it also has a geometrically irreducible component, and we can then finish as before.

It is now clear that the methods of this section together with (4.6) imply the following:

Theorem 7.7. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth SRC hypersurface of degree $m \le n+1$ defined over a finite field $\mathbb{F}_q$. Then there is a $C(n)$ such that if $q > C(n)$ then through every point in $X(\mathbb{F}_q)$ there is a rational curve defined over $\mathbb{F}_q$.* □

Exercise 7.8. Prove the following consequence of (7.1):

Let $f : U \to W$ be a dominant morphism over $\mathbb{F}_q$. Assume that W and the generic fiber of f are both geometrically irreducible. Then there is a dense open set W^0 such that $f\big(U(\mathbb{F}_{q^m})\big) \supset W^0(\mathbb{F}_{q^m})$ for $m \gg 1$.

8. Rational curves through two points and Lefschetz-type theorems

8.1 (How not to find rational curves through two points)**.** Let us see what happens if we try to follow the method of (6.7) for 2 points. Assume that over $\overline{K}$ we have a rational curve C_1 through p, p'. Then C_1 is already defined over a finite Galois extension K' of K. As before, consider its conjugates of $C_2, \dots, C_m$ under $G := \mathrm{Gal}(K'/K)$, and attach copies $C'_1, \dots, C'_m$ to *two* copies of $\mathbb{P}^1$, one over p and one over p'. This results in a curve Y_0 which is defined over K and may be deformed to a smooth curve Y_ε, still passing through p, p'.

The problem is that although all the $\overline{K}$-irreducible components of Y_0 are rational, it has arithmetic genus $m-1$, hence the smooth curve Y_ε has genus $m-1$.

Note that finding curves of higher genus through p, p' is not very interesting. Such a curve can easily be obtained by taking the intersection of X with hyperplanes through p, p'.

In fact, no other choice of Y_0 would work, as shown by the next exercise.

Exercise 8.2. Let C be a reduced, proper, connected curve of arithmetic genus 0 defined over K. Let $p \neq p' \in C(K)$ be 2 points. Then there is a closed sub-curve $p, p' \in C' \subset C$ such that C' is connected and every K-irreducible component of C' is isomorphic to $\mathbb{P}^1_K$.

In this section we first connect the existence of rational curves through two points with Lefschetz-type results about the fundamental groups of open subsets of X and then use this connection to find such rational curves in certain cases.

Definition 8.3. Let K be a field, X a normal, projective variety and

$$\begin{array}{ccc} C_U & \stackrel{\phi}{\to} & X \\ \pi \downarrow\uparrow s & & \\ U & & \end{array} \tag{8.3.1}$$

a smooth family of reduced, proper, connected curves mapping to X with a section s. For $x \in X$, set $U_{s\to x} := s^{-1}\phi^{-1}(x)$, parametrizing those maps that send the marked point to x, and

$$\begin{array}{ccc} C_{U_{s\to x}} & \stackrel{\phi_x}{\to} & X \\ \pi_x \downarrow\uparrow s_x & & \\ U_{s\to x} & & \end{array} \tag{8.3.2}$$

the corresponding family.

We say that the family (8.3.1) satisfies the *Lefschetz condition* if, for general $x \in X(\bar{K})$, the map ϕ_x is dominant with geometrically irreducible generic fiber.

Sometimes it is more convenient to give just

$$U \stackrel{\pi}{\leftarrow} C_U \stackrel{\phi}{\to} S, \tag{8.3.3}$$

without specifying the section $s : U \to C_U$. In this case, we consider the family obtained from the universal section. That is,

$$\begin{array}{ccc} C_U \times_U C_U & \stackrel{\phi_1}{\to} & X \\ \pi_2 \downarrow\uparrow s_1 & & \\ C_U & & \end{array} \tag{8.3.4}$$

where $\pi_2(c, c') = c', \phi_1(c, c') = \phi(c)$ and $s_1(c) = (c, c)$.

If $x \in X$ then $U_x = \phi^{-1}(x) = s_1^{-1}\phi_1^{-1}(x)$ is the set of triples $(C, c, \phi|_C)$ where C is a fiber of π and c a point of C such that $\phi(c) = x$.

