Beilinson Conjectures in the non-commutative setting

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Abstract. We discuss a p-adic version of Beilinson's conjecture and its relationship with noncommutative geometry.

Introduction

Hodge theory is one of the most important computation tools in modern algebraic geometry, and for many reasons; in these lectures, we will be concerned with only one facet of the story – the properties of the so-called *regulator map*. This actually has a long history which predates both Hodge theory and algebraic geometry and includes, for instance, the well-known Dirichlet Unit Theorem. However, from the modern viewpoint – and we will adopt the modern viewpoint – the regulator map is a gadget which compares *algebraic K-theory* and certain *cohomology groups* of algebraic varieties constructed by means of the Hodge theory. Algebraic K-groups of a variety contain a lot of valuable information, but are notoriously hard to compute; cohomology, on the other hand, is easily computable in most cases. Thus it would be very important to be able to express one through the other. This is what the regulator map does.

Of course, one needs to know that the comparison is exact, so that no information is lost in the process. This is essentially the content of the first of the famous Beilinson conjectures made about twenty years ago ([B], [RSS]).

At the time of writing, there is still no significant progress in *proving* the conjectures. However, we now *understand* them somewhat better. In particular, while Beilinson was working with algebraic varieties defined over $\mathbb{Q}$ and their cohomology with real coefficients, we now have a p-adic version of the story. The goal of these lectures is to give a very brief introduction to a still more recent discovery – it turns out that the p-adic version of the first of Beilinson conjectures can be transfered to the setting of *non-commutative varieties*. We still cannot prove anything; however, since the p-adic conjecture can be now formulated in much larger generality, it becomes more accessible, and a lot of structure used in the original version can be removed as redundant. Hopefully, this will allow someone to concentrate on the essential heart of the problem, and maybe finally solve it.

The paper follows very closely two lectures I gave at a summer school in Goettingen in June 2007. The exposition is very threadbare – we only indicate proofs, with details given elsewhere, and we try to concentrate on the ideas by cutting a lot of technical corners. We follow the most direct path we could find from the definitions, to Beilinson conjectures, to the p-adic analog, to the non-commutative p-adic version. For better or for worse, we choose brevity over completeness at every turn.

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1. Regulator maps and Beilinson conjectures

Let X be a smooth projective algebraic variety over $\mathbb{C}$. Let

$$\mathrm{ch} : K^0(X) \longrightarrow \bigoplus_i H^{2i}(X_{an}, \mathbb{Q})$$

be the Chern character map from the algebraic K-group $K^0(X)$ to the cohomology of the underlying analytic space of X with rational coefficients. For most varieties, the map is of course not even close to being surjective. What can be said about its image? One constraint is well-known: if we denote by

$$H^i(X_{an}, \mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}(X)$$

the Hodge decomposition, then every class $[\mathcal{E}] \in K^0(X)$ satisfies

$$\mathrm{ch}([\mathcal{E}]) \in \bigoplus_i H^{i,i}(X).$$

Definition 1.1. *A pure $\mathbb{Q}$-Hodge structure of weight n consists of*

(i) *a $\mathbb{Q}$-vector space $V_{\mathbb{Q}}$, and*
(ii) *a decreasing filtration $F^i V_{\mathbb{C}}$ on its complexification $V_{\mathbb{C}} := V_{\mathbb{Q}} \otimes \mathbb{C}$,*

such that the following two conditions are satisfied:

(i) *$F^i V_{\mathbb{C}} \cap \overline{F^j V_{\mathbb{C}}} = 0$ whenever $i + j > n$, and*
(ii) *$V_{\mathbb{C}} = \bigoplus_i F^i V_{\mathbb{C}} \cap \overline{F^{n-i} V_{\mathbb{C}}}$.*

Here the overline in $\overline{V}$ denotes complex conjugation on $V_{\mathbb{C}}$, i. e. the tensor product $V_{\mathbb{C}} \to V_{\mathbb{C}}$ of the identity on $V_{\mathbb{Q}}$ and the complex conjugation on $\mathbb{C}$.

Example 1.2. The vector space $H^n(X, \mathbb{Q})$ carries a natural pure $\mathbb{Q}$-Hodge structure of weight n, given by the Hodge decomposition:

$$F^i H^n(X, \mathbb{C}) = \bigoplus_{j \geq i} H^{j,n-j}(X).$$

Assume that V is a pure $\mathbb{Q}$-Hodge structure of weight $2i$. Then the space of (i,i)-classes in $V_\mathbb{Q}$ is the kernel of the map

$$F^i V_\mathbb{C} \oplus V_\mathbb{Q} \longrightarrow V_\mathbb{C}$$

which is the difference of the two inclusions $F^i V_\mathbb{C} \hookrightarrow V_\mathbb{C}$ and $V_\mathbb{Q} \hookrightarrow V_\mathbb{C}$.

Applying this to the vector spaces $V = H^i(X, \mathbb{Q})$, we see that the Chern character maps into the direct sum of these kernels for all $i \geq 0$.

