De Jong-Oort Purity for p-Divisible Groups

Thomas Zink

March 6, 2007

Introduction: De Jong-Oort purity states that for a family of p-divisible groups $X \to S$ over a noetherian scheme S the geometric fibres have all the same Newton polygon if this is true outside a set of codimension bigger than 2. A more general result was first proved in [JO] and an alternative proof is given in [V1]. We present here a short proof which is based on the fact that a formal p-divisible group may be defined by a display. There are two other ingredients of the proof which are known for a long time. One is the boundedness principal for crystals over an algebraically closed field and the other is the existence of a slope filtration for a p-divisible group over a non-perfect field. The last fact was already mentioned in a letter of Grothendieck to Barsotti. The boundedness property is also an important ingredient in the proof given by Vasiu in [V1].

We discuss in detail some elementary consequences of the display structure. The other two ingredients can be found in the literature. Therefore we discuss them only briefly.

Let R be a commutative ring. We fix a prime number p. The ring of Witt vectors with respect to p is denoted by $W(R)$. We write $I_R = VW(R)$ for the Witt vectors whose first component is 0. The Witt polynomials are denoted by $\mathbf{w}_n : W(R) \to R$. The truncated Witt vectors of length n are denoted by $W_n(R)$. If $pR = 0$ the Frobenius endomorphism F of the ring $W(R)$ induces an endomorphism $F : W_n(R) \to W_n(R)$.

Definition 1 *A Frobenius module over R is a pair (M, F), where M is a projective finitely generated $W(R)$-module of some fixed rank h and $F : M \to M$ is a Frobenius-linear homomorphism, such that* $\det F = p^d \epsilon$ *locally for the Zariski topology on R, where $\epsilon : \det M \to \det M$ is a Frobenius-linear* *isomorphism and* $d \geq 0$ *is some integer. We call* h *the height of the Frobenius module and* d *the dimension.*

This definition implies that the decomposition $\det F = p^d \epsilon$ exists even globally. We will often consider the case where M is a free $W(R)$-module. If we choose a basis of M we may view $\det F$ as an element of $W(R)$. This element is at least unique up to multiplication by a unit in $W(R)$. In proofs we take this point of view without metionning.

In this article a display over R is a 3n-display in the sense of [Z1]. The displays of [Z1] are called nilpotent displays. If $\mathcal{P} = (P, Q, F, F_1)$ is a display over R then (P, F) is a Frobenius module over R.

Let X be a p-divisible over R and assume that p is nilpotent in R. If we evaluate the Grothendieck-Messing crystal of X at $W(R)$ we obtain a finitely generated locally free $W(R)$-module M_X, which is endowed with a Frobenius linear map $F : M_X \to M_X$. If X is the formal p-divisible group associated to a nilpotent display $\mathcal{P}$ then $(M_X, F) = (P, F)$ is a Frobenius module. The pair (M_Y, F) is also a Frobenius module, if Y is an extension of an étale p-divisible group by X.

If we assume moreover that R is a complete local noetherian ring (M_X, F) is a Frobenius module for an arbitrary p-divisible group X over R. By these remarks any (M_X, F) appearing in this work are Frobenius modules. Indeed if the special fibre of X has no étale part, then (M, F)

We add that Lau [L] associated a display to any p-divisible group over a ring R, where p is nilpotent and therefore also a Frobenius module.

The following lemma is mainly a motivation for the definitions we are going to make:

Lemma 2 *Let* $\mathcal{P}$ *and* $\mathcal{P}'$ *be displays over a ring* R *of the same height and dimension. Let* $\alpha : \mathcal{P} \to \mathcal{P}'$ *be a homomorphism.*

Locally on $\operatorname{Spec} R$ *the element* $\det \alpha \in W(R)$ *satisfies an equation:*

$$^F\!\det \alpha = \varepsilon \cdot \det \alpha,$$

where $\varepsilon \in W(R)^*$ *is a unit.*

Proof: We choose normal decompositions

$$\begin{array}{ll} P = L \oplus T, & Q = L \oplus I_R T \\ P' = L' \oplus T', & Q' = L' \oplus I_R T'. \end{array}$$