Similarly, if $x, x' \in X$ then $U_{x,x'} := \phi^{-1}(x) \times_U \phi^{-1}(x')$ is the set of all $(C, c, c', \phi|_C)$ where C is a fiber of π and c, c' points of C such that $\phi(c) = x$ and $\phi(c') = x'$. Informally (and somewhat imprecisely) $U_{x,x'}$ is the family of curves in U that pass through both x and x'.

Thus the family (8.3.4) satisfies the Lefschetz condition iff $(C_U)_{x,x'}$ is geometrically irreducible for general $x, x' \in X(\bar{K})$.

Exercise 8.4 (Stein factorization)**.** Let $g : U \to V$ be a morphism between irreducible and normal varieties. Then g can be factored as

$$g : U \stackrel{c}{\to} W \stackrel{h}{\to} V$$

where W is normal, h is finite and generically étale and there is an open and dense subset W^0 such that $c^{-1}(w)$ is geometrically irreducible for every $w \in W^0$.

Thus $g : U \to V$ is dominant with geometrically irreducible generic fiber iff g can not be factored through a nontrivial finite and generically étale map $W \to V$.

The Appendix explains how the Lefschetz condition connects with the Lefschetz theorems on fundamental groups of hyperplane sections. For now let us prove that a family satisfying the Lefschetz condition leads to rational curves through 2 points.

Example 8.5. Let $S \subset \mathbb{P}^3$ be a smooth cubic surface. Let $U \leftarrow C_U \to S$ be the family of rational hyperplane sections. Note that $C_U \to S$ is dominant with geometrically irreducible generic fiber. Furthermore, for general $p \in S(\bar{K})$, the map ϕ_p is dominant, generically finite and has degree 12.

On the other hand, let U be an irreducible family of twisted cubics on S. Then U satisfies the Lefschetz condition. As discussed in (5.4), U corresponds to the family of lines in $\mathbb{P}^2$ not passing through the 6 blown-up points. Thus U_x consists of lines in $\mathbb{P}^2$ through x, hence $\phi_x : C_{U_x} \to S$ is birational. Thus it cannot factor through a nontrivial finite cover.

Theorem 8.6. *Let X be a smooth projective variety over $\mathbb{F}_q$. Let $U \subset \operatorname{Hom}^{\text{free}}(\mathbb{P}^1, X)$ be a geometrically irreducible smooth subset, closed under $\operatorname{Aut}(\mathbb{P}^1)$. Assume that $U \stackrel{\pi}{\leftarrow} U \times \mathbb{P}^1 \stackrel{\phi}{\to} X$ satisfies the Lefschetz condition.*

Then there is an open subset $Y^0 \subset X \times X$ such that for $m \gg 1$ and $(x, x') \in Y^0(\mathbb{F}_{q^m})$ there is a point $u \in U(\mathbb{F}_{q^m})$ giving a rational curve

$$\phi_u : \mathbb{P}^1 \to X \quad \textit{such that} \quad \phi_u(0) = x, \ \phi_u(\infty) = x'.$$

Proof. Set $s(u) = (u, 0)$ and consider the map

$$\Phi_2 := (\phi \circ s \circ \pi, \phi) : U \times \mathbb{P}^1 \to X \times X.$$

Note that on $U_{s \to x} \times \mathbb{P}^1$ this is just ϕ_x followed by the injection $X \cong \{x\} \times X \hookrightarrow X \times X$.

If the generic fiber of Φ_2 is geometrically irreducible, then by (8.4) and (7.8), there is an open subset $Y^0 \subset X \times X$ such that for $m \gg 1$ and for every $(x, x') \in$

$Y^0(\mathbb{F}_{q^m})$ there is a $(u,p) \in U(\mathbb{F}_{q^m}) \times \mathbb{P}^1(\mathbb{F}_{q^m})$ such that $\Phi_2(u,p) = (x, x')$. This means that $\phi_u(0) = x$ and $\phi_u(p) = x'$. A suitable automorphism γ of $\mathbb{P}^1$ sends $(0,\infty)$ to $(0,p)$. Thus $\phi_u \circ \gamma$ is the required rational curve.

If the generic fiber of Φ_2 is geometrically reducible, then Φ_2 factors through a nontrivial finite cover $W \to X \times X$. For general $x \in X$, the restriction $\operatorname{red} W_x \to \{x\} \times X$ is nontrivial and $U_{s \to x} \to \operatorname{red} W_x$ is dominant. This is impossible by the Lefschetz condition. □

Next we discuss how to construct families that satisfy the Lefschetz condition.

