Correspondences of a K3 surface with itself via moduli of sheaves. I

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Abstract

Let X be an algebraic K3 surface, $v = (r, H, s)$ a primitive isotropic Mukai vector on X and $M_X(v)$ the moduli of sheaves over X with v. Let $N(X)$ be the Picard lattice of X.

In [14] and [3], all divisors in moduli of (X, H) (i. e. pairs $H \in N(X)$ with rk $N(X) = 2$) implying $M_X(v) \cong X$ were described. They give some Mukai's correspondences of X with itself.

Applying these results, we show that there exists v and a codimension 2 submoduli in moduli of (X, H) (i. e. a pair $H \in N(X)$ with rk $N(X) = 3$) implying $M_X(v) \cong X$, but this submoduli cannot be extended to a divisor in the moduli with the same property. There are plenty of similar examples.

We discuss the general problem of description of all similar submoduli (to generalize results of [14] and [3] to the general case), and defined by them Mukai's correspondences of X with itself and their compositions, trying to outline a possible general theory.

1 Introduction

To 70th Birthday of Yuri Ivanovich Manin

We consider algebraic K3 surfaces X over $\mathbb{C}$. We remind that a non-singular projective algebraic (or Kählerian compact) surface X is a K3 surface if its canonical class K_X is equal to zero and the irregularity $q = \dim \Omega^1[X] = 0$.

Further, $N(X)$ denotes the Picard lattice of X, and $T(X)$ the transcendental lattice. Further, $\rho(X) = \mathrm{rk}\, N(X)$ denotes the Picard number of X.

We consider primitive isotropic Mukai vectors

$$v = (r, l, s), \ r \in \mathbb{N}, \ s \in \mathbb{Z}, \ l \in N(X), \ l^2 = 2rs. \tag{1.1}$$

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on X. We denote by $Y = M_X(v) = M_X(r, l, s)$ the K3 surface which is the minimal resolution of singularities of the moduli space of sheaves over X with the Mukai vector v. Clarify in Mukai [4] — [7] and Yoshioka [21]. In this case, the corresponding quasi-universal sheaf on $X \times Y$, and its Chern class define a 2-dimensional algebraic cycle on $X \times Y$ and a correspondence between X and Y according to Mukai [5]. It has very nice geometric properties. For more details, see Sect. 6.

If $Y \cong X$, then we obtain an important 2-dimensional algebraic cycle on $X \times X$, and a correspondence of X with itself. Thus, the question, when $Y \cong X$ is very interesting.

If $\rho(X) = 1$, we give the corresponding results in Sect. 3. The author believes that they should be known to specialists.

Let $D \in N(X)$. Then one has the natural isomorphism given by the tensor product

$$T_D : M_X(r, l, s) \cong M_X(r, l + rD, s + r(D^2/2) + D \cdot l), \quad \mathcal{E} \mapsto \mathcal{E} \otimes \mathcal{O}(D).$$

For r, $s > 0$, one has the isomorphism which is called *reflection*

$$\delta : M_X(r, l, s) \cong M_X(s, l, r).$$

E. g. see [5] and [18], [19], [22].

For integers d_1, $d_2 > 0$ such that $(d_1, d_2) = (d_1, s) = (r, d_2) = 1$, one has the isomorphism

$$\nu(d_1, d_2) : M_X(r, l, s) \cong M_X(d_1^2 r, d_1 d_2 l, d_2^2 s)$$

and its inverse $\nu(d_1, d_2)^{-1}$. See [5], [6], [14], [3].

In Theorem 3.1 and Corollary 3.1, we show that if $\rho(X) = 1$ and X is general, then for two primitive isotropic Mukai vectors v_1 and v_2, moduli $M_X(v_1)$ and $M_X(v_2)$ are isomorphic if and only if there exists their isomorphism which is a composition of the three isomorphisms described above. They and their specializations for higher Picard numbers give *universal isomorphisms* between moduli of sheaves over X.

It is known (e.g. see [19]) that for $l \in N(X)$ and $\pm l^2 > 0$, one has Tyurin's isomorphism

$$Tyu = Tyu(\pm l) \, : \, M_X(\pm l^2/2, l, \pm 1) \cong X \, . \qquad (1.2)$$

In Corollary 3.2 (see also Remark 4.1) we show that, if $\rho(X) = 1$, then $M_X(r, H, s)$ and X are isomorphic if and only if there exists their isomorphism which is a composition of the three universal isomorphisms above between moduli of sheaves, and the Tyurin's isomorphism. See also [14] for a similar result.

It was shown in [14] and geometrically interpreted in [3] (together with Carlo Madonna) that analogous results are valid for $\rho(X) = 2$ and X which is general for its Picard lattice, i. e. the automorphism group of the transcendental periods is trivial: $\mathrm{Aut}\,(T(X), H^{2,0}(X)) = \pm 1$. (See also [1], [2], [13] about important particular cases of these results.) We review these results in Sect. 4, see Theorems 4.1, 4.2 and 4.3 for exact formulations. They show that in this case (i. e. when $\rho(X) = 2$ and X is general for its Picard lattice), $M_X(r, H, s) \cong X$

if and only if there exists an isomorphism between $M_X(r, H, s)$ and X which is given by a composition of the universal isomorphisms T_D, δ and $\nu(d_1, d_2)$ between moduli of sheaves over X and by Tyurin's isomorphism between moduli of sheaves over X and X itself. The results for $\rho(X) = 1$ above clarify appearance of the natural isomorphisms T_D, δ, $\nu(d_1, d_2)$, Tyu in these results for Picard number 2.

Importance of the results for $\rho(X) = 2$ and general X is that they describe all divisorial conditions on moduli of algebraic polarized K3 surfaces (X, H) which imply that $M_X(r, H, s) \cong X$. More exactly, the results for $\rho(X) = 2$ describe all abstract polarized Picard lattices $H \in N$ of the rank rk $N = 2$ such that if $H \in N \subset N(X)$, then $M_X(r, H, s) \cong X$. We remind that the codimension of moduli of such X in the 19-dimensional moduli of algebraic polarized K3 surfaces is then 1. Applying these results, in Theorems 4.4 and 4.5, we give a necessary condition on Mukai vector (r, H, s) and a K3 surface X, for the isomorphism $M_X(r, H, s) \cong X$ would follow from a divisorial condition on moduli of polarized K3 surfaces. In Example 4.1, we give an exact numerical case when this necessary condition is not satisfied. Thus, for K3 surfaces X of this example, the isomorphism $M_X(r, H, s) \cong X$ cannot follow from any divisorial condition on moduli of polarized K3 surfaces which implies the isomorphism $M_{X'}(r, H, s) \cong X'$ for K3 surfaces X' satisfying this divisorial condition.

Applying these results, in Sect. 5, Theorem 5.1, we give an exact example of the type of a primitive isotropic Mukai vector (r, H, s) and a pair $H \in N$ of a (abstract) polarized K3 Picard lattice of the rank rk $N = 3$ such that for any polarized K3 surface (X, H) with $H \in N \subset N(X)$ one has $M_X(r, H, s) \cong X$, but this isomorphism does not follow from any divisorial condition (i. e. from Picard number 2) on moduli of polarized K3 surfaces. Thus, moduli of these polarized K3 surfaces have codimension 2, and they cannot be extended to a divisor in moduli of polarized K3 surfaces preserving the isomorphism $M_X(r, H, s) \cong X$. This is the *main result of this paper*. In Sect. 5, we give many similar examples for Picard number $\rho(X) \geq 3$.

