CUBIC SURFACES AND CUBIC THREEFOLDS, JACOBIANS AND INTERMEDIATE JACOBIANS

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In this paper we study principally polarized abelian varieties that admit an automorphism of order 3. It turns out that certain natural conditions on the multiplicities of its action on the differentials of the first kind do guarantee that those polarized varieties are not jacobians of curves. As an application, we get another proof of the (already known) fact that intermediate jacobians of certain cubic threefolds are not jacobians of curves.

1. Principally polarized abelian varieties that admit an automorphism of order 3

Let $\zeta_3 = \frac{-1+\sqrt{-3}}{2}$ be a primitive (complex) cubic root of unity. It generates the multiplicative order 3 cyclic group μ_3 of cubic roots of unity.

Let $g > 1$ be an integer and (X, λ) a principally polarized g-dimensional abelian variety over the field $\mathbf{C}$ of complex numbers, δ an automorphism of (X, λ) that satisfies the cyclotimic equation $\delta^2 + \delta + 1 = 0$ in $\mathrm{End}(X)$. In other words, δ is a periodic automorphism of order 3, whose set of fixed points is finite. This gives rise to the embeddings

$$\mathbf{Z}[\zeta_3] \hookrightarrow \mathrm{End}(X), 1 \mapsto 1_X,\ \zeta_3 \mapsto \delta,$$

$$\mathbf{Q}(\zeta_3) \hookrightarrow \mathrm{End}^0(X), 1 \mapsto 1_X,\ \zeta_3 \mapsto \delta.$$

By functoriality, $\mathbf{Q}(\zeta_3)$ acts on the g-dimensional complex vector space $\Omega^1(X)$ of differentials of the first kind on X. This provides $\Omega^1(X)$ with the structure of $\mathbf{Q}(\zeta_3) \otimes_{\mathbf{Q}} \mathbf{C}$-module. Clearly,

$$\mathbf{Q}(\zeta_3) \otimes_{\mathbf{Q}} \mathbf{C} = \mathbf{C} \oplus C$$

where the summands correspond to the embeddings $\mathbf{Q}(\zeta_3) \to \mathbf{C}$ that send ζ_3 to ζ_3 and ζ_3^{-1} respectively. So, $\mathbf{Q}(\zeta_3)$ acts on $\Omega^1(X)$ with multiplicities a and b that correspond to two embeddings of $\mathbf{Q}(\zeta_3)$ into $\mathbf{C}$. Clearly, a and b are non-negative integers with $a + b = g$.

Theorem 1.1. *If $g + 2 > 3 \mid a - b \mid$ then (X, λ) is not the jacobian of a smooth projective irreducible genus g curve with canonical principal polarization.*

Proof. Suppose that $(X, \lambda) \cong (J(C), \Theta)$ where C is an irreducible smooth projective genus g curve, $J(C)$ its jacobian with canonical principal polarization Θ. It follows from Torelli theorem in Weil's form [10, 11] that there exists an automorphism $\phi : C \to C$, which induces (by functoriality) either δ or $-\delta$ on $J(C) = X$. Replacing ϕ by ϕ^4 and taking into account that δ^3 is the identity automorphism of $X = J(C)$, we may and will assume that ϕ does induces δ. Clearly, ϕ^3 is the identity automorphism

of C, because it induces the identity map on $J(C)$ and $g > 1$. The action of ϕ on C gives rise to the embedding

$$\mu_3 \hookrightarrow \mathrm{Aut}(C),\ \zeta_3 \mapsto \phi.$$

Let $P \in C$ be a fixed point of ϕ. Then ϕ induces the automorphism of the corresponding (one-dimensional) tangent space $T_P(C)$, which is multiplication by a complex number c_P. Clearly, c_P is a cubic root of unity.

Lemma 1.2. *Every fixed point P of ϕ is nondegenerate, i.e., $c_P \ne 1$.*

Proof of Lemma 1.2. The result is pretty well-known. However, I failed to find a proper reference.