Without loss of generality we may assume that L, L', T, T' are free $W(R)$-modules. We choose identifications

$$L \simeq W(R)^l \simeq L', \quad T \simeq W(R)^t \simeq T'.$$

Then operators F_1 and F_1' are given by invertible block-matrices with coefficient in $W(R)$:

$$F_1 \begin{pmatrix} \underline{x} \\ {}^V\underline{y} \end{pmatrix} = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} {}^F\underline{x} \\ \underline{y} \end{pmatrix}$$

$$F_1' \begin{pmatrix} \underline{x} \\ {}^V\underline{y} \end{pmatrix} = \begin{pmatrix} X' & Y' \\ Z' & W' \end{pmatrix} \begin{pmatrix} {}^F\underline{x} \\ \underline{y} \end{pmatrix}$$

The block-matrices are invertible by the definition of a display. We also represent α by a block matrix

$$\alpha \begin{pmatrix} \underline{x} \\ {}^V\underline{y} \end{pmatrix} = \begin{pmatrix} A & B \\ {}^VC & D \end{pmatrix} \begin{pmatrix} \underline{x} \\ {}^V\underline{y} \end{pmatrix}$$

Since α commutes with the operators F_1 and F_1' we find

$$\begin{pmatrix} X' & Y' \\ Z' & W' \end{pmatrix} \begin{pmatrix} {}^FA & p\,{}^FB \\ C & {}^FD \end{pmatrix} = \begin{pmatrix} A & B \\ {}^VC & D \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \tag{1}$$

We see that

$${}^F\begin{pmatrix} A & B \\ {}^VC & D \end{pmatrix} = \begin{pmatrix} {}^FA & {}^FB \\ pC & {}^FD \end{pmatrix}$$

has the same determinant as

$$\begin{pmatrix} {}^FA & p\,{}^FB \\ C & {}^FD \end{pmatrix}$$

But then taking determinants in (1) gives the result. *Q.E.D.*

Proposition 3 *Let R be a ring such that* $\operatorname{Spec} R$ *is connected. Let $\alpha : \mathcal{P} \to \mathcal{P}'$ be a homomorphism of displays of the same height h and the same dimension d.*

If $\det \alpha \neq 0$ *then there is a number u, such that locally on* $\operatorname{Spec} R$ *the following equation holds:*

$$\det \alpha = p^u \varepsilon, \quad \textit{where} \quad \varepsilon \in W(R)^*,\ u \in \mathbb{Z}_{\geq 0}.$$

Proof: We set $\eta = \det \alpha$. By the last proposition we find:

$$ {}^{F}\eta = \zeta \cdot \eta \text{ for some } \zeta \in W(R)^*. \tag{2} $$

We write $\eta = {}^{V^h}\xi$, such that $\mathbf{w}_0(\xi) \neq 0$. We claim that (2) implies:

$$ {}^{F}\xi = {}^{F^h}\zeta \cdot \xi. \tag{3} $$

To verify this we may assume that $h > 0$. We obtain:

$$ {}^{FV^h}\xi = \zeta\ {}^{V^h}\xi = {}^{V^h}({}^{F^h}\zeta\xi) $$

We deduce that:

$$ p\xi = {}^{V}({}^{F^h}\zeta\xi) \tag{4} $$

Let $y_0 = \mathbf{w}_0(\xi)$ be the first component of the Witt vector ξ. By (4) we obtain $py_0 = 0$. But this implies $p\xi = {}^{VF}\xi$ and therefore (3).

Let $\mathbf{w}_0(\xi) = x$ and $\mathbf{w}_0({}^{F^h}\zeta) = e \in R^*$. We find

$$ x^p = ex. \tag{5} $$

Since the product

$$ x(x^{p-1} - e) = 0 $$

has relatively prime factors, it follows that

$$ \begin{aligned} D(x) \cup D(x^{p-1} - e) &= \operatorname{Spec} R \\ D(x) \cap D(x^{p-1} - e) &= \emptyset. \end{aligned} $$