These results give important corollaries for higher Picard number $\rho(X) \geq 3$ of the results for Picard number 1 and 2, described above. They also show that the case of $\rho(X) \geq 3$ is very non-trivial. These are the main subjects of this paper.

Another important subject of this paper is to formulate some general concepts and predict the general structure of possible results for higher Picard number $\rho(X) \geq 3$.

At the end of Sect. 5, for a type (r, H, s) of a primitive isotropic Mukai vector, we introduce the concept of a *critical polarized K3 Picard lattice $H \in N$* (for the problem of correspondences of a K3 surface with itself). Roughly speaking, it means that $M_X(r, H, s) \cong X$ for a polarized K3 surface X such that $H \in N \subset N(X)$, but the same is not valid for any primitive sublattice $H \in N_1 \subset N$ of strictly smaller rank. Thus, the corresponding moduli of K3 surfaces have dimension $20 - \text{rk } N$, and they are not specialization of moduli of higher dimension of analogous K3 surfaces.

Classification of critical polarized K3 Picard lattices is the main problem of

correspondences of a K3 surface with itself via moduli of sheaves. Our results for $\rho = 1$ and $\rho = 2$ can be interpreted as classification of all critical polarized K3 Picard lattices of the rank one and two. The described above example of the lattice N of the rank three from Theorem 5.1, gives an example of a critical polarized K3 Picard lattice of the rank three. In Theorem 5.4 we prove that rk $N \leq 12$ for a critical polarized K3 Picard lattice N. In Problem 5.2, we raise a problem of exact estimate of the rank of critical polarized K3 Picard lattices for the fixed type of a primitive isotropic Mukai vector. This problem is now solved only for very special types, when the rank is equal to one.

In Sect. 6, we interpret the results above in terms of the action of correspondences as isometries in $H^2(X, \mathbb{Q})$, and their compositions. For example, Tyurin's isomorphisms (1.2) give reflections in elements $l \in N(X)$ and generate the full automorphism group $O(N(X) \otimes \mathbb{Q})$. Each primitive isotropic Mukai vector (r, H, s) on X with $M_X(r, H, s) \cong X$ generates some class of isometries from $O(N(X) \otimes \mathbb{Q})$. See Sect. 6 for exact formulations. Thus, the main problem of correspondences of X with itself via moduli of sheaves is to find all these generators and their relations. We formulate the corresponding problems (1), (2), (3) and (4) at the end of Sect. 6. They show that, in principle, general results for any $\rho(X)$ should look similar to the now known results for $\rho(X) = 1$, 2.

A reader can see that our general idea is that a very complicated structure of correspondences of a general (for its Picard lattice) K3 surface X with itself via moduli of sheaves is hidden inside of the abstract Picard lattice $N(X)$, and we try to recover this structure. This should lead to some non-trivial constructions related to the abstract Picard lattice $N(X)$ and more closely relate it to geometry of the K3 surface.

2 Preliminary notations about lattices

We use notations and terminology from [10] about lattices, their discriminant groups and forms. A *lattice* L is a non-degenerate integral symmetric bilinear form. I. e. L is a free $\mathbb{Z}$-module equipped with a symmetric pairing $x \cdot y \in \mathbb{Z}$ for $x, y \in L$, and this pairing should be non-degenerate. We denote $x^2 = x \cdot x$. The *signature* of L is the signature of the corresponding real form $L \otimes \mathbb{R}$. The lattice L is called *even* if x^2 is even for any $x \in L$. Otherwise, L is called *odd*. The *determinant* of L is defined to be $\det L = \det(e_i \cdot e_j)$ where $\{e_i\}$ is some basis of L. The lattice L is *unimodular* if $\det L = \pm 1$. The *dual lattice* of L is $L^* = Hom(L, \mathbb{Z}) \subset L \otimes \mathbb{Q}$. The *discriminant group* of L is $A_L = L^*/L$. It has the order $|\det L|$. The group A_L is equipped with the *discriminant bilinear form* $b_L : A_L \times A_L \to \mathbb{Q}/\mathbb{Z}$ and the *discriminant quadratic form* $q_L : A_L \to \mathbb{Q}/2\mathbb{Z}$ if L is even. To get this forms, one should extend the form of L to the form on the dual lattice L^* with values in $\mathbb{Q}$.

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3 Isomorphisms between $M_X(v)$ for a general K3 surface X and a primitive isotropic Mukai vectors v

We consider algebraic K3 surfaces X over $\mathbb{C}$. We remind that a non-singular projective algebraic (or Kählerian compact) surface X is a K3 surface if the canonical class K_X is equal to zero and the irregularity $q = \dim \Omega^1[X] = 0$.

Further $N(X)$ denotes the Picard lattice of X, and $T(X)$ the transcendental lattice of X.

We consider primitive isotropic Mukai vectors

$$v = (r, l, s),\ r \in \mathbb{N},\ s \in \mathbb{Z},\ l \in N(X),\ l^2 = 2rs. \qquad (3.1)$$

on X. We denote by $Y = M_X(v) = M_X(r, l, s)$ the K3 surface which is the minimal resolution of singularities of the moduli space of sheaves over X with the Mukai vector v. Clarify in Mukai [4] — [7] and Yoshioka [21].

In this section, we call an *algebraic K3 surface* to be *general* if the Picard number $\rho(X) = \mathrm{rk}\ N(X) = 1$ and the automorphsim group of the transcendental periods of X is trivial over $\mathbb{Q}$: $Aut(T(X) \otimes \mathbb{Q},\ H^{2,0}(X)) = \pm 1$.

Here we consider the following question: When for a general algebraic K3 surface X and two its primitive isotropic Mukai vectors $v_1 = (r_1, l_1, s_1)$ and $v_2 = (r_2, l_2, s_2)$, the moduli spaces $M_X(v_1)$ and $M_X(v_2)$ are isomorphic?

We have the following three *universal isomorphisms* between moduli of sheaves over a K3 surface :

Let $D \in N(X)$. Then one has the natural isomorphism given by the tensor product

$$T_D : M_X(r, l, s) \cong M_X(r, l + rD, s + r(D^2/2) + D \cdot l),\ \ \mathcal{E} \mapsto \mathcal{E} \otimes \mathcal{O}(D).$$

Moreover, here Mukai vectors

$$v = (r, l, s),\ \ T_D(v) = (r, l + rD, s + r(D^2/2) + D \cdot l)$$

have the same general common divisor and the same square under Mukai pairing. In particular, they are primitive and isotropic simultaneously.

Taking $D = kH$ where H is a hyperplane section and $k > 0$, using the isomorphisms T_D, we can always replace $M_X(r, l, s)$ by an isomorphic $M_X(r, l', s')$ where l' is ample, and then $l'^2 > 0$. Thus, in our problem, we can further assume that $v = (r, l, s)$ where $r > 0$ and l is ample. Then $l^2 = 2rs > 0$ and $s > 0$ either. Further we assume that.