Note that we can turn a pure $\mathbb{Q}$-Hodge structure of weight n into one of weight $n+2$ by just renumbering the filtration. Thus we can turn the pure $\mathbb{Q}$-Hodge structure $H^{2i}(X, \mathbb{Q})$ into one of any given even weight; we denote by

$$H^{2i}(X, \mathbb{Q}(j))$$

the pure $\mathbb{Q}$-Hodge structure of weight $2i - 2j$ thus obtained. In particular, this produces a pure $\mathbb{Q}$-Hodge structure $H^{2i}(X, \mathbb{Q}(i))$ of weight 0. Altogether, we get a factorisation of the Chern character

$$K^0(X) \longrightarrow \ker\left[F^0 V_\mathbb{C} \oplus V_\mathbb{Q} \longrightarrow V_\mathbb{C}\right] \subseteq V := \bigoplus_i H^{2i}(X, \mathbb{Q}(i)).$$

The famous Hodge conjecture states that this arrow is surjective.

To proceed further, recall that $K^0(X)$ is a part of Quillen's higher K-theory $K^\bullet(X)$, which behaves like a cohomology theory (has Mayer-Vietoris sequences, excision etc.) Can we extend the Chern character ch to $K^\bullet(X)$? Yes – as shown in [B], based on earlier work by other people, there exists a *regulator map*

$$r : K^\bullet(X) \longrightarrow \operatorname{cone}\left(F^0 V^\bullet_\mathbb{C} \oplus V^\bullet_\mathbb{Q} \longrightarrow V^\bullet_\mathbb{C}\right), \qquad V^\bullet := \bigoplus_j H^{2j+\bullet}(X, \mathbb{Q}(j)).$$

where, just as one would expect for a cohomology theory, we have replaced the kernel above by a mapping cone. This cone

$$\operatorname{cone}\left(F^0 V^\bullet_\mathbb{C} \oplus V^\bullet_\mathbb{Q} \longrightarrow V^\bullet_\mathbb{C}\right), \qquad V^\bullet := \bigoplus_j H^{2j+\bullet}(X, \mathbb{Q}(j))$$

has a name: it is called *Deligne cohomology* and denoted by $H^{2j+\bullet}_\mathcal{D}(X, \mathbb{Q}(j))$.

(We note that the Deligne cohomology is usually defined as the hypercohomology $\mathbb{H}^\bullet(X, \mathbb{Q}(j))$ of the complex

$$\mathbb{Q}(j): \quad \mathbb{Q} \longrightarrow \mathcal{O}_X \longrightarrow \Omega^1_X \longrightarrow \Omega^2_X \longrightarrow \ldots \longrightarrow \Omega^j_X \longrightarrow 0.$$

In order to compare this definition with the one given above, one notes that the first term $\mathbb{Q}$ in this complex yields $V^\bullet_\mathbb{Q}$, and that the rest of the complex yields up to quasi-isomorphism the mapping cone of $F^0 V^\bullet_\mathbb{C} \to V^\bullet_\mathbb{C}$.)

Replacing the $\mathbb{Q}$-lattices in the Deligne cohomology by $\mathbb{R}$-vector spaces, we obtain a version

$$r : K^\bullet(X) \otimes \mathbb{R} \longrightarrow \bigoplus_j H_{\mathcal{D}}^{2j+\bullet}(X, \mathbb{R}(j)) \tag{1}$$

of the regulator map above. Roughly speaking, the Beilinson conjecture asserts that this is an isomorphism if X is defined over $\mathbb{Q}$.

More precisely, there is one necessary modification: if a smooth projective variety X is defined over $\mathbb{Q}$, or even over $\mathbb{R} \subset \mathbb{C}$, then it has a real structure — that is, an anti-complex involution $\iota : X \to \overline{X}$. This involution acts on everything in the story above, and in particular, on Deligne cohomology; one twists the action of ι on $H_{\mathcal{D}}^\bullet(X, \mathbb{R}(j))$ by $(-1)^j$ and replaces the right-hand side of (1) with the subspace of ι-invariant vectors.

However, even with this modification, the Beilinson conjecture is false for a stupid reason, since the left hand side of (1) is zero in negative degrees, and the right-hand side is not. To kill off the parasitic cohomology classes, one has to replace Deligne cohomology by the so-called Deligne-Beilinson cohomology.

Definition 1.3. *A* mixed $\mathbb{R}$-Hodge structure *consists of*

(i) *an $\mathbb{R}$-vector space $V_\mathbb{R}$,*
(ii) *an increasing filtration $W_\bullet V_\mathbb{R}$, called the* weight filtration*, and*
(iii) *a decreasing filtration $F^\bullet V_\mathbb{C}$ on $V_\mathbb{C} := V_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$, called the* Hodge filtration,

such that the graded piece $\mathrm{gr}_i^W(V)$ with the induced filtration $F^\bullet$ is a pure Hodge structure of weight i for all i.