Hence by connectedness either $D(x) = \operatorname{Spec} R$ or $D(x) = \emptyset$. In the first case x is nilpotent. But then we find $x = 0$, by iterating the equation (5). This is a contradiction to our choices. Therefore $D(x) = \operatorname{Spec} R$ and x is a unit. Then ξ is a unit too. We find

$$ {}^{F^h}\eta = {}^{F^hV^h}\xi = p^h\xi. $$

But by (2) ${}^{F^h}\eta$ may be expressed as the product of η by a unit. This proves the result. *Q.E.D.*

Definition 4 *A homomorphism as in the proposition is called an isogeny of displays.*

Let $\alpha : \mathcal{P} \to \mathcal{P}'$ be a homomorphism of nilpotent displays of the same height and dimension. By the functor from the category of nilpotent displays to the category of formal p-divisible groups we obtain from α a morphism $\phi : X \to X'$ of p-divisible groups. It follows from [Z1] that α is an isogeny, iff ϕ is an isogeny of p-divisible groups.

Let R be a ring such that $pR = 0$. Then the Frobenius endomorphism on $W(R)$ induces a Frobenius endomorphism on the truncated Witt vectors $F : W_n(R) \to W_n(R)$. Therefore we may consider truncated Frobenius modules. We are going to prove a version of Proposition 3 for truncated Frobenius modules.

Definition 5 *Let R be a ring such that $pR = 0$. A truncated Frobenius module of level n, dimension d, and height h over R is a finitely generated projective $W_n(R)$-module M of rank h equipped with a Frobenius linear operator $F : M \to M$, such that locally on* $\operatorname{Spec} R$ *the determinant has the form*

$$\det F = p^d \varepsilon, \tag{6}$$

where $\varepsilon : \det M \to \det M$ is a Frobenius linear isomorphism.

A Frobenius module M over R induces a truncated Frobenius module, if we tensor it by $W_n(R)$.

Definition 6 *Let M and N be truncated Frobenius modules of level n and of the same dimension d and height h. A morphism of Frobenius modules $\alpha : M \to N$ is called an isogeny if there is a natural number $u < n$ such that the determinant of α has locally on* $\operatorname{Spec} R$ *the form:*

$$^{F^d} \det \alpha = p^u \varepsilon, \quad \varepsilon \in W_n(R)^*.$$

The number u is called the height of the isogeny.

Proposition 7 *Let M and N be truncated Frobenius modules of level n and of the same dimension d and height h over a ring R such that* $\operatorname{Spec} R$ *is connected.*

Let $u > 0$ be an integer, such that $n > u + d$. Let $\alpha : M \to N$ be a homomorphism of Frobenius modules such that

$$^{F^d} \det \alpha \notin V^{u+1} W_{n-u-1}(R).$$

Then α becomes an isogeny if we truncate it to level $n - d$:

$$\alpha[n-d] : M[n-d] \to N[n-d].$$

Proof: We may assume that M and N are free $W_n(R)$-modules. We choose isomorphisms

$$\det M \simeq W_n(R) \simeq \det N$$

and view $\theta := \det\alpha$ as an element of $W(R)$. Then we obtain a commutative diagram

$$\begin{array}{ccc} \det M & \xrightarrow{\theta} & \det N \\ {\scriptstyle p^d\tau_M F}\downarrow & & \downarrow{\scriptstyle p^d\tau_N F} \\ \det M & \xrightarrow{\theta} & \det N, \end{array}$$

where $\tau_M, \tau_N \in W_n(R)^*$ are units. We obtain

$$p^d\tau_N\ {}^F\theta = \theta p^d\tau_M. \tag{7}$$

Using $p^d = V^dF^d$ in $W(R)$, we can divide (7) by V^d. We then obtain an equality in $W_{n-d}(R)$:

$${}^{F^{d+1}}\theta[n-d] = {}^{F^d}\theta[n-d]\rho. \tag{8}$$

Here $\theta[n-d]$ denotes the image of θ by the natural restriction $W_n(R) \to W_{n-d}(R)$ and $\rho \in W_{n-d}(R)^*$ is a unit.