For $r,\ s > 0$, one has the isomorphism which is called *reflection*

$$\delta : M_X(r, l, s) \cong M_X(s, l, r).$$

E. g. see [5] and [18], [19], [22]. Thus, using the reflection, we can further assume that $0 < r \leq s$.

For integers d_1, $d_2 > 0$ such that $(d_1, d_2) = (d_1, s) = (r, d_2) = 1$, one has the isomorphism

$$\nu(d_1, d_2) : M_X(r, l, s) \cong M_X(d_1^2 r,\ d_1 d_2 l,\ d_2^2 s)$$

and its inverse $\nu(d_1, d_2)^{-1}$. See [5], [6], [14], [3]. Using the isomorphisms $\nu(d_1, d_2)$, $\nu(d_1, d_2)^{-1}$ and the reflection δ, we can always assume that the primitive isotropic Mukai vector $v = (r, l, s)$ satisfies:

$v = (r, l, s)$ has $0 < r \le s$, $l^2 = 2rs$ and $l \in N(X)$ is primitive and ample.
$$(3.2)$$

We call such a primitive isotropic Mukai vector as a *reduced primitive isotropic Mukai vector* (for $\rho(X) = 1$).

We have the following result.

Theorem 3.1. *Let X be a general algebraic K3 surface, i. e. $N(X) = \mathbb{Z}H$ where H is a primitive polarization of X and $Aut(T(X) \otimes \mathbb{Q}, H^{2,0}(X)) = \pm 1$. Let $v = (r, H, s)$ and $v' = (r', H, s')$ are two reduced primitive isotropic Mukai vectors on X (see (3.2)), i. e. $0 < r \le s$ and $0 < r' \le s'$.*

Then $M_X(v) \cong M_X(v')$ if and only if $v = v'$, i. e. $r' = r$, $l' = l$.

It follows that the described above universal (i. e. valid for all algebraic K3 surfaces, even general ones) isomorphisms T_D, δ and $\nu(d_1, d_2)$ are sufficient to find isomorphic moduli of sheaves with primitive isotropic Mukai vectors for a general K3 surface.

Corollary 3.1. *Let X be a general algebraic K3 surface and v, v' are primitive isotropic Mukai vectors on X.*

Then $M_X(v) \cong M_X(v')$ if and only if there exists an isomorphism between $M_X(v)$ and $M_X(v')$ which is a composition of the universal isomorphisms T_D, δ and $\nu(d_1, d_2)$.

Proof. Considerations below are very similar to much more general and difficult considerations in ([14], Sect. 2.3).

We have

$$N(X) = \mathbb{Z}H = \{x \in H^2(X, \mathbb{Z}) \mid x \cdot H^{2,0}(X) = 0\},$$

and the transcendental lattice of X is

$$T(X) = N(X)^{\perp}_{H^2(X, \mathbb{Z})}.$$

The lattices $N(X)$ and $T(X)$ are orthogonal complements to one another in the unimodular lattice $H^2(X, \mathbb{Z})$. We have that $N(X) \oplus T(X) \subset H^2(X, \mathbb{Z})$ is a sublattice of a finite index. Here and in what follows $\oplus$ denotes the orthogonal sum. Since $H^2(X, \mathbb{Z})$ is unimodular and $N(X) = \mathbb{Z}H$ is its primitive sublattice, there exists $u \in H^2(X, \mathbb{Z})$ such that $u \cdot H = 1$.

We denote by $N(X)^* = \mathbb{Z}H/(2rs) \subset N(X) \otimes \mathbb{Q}$ and $T(X)^* \subset T(X) \otimes \mathbb{Q}$ the dual lattices. Then $H^2(X, \mathbb{Z}) \subset N(X)^* \oplus T(X)^*$, and

$$u = H/(2rs) \oplus t^*(H),\ t^*(H) \in T(X)^*.$$

The element
$$t^*(H) \mod T(X) \in T(X)^*/T(X) \cong \mathbb{Z}/2rs\mathbb{Z}$$
is defined canonically by the primitive element $H \in H^2(X, \mathbb{Z})$. We evidently have
$$H^2(X, \mathbb{Z}) = [N(X), T(X), u = H/(2rs) + t^*(H)]$$
where $[\,\cdot\,]$ means "generated by" $\cdot$. The element $t^*(H) \mod T(X)$ evidently distinguishes between different polarized K3 surfaces with Picard number one and the same transcendental periods. More exactly, for another polarized K3 surface (X', H') and its transcendental periods $(T(X'), H^{2,0}(X'))$, the periods of X and X' are isomorphic (and then $X \cong X'$ by the Global Torelli Theorem [15]) if and only if there exists an isomorphism $\phi : T(X) \cong T(X')$ of the transcendental lattices such that $(\phi \otimes \mathbb{C})(H^{2,0}(X)) = H^{2,0}(X')$ and $(\phi \otimes \mathbb{Q})(t^*(H)) \mod T(X) = t^*(H') \mod T(X')$.

Thus, calculation of the periods of X in terms of its transcendental periods is contained in the following statement.

Proposition 3.1. *Let (X, H) be a polarized K3 surface with a primitive polarization H such that $H^2 = 2rs$. Assume that $N(X) = \mathbb{Z}H$ (i. e. $\rho(X) = 1$).*
Then
$$H^2(X, \mathbb{Z}) = [N(X) = \mathbb{Z}H, T(X), H/(2rs) + t^*(H)]$$
where $t^*(H) \in T(X)^*$. The element $t^*(H) \mod T(X)$ is defined uniquely.
Moreover, $H^{2,0}(X) \subset T(X) \otimes \mathbb{C}$.
(In general, when $\rho(X) \geq 1$, one should replace $T(X)$ by $H^{\perp}_{H^2(X,\mathbb{Z})}$.)

Let $Y = M_X(r, H, s)$. Let us calculate periods of Y.
We denote by
$$\widetilde{H}(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) + H^2(X, \mathbb{Z}) + H^4(X, \mathbb{Z}) = U \oplus H^2(X, \mathbb{Z})$$

(it is the direct sum) the Mukai lattice of X. Here $U = \mathbb{Z}e_1 + \mathbb{Z}e_2$ is the hyperbolic plane where canonically $\mathbb{Z}e_1 = H^0(X, \mathbb{Z})$ and $\mathbb{Z}e_2 = H^4(X, \mathbb{Z})$ with the Mukai pairing $e_1^2 = e_2^2 = 0$ and $e_1 \cdot e_2 = -1$. Here $H^2(X, \mathbb{Z})$ is the cohomology lattice of X with the intersection pairing. Here $\oplus$ denotes the orthogonal sum of lattices. We have
$$v = re_1 + se_2 + H. \tag{3.3}$$

By Mukai [5], we have
$$H^2(Y, \mathbb{Z}) = v^{\perp}/\mathbb{Z}v, \tag{3.4}$$

and $H^{2,0}(Y) = H^{2,0}(X)$ by the canonical projection. This defines periods of Y and the isomorphism class of the K3 surface Y (by Global Torelli Theorem [15]). Let us calculate periods of Y similarly to Proposition 3.1.