Although the category of filtered vector spaces is not abelian, we have:

Fact 1.4. (i) *Mixed $\mathbb{R}$-Hodge structures form an abelian category.*
(ii) *This abelian category has homological dimension* 1*.*

Example 1.5. The mixed $\mathbb{R}$-Hodge structure $\mathbb{R}(j)$ consists of

(i) the vector space $V_\mathbb{R} := \mathbb{R}$,
(ii) the weight filtration $W_{-1}V_\mathbb{R} = 0$, $W_0 V_\mathbb{R} = V_\mathbb{R}$, and
(iii) the Hodge filtration $F^{-j}V_\mathbb{C} = V_\mathbb{C}$, $F^{1-j}V_\mathbb{C} = 0$.

For any mixed $\mathbb{R}$-Hodge structure V, one can check that

$$\mathrm{RHom}^\bullet(\mathbb{R}(0), V)$$

is (quasi-isomorphic to) the mapping cone of

$$(W_0 V_\mathbb{C} \cap F^0 V_\mathbb{C}) \oplus W_0 V_\mathbb{R} \longrightarrow W_0 V_\mathbb{C}.$$

This is the Deligne-Beilinson cohomology – it both gives a conceptual explanation for the Deligne cohomology, and refines it by removing the "parasitic" terms. As shown by Beilinson, the regulator map factors through a map from $K^\bullet(X) \otimes \mathbb{R}$ to the Deligne-Beilinson cohomology; what he actually conjectured was that this refined regulator map is an isomorphism onto the subspace of ι-invariant vectors when X is a smooth projective variety over $\mathbb{Q}$.

To finish the section, here are some additional comments.

(i) The Beilinson conjecture comprises both a Hodge-type conjecture which says that the regulator map is surjective, and a generalization of a conjecture by S. Bloch which says that the map is injective. Hodge-type conjecture has a chance of being true even for varieties defined over $\mathbb{R}$, but the injectivity certainly fails unless X is defined over $\mathbb{Q}$.
(ii) There are further conjectures about the determinant of the regulator map in some appropriate basis and its relation to values of L-functions at integral points, but this lies outside the scope of the present paper (and a reader who does not know or does not wish to know what an L-function is may safely read on).
(iii) In the usual definition of the Hodge structure $\mathbb{R}(i)$, one modifies the embedding $V_{\mathbb{R}} \to V_{\mathbb{C}}$ by multiplying it by $(2\pi\sqrt{-1})^i$. This makes no sense in our definition of Hodge structure; the only place where $\sqrt{-1}^i$ actually appears is in the action of the additional complex conjugation ι (this explains the twist by $(-1)^j$ on $H^\bullet_{\mathcal{D}}(X, \mathbb{R}(j))$). The multiplier 2π is important for the further Beilinson conjectures on special values; for the purposes of the present paper, it can be ignored.

2. A p-adic version

There is a p-adic version of the above theory, due to Fontaine and Laffaille [FL], Fontaine and Messing [FM], M. Gros [G1,G2].

We work over the ring of Witt vectors $W := W(\overline{\mathbb{F}}_p)$, which is the maximal unramified extension of $\mathbb{Z}_p$. Let $\mathrm{Fr}_W : W \to W$ be the unique lift of the Frobenius automorphism on $\overline{\mathbb{F}}_p$. Given a W-module M, we denote its Frobenius twist by

$$M^{(1)} := M \otimes_{W, \mathrm{Fr}_W} W.$$

Thus a W-linear map $M^{(1)} \to M'$ is the same thing as a Fr_W-semilinear map $M \to M'$.

Definition 2.1 ([FL]). *A* filtered Dieudonné module *M consists of*

(i) *a finitely generated module M over the ring $W(\overline{\mathbb{F}}_p)$,*
(ii) *a decreasing filtration $F^\bullet M$ of M, and*
(iii) *W-linear maps $\varphi_i : F^i M^{(1)} \to M$ for all i,*

such that the following two conditions are satisfied:

(i) *$\varphi_i|_{F^{i+1}M} = p\varphi_{i+1}$ for all i, and*
(ii) *the direct sum $\bigoplus_i \varphi_i : \bigoplus_i F^i M^{(1)} \to M$ is surjective.*

Note that for any i and $j \geq i$, φ_j is determined by φ_i if M is torsion-free. However, it is useful also to include modules M with torsion, and then we need all the φ_i.

Fact 2.2 ([FL]). (i) *Filtered Dieudonné modules form an abelian category.*
(ii) *This abelian category has homological dimension* 1.

Example 2.3. The filtered Dieudonné module $\mathbb{Z}_p(0)$ consists of

(i) the free W-module $M := W$,
(ii) the trivial filtration $F^0M = M$, $F^1M = 0$, and
(iii) the Frobenius map $\varphi_0 := \mathrm{Fr}_W : F^0M^{(1)} \to M$.

From now on, let X be a smooth projective variety over W; we assume $p > \dim(X)$ (in order to be able to divide by $i!$ when dealing with exterior i-forms on X). Recall that the de Rham cohomology $H^\bullet_{\mathrm{DR}}(X)$ of X is by definition the hypercohomology of its de Rham complex.

Example 2.4. $M := H^\bullet_{\mathrm{DR}}(X)$, together with the filtration $F^\bullet M$ given by the stupid filtration of the de Rham complex, is a filtered Dieudonné module.