Lemma 8.7. *Let $U \stackrel{\pi}{\leftarrow} C_U \stackrel{\phi}{\to} X$ be a smooth family of reduced, proper, generically irreducible curves over $\bar{K}$ such that U_x is irreducible for general $x \in X$. Let $W \subset U$ be a locally closed smooth subset and $W \times \mathbb{P}^1 \cong D_W \subset C_W$ a subfamily. Let $U^0 \subset U$ be an open dense subset. If*

$$W \stackrel{\pi}{\leftarrow} D_W \stackrel{\phi}{\to} X$$

satisfies the Lefschetz condition, then so does

$$U^0 \stackrel{\pi}{\leftarrow} C_{U^0} \stackrel{\phi}{\to} X.$$

Proof. Assume that contrary. Then there is a nontrivial finite and generically étale map $Z \to X$ such that the restriction $\phi|_{C_{U_x^0}} : C_{U_x^0} \to X$ factors through Z. Since U_x is irreducible, so is Z.

Let $g_C : C_U \times_X Z \to C_U$ be the projection. By assumption, there is a rational section $s : C_{U_x^0} \to C_U \times_X Z$. Let $B \subset C_U \times_X Z$ be the closure of its image. Then $g_C|_B : B \to C_U$ is finite and an isomorphism over $C_{U_x^0}$. Thus $g_C|_B : B \to C_U$ is an isomorphism at every point where C_U is smooth (or normal). In particular, s restricts to a rational section $s_W : D_W \dashrightarrow C_U \times_X Z$.

Repeating the previous argument, we see that s_W is an everywhere defined section, hence $\phi|_{D_W}$ factors through Z, a contradiction. □

Corollary 8.8. *Let X be a smooth projective variety over a perfect field K. If there is a $\bar{K}$-family of free curves*

$$U_1 \stackrel{\pi_1}{\leftarrow} U_1 \times \mathbb{P}^1 \stackrel{\phi_1}{\to} X$$

satisfying the Lefschetz condition then there is a K-family of free curves

$$U \stackrel{\pi}{\leftarrow} U \times \mathbb{P}^1 \stackrel{\phi}{\to} X$$

satisfying the Lefschetz condition.

Proof. As usual, the first family is defined over a finite Galois extension; let $U_1, \dots, U_m$ be its conjugates.

Consider the family of all $\bar{K}$-combs

$$\text{Comb}(U) := \{\text{Comb}(\phi_{1,u_1}, \ldots, \phi_{m,u_m})\}$$

where $u_i \in U_i$ and $\phi_1(u_1, 0) = \cdots = \phi_m(u_m, 0)$ with 0 a marked point on the handle. (We do not assume that the u_i are conjugates of each other.) Each comb is defined by choosing $u_1, \ldots, u_m$ as above and m distinct points in $\mathbb{P}^1 \setminus \{0\}$.

Thus $\text{Comb}(U) \subset \overline{M}_{0,1}(X)$ is defined over K. Furthermore, for each $x \in X$, $\text{Comb}(U)_x \subset \overline{M}_{0,1}(X, 0 \mapsto x)$ is isomorphic to an open subset of

$$(\mathbb{P}^1)^m \times U_{1,x} \times \cdots \times U_{m,x},$$

hence irreducible.

By (6.4), there is a unique irreducible component $\text{Smoothing}(U) \subset \overline{M}_{0,1}(X)$ containing $\text{Comb}(U)$ and $\text{Smoothing}(U)$ is defined over K.

We can now apply (8.7) with $W := \text{Comb}(U)$ and $D_W \to W$ the first tooth of the corresponding comb. This shows that $\text{Smoothing}(U)$ satisfies the Lefschetz condition. □

Example 8.9 (Cubic hypersurfaces)**.** We have already seen in (8.5) how to get a family of rational curves on a smooth cubic surface S that satisfies the Lefschetz condition:

For general $p \in S$, there are 72 one-parameter families of twisted cubics $C_1, \ldots, C_{72}$ through p. Assemble these into a 1-pointed comb and smooth them to get a family $U(S)$ of degree 216 rational curves. (In fact, the family of degree 216 rational curves on S that are linearly equivalent to $\mathcal{O}_S(72)$ is irreducible, and so equals $U(S)$, but we do not need this.)