Any element f of $\widetilde{H}(X, \mathbb{Z})$ can be uniquely written as
$$f = xe_1 + ye_2 + \alpha H/(2rs) + t^*, \ x, y, \alpha \in \mathbb{Z}, \ t^* \in T(X)^*.$$

We have $f \cdot v = -sx - ry + \alpha$, and $f \in v^{\perp}$ if and only if $-sx - ry + \alpha = 0$, and then
$$f = xe_1 + ye_2 + (sx + ry)(H/(2rs)) + t^*.$$

By Proposition 3.1, $f \in \widetilde{H}(X, \mathbb{Z})$ if and only if $t^* = (sx + ry)t^*(H) \mod T(X)$. Since $T(X) \subset v^{\perp}$, we can write
$$f = xe_1 + ye_2 + (sx + ry)\left(H/(2rs) + t^*(H)\right) \mod T(X), \ x, y \in \mathbb{Z}.$$

We denote
$$c = (r, s), \ a = r/c, \ b = s/c.$$

Then $(a, b) = 1$. We have $h = -ae_1 + be_2 \in v^{\perp}$ and $h^2 = 2ab = 2rs/c^2$. Moreover, $h \perp T(X)$ and then $h \perp H^{2,0}(X)$. Thus,
$$h \mod \mathbb{Z}v = -ae_1 + be_2 \mod \mathbb{Z}v \qquad (3.5)$$

gives an element of the Picard lattice $N(Y)$. We have
$$e_1 = \frac{v - ch - H}{2r}, \ e_2 = \frac{v + ch - H}{2s}.$$

It follows that
$$f = \frac{sx + ry}{2rs}v + \frac{c(-sx + ry)}{2rs}h + (sx + ry)t^*(H) \mod T(X), \ x, y \in \mathbb{Z}. \ (3.6)$$

Here $f \mod \mathbb{Z}v$ gives all elements of $H^2(Y, \mathbb{Z})$ and $H^{2,0}(Y) = H^{2,0}(X) \subset T(X) \otimes \mathbb{C}$.

It follows that $f \mod \mathbb{Z}v \in T(Y)$ (where $\mathbb{Z}v$ gives the kernel of $v^{\perp}$ and $H^2(Y, \mathbb{Z}) = v^{\perp}/\mathbb{Z}v$) if and only if $-sx + ry = 0$, equivalently $-bx + ay = 0$, equivalently (since $(a, b) = 1$) $x = az$, $y = bz$ where $z \in \mathbb{Z}$, and then $(sx + ry)t^*(H) = z(sa + rb)t^*(H) = z\,2abc\,t^*(H)$ where $z \in \mathbb{Z}$. It follows that
$$T(Y) = [T(X), 2abc\,t^*(H)]. \qquad (3.7)$$

Since $t^*(H) \mod T(X)$ has the order $2rs = 2abc^2$ in $T(X)^*/T(X) \cong \mathbb{Z}/2rs\mathbb{Z}$, it follows that $[T(Y) : T(X)] = c$ (this is the result of Mukai, [5]).

By (3.6), (3.7), we have $f \perp H^{2,0}(Y) = H^{2,0}(X)$, equivalently $f \mod \mathbb{Z}v \in N(Y)$, if and only if
$$f = \frac{sx + ry}{2rs}v + \frac{c(-sx + ry)}{2rs}h$$

where $sx + ry \equiv 0 \mod 2abc$. Thus, $acx + bcy \equiv 0 \mod 2abc$ and $ax + by \equiv 0 \mod 2ab$. Since $(a, b) = 1$, it follows that $x = b\widetilde{x}$, $y = a\widetilde{y}$ where $\widetilde{x}, \widetilde{y} \in \mathbb{Z}$, and $\widetilde{x} + \widetilde{y} \equiv 0 \mod 2$. Thus, $\widetilde{y} = -\widetilde{x} + 2k$ where $k \in \mathbb{Z}$. It follows that
$$f = \frac{k}{c}v + (-\widetilde{x} + k)h, \ \widetilde{x}, k \in \mathbb{Z}.$$

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Thus, $h \mod \mathbb{Z}v$ generates the Picard lattice $N(Y)$, and $h \mod \mathbb{Z}v$ can be considered as the polarization of Y (or $-h \mod \mathbb{Z}v$ which does not matter from the point of view of periods and the isomorphism class of Y).

Let us calculate $t^*(h) \in T(Y)^*$. Then in (3.6) we should take an element f with $c(-sx + ry)/(2rs) = 1/(2ab)$. Thus, $-sx + ry = c$ or $-bx + ay = 1$. Then

$$t^*(h) = (sx + ry)t^*(H) \mod T(Y).$$

By (3.7), $T(Y)^* = [T(X), ct^*(H)]$ and $T(Y)^*/T(Y) \cong \mathbb{Z}/2ab\mathbb{Z}$.

Thus, $t^*(h) = (bx + ay)(ct^*(H)) \mod [T(X), 2ab(ct^*(H))]$ is defined by $m \equiv bx + ay \mod 2ab$. Since $-bx + ay = 1$, we have $m \equiv 2ay - 1 \equiv -1 \mod 2a$ and $m \equiv 2bx + 1 \equiv 1 \mod 2b$. This defines $m \mod 2ab$ uniquely. We call such $m \mod 2ab$ as Mukai element (compare with [6]). Thus, $m(a,b) \mod 2ab$ is called *Mukai element* if

$$m(a,b) \equiv -1 \mod 2a \quad \text{and} \quad m(a,b) \equiv 1 \mod 2b. \tag{3.8}$$

Thus, $t^*(h) = m(a,b)\, ct^*(H) \mod [T(X), 2abc\, t^*(H)]$.

Thus, finally, we finished the calculation of periods of Y in terms of periods of X (see Proposition 3.1).

Proposition 3.2. *Let (X, H) be a polarized K3 surface with a primitive polarization H such $H^2 = 2rs$, $r, s > 0$. Assume that $N(X) = \mathbb{Z}H$ (i. e. $\rho(X) = 1$). Let $Y = M_X(r, H, s)$. Let $c = (r, s)$ and $a = r/c$, $b = s/c$.*

Then $N(Y) = \mathbb{Z}h$ where $h^2 = 2ab$, $T(Y) = [T(X), 2abc\, t^*(H)]$, $T(Y)^* = [T(X), ct^*(H)]$ and $t^*(h) \mod T(Y) = m(a,b)ct^*(H) \mod T(Y)$ where $m(a,b) \mod 2ab$ is the Mukai element: $m(a,b) \equiv -1 \mod 2a$, $m(a,b) \equiv 1 \mod 2b$. Thus,

$$H^2(Y, \mathbb{Z}) = [N(Y), T(Y), h/(2ab) + t^*(h)] =$$

$$[\mathbb{Z}h,\ [T(X), 2abc\, t^*(H)],\ h/(2ab) + m(a,b)ct^*(H)].$$

(In general, when $\rho(X) \geq 1$, one should replace $T(X)$ by $H^{\perp}_{H^2(X,\mathbb{Z})}$ and $T(Y)$ by $h^{\perp}_{H^2(Y,\mathbb{Z})}$.)

Now let us prove Theorem 3.1. We need to recover r and s from periods of Y. By Proposition 3.2, we have $N(Y) = \mathbb{Z}h$ where $h^2 = 2ab$. Thus, we recover ab. Since $c^2 = 2rs/2ab$, we recover c.