About the proof. The main point is to construct the maps $\varphi_i : F^iM^{(1)} \to M$. Let $X^{(1)}$ be the Frobenius twist of X, i. e. the fibered product

$$\begin{array}{ccc} X^{(1)} & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spec} W(\overline{\mathbb{F}}_p) & \xrightarrow{\mathrm{Fr}_W^*} & \operatorname{Spec} W(\overline{\mathbb{F}}_p). \end{array}$$

We make the simplifying assumption that there is a map

$$\widetilde{\mathrm{Fr}} : X \longrightarrow X^{(1)} \tag{2}$$

which lifts the (absolute) Frobenius map at the special fiber. Then we get a map

$$\widetilde{\mathrm{Fr}}^* : \Omega^\bullet_{X^{(1)}} \longrightarrow \Omega^\bullet_X.$$

The induced map on hypercohomology is our φ_0.

We claim that $\widetilde{\mathrm{Fr}}^* : \Omega^1_{X^{(1)}} \longrightarrow \Omega^1_X$ vanishes modulo p. Indeed, the sheaf Ω^1 is locally generated by exact forms df, and

$$\widetilde{\mathrm{Fr}}^*(df) = d(\widetilde{\mathrm{Fr}}^*(f)) = d(f^p + pf') = pf^{p-1}df + pdf' \equiv 0 \bmod p$$

for some function f'. This shows that $\widetilde{\mathrm{Fr}}$ is divisible by p on Ω^1; by multiplicativity, it is then divisible by p^i on Ω^i. Thus we get a map $1/p^i\widetilde{\mathrm{Fr}}$ on the truncated de Rham complex $\Omega^{\geq i}$; the induced map on hypercohomology is our φ_i.

This constructs the filtered Dieudonné module structure on $H^\bullet_{\mathrm{DR}}(X)$ in the case where a lift $\widetilde{\mathrm{Fr}}$ of the absolute Frobenius at the special fiber exists. In general, there are obstructions against such a lift. There is a way of deducing the general case from the special case treated here. However, we can then no longer guarantee that φ_i maps $F^iM^{(1)}$ to F^iM; all we can say is that it maps $F^iM^{(1)}$ to M. □

For any filtered Dieudonné module M, it is easy to check that

$$\mathrm{RHom}(\mathbb{Z}_p(0), M)$$

is (quasi-isomorphic to) the mapping cone of

$$F^0 M^{(1)} \xrightarrow{\mathrm{Id} - \varphi_0} M.$$

E. g. for $\mathrm{RHom}(\mathbb{Z}_p(0), \mathbb{Z}_p(0))$, we get $W \xrightarrow{\mathrm{Id} - \mathrm{Fr}_W} W$, which is just $\mathbb{Z}_p$ in degree 0.

The analogy with Beilinson's definition of the Deligne cohomology leads to:

Definition 2.5 ([FM])**.** *The* syntomic cohomology *of* X *is*

$$\mathrm{RHom}(\mathbb{Z}_p(0), H^{\bullet}_{\mathrm{DR}}(X)).$$

One immediate problem with this definition is that our Dieudonné modules lack the weight filtration – thus what we get is a version of Deligne cohomology, not of Deligne-Beilinson cohomology, and we cannot expect a version of Beilinson conjectures to hold for the same stupid reason as in $\mathsf{char}\, 0$. At present, it is not known how to cure this. The best we can do is to introduce the following.

Definition 2.6. *Assume that the operations φ^i on the de Rham cohomology groups $H^{\bullet}_{\mathrm{DR}}(X)$ preserve the Hodge filtration, $\phi^i(F^i) \subset F^i$ (for instance, this is the case when X admits a lifting of the Frobenius, as in* (2)*). The* reduced syntomic cohomology *of X is the mapping cone of the natural map*

$$F^0 M^{(1)} \xrightarrow{\mathrm{Id} - \varphi_0} F^0 M,$$

where $M = H^{\bullet}_{\mathrm{DR}}(X)$.

Unfortunately, the assumption needed to define reduced syntomic cohomology is only rarely satisfied; in general, one has to deal with the full syntomic cohomology which contains parasitic classes. Be it as it may, Michel Gros [G1,G2] has constructed a regulator map

$$r : K^{\bullet}(X) \longrightarrow \text{ syntomic cohomology.}$$

He also formulated a precise version of the Beilinson conjectures in this p-adic setting (including those that deal with special values of L-functions).

3. The non-commutative setting

We still work over the ring of Witt vectors $W = W(\overline{\mathbb{F}}_p)$. Let A be a flat W-algebra; all our algebras are associative and unital, but not necessarily commutative. Our goal is to construct

(i) an analogue of the de Rham cohomology for A,
(ii) a filtered Dieudonné module structure on it, and
(iii) a regulator map.

At this point I should add a disclaimer. While my source for the material here is [K1], a large part of it is an independent rediscovery of things discovered by algebraic topogists about 15 years ago — mostly within the theory of the so-called *Topological Cyclic Homology* and *cyclotomic trace* of Bökstedt, Hsiang and Madsen (see [BHM], or a very good exposition in [HM]). However, at the moment I don't completely understand the precise relation to the topological story, and I prefer to completely ignore this in these lectures.