Let us go now to a higher dimensional cubic $X \subset \mathbb{P}^{n+1}$. Let G denote the Grassmannian of 3-dimensional linear subspaces in $\mathbb{P}^{n+1}$. Over G we have $\mathbf{S} \to G$, the universal family of cubic surface sections of X. For any fiber $S = L^3 \cap X$ we can take $U(S)$. These form a family of rational curves $\mathbf{U}(\mathbf{S})$ on X and we obtain

$$\mathbf{U}(\mathbf{S}) \overset{\pi}{\leftarrow} \mathbf{U}(\mathbf{S}) \times \mathbb{P}^1 \overset{\phi}{\to} X.$$

We claim that it satisfies the Lefschetz condition. Indeed, given $x, x' \in X$, the family of curves in $\mathbf{U}(\mathbf{S})$ that pass through x, x' equals

$$\mathbf{U}(\mathbf{S})_{x,x'} = \bigcup_{x,x' \in L^3} \big(U(L^3 \cap X)\big)_{x,x'}.$$

The set of all such L^3-s is parametrized by the Grassmannian of lines in $\mathbb{P}^{n-1}$, hence geometrically irreducible. The general $L^3 \cap X$ is a smooth cubic surface, hence we already know that the corresponding $U(L^3 \cap X)_{x,x'}$ is irreducible. Thus $\mathbf{U}(\mathbf{S})_{x,x'}$ is irreducible.

Although we did not use it for cubics, let us note the following.

Theorem 8.10. *[Kol00,Kol03] Let X be a smooth, projective SRC variety over a field K. Then there is a family of rational curves defined over K*

$$U \overset{\pi}{\leftarrow} U \times \mathbb{P}^1 \overset{\phi}{\to} X$$

that satisfies the Lefschetz condition.

8.11 (Going from 2 points to many points)**.** It turns out that going from curves passing through 2 general points to curves passing through m arbitrary points does not require new ideas.

Let us see first how to find a curve through 2 arbitrary points $x, x' \in X$.

We have seen in Section 5 how to produce very free curves in $\mathrm{Hom}(\mathbb{P}^1, X, 0 \mapsto x)$ and in $\mathrm{Hom}(\mathbb{P}^1, X, 0 \mapsto x')$. If $m \gg 1$ then we can find $\psi \in \mathrm{Hom}(\mathbb{P}^1, X, 0 \mapsto x)$ and $\psi' \in \mathrm{Hom}(\mathbb{P}^1, X, 0 \mapsto x')$ such that (8.6) produces a rational curve $\phi : \mathbb{P}^1 \to X$ passing through $\psi(\infty)$ and $\psi'(\infty)$.

We can view this as a length 3 chain

$$(\psi, \phi, \psi') : \mathbb{P}^1 \vee \mathbb{P}^1 \vee \mathbb{P}^1 \to X$$

through x, x'. Using (6.4), we get a family of free rational curves through x, x' and, again for $m \gg 1$ a single free curve through x, x'.

How to go from 2 points to m points $x_1, \dots, x_m$? For each $i > 1$ we already have very free curves a $g_i : \mathbb{P}^1 \to X$ such that $g_i(0) = x_1$ and $g_i(\infty) = x_i$. We can assemble a comb with $(m-1)$ teeth $\phi : \mathrm{Comb}(g_2, \dots, g_m) \to X$.

By (6.4), we can smooth it in

$$\overline{M}_{0,m}(X, p_1 \mapsto x_1, \cdots, p_m \mapsto x_m)$$

to get such rational curves.

Appendix. The Lefschetz condition and fundamental groups

The classical Lefschetz theorem says that if X is a smooth, projective variety over $\mathbb{C}$ and $j : C \hookrightarrow X$ is a smooth curve obtained by intersecting X with hypersurfaces, then the natural map

$$j_* : \pi_1(C) \to \pi_1(X) \quad \text{is onto.}$$

Later this was extended to X quasi-projective. Here j_* need not be onto for every curve section C, but j_* is onto for general curve sections. In particular we get the following. (See [GM88] for a general discussion and further results.)

Theorem 8.12. *Let X^n be a smooth, projective variety over $\mathbb{C}$ and $|H|$ a very ample linear system. Then, for every open subset $X^0 \subset X$ and general $H_1, \dots, H_{n-1} \in |H|$,*

$$\pi_1\big(X^0 \cap H_1 \cap \cdots \cap H_{n-1}\big) \to \pi_1(X) \quad \text{is onto.}$$

It should be stressed that the notion of "general" depends on X^0.