We have $(T(X) \otimes \mathbb{Q}, H^{2,0}(X)) \cong (T(Y) \otimes \mathbb{Q}, H^{2,0}(Y))$. Since X is general, there exists only one such isomorphism up to multiplication by ± 1. It follows that there exists only one (up to multiplication by ± 1) embedding $T(X) \subset T(Y)$ of lattices which identifies $H^{2,0}(X)$ and $H^{2,0}(Y)$. By Proposition 3.2, then $t^*(h) \mod T(Y) = \widetilde{m}(a,b)ct^*(H) \mod T(Y)$ where $\widetilde{m}(a,b) \equiv \pm m(a,b) \mod 2ab$ and $m(a,b)$ is the Mukai element. Assume $p^{\alpha}|ab$ and $p^{\alpha+1}$ does not divide ab where p is prime and $\alpha > 0$. Then $\widetilde{m}(a,b) \equiv \pm 1 \mod 2p^{\alpha}$. Evidently, here only one sing ± 1 is possible, and we denote by a the product of all such p^{α} having $\widetilde{m}(a,b) \equiv -1 \mod 2p^{\alpha}$, and by by b the product of all such remaining p^{α} having $\widetilde{m}(a,b) \equiv 1 \mod 2p^{\alpha}$. If $a > b$, we should change a and b places. Thus,

we recover a and b and the reduced primitive Mukai vector $(r, H, s) = (ac, H, bc)$ such that periods of $M_X(r, H, s)$ are isomorphic to the periods of Y.

This finishes the proof. □

Remark 3.1. The same Propositions 3.1, 3.2 and the proofs are valid for an algebraic K3 surface X and a primitive element $H \in N(X)$ with $H^2 = 2rs \neq 0$, if one replaces $T(X)$ by the orthogonal complement $H^\perp_{H^2(X, \mathbb{Z})}$.

As an example of an application of Theorem 3.1, let us consider the case when $M_X(r, l, s) \cong X$. It is known (e.g. see [19]) that for $l \in N(X)$ and $\pm l^2 > 0$, one has the Tyurin isomorphism

$$Tyu = Tyu(\pm l) \; : \; M_X(\pm l^2/2, l, \pm 1) \cong X \,. \tag{3.9}$$

Existence of such an isomorphism follows at once from Global Torelli Theorem for K3 surfaces [15] using Propositions 3.1, 3.2 and Remark 3.1.

Thus, for a general K3 surface X and a reduced primitive isotropic Mukai vector $v = (r, H, 1)$ where $r = H^2/2$, we have $M_X(r, H, 1) \cong X$. By Theorem 3.1, we then obtain the following result where we also use the well-known fact that $\mathrm{Aut}\,(T(X), H^{2,0}(X)) = \pm 1$ if $\rho(X) = 1$ (see (4.25) below); it is sufficient to consider the automorphism group over $\mathbb{Z}$ for this result.

Corollary 3.2. *Let X be an algebraic K3 surface with $\rho(X) = 1$, i. e. $N(X) = \mathbb{Z}H$ where H is a primitive polarization of X. Let $v = (r, H, s)$ be a reduced primitive isotropic Mukai vector on X (see (3.2)), i. e. $0 < r \le s$.*

Then $M_X(v) \cong X$ if and only if $v = (1, H, H^2/2)$, i. e. $r = 1$, $s = H^2/2$.

4 Isomorphisms between $M_X(v)$ and X for a general K3 surface X with $\rho(X) = 2$

Here we consider general K3 surfaces X with $\rho(X) = \mathrm{rk}\, N(X) = 2$. *Here a K3 surface X will be called general* if the group of automorphisms of the transcendental periods is trivial: $\mathrm{Aut}\,(T(X), H^{2,0}(X)) = \pm 1$.

For $\rho(X) \ge 2$, we don't know when $M_X(v_1) \cong M_X(v_2)$ for primitive isotropic Mukai vectors v_1 and v_2 on X. But we still have the universal isomorphisms T_D, $D \in N(X)$, the reflection δ, the isomorphism $\nu(d_1, d_2)$ and the Tyurin isomorphism Tyu considered in Section 3. They are *universal isomorphisms,* i. e. they are defined for all K3 surfaces, even with Picard number one.

First, we review results of [14] and [3] where for general K3 surfaces X with $\rho(X) = 2$ all primitive isotropic Mukai vectors v with $M_X(v) \cong X$ were found. In particular, we know when $M_X(v_1) \cong M_X(v_2)$ in the case when both moduli are isomorphic to X. The result is that $M_X(v) \cong X$ if and only if there exists such an isomorphism which is a composition of the universal isomorphisms δ, T_D and $\nu(d_1, d_2)$ between moduli of sheaves over X and the Tyurin isomorphism

Tyu between moduli of sheaves over X and X itself. More exactly, the results are as follows.

Using universal isomorphisms T_D, we can assume that the primitive isotropic Mukai vector is
$$v = (r, H, s), \quad r > 0, \ s > 0, \ H^2 = 2rs.$$
(We can even assume that H is ample.) We are interested in the case when $Y = M_X(r, H, s) \cong X$.

We denote $c = (r, s)$ and $a = r/c$, $b = s/c$. Then $(a, b) = 1$. Let H is divisible by $d \in \mathbb{N}$ where $\widetilde{H} = H/d$ is primitive in $N(X)$. Primitivity of $v = (r, H, s)$ means that $(r, d, s) = (c, d) = 1$. Since $\widetilde{H}^2 = 2abc^2/d^2$ is even, we have $d^2 | abc^2$. Since $(a, b) = (c, d) = 1$, it follows that $d = d_a d_b$ where $d_a = (d, a)$, $d_b = (d, b)$, and we can introduce integers

$$a_1 = \frac{a}{d_a^2}, \quad b_1 = \frac{b}{d_b^2}.$$

Then we obtain that $\widetilde{H}^2 = 2a_1 b_1 c^2$.

Let $\gamma = \gamma(\widetilde{H})$ is defined by $\widetilde{H} \cdot N(X) = \gamma \mathbb{Z}$, i.e. $H \cdot N(X) = \gamma d \mathbb{Z}$. Clearly, $\gamma | \widetilde{H}^2 = 2a_1 b_1 c^2$.

We denote
$$n(v) = (r, s, d\gamma) = (r, s, \gamma). \tag{4.1}$$

By Mukai [5], we have $T(X) \subset T(Y)$, and

$$n(v) = [T(Y) : T(X)] \tag{4.2}$$

where $T(X)$ and $T(Y)$ are transcendental lattices of X and Y. Thus,

$$Y \cong X \implies n(v) = (r, s, d\gamma) = (c, d\gamma) = (c, \gamma) = 1. \tag{4.3}$$

Assuming that $Y \cong X$ and then $n(v) = 1$, we have $\gamma | 2a_1 b_1$, and we can introduce

$$\gamma_a = (\gamma, a_1), \ \gamma_b = (\gamma, b_1), \ \gamma_2 = \frac{\gamma}{\gamma_a \gamma_b}. \tag{4.4}$$

Clearly, $\gamma_2 | 2$.

In ([14], Theorem 4.4) the following general theorem had been obtained (about its important particular cases see also [1], [2] and [13]). In the theorem, we use notations c, a, b, d, d_a, d_b, a_1, b_1 introduced above. The same notations γ, γ_a, γ_b and γ_2 as above are used if one replaces $N(X)$ by a 2-dimensional primitive sublattice $N \subset N(X)$, e. g. $\widetilde{H} \cdot N = \gamma \mathbb{Z}$, $\gamma > 0$. We denote $\det N = -\gamma\delta$ and $\mathbb{Z}f(\widetilde{H})$ denotes the orthogonal complement to $\widetilde{H}$ in N.