3.1. Non-commutative de Rham cohomology

The main reference for this subsection is Loday's book [L] (which is in particular a reliable source for the many signs involved).

We consider A as a bimodule under left and right multiplication by A, or in other words as a (left) $A \otimes A^{\mathrm{opp}}$-module (where A^{opp} denotes the opposite algebra). This bimodule is in general not flat. E. g. if A is commutative, then this bimodule A corresponds to the structure sheaf of the diagonal as a module over the structure sheaf of $\mathrm{Spec}(A) \times \mathrm{Spec}(A)$.

Definition 3.1. *The* Hochschild homology *of A is*

$$HH_\bullet(A) := \mathrm{Tor}_\bullet^{A \otimes A^{\mathrm{opp}}}(A, A).$$

The diagonal bimodule A has a standard flat resolution $C_\bullet(A) \to A$, namely

$$\ldots \xrightarrow{b'} A \otimes A \otimes A \xrightarrow{b'} A \otimes A \xrightarrow{b'} A, \tag{3}$$

whose differential $b' := \sum_{i=1}(-1)^i m_i$ involves the multiplication $m : A \otimes A \to A$; m_i is the multiplication at the i-th tensor sign.

Using this flat resolution $C_\bullet(A) \to A$ to compute $HH_\bullet(A)$, we get the complex

$$\ldots \xrightarrow{b} A \otimes A \otimes A \xrightarrow{b} A \otimes A \xrightarrow{b} A \tag{4}$$

whose differential $b := b' + (-1)^n m_0$ contains the extra summand $m_0 := m_1 \circ \sigma$, where n is the number of A's, and σ is the cyclic permutation

$$\sigma(a_1 \otimes \ldots \otimes a_{n-1} \otimes a_n) := (-1)^{n-1} a_n \otimes a_1 \otimes \ldots \otimes a_{n-1}.$$

The two complexes (3) and (4) can be put together to a periodic bicomplex

$$
\begin{tikzcd}
{} \arrow[r] & A \arrow[r, "1"] & A \arrow[r, "1-\sigma"] & A \arrow[r] & {} \\
{} \arrow[r] & A \otimes A \arrow[u, "b"] \arrow[r, "1+\sigma"] & A \otimes A \arrow[u, "b'"] \arrow[r, "1-\sigma"] & A \otimes A \arrow[u, "b"'] \arrow[r] & {} \\
{} \arrow[r] & A \otimes A \otimes A \arrow[u, "b"] \arrow[r, "1+\sigma+\sigma^2"] & A \otimes A \otimes A \arrow[u, "b'"] \arrow[r, "1-\sigma"] & A \otimes A \otimes A \arrow[u, "b"'] \arrow[r] & {} \\
& \cdots \arrow[u, "b"] & \cdots \arrow[u, "b'"] & \cdots \arrow[u, "b"'] & \\
{} \arrow[r] & A^{\otimes n} \arrow[u, "b"] \arrow[r, "1+\sigma+\cdots+\sigma^{n-1}"] & A^{\otimes n} \arrow[u, "b'"] \arrow[r, "1-\sigma"] & A^{\otimes n} \arrow[u, "b"'] \arrow[r] & {} \\
& {} \arrow[u, "b"] & {} \arrow[u, "b'"] & {} \arrow[u, "b"'] &
\end{tikzcd}
\tag{5}
$$

which we denote by $\mathrm{Per}_\bullet(A)$.

Definition 3.2. *$HP_\bullet(A)$ is the homology of the total complex of* $\mathrm{Per}_\bullet(A)$.

Remark 3.3. Since the bicomplex $\mathrm{Per}_\bullet(A)$ is unbounded in one direction, there are two ways of forming its total complex: one that involves infinite direct sums of its entries, and one that involves infinite products. We use products. This is important – for instance, were the base field to have characteristic 0, the total complex understood as a sum would have been acyclic.

Example 3.4. Suppose that A is commutative, that $X := \mathrm{Spec}(A)$ is smooth and that $p > \dim(X)$. Then

$$HH_i(A) = \Omega^i(X) \qquad \text{and} \qquad HP_i(A) = \bigoplus_j H^{2j+i}_{\mathrm{DR}}(X).$$

Thus $HP_\bullet(A)$ contains less information then $H^\bullet_{\mathrm{DR}}(X)$ in the commutative case — we can only recover certain direct sums of $H^n_{\mathrm{DR}}(X)$ from the $HP_i(A)$, not all the $H^n_{\mathrm{DR}}(X)$ themselves. However, as the reader will easily notice, it is exactly these direct sums that are relevant for our story.

3.2. The filtered Dieudonné module structure.

The aim of this subsection is to turn $HP_\bullet(A)$ into a filtered Dieudonné module. Here the main problem is to find a non-commutative analogue of the Frobenius.