On the other hand we may write by assumption:

$${}^{F^d}\theta = {}^{V^{u_1}}\sigma, \tag{9}$$

where $u_1 \leq u$, and $\mathbf{w}_0(\sigma) = s_0 \neq 0$. Clearly we may assume $u = u_1$. Since $n-d > u$ we obtain from equation (8)

$$s_0^p = s_0 e$$

for some unit $e \in R^*$. As in the proof of Proposition 3 (see: (5)) we conclude that s_0 is a unit. Then σ is a unit too. From (9) we obtain

$${}^{F^{d+u}}\theta = p^u\sigma.$$

We truncate this equation to $W_{n-d}(R)$ and use (8) to obtain

$${}^{F^d}\theta[n-d] = p^u\varepsilon$$

for some unit $\varepsilon \in W_{n-d}(R)^*$. *Q.E.D.*

Let $n > u$ be natural numbers. It is clear that a morphism of displays $\alpha : \mathcal{P} \to \mathcal{P}'$ is an isogeny of height u, iff the map of the truncated Frobenius modules $\alpha[n] : (P[n], F) \to (P'[n], F)$ is an isogeny of height u.

For the proof of the purity theorem of de Jong and Oort for p-divisible groups we need to recall a few facts on completely slope divisible p-divisible groups (abbreviated: c.s.d. groups) from [Z2] and [OZ]. We will use truncated Frobenius modules of p-divisible groups over any scheme U. These are locally free $W_n(\mathcal{O}_U)$-modules.

Lemma 8 *Let Y be a c.s.d. group over a normal noetherian scheme U over $\bar{\mathbb{F}}_p$. Let n be a natural number. Then there is a finite morphism $U' \to U$, such that the truncated Frobenius module $M_Y[n]$ of Y over U' is obtained by base change from a truncated Frobnius module over $\bar{\mathbb{F}}_p$, i.e. we can find a Frobenius module N over $\bar{\mathbb{F}}_p$ such that there is an isomorphism of Frobenius modules*

$$W_n(\mathcal{O}_{U'}) \otimes_{W_n(\mathcal{O}_U)} M_Y[n] \simeq W_n(\mathcal{O}_{U'}) \otimes_{W(\bar{\mathbb{F}}_p)} N \qquad (10)$$

This is an immediate consequence of [OZ] Proposition 1.3, since it says that this is true if we take for U' the perfect hull of the universal pro-étale cover of U. Another proof is obtained by substituting in the proof of loc.cit. Frobenius modules.

Proposition 9 *Let T be a regular connected 1-dimensional scheme over $\mathbb{F}_p$. Then any p-divisible group X with constant Newton polygon over T is isogenous to a c.s.d. group.*

Proof: This follows from the main result of [OZ] Thm. 2.1. for any normal noetherian scheme T. But under under the assumptions made the proof is much easier (compare [Z2] proof of Thm. 7). Indeed let $K = K(T)$ be the function field of T. Then we find by over K an isogeny to a c.s.d. group:

$$X_K \to \overset{\circ}{Y} \qquad (11)$$

Let $\overset{\circ}{G}$ be the finite group scheme which is the kernel of (11) and let $G \subset X$ be its scheme theoretic closure. We set $Y = X/G$. Using the fact that X has constant Newton polygon one proves that Y is c.s.d. *Q.E.D.*

The third ingredient is the boundedness principle, which seems to be known for a long time [M]. But it importance was rediscovred by Vasiu only recently. It is also an ingredient of Vasiu's proof [V1] of the de Jong-Oort purity.

Proposition 10 *Let k be an algebraic closed field. Let h be a natural number. Then there is a constant $c \in \mathbb{N}$ with the following property:*

Let M_1 and M_2 be Frobenius modules of height $\leq h$ over k. Let $n \in \mathbb{N}$ be arbitrary and let $\bar{\alpha} : M_1/p^n M_1 \to M_s/p^n M_2$ be a morphism of truncated Frobenius modules which lifts to a morphism of truncated Frobenius modules $M_1/p^{n+c}M_1 \to M_2/p^{n+c}M_2$. Then $\bar{\alpha}$ lifts to a morphism of Frobenius modules $\alpha : M_1 \to M_2$.