Theorem 4.1. *Let X be a K3 surface and H a polarization of X such that $H^2 = 2rs$ where $r, s \in \mathbb{N}$. Assume that the Mukai vector (r, H, s) is primitive. Let $Y = M_X(r, H, s)$ be the K3 surface which is the moduli of sheaves over X with the isotropic Mukai vector $v = (r, H, s)$. Let $\widetilde{H} = H/d$, $d \in \mathbb{N}$, be the corresponding primitive polarization.*

We have $Y \cong X$ if there exists $\widetilde{h}_1 \in N(X)$ such that $\widetilde{H}$ and $\widetilde{h}_1$ belong to a 2-dimensional primitive sublattice $N \subset N(X)$ such that $\widetilde{H} \cdot N = \gamma \mathbb{Z}$, $\gamma > 0$, $(c, d\gamma) = 1$, and the element $\widetilde{h}_1$ belongs to the a-series or to the b-series described below:

$\widetilde{h}_1$ belongs to the a-series if

$$\widetilde{h}_1^2 = \pm 2b_1 c, \quad \widetilde{H} \cdot \widetilde{h}_1 \equiv 0 \mod \gamma(b_1/\gamma_b)c, \quad f(\widetilde{H}) \cdot \widetilde{h}_1 \equiv 0 \mod \delta b_1 c \quad (4.5)$$

(where $\gamma_b = (\gamma, b_1)$);

$\widetilde{h}_1$ belongs to the b-series if

$$\widetilde{h}_1^2 = \pm 2a_1 c, \quad \widetilde{H} \cdot \widetilde{h}_1 \equiv 0 \mod \gamma(a_1/\gamma_a)c, \quad f(\widetilde{H}) \cdot \widetilde{h}_1 \equiv 0 \mod \delta a_1 c \quad (4.6)$$

(where $\gamma_a = (\gamma, a_1)$).

These conditions are necessary to have $Y \cong X$ if $\rho(X) \leq 2$ and X is a general K3 surface with its Picard lattice, i. e the automorphism group of the transcendental periods $(T(X), H^{2,0}(X))$ is ± 1.

In [3], the theorem 4.1 was geometrically interpreted as follows.

Theorem 4.2. *Let X be a K3 surface and H a polarization of X such that $H^2 = 2rs$ where $r, s \in \mathbb{N}$. Assume that the Mukai vector (r, H, s) is primitive. Let $Y = M_X(r, H, s)$ be the K3 surface which is the moduli of sheaves over X with the isotropic Mukai vector $v = (r, H, s)$. Let $\widetilde{H} = H/d$, $d \in \mathbb{N}$, be the corresponding primitive polarization.*

Assume that there exists $\widetilde{h}_1 \in N(X)$ such that $\widetilde{H}$ and $\widetilde{h}_1$ belong to a 2-dimensional primitive sublattice $N \subset N(X)$ such that $\widetilde{H} \cdot N = \gamma \mathbb{Z}$, $\gamma > 0$, $(c, d\gamma) = 1$, and the element $\widetilde{h}_1$ belongs to the a-series or to the b-series described below:

$\widetilde{h}_1$ belongs to the a-series if

$$\widetilde{h}_1^2 = \pm 2b_1 c, \quad \widetilde{H} \cdot \widetilde{h}_1 \equiv 0 \mod \gamma(b_1/\gamma_b)c, \quad f(\widetilde{H}) \cdot \widetilde{h}_1 \equiv 0 \mod \delta b_1 c \quad (4.7)$$

(where $\gamma_b = (\gamma, b_1)$);

$\widetilde{h}_1$ belongs to the b-series if

$$\widetilde{h}_1^2 = \pm 2a_1 c, \quad \widetilde{H} \cdot \widetilde{h}_1 \equiv 0 \mod \gamma(a_1/\gamma_a)c, \quad f(\widetilde{H}) \cdot \widetilde{h}_1 \equiv 0 \mod \delta a_1 c \quad (4.8)$$

(where $\gamma_a = (\gamma, a_1)$).

Then we have:

If $\widetilde{h}_1$ belongs to the a-series, then

$$\widetilde{h}_1 = d_2 \widetilde{H} + b_1 cD \text{ for some } d_2 \in \mathbb{N}, \ D \in N, \quad (4.9)$$

which defines the isomorphism

$$Tyu(\pm \widetilde{h}_1) \cdot T_D \cdot \nu(1, d_2) \cdot \delta \cdot \nu(d_a, d_b)^{-1} : Y = M_X(r, H, s) \cong X. \quad (4.10)$$

If $\widetilde{h}_1$ belongs to the b-series, then

$$\widetilde{h}_1 = d_2\widetilde{H} + a_1 cD \ \textit{for some } d_2 \in \mathbb{N}, \ D \in N, \qquad (4.11)$$

which defines the isomorphism

$$Tyu(\pm\widetilde{h}_1) \cdot T_D \cdot \nu(1, d_2) \cdot \nu(d_a, d_b)^{-1} : Y = M_X(r, H, s) \cong X. \qquad (4.12)$$

Since conditions of Theorems 4.1, 4.2 are necessary for general K3 surfaces with $\rho(X) \le 2$, we obtain

Theorem 4.3. *Let X be a K3 surface with a polarization H such that $H^2 = 2rs$, $r, s \ge 1$, the Mukai vector (r, H, s) be primitive, and $Y = M_X(r, H, s)$ be the moduli of sheaves over X with the isotropic Mukai vector (r, H, s). Assume that $\rho(X) \le 2$ and X is general with its Picard lattice (i. e. the automorphism group of the transcendental periods $\operatorname{Aut}(T(X), H^{2,0}(X)) = \pm 1$). Let $\widetilde{H} = H/d$, $d \in \mathbb{N}$, be the corresponding primitive polarization.*

Then $Y = M_X(r, H, s)$ is isomorphic to X if and only if there exists $d_2 \in \mathbb{N}$ and $D \in N = N(X)$ such that
   either

$$\widetilde{h}_1 = d_2\widetilde{H} + b_1 cD \ \textit{has } \widetilde{h}_1^2 = \pm 2b_1 c, \qquad (4.13)$$

which defines the isomorphism

$$Tyu(\pm\widetilde{h}_1) \cdot T_D \cdot \nu(1, d_2) \cdot \delta \cdot \nu(d_a, d_b)^{-1} : Y = M_X(r, H, s) \cong X, \qquad (4.14)$$

   or

$$\widetilde{h}_1 = d_2\widetilde{H} + a_1 cD \ \textit{has } \widetilde{h}_1^2 = \pm 2a_1 c, \qquad (4.15)$$

which defines the isomorphism

$$Tyu(\pm\widetilde{h}_1) \cdot T_D \cdot \nu(1, d_2) \cdot \nu(d_a, d_b)^{-1} : Y = M_X(r, H, s) \cong X. \qquad (4.16)$$

Theorem 3.1 clarifies appearance of the isomorphisms T_D, δ, $\nu(d_1, d_2)$ and Tyu in these results for Picard number 2. They are universal and exist for all K3 surfaces; moreover, they are all isomorphisms which one needs to distinguish isomorphic moduli $M_X(v)$ for isotropic Mukai vectors v on a general K3 surface X. Thus, appearance of the isomorphisms T_D, δ, $\nu(d_1, d_2)$ and Tyu is very natural in the results above.