Even if we reduce everything $\mod p$ and replace A with the $\overline{\mathbb{F}}_p$-algebra A/p, the naive guess does not work: the map $x \mapsto x^p$ is not even additive modulo p. However, we can analyze the difficulty by decomposing it into two maps

$$A \xrightarrow{\varphi} A^{\otimes p} \xrightarrow{m} A, \quad \varphi(a) := a \otimes a \ldots \otimes a, \qquad m(a_1 \otimes \ldots \otimes a_p) := a_1 \cdot \ldots \cdot a_p. \tag{6}$$

The first map is awful (not additive, etc.), but it is equally awful in the commutative case; it is the multiplication map m which stops being an algebra map in the non-commutative case and creates difficulties.

Fortunately — and this is the main idea — while this map m cannot be made into an algebra map, it can be made to act on Hochschild and cyclic homology.

More precisely, we have two morphisms of complexes

$$\begin{array}{ccccccc} \xrightarrow{b'} & A^{\otimes 3} & \xrightarrow{b'} & A^{\otimes 2} & \xrightarrow{b'} & A \\ & m\uparrow & & m\uparrow & & \uparrow m \\ \xrightarrow{b'_p} & (A^{\otimes 3})^{\otimes p} & \xrightarrow{b'_p} & (A^{\otimes 2})^{\otimes p} & \xrightarrow{b'_p} & A^{\otimes p} \end{array} \tag{7}$$

and

$$\begin{array}{ccccccc} \xrightarrow{b} & A^{\otimes 3} & \xrightarrow{b} & A^{\otimes 2} & \xrightarrow{b} & A \\ & m\uparrow & & m\uparrow & & \uparrow m \\ \xrightarrow{b_p} & (A^{\otimes 3})^{\otimes p} & \xrightarrow{b_p} & (A^{\otimes 2})^{\otimes p} & \xrightarrow{b_p} & A^{\otimes p} \end{array} \tag{8}$$

where the differentials $b'_p := \sum_{i=1}(-1)^i m_i^{\otimes p}$ and $b_p := \sum_{i=0}(-1)^i m_i^{\otimes p}$ involve the same multiplications m_i as in (3) and (4) above, but raised to the p-th tensor power, and

$$m : A^{\otimes pn} = \underbrace{A^{\otimes n} \otimes \ldots \otimes A^{\otimes n}}_{p \text{ terms}} \longrightarrow A^{\otimes n}$$

is the identity on the first $n-1$ tensor factors A, and the multiplication on the remaining $pn - n + 1$ factors. (If for example $n = 2$ and $p = 3$, then

$$m : (A \otimes \underbrace{A) \otimes (A \otimes A) \otimes (A \otimes A)}_{\text{multiply}} \longrightarrow A \otimes A \tag{9}$$

sends $(a_{11} \otimes a_{12}) \otimes (a_{21} \otimes a_{22}) \otimes (a_{31} \otimes a_{32})$ to $a_{11} \otimes a_{12}a_{21}a_{22}a_{31}a_{32}$.) We can also form a periodic bicomplex $\mathrm{Per}^p_\bullet(A)$, whose vertical differentials are b^p and $(b')^p$, and whose horizontal differentials are the same as in $\mathrm{Per}_\bullet(A)$ (with σ on $A^{\otimes np}$ being the cyclic permutation of order np).

We define the Hodge filtration $F^\bullet$ on the complexes $\mathrm{Per}_\bullet(A)$ and $\mathrm{Per}^p_\bullet(A)$ by the following rule

(i) For the 0-th filtration piece $F^0 HP_\bullet(A)$, we take everything in the bicomplex $\mathrm{Per}_\bullet(A)$ to the right of the column with b.
(ii) For the i-th filtration piece $F^i HP_\bullet(A)$, we shift this to the right by $2i$ columns.

Fact 3.5. *The morphisms of complexes* (7) *and* (8) *are quasi-isomorphisms, and they extend to a quasiisomorphism of filtered complexes* $m : \mathrm{Per}^p_\bullet(A) \to \mathrm{Per}_\bullet(A)$.

About the proof:. The upper rows both in (7) and (8) come from a simplicial abelian group. The first statement is a very general property of simplicial abelian groups, similar to barycentric subdivision. To obtain the second statement, one has to use the notion of *cyclic object* which extends that of a simplicial object, see e. g. [L, Ch. 6]. We say no more and refer the reader to [K2, Lemma 2.2] and [K1, Lemma 1.14] for actual proofs. □

As for the very bad map φ in (6), it turns out that it can be modified quite a bit without changing anything – in particular, it can sometimes be replaced with an actual algebra map. Namely, consider $A^{\otimes p}$ as a representation of $\mathbb{Z}/p\mathbb{Z}$, the generator $\tau \in \mathbb{Z}/p\mathbb{Z}$ acting by cyclic permutation of the tensor factors. The group cohomology of this representation is computed by the periodic complex

$$\longrightarrow A^{\otimes p} \xrightarrow{1+\tau+\dots+\tau^{p-1}} A^{\otimes p} \xrightarrow{1-\tau} A^{\otimes p} \longrightarrow .$$

The above map $\varphi : A \to A^{\otimes p}$, $a \mapsto a \otimes \ldots \otimes a$, induces an isomorphism

$$\varphi : A^{(1)}/p \longrightarrow H_{\mathrm{odd}}(\mathbb{Z}/p\mathbb{Z}, A^{\otimes p}) \tag{10}$$

where $A^{(1)} := A \otimes_{W,\mathrm{Fr}_W} W$ is again the Frobenius twist.