A weaker version of this is contained in [O], where the existence of the constant c is only asserted for given modules M_1 and M_2. But one can show that for given modules N_1 resp. N_2 in the isogeny class of M_1 resp. M_2, there are always isogenies $N_1 \to M_1$ resp. $N_1 \to M_1$ whose degrees are bounded by a constant only depending on h. This is another well-known boundedness principle. As an alternative to this proof the reader may use the much stronger results of [V2].

Theorem 11 *(de Jong-Oort) Let R be a noetherian local ring of Krull dimension ≥ 2 with $p \cdot R = 0$. Let $U = \operatorname{Spec} R \setminus \{\mathfrak{m}\}$, the complement of the closed point. A p-divisible group X over SpecR, which has constant Newton polygon over U has constant Newton polygon over* $\operatorname{Spec} R$.

Proof: It is not difficult to reduce to the case where R is complete, normal of Krull dimension 2 with algebraically closed residue class field $k = R/\mathfrak{m}$. Then U is a 1-dimensional regular scheme. We find by Proposition 9 a c.s.d. group Y over U and an isogeny

$$\alpha : Y \to X_{|U}, \tag{12}$$

Let d be the dimension of X let u be the height of α and let c be the number from Proposition 10. We choose a natural number $n > c+u+d$.. After a finite extension of R we may assume by Lemma 8 that the truncated Frobenius module of Y is constant

$$M_Y[n] \simeq W_n(\mathcal{O}_U) \otimes_{W(\bar{\mathbb{F}}_p)} N \tag{13}$$

where N is a Frobenius module over $\bar{\mathbb{F}}_p$. In particular the Newton polygons of N and Y must be the same by the boundedness principle applied to the field $\bar{K}$, where K is the field of fractions of R.

Combining (12) and (13) we find an isogeny of height u of truncated Frobenius modules

$$W_n(\mathcal{O}_U) \otimes_{W(\bar{\mathbb{F}}_p)} N \to W_n(\mathcal{O}_U) \otimes_R M_X[n]. \tag{14}$$

By the normality of R we find $\Gamma(U, W_n(\mathcal{O}_U)) = W_n(R)$. Taking the global section of (14) over U we obtain a morphism of truncated Frobenius modules

$$W_n(R) \otimes_{W(\bar{\mathbb{F}}_p)} N \to M_X[n]. \tag{15}$$

We know that (15) is an isogeny over K of height u. Therefore Proposition 3 is applicable to the morphism (15). We obtain therefore an isogeny of height u of truncated Frobenius modules over R:

$$W_{n-d}(R) \otimes_{W(\bar{\mathbb{F}}_p)} N \to M_X[n-d],$$

It is clear that the base change of an isogeny of truncated Frobenius modules is again an isogeny. Making the base change $R \to k$ we obtain an isogeny:

$$W_{n-d}(k) \otimes_{W(\bar{\mathbb{F}}_p)} N \to W_{n-d}(k) \otimes_{W(R)} M_X[n-d] = M_{X_k}[n-d].$$

The boundedness principle shows that X_k and N habe the same Newton polygon. *Q.E.D.*

References

[JO] A.J.de Jong and F.Oort, Purity of the stratification by Newton polygons, J.Amer.Math.Soc.13(2000).

[L] E.Lau in preparation.

[M] Yu. I. Manin The theory of commutative formal groups over fields of finite characteristic. Usp. Math. 18 (1963), 3-90, Russ. Math. Surveys 18, (1963), 1-80.

[Me] Messing, W.: The crystals associated to Barsotti-Tate groups, LNM **264**, Springer 1972.

[O] F.Oort, Foliations of moduli spaces of abelian varieties J.Amer.Math.Soc 17 (2004).

[OZ] F.Oort, T.Zink, Families of p-divisible groups with constant Newton polygon. Doc. Math. 7 (2002), 183–201

[V1] A.Vasiu, Crystalline boundedness principle, Ann. Sci. cole Norm. Sup.(4) 39 (2006), 245–300.

[V2] A.Vasiu, Reconstructing p-divisible groups from their truncations of small level.

[Z1] Th. Zink, The display of a formal p-divisible group, in: Cohomologies p-adiques et applications arithmtiques, I. Astrisque No. 278 (2002), 127–248.

[Z2] Th. Zink, On the slope filtration, Duke Math. J. Vol.109 (2001), 79-95.