Remark 4.1. For The Picard number $\rho(X) = 1$, Theorems 4.1, 4.2 and 4.3 are formally equivalent to Corollary 3.2. Really, for $\rho(X) = 1$ we have $\gamma = 2a_1 b_1 c^2$. Thus, $(\gamma, c) = 1$ implies that $c = 1$. Then $\gamma = 2a_1 b_1$ and $\gamma_2 = 2$, $\gamma_a = a_1$, $\gamma_b = b_1$. Conditions of Theorem 4.1 can be satisfied only for $\widetilde{h}_1 = \widetilde{H}$ which implies that $a_1 = 1$ for the a-series, and $b_1 = 1$ for the b-series (one can formally put $f(\widetilde{H}) = 0$). Thus, for $\rho(X) = 1$ and general X, we have $Y \cong X$ if and only if $c = 1$ and either $a_1 = 1$ or $b_1 = 1$. This is equivalent to Corollary 3.2.

Under conditions of Theorem 4.1, let us assume that for a primitive 2-dimensional sublattice $N \subset N(X)$ an element $\widetilde{h}_1 \in N$ with $\widetilde{h}_1^2 = \pm 2b_1c$ belongs to the a-series. This is equivalent to the condition (4.9) of Theorem 4.2. Replacing $\widetilde{h}_1$ by $-\widetilde{h}_1$ if necessary, we see that (4.9) is equivalent to

$$\widetilde{h}_1 = d_2\widetilde{H} + b_1cD, \quad d_2 \in \mathbb{Z}, \ D \in N. \tag{4.17}$$

Since $\widetilde{H}$ is primitive, the lattice N has a basis $\widetilde{H}, D \in N$, i. e. $N = [\widetilde{H}, D]$. Since $\widetilde{H} \cdot N = \gamma\mathbb{Z}$ where $(\gamma, c) = 1$, the matrix of N in this basis is

$$\begin{pmatrix} \widetilde{H}^2 & \widetilde{H} \cdot D \\ \widetilde{H} \cdot D & D^2 \end{pmatrix} = \begin{pmatrix} 2a_1b_1c^2 & \gamma k \\ \gamma k & 2t \end{pmatrix} \tag{4.18}$$

where $k, t \in \mathbb{Z}$ and $\gamma | 2a_1b_1$, $(\gamma, c) = 1$ and $(2a_1b_1c^2/\gamma, k) = 1$.

The condition of a-series (4.17) is then equivalent to existence of $\widetilde{h}_1 \in [\widetilde{H}, b_1cN] = [\widetilde{H}, b_1cD]$ with $\widetilde{h}_1^2 = \pm 2b_1c$. Thus, the lattice $N_1 = [\widetilde{H}, b_1cD]$ with the matrix

$$\begin{pmatrix} 2a_1b_1c^2 & b_1c\gamma k \\ b_1c\gamma k & b_1^2c^22t \end{pmatrix} \tag{4.19}$$

must have $\widetilde{h}_1$ with $\widetilde{h}_1^2 = \pm 2b_1c$. Writing $\widetilde{h}_1$ as $\widetilde{h}_1 = x\widetilde{H} + yb_1cD$, we obtain that the quadratic equation $a_1cx^2 + \gamma kxy + b_1cty^2 = \pm 1$ must have an integral solution. Similarly, for b-series we obtain the equation $b_1cx^2 + \gamma kxy + a_1cty^2 = \pm 1$. Thus, we finally obtain a very elementary reformulation of the results above.

Lemma 4.1. *For the matrix* (4.18) *of the lattice N in Theorems 4.1, 4.2 and 4.3, the conditions of a-series are equivalent to existence of an integral solution of the equation*

$$a_1cx^2 + \gamma kxy + b_1cty^2 = \pm 1, \tag{4.20}$$

and for b-series of the equation

$$b_1cx^2 + \gamma kxy + a_1cty^2 = \pm 1. \tag{4.21}$$

This calculation has a very important corollary. Let us assume that a prime $p | \gamma_b = (\gamma, b_1)$. Then for the equation (4.20) we obtain a congruence $a_1cx^2 \equiv \pm 1$ mod p. Thus, $\pm a_1c$ is a square mod p. Similarly, for the equation (4.21), we obtain that $\pm b_1c$ is a square mod p for a prime $p | \gamma_a = (\gamma, a_1)$.

Thus, we obtain an important necessary condition of $Y = M_X(v) \cong X$ for $\rho(X) = 2$.

Theorem 4.4. *Let X be a K3 surface with a polarization H such that $H^2 = 2rs$, $r, s \geq 1$, the Mukai vector (r, H, s) be primitive, and $Y = M_X(r, H, s)$ be the moduli of sheaves over X with the isotropic Mukai vector (r, H, s). Assume that $\rho(X) \leq 2$ and X is general with its Picard lattice (i. e. the automorphism group of the transcendental periods $\operatorname{Aut}(T(X), H^{2,0}(X)) = \pm 1$). Let $\widetilde{H} = H/d$, $d \in \mathbb{N}$, be the corresponding primitive polarization, $\widetilde{H} \cdot N(X) = \gamma\mathbb{Z}$ and $(\gamma, c) = 1$.*

Then $Y = M_X(r, H, s) \cong X$ implies that for one of $\pm$ either

$$\forall\, p | \gamma_b \implies \left(\frac{\pm a_1 c}{p}\right) = 1 \qquad (4.22)$$

or

$$\forall p | \gamma_a \implies \left(\frac{\pm b_1 c}{p}\right) = 1. \qquad (4.23)$$

Here p means any prime, and $\left(\frac{x}{2}\right) = 1$ means that $x \equiv 1 \mod 8$.
  Thus, if

$$\forall\, \pm \left(\exists\, p | \gamma_b \text{ such that } \left(\frac{\pm a_1 c}{p}\right) = -1 \text{ and } \exists p | \gamma_a \text{ such that } \left(\frac{\pm b_1 c}{p}\right) = -1\right)$$
$$(4.24)$$

then $Y = M_X(r, H, s)$ is not isomorphic to X for a general (for its Picard lattice) K3 surface X with $\rho(X) \leq 2$.

Example 4.1. Assume that $a_1 = 5$, $b_1 = 13$, $c = 1$ and $\gamma = 5 \cdot 13$ (or $\gamma = 2 \cdot 5 \cdot 13$). Obviously, then (4.24) is valid. Thus, for

$$v = (5, H, 13), \quad H^2 = 2 \cdot 5 \cdot 13, \text{ and } \gamma = 5 \cdot 13 \text{ or } 2 \cdot 5 \cdot 13$$

(then H is always primitive), for any general K3 surface X with $\rho(X) = 2$ and any $H \in N(X)$ with $H^2 = 2 \cdot 5 \cdot 13$ and $H \cdot N(X) = \gamma \mathbb{Z}$, the moduli $Y = M_X(v)$ are not isomorphic to X. There are a lot of such Picard lattices given by (4.18).