Definition 3.6. *A* quasi-Frobenius map for *A is a $\mathbb{Z}/p\mathbb{Z}$-equivariant algebra homomorphism*

$$\varphi : A^{(1)} \longrightarrow A^{\otimes p}$$

which induces the standard isomorphism (10).

Example 3.7. Let $A = W[G]$ be the group algebra of some (discrete) group G. Then the map $\varphi : A^{(1)} \to A^{\otimes p}$ induced by the diagonal embedding $G \to \underbrace{G \times \ldots \times G}_{p \text{ times}}$ is a quasi-Frobenius map for $A = W[G]$.

If we are given a quasi-Frobenius map φ for A, then we can construct the filtered Dieudonné module structure on $HP_\bullet(A)$ as follows:

(i) The Hodge filtration $F^\bullet$ is as above.
(ii) The required map $\varphi_0 : F^0 HP_\bullet(A^{(1)}) \to HP_\bullet(A)$ is induced by the following morphism of bicomplexes $\varphi_0 : F^0 \operatorname{Per}_\bullet(A) \longrightarrow \operatorname{Per}^p_\bullet(A)$:
  - On F^0/F^1, φ_0 is given by powers of the quasi-Frobenius map φ.
  - On F^i/F^{i+1}, the same times p^i.

(iii) The required maps $\varphi_i : F^i HP_\bullet(A^{(1)}) \to HP_\bullet(A)$ are again obtained by dividing an appropriate restriction of the morphism of bicomplexes φ_0 by p^i.

It is easy to see that this is well-defined. Indeed, the power $\phi^n : A^{\otimes n(1)} \to A^{\otimes pn}$ of the quasi-Frobenius map ϕ commutes with the horizontal differential $1-\sigma$ in the complexes $\operatorname{Per}_\bullet(A)$, $\operatorname{Per}^p_\bullet(A)$ on the nose; to make it send the differential $1 + \sigma + \cdots + \sigma^{n-1}$ to

$$1 + \sigma + \cdots + \sigma^{np-1} = (1 + \sigma + \cdots + \sigma^{n-1})(1 + \tau + \cdots + \tau^{p-1}),$$

we have to multiply it by $(1+\tau+\cdots+\tau^{p-1})$, where τ is the generator of the $\mathbb{Z}/p\mathbb{Z}$-action on $A^{\otimes pn}$. But since $\phi^{\otimes n} : A^{\otimes n} \to A^{\otimes np}$ is $\mathbb{Z}/p\mathbb{Z}$-equivariant with respect to the trivial $\mathbb{Z}/p\mathbb{Z}$-action on the left-hand side, this is equivalent to multiplying by p.

In general, there is no quasi-Frobenius map, but there is a complicated procedure to still obtain such a filtered Dieudonné module structure, cf. [K1], [K3]. It yields the following:

(i) φ_0 exists and is unique up to a quasi-isomorphism if the homological dimension of A is less than $2p$, $\mathrm{hom.dim.}(A \otimes A^{\mathrm{opp}}) < 2p$.
(ii) This structure is functorial.

We note that in general, the maps $\varphi^{\bullet}$ do not preserve the filtration $F^{\bullet}$, but if A admits a quasi-Frobenius map, then they do (this is similar to the case of commutative algebraic varieties, where a similar role is palyed by the lifting map (2)). Using the filtered Dieudonné module structure on $HP_{\bullet}(A)$, we can define the *syntomic homology* $HP_{\bullet}^{\mathrm{synt}}(A)$ as the mapping cone of

$$F^0 HP_{\bullet}(A^{(1)}) \xrightarrow{\mathrm{Id} - \varphi_0} HP_{\bullet}(A).$$

If there is a quasi-Frobenius map φ, then we can define *reduced syntomic homology* $\overline{HP}_{\bullet}^{\mathrm{synt}}(A)$, namely as the mapping cone of

$$F^0 HP_{\bullet}(A^{(1)}) \xrightarrow{\mathrm{Id} - \varphi_0} F^0 HP_{\bullet}(A).$$

Of course, the natural embedding $F^0 HP_{\bullet}(A) \to HP_{\bullet}(A)$ induces a natural map $\overline{HP}_{\bullet}^{\mathrm{synt}}(A) \to HP_{\bullet}^{\mathrm{synt}}(A)$.

3.3. The regulator map.

We now turn to the construction of a regulator map. First, let us consider the case $A = W[G]$ for a (discrete) group G. Here we have

$$C_{\bullet}(W[G]) = C_{\bullet}(\widetilde{BG}, W)$$

where the left-hand side is the standard complex which computes $HH_{\bullet}$, and the right-hand side is the chain complex of the simplicial nerve $\widetilde{BG}$ of the groupoid G/G_{ad}, the quotient of the set G modulo the conjugation action of G.