If X is a hypersurface of degree ≥ 3 then the genus of the curves $H_1 \cap \cdots \cap H_{n-1}$ is at least 1. We would like to get a similar result where $\{H_1 \cap \cdots \cap H_{n-1}\}$ is replaced by some family of rational curves.

The following argument shows that if a family of curves satisfies the Lefschetz condition, then (8.12) also holds for that family.

Pick a family of curves $U \stackrel{\pi}{\leftarrow} C_U \stackrel{\phi}{\rightarrow} X$ with a section $s : U \rightarrow C_U$ that satisfies the Lefschetz condition.

Given a generically étale $g : Z \rightarrow X$, there is an open $X^0 \subset X$ such that $Z^0 := g^{-1}(X^0) \rightarrow X^0$ is finite and étale.

Pick a general point $p \in X^0$. There is an open subset $U_p^0 \subset U_p$ such that $\phi_p^{-1}(X^0) \rightarrow U_p$ is topologically a locally trivial fiber bundle over $C_{U_p}^0 \rightarrow U_p^0$ with typical fiber $C_u^0 = C_u \cap \phi^{-1}(X^0)$ where $u \in U$ is a general point.

Thus there is a right split exact sequence

$$\pi_1(C_u^0) \rightarrow \pi_1(C_{U_p}^0) \leftrightarrows \pi_1(U_p^0) \rightarrow 1,$$

where the splitting is given by the section s. Since $s(U_p)$ gets mapped to the point p by ϕ, $\pi_1(C_{U_p}^0)$ gets killed in $\pi_1(X^0)$. Hence

$$\operatorname{im}[\pi_1(C_u^0) \rightarrow \pi_1(X^0)] = \operatorname{im}[\pi_1(C_{U_p}^0) \rightarrow \pi_1(X^0)].$$

Since $C_{U_p} \rightarrow X$ is dominant, $\operatorname{im}[\pi_1(C_{U_p}^0) \rightarrow \pi_1(X^0)]$ has finite index in $\pi_1(X^0)$. We are done if the image is $\pi_1(X^0)$. Otherwise the image corresponds to a non-trivial covering $Z^0 \rightarrow X^0$ and ϕ_p factors through Z^0. This, however, contradicts the Lefschetz condition. □

A more detailed consideration of the above argument shows that (8.12) is equivalent to the following weaker Lefschetz-type conditions:

1. The generic fiber of $C_U \rightarrow X$ is geometrically irreducible, and
2. for general $x \in X$, U_x is geometrically irreducible and $C_{U_x} \rightarrow X$ is dominant.

In positive characteristic the above argument has a problem with the claim that something is "topologically a locally trivial fiber bundle" and indeed the two versions are not quite equivalent. In any case, the purely algebraic version of (8.3) works better for us.

9. Descending from $\mathbb{F}_{q^2}$ to $\mathbb{F}_q$

Our methods so far constructed rational curves on hypersurfaces over $\mathbb{F}_q$ for $q \gg 1$. Even for cubics, the resulting bounds on q are huge. The aim of this section is to use the third intersection point map to construct rational curves on cubic hypersurfaces over $\mathbb{F}_q$ from rational curves on cubic hypersurfaces over $\mathbb{F}_{q^2}$. The end result is a proof of (1.1). The price we pay is that the degrees of the rational curves become larger as q gets smaller.

9.1 (Descent method)**.** Let X be a cubic hypersurface, C a smooth curve and $\phi : C(\mathbb{F}_q) \to X(\mathbb{F}_q)$ a map of sets.

Assume that for each $p \in C(\mathbb{F}_q)$ there is a line L_p through $\phi(p)$ which intersects X in two further points $s(p), s'(p)$. These points are in $\mathbb{F}_{q^2}$ and we assume that none of them is in $\mathbb{F}_q$, hence $s(p), s'(p)$ are conjugate over $\mathbb{F}_q$. This gives a lifting of ϕ to $\phi_2 : C(\mathbb{F}_q) \to X(\mathbb{F}_{q^2})$ where $\phi_2(p) = s(p)$. (This involves a choice for each p but this does not matter.)

Assume that over $\mathbb{F}_{q^2}$ there is an extension of ϕ_2 to $\Phi_2 : C \to X$. If $\bar{\Phi}_2$ denotes the conjugate map, then $\bar{\Phi}_2(p) = s'(p)$.