So, suppose that $c_P = 1$. Let $\mathcal{O}_P$ be the local ring at P and $\mathfrak{m}_P$ its maximal ideal. We write ϕ_* for the automorphism of $\mathcal{O}_P$ induced by ϕ. Clearly, ϕ_*^3 is the identity map. Since ϕ is *not* the identity map, there are no ϕ_*-invariant local parameters at P. Clearly, $\phi_*(\mathfrak{m}_P) = \mathfrak{m}_P, \phi_*(\mathfrak{m}_P^2) = \mathfrak{m}_P^2$. Since $T_P(C)$ is the dual of $\mathfrak{m}_P/\mathfrak{m}_P^2$ and $c_p = 1$, we conclude that ϕ_* induced the identity map on $\mathfrak{m}_P/\mathfrak{m}_P^2$. This implies that if $t \in \mathfrak{m}_P$ is a local parameter at t (i.e., its image $\bar{t}$ in $\mathfrak{m}_P/\mathfrak{m}_P^2$ is *not* zero) then $t' := t + \phi_*(t) + \phi_*^2(t)$ is ϕ_*-invariant and its image in $\mathfrak{m}_P/\mathfrak{m}_P^2$ equals $3\bar{t} \ne 0$. This implies that t' ia a ϕ_*-invariant local parameter at P. Contradiction.

□

Corollary 1.3. *$D := C/\mu_3$ is a smooth projective irreducible curve. The map $C \to D$ has degree 3, its ramification points are exactly the images of fixed points of ϕ and all the ramification indices are 3.*

Lemma 1.4. *D is biregularly isomorphic to the projective line.*

Proof of Lemma 1.4. The map $C \to D$ induces, by Albanese functorialy, the surjective homomorphism of the corresponding jacobians $J(C) \to J(D)$ that kills all the divisors classes of the form $(Q) - (\phi(Q))$ $(Q \in C)$. This implies that it kills $(1 - \delta)J(C)$. On the other hand, $1 - \delta : J(C) \to J(C)$ is, obviously, an isogeny. This implies that the image of $J(C)$ in $J(D)$ is zero and the surjectiveness implies that $J(D) = 0$. This means that the genus of D is 0. □

Corollary 1.5. *The number h of fixed points of ϕ is $g + 2$.*

Proof of Corollary 1.5. Applying Hurwitz formula to $C \to D$, we get

$$2g - 2 = 3 \cdot (-2) + 2 \cdot h.$$

□

Lemma 1.6. *Let $\phi^* : \Omega^1(C) \to \Omega^1(C)$ be the automorphism of $\Omega^1(C)$ induced by ϕ and τ its trace. Then*

$$\tau = a\zeta_3 + b\zeta_3^{-1}.$$

Proof of Lemma 1.6. Pick a ϕ-invariant point P_0 and consider the regular map

$$\alpha : C \to J(C), Q \mapsto \mathrm{cl}((Q) - (P_0)).$$

It is well-known that α induces an isomorphism of complex vector spaces

$$\alpha^* : \Omega^1(X) \cong \Omega^1(C).$$

Clearly,

$$\phi^* = \alpha^* \delta^* {\alpha^*}^{-1}$$

where $\delta^* : \Omega^1(J(C)) = \Omega^1(J(C))$ is the automorphism induced by δ. This implies that the traces of α^* and δ^* do coincide. Now the very definition of a and b implies that the trace of α^* equals $a\zeta_3 + b\zeta_3^{-1}$. □

End of proof of Theorem 1.1. Let B be the set of fixed points of ϕ. We know that $\#(B) = g + 2$. By holomorphic Lefschetz fixed point formula [2, Th. 2], [6, Ch. 2, Sect. 4] (see also [9, Sect. 12.2 and 12.5]) applied to ϕ,

$$1 - \bar{\tau} = \sum_{P \in B} \frac{1}{1 - c_P}$$

where $\bar{\tau}$ is the complex-conjugate of τ. Recall that every c_P is a (primitive) cubic root of unity and therefore

$$| \, 1 - c_P \, | = \sqrt{3}, \; | \, \frac{1}{1 - c_P} \, | = \frac{1}{\sqrt{3}}$$

and

$$| \, 1 - \bar{\tau} \, | \leq \frac{g+2}{\sqrt{3}}.$$

Now

$$| \, 1 - \bar{\tau} \, |^2 = \frac{(a+b+2)^2 + 3(a-b)^2}{4} = \frac{(g+2)^2 + 3(a-b)^2}{4}.$$