Inside G/G_{ad}, we have the "unity component" $1/G_{\mathrm{ad}} \subset G/G_{\mathrm{ad}}$; its nerve $BG \subset \widetilde{BG}$ is the usual classifying simplicial set of the group G, and we obtain the inclusion $C_{\bullet}(BG) \subset C_{\bullet}(\widetilde{BG})$. One checks easily that $BG \subset \widetilde{BG}$ is preserved by the cyclic permutation σ needed to define the periodic cyclic complex (the scientific formulation is "BG is a cyclic subset in $\widetilde{BG}$", see [L, Ch. 7]). Thus one can define $HP_{\bullet}(BG)$ together with a map $HP_{\bullet}(BG) \to HP_{\bullet}(\widetilde{BG}) = HP_{\bullet}(W[G])$.

Moreover, the quasi-Frobenius map of Example 3.7 preserves $BG \subset \widetilde{BG}$, so that we can define the reduced syntomic homology

$$\overline{HP}^{\mathrm{synt}}_{\bullet}(BG).$$

Lemma 3.8. *We have* $\overline{HP}^{\mathrm{synt}}_{\bullet}(BG) \cong H_{\bullet}(G, \mathbb{Z}_p)$.

Proof. On F^1, φ_0 is divisible by p, so $\mathrm{Id} - \varphi_0$ is invertible. Thus it suffices to consider the cone of

$$F^0/F^1 \xrightarrow{\mathrm{Id} - \varphi_0} F^0/F^1.$$

These two complexes have entries $A^{\otimes n} = W[G^n]$ for various n. Moreover, by definition, the map φ_0 is induced by the identity map on the sets G^n, the components of the simplicial set BG. Indeed, the inclusion $BG \subset \widetilde{BG}$ identifies G^n with the subset of elements

$$\langle g_0, g_1, \ldots, g_n \rangle \in G^{n+1}$$

such that $g_0 \cdot g_1 \cdot \cdots \cdot g_n = 1$, and

$$m(\phi(\langle g_0, g_1, \ldots, g_n \rangle)) = m(\langle g_0, g_1, \ldots, g_n, g_0, g_1, \ldots, g_n, \ldots, g_0, g_1, \ldots, g_n \rangle),$$

with p factors in the right-hand side; plugging this into (9), we obtain

$$m(\phi(\langle g_0, g_1, \ldots, g_n \rangle)) = \langle g_0, g_1, \ldots, g_n (g_0 \cdot g_1 \cdot \ldots \cdot g_n)^{p-1} \rangle = \langle g_0, g_1, \ldots, g_n \rangle.$$

But in the difference $\mathrm{Id} - \varphi_0$, one term is Fr_W-semilinear, whereas the other is W-linear. Thus we obtain many copies of the cone of $\mathrm{Id} - \mathrm{Fr}_W : W \to W$. Replacing each of these copies (quasi-isomorphically) by $\mathbb{Z}_p$, we obtain a complex that computes $H_{\bullet}(G, \mathbb{Z}_p)$. $\square$

This Lemma, astonishingly trivial as it may be, is the crucial part in the construction. The rest is a standard and well-known procedure, see e. g. [L]. For any $n \geq 1$, we let $M_n(A)$ be the ring of $(n \times n)$ quadratic matrices over A, and we let $\mathrm{GL}_n(A) \subseteq M_n(A)$ be its group of invertible elements. Then the canonical ring homomorphism

$$W[\mathrm{GL}_n(A)] \longrightarrow M_n(A)$$

and the canonical maps $BG \subset \widetilde{BG}$ etc. yield two maps

$$\overline{HP}^{\mathrm{synt}}_{\bullet}(B\,\mathrm{GL}_n(A)) \longrightarrow HP^{\mathrm{synt}}_{\bullet}\big(W[\mathrm{GL}_n(A)]\big) \longrightarrow HP^{\mathrm{synt}}_{\bullet}(M_n(A)).$$

Due to the Morita invariance of $HP^{\mathrm{synt}}_{\bullet}$, the right-hand side does not depend on n, so that we can pass to the limit with respect to the natural embeddings $GL_n(A) \to GL_{n+1}(A)$ and obtain a map

$$\overline{HP}^{\mathrm{synt}}_{\bullet}(B\,\mathrm{GL}_\infty(A)) \longrightarrow HP^{\mathrm{synt}}_{\bullet}(A).$$

This is the desired regulator map. Indeed, according to the Lemma 3.8, its source is

$$\overline{HP}^{\mathrm{synt}}_{\bullet}(B\,\mathrm{GL}_\infty(A)) \cong H_\bullet(B\,\mathrm{GL}_\infty(A), \mathbb{Z}_p) \cong H_\bullet(B\,\mathrm{GL}^+_\infty(A), \mathbb{Z}_p),$$

Quillen's plus-construction for K-theory, and its target is $HP^{\mathrm{synt}}_{\bullet}(A)$.

Remark 3.9. To re-iterate: since the weight filtration is missing here, we cannot expect this regulator map to be an isomorphism. In the commutative setting and in $\mathsf{char}\,0$, this was healed by the passage from Deligne to Deligne-Beilinson cohomology. We don't know yet how to do this here. A reader who has an idea is kindly requested to contact the author.

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