Applying the third intersection point map (3.5) to the Weil restriction (3.8) we get an $\mathbb{F}_q$-map

$$h : \mathfrak{R}_{\mathbb{F}_{q^2}/\mathbb{F}_q} C \to X.$$

Since C is defined over $\mathbb{F}_q$, the Weil restriction has a diagonal

$$j : C \hookrightarrow \mathfrak{R}_{\mathbb{F}_{q^2}/\mathbb{F}_q} C$$

and $\Phi := h \circ j : C \to X$ is the required lifting of ϕ.

Thus, in order to prove (1.1), we need to show that

1. (1.1) holds for $q \gg 1$, and
2. for every $x \in X(\mathbb{F}_q)$ there is a line L as required.

Remark 9.2. In trying to use the above method over an arbitrary field K, a significant problem is that for each point p we get a degree 2 field extension $K(s(p))/K$ but we can use these only if they are all the same. A finite field has a unique extension of any given degree, hence the extensions $K(s(p))/K$ are automatically the same.

There are a few other fields with a unique degree 2 extension, for instance $\mathbb{R}, \overline{\mathbb{Q}}((t))$ or $\overline{\mathbb{F}}_p((t))$ for $p \neq 2$.

If we have only 1 point p, then the method works over any field K. This is another illustration that the 1 point case is much easier.

In the finite field case, the method can also deal with odd degree points of C but not with even degree points.

9.3 (Proof of (9.1.1))**.** We could just refer to (8.11) or to [KS03, Thm.2], but I rather explain how to prove the 2 point case using (8.6) and the above descent method.

Fix $c, c' \in C(\mathbb{F}_q)$. By (8.6), there is an open subset $Y^0 \subset X \times X$ such that the following holds

(∗) If $\mathbb{F}_{q^m} \supset \mathbb{F}_q$ is large enough then for every $(x, x') \in Y^0(\mathbb{F}_{q^m})$ there is an $\mathbb{F}_{q^m}$-map $\Psi : C \to X$ such that $\Psi(c) = x$ and $\Psi(c') = x'$.

Assume now that we have any $x, x' \in X(\mathbb{F}_{q^m})$. If we can choose the lines L through x and L' through x' such that $(s(x), s(x')) \in Y^0$, then the descent method produces the required extension $\Psi : C \to X$ over $\mathbb{F}_{q^m}$.

By the Lang-Weil estimates, $Y^0(\mathbb{F}_{q^m})$ has about q^{2nm} points. If, for a line L through x, one of the other two points of $X \cap L$ is in $\mathbb{F}_{q^m}$ then so is the other point.

Thus we have about $\frac{1}{2}q^{nm}$ lines where $s(x), s'(x)$ are in $X(\mathbb{F}_{q^m})$. Accounting for the lines tangent to X gives a contribution $O(q^{(n-1)m})$. Thus about $\frac{1}{4}$ of all line pairs (L, L') work for us.

The proof of (9.1.2) is an elaboration of the above line and point counting argument.

Lemma 9.4. *Let $X \subset \mathbb{P}^{n+1}$ be a normal cubic hypersurface and $p \in X(\mathbb{F}_q)$ a smooth point. Assume that $n \geq 1$ and $q \geq 8$. Then*

1. *either there is a line defined over $\mathbb{F}_q$ through p but not contained in X that intersects X in two further smooth points $s, s' \in X(\mathbb{F}_{q^2}) \setminus X(\mathbb{F}_q)$,*
2. *or projecting X from p gives an inseparable degree 2 map $X \dashrightarrow \mathbb{P}^n$. In this case $q = 2^m$ and X is singular.*

Proof. Start with the case $n = 1$. Thus $C := X$ is plane cubic which we allow to be reducible.

Consider first the case when $C = L \cup Q$ a line through p and a smooth conic Q. There are $q+1$ $\mathbb{F}_q$-lines through p, one is L and at most 2 of them are tangent to Q, unless projecting Q from p is purely inseparable. If all the remaining $q-2$ lines intersect Q in two $\mathbb{F}_q$-points, then Q has $2 + 2(q-2) = 2q - 2$ points in $\mathbb{F}_q$. This is impossible for $q > 3$. In all other reducible cases, C contains a line not passing through p. (Since C is smooth at p, C can not consist of 3 lines passing through p.)