This implies that

$$\frac{(g+2)^2}{3} \geq \frac{(g+2)^2 + 3(a-b)^2}{4}.$$

It follows that $(g+2)^2 \geq 9(a-b)^2$ and we are done. □

2. Cubic threefolds

Let $S : F(x_0, x_1, x_2, x_3) = 0 \subset \mathbf{P}^3$ be a smooth projective cubic surface over $\mathbf{C}$ [7]. (In particular, F is an irreducible homogeneous cubic polynomial in x_0, x_1, x_2, x_3 with complex coefficients.) Then the equation

$$y^3 = F(x_0, x_1, x_2, x_3)$$

defines a smooth projective threefold $T \subset \mathbf{P}^4$ provided with the natural action of μ_3 that arises from multiplication of y by cubic roots of unity [1] (see also [3, 8]). We have the μ_3-invariant Hodge decomposition

$$\mathrm{H}^3(T, \mathbf{C}) = \mathrm{H}^3(T, \mathbf{Z}) \otimes \mathbf{C}) = \mathrm{H}^{1,2}(T) \oplus \mathrm{H}^{2,1}(T)$$

and the μ_3-invariant non-degenerate alternating intersection pairing

$$(\, , \,) : \mathrm{H}^3(T, \mathbf{C}) \times \mathrm{H}^3(T, \mathbf{C}) \to \mathbf{C}.$$

In addition, both $\mathrm{H}^{1,2}(T)$ and $\mathrm{H}^{2,1}(T)$ are 5-dimensional isotropic subspaces and μ_3 acts on $\mathrm{H}^{2,1}(T)$ with multiplicities $(4, 1)$, i.e. $\zeta_3 \in \mu_3$ acts as diagonalizable linear operator in $\mathrm{H}^{2,1}(T)$ with eigenvalue ζ_3 of multiplicity 4 and eigenvalue ζ_3^{-1} of multiplicity 1 ([3, Sect. 5], [1, Sect. 2.2 and Lemma 2.6]). (The proof is based on [5, Th. 8.3 on p. 488]; see also [4, pp. 338–339].)

Since both $\mathrm{H}^{1,2}(T)$ and $\mathrm{H}^{2,1}(T)$ are isotropic and the intersection pairing is non-degenerate, its restriction to $\mathrm{H}^{1,2}(T) \times \mathrm{H}^{2,1}(T)$ gives rise to the non-degenerate μ_3-invariant $\mathbf{C}$-bilinear pairing

$$(\, , \,) : \mathrm{H}^{1,2}(T) \times \mathrm{H}^{2,1}(T) \to \mathbf{C}. \tag{1}$$

It follows that μ_3 acts on $\mathrm{H}^{1,2}(T)$ also with multiplicities $(1,4)$. (This assertion also follows from the fact that $\mathrm{H}^{1,2}(T)$ is the complex-conjugate of $\mathrm{H}^{2,1}(T)$.) In particular, the action of μ_3 on $\mathrm{H}^{1,2}(T)$ extends to the embedding

$$\mathbf{Z}[\mu_3] \hookrightarrow \mathrm{End}_{\mathbf{C}}(\mathrm{H}^{1,2}(T)). \tag{2}$$

3. Intermediate jacobians

Let $(J(T), \theta_T)$ be the *intermediate jacobian* of the cubic threefold T [4, Sect. 3]; it is a principally polarized five-dimensional complex abelian variety. By functoriality, μ_3 acts on $J(T)$ and respects the principal polarization θ_T. As complex torus,

$$J(T) = \mathrm{H}^{1,2}(T)/p(\mathrm{H}^3(T, \mathbf{Z})), \tag{3}$$

where

$$p : \mathrm{H}^3(T, \mathbf{C}) = \mathrm{H}^3(T, \mathbf{Z}) \otimes \mathbf{C} = \mathrm{H}^{1,2}(T) \oplus \mathrm{H}^{2,1}(T) \to \mathrm{H}^{1,2}(T)$$

is the projection map that kills $\mathrm{H}^{2,1}(T)$. The imaginary part of the Riemann form of the polarization coincides with the intersection pairing on $\mathrm{H}^3(T, \mathbf{Z}) \cong p(\mathrm{H}^3(T, \mathbf{Z}))$.

It follows from (2) that the action of μ_3 on $J(T)$ extends to the embedding

$$\mathbf{Z}[\mu_3] \hookrightarrow \mathrm{End}(J(T)).$$

Combining (1) and (3), we conclude that the μ_3-modules $\Omega^1(J(T)) = \mathrm{Hom}_{\mathbf{C}}(\mathrm{H}^{1,2}(T), \mathbf{C})$ and $\mathrm{H}^{2,1}(T)$ are canonically isomorphic. Now the assertions of Sect. 3 about multiplicities imply that $\mathbf{Z}[\zeta_3]$ acts on $\Omega^1(J(T))$ with multiplicities $(4,1)$.

Since $3\times \mid 4-1 \mid > 5+2$, it follows from Theorem 1.1 that $(J(T), \theta_T)$ is not isomorphic to the canonically polarized jacobian of a curve. Of course, this assertion was proven by completely different method in [4] for arbitrary smooth projective cubic threefolds.

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