Assume next that C is irreducible and smooth. If projection from p is separable, then at most 4 lines through p are tangent to C away from p and one is tangent at p. If all the remaining $q-4$ lines intersect C in two $\mathbb{F}_q$-points, then C has $5 + 2(q-4) = 2q - 3$ points in $\mathbb{F}_q$. For $q \geq 8$ this contradicts the Hasse-Weil estimate $\#C(\mathbb{F}_q) \leq q + 1 + 2\sqrt{q}$. The singular case works out even better.

Now to the general case. Assume that in affine coordinates p is the origin and write the equation as

$$L(x_1, \ldots, x_{n+1}) + Q(x_1, \ldots, x_{n+1}) + C(x_1, \ldots, x_{n+1}) = 0.$$

Let us show first that there is a line defined over $\mathbb{F}_q$ through p but not contained in X that intersects X in two further smooth points s, s'.

If the characteristic is 2, then projection from p is inseparable iff $Q \equiv 0$. If Q is not identically zero, then for $q \geq 3$ there are $a_1, \ldots, a_{n+1} \in \mathbb{F}_q$ such that $(L \cdot Q)(a_1, \ldots, a_{n+1}) \neq 0$. The corresponding line intersects X in 2 further distinct points, both necessarily smooth.

If the characteristic is $\neq 2$, then the line corresponding to $a_1, \ldots, a_{n+1} \in \mathbb{F}_q$ has a double intersection iff the discriminant $Q^2 - 4LC$ vanishes. Note that $Q^2 - 4LC$ vanishes identically only if X is reducible. Thus, for $q \geq 5$ there are $a_1, \ldots, a_{n+1} \in \mathbb{F}_q$ such that $(L \cdot (Q^2 - 4LC))(a_1, \ldots, a_{n+1}) \neq 0$. As before, the corresponding line intersects X in 2 further distinct points, both necessarily smooth.

It is possible that for this line $s, s' \in X(\mathbb{F}_{q^2}) \setminus X(\mathbb{F}_q)$ and we are done. If not then $s, s' \in X(\mathbb{F}_q)$. We can choose the line to be $(x_1 = \cdots = x_n = 0)$ and write

$s = (0, \ldots, 0, s_{n+1})$ and $s' = (0, \ldots, 0, s'_{n+1})$. Our aim now is to intersect X with the planes

$$P(a_1, \ldots, a_n) := \langle (0, \ldots, 0, 1), (a_1, \ldots, a_n, 0) \rangle$$

for various $a_1, \ldots, a_n \in \mathbb{F}_q$ and show that for one of them the intersection does not contain a line not passing through p. Then the curve case discussed above finishes the proof.

Set $x'_{n+1} = x_{n+1} - s_{n+1}$. At s the equation of X is

$$L_s(x_1, \ldots, x'_{n+1}) + Q_s(x_1, \ldots, x'_{n+1}) + C_s(x_1, \ldots, x'_{n+1}) = 0.$$

Since X is irreducible, L_s does not divide either Q_s or C_s. L_s contains x'_{n+1} with nonzero coefficient since the vertical line has intersection number 1 with X. We can use L_s to eliminate x'_{n+1} from Q_s and C_s. As we saw, one of these is nonzero, let it be $B_s(x_1, \ldots, x_n)$. Similarly, at s' we get $B'_s(x_1, \ldots, x_n)$.

If $X \cap P(a_1, \ldots, a_n)$ contains a line through s (resp. s') then $B_s(a_1, \ldots, a_n) = 0$ (resp. $B'_s(a_1, \ldots, a_n) = 0$). Thus we have the required $(a_1, \ldots, a_n)$, unless $B_s \cdot B'_s$ is identically zero on $\mathbb{P}^{n-1}(\mathbb{F}_q)$. This happens only for $q \leq 5$. □

Exercise 9.5. Let $H(x_1, \ldots, x_n)$ be a homogeneous polynomial of degree d. If H vanishes on $\mathbb{F}_q^n$ and $q \geq d$ then H is identically zero.

Exercise 9.6. Set $F(x_0, \ldots, x_m) = \sum_{i \neq j} x_i^{2^n} x_j$. Show that F vanishes on $\mathbb{P}^m(\mathbb{F}_{2^n})$ and for m odd it defines a smooth hypersurface.

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