In [13], it is shown that any primitive isotropic Mukai vector $v = (r, H, s)$ with $H^2 = 2rs$ and $\gamma = 1$ can be realized by a general K3 surface X with Picard number 2 and $Y = M_X(v) \cong X$. It is possible that Theorem 4.4 gives all necessary conditions to have similar result for any γ. We hope to consider this problem later.

Importance of these results for general K3 surfaces X with $\rho(X) = 2$ is that these results describe *all divisorial conditions on moduli of polarized K3 surfaces which imply that $Y = M_X(r, H, s) \cong X$.* Let us consider the corresponding simple general arguments.

It is well-known (see [9] and [11] where it was observed first, it seems) that $Aut(T(X), H^{2,0}(X)) \cong C_m$ is a finite cyclic group of the order m, and its representation in $T(X) \otimes \mathbb{Q}$ is the sum of irreducible representations of the dimension $\phi(m)$ ($\phi(m)$ is the Euler function), and $H^{2,0}(X)$ is a line in one of eigen-spaces of C_m. In particular, $\phi(m) | rkT(X)$ and the dimension of moduli of these X is equal to

$$\dim Mod(X) = \frac{\operatorname{rk} T(X)}{\phi(m)} - 1 \qquad (4.25)$$

if $m > 2$. If $m = 1, 2$, then $\dim Mod(X) = \operatorname{rk} T(X) - 2$.

Let us consider polarized K3 surfaces (X, H) with the polarization $H^2 = 2rs$ and a primitive isotropic Mukai vector (r, H, s), $r, s > 0$. Let us assume that $Y = M_X(r, H, s) \cong X$.

If $\rho(X) = 1$, then rk $T(X) = 21$ and $\phi(m)|21$. It follows that $m = 1$ or $m = 2$ because 21 is odd. Thus $Aut(T(X), H^{2,0}(X)) = \pm 1$, and then $c = 1$ and either $a_1 = 1$ or $b_1 = 1$ by Corollary 3.2 (or Remark 4.1). By specialization principle (see Lemma 2.1.1 in [14]), then $Y \cong M_X(r, H, s)$ for all K3 surfaces X and the Mukai vector with these invariants:

$$c = 1, \text{ and either } a_1 = 1 \text{ or } b_1 = 1. \tag{4.26}$$

Now let us assume that (r, H, s) does not satisfy (4.26), but $Y = M_X(r, H, s)$ $\cong X$. By Corollary 3.2, then $\rho(X) \neq 1$. Then $\rho(X) \geq 2$ and $\dim Mod(X) \leq 20 - \rho(X) \leq 18$. Thus, a divisorial condition on moduli or polarized K3 surfaces (X, H) to have $Y = M_X(r, H, s) \cong X$ means that $\rho(X) = 2$ for a general K3 surface satisfying this condition. All these conditions are described by the isomorphism classes of $H \in N(X)$ where rk $N(X) = 2$ and $H \in N(X)$ satisfies the equivalent (for this case) Theorems 4.1, 4.2 or 4.3. If $H \in N \subset N(X)$ is a primitive sublattice of the rank two and $H \in N$ satisfies equivalent Theorems 4.1 4.2, then $Y = M_X(r, H, s) \cong X$ by the specialization principle. This means that X belongs to the closure of the divisor defined by the moduli of polarized K3 surfaces (X', H) with the Picard lattice $N(X') = N$ of the rank two. Thus, $Y' = M_X(r, H, s) \cong X'$ because X' satisfies the divisorial condition $H \in N$ where $H \in N \subset N(X')$.

By Theorem 4.4 we obtain

Theorem 4.5. *Let for $r, s \geq 1$,*

$$v = (r, H, s), \ H^2 = 2rs$$

be a type of a primitive isotropic Mukai vector, and $\gamma | 2a_1 b_1$, and $(\gamma, c) = 1$.

Then, if (4.24) is valid, there does not exist a divisorial condition on moduli of polarized K3 surfaces (X, H) which implies $Y = M_X(r, H, s) \cong X$ and $H \cdot N(X) = \gamma \mathbb{Z}$. Thus, these K3 surfaces have codimension at least 2 in 19-dimensional moduli of polarized K3 surfaces (X, H).

For example, this is valid for $r = 5$, $s = 13$ (then H is primitive and $d = 1$), and $\gamma = 5 \cdot 13$ (or $\gamma = 2 \cdot 5 \cdot 13$).

In the section below, we will show that the numerical example of Theorem 4.5 can be satisfied by K3 surfaces X with $\rho(X) = 3$. Thus, these K3 surfaces define a 17-dimensional submoduli in 19-dimensional moduli of polarized K3 surfaces. This submoduli cannot be extended to a divisor in moduli preserving the condition $Y = M_X(r, H, s) \cong X$.

5 Isomorphisms between $M_X(v)$ and X for a general K3 surface X with $\rho(X) \geq 3$

Here we show that it is interesting and non-trivial to generalize results of the previous section to $\rho(X) \geq 3$.

Let $K = [e_1, e_2, (e_1 + e_2)/2]$ be a negative definite 2-dimensional lattice with $e_1^2 = -6$, $e_2^2 = -34$ and $e_1 \cdot e_2 = 0$. Then $((e_1 + e_2)/2)^2 = (-6 - 34)/4 = -10$ is even, and the lattice K is even. Since $6x^2 + 34y^2 = 8$ has no integral solutions, it follows that K has no elements $\delta \in K$ with $\delta^2 = -2$. Let us consider the lattice

$$S = \mathbb{Z}H \oplus K$$

which is the orthogonal sum of $\mathbb{Z}H$ with $H^2 = 2 \cdot 5 \cdot 13$ and the lattice K. By standard results about K3 surfaces, there exists a polarized K3 surface (X, H) with the Picard lattice S and the polarization $H \in S$. We then have $H \cdot S = 2 \cdot 5 \cdot 13\,\mathbb{Z}$. Thus, $\gamma = 2 \cdot 5 \cdot 13$.

Let $Y = M_X(5, H, 13)$. We have the following result. Perhaps, it gives the main result of the paper.

Theorem 5.1. *For any polarized K3 surface (X, H) with $N(X) = S$ where S is the hyperbolic lattice of the rank 3 defined above, one has $Y = M_X(5, H, 13) \cong X$ which gives a 17-dimensional moduli M_S of polarized K3 surfaces (X, H) with $Y = M_X(5, H, 13) \cong X$.*

On the other hand, M_S is not contained in any 18-dimensional moduli M_N of polarized K3 surfaces (X', H) where $H \in N(X') = N \subset S$ is a primitive sublattice of rk $N = 2$ and $M_{X'}(5, H, 13) \cong X'$. Thus, M_S is not defined by any divisorial condition on moduli of polarized K3 surfaces (X, H) (or, it is not a specialization of) implying $M_X(5, H, 13) \cong X$.

Proof. For this case, $c = (5, 13) = 1$ and $(\gamma, c) = 1$. By Mukai results (3.4) and (4.2), then the transcendental periods $(T(X), H^{2,0}(X))$ and $(T(Y), H^{2,0}(Y))$ are isomorphic. The discriminant group $A_S = S^*/S$ of the lattice $S = T(X)^\perp$ is a cyclic group $\mathbb{Z}/(2 \cdot 5 \cdot 13 \cdot 3 \cdot 17\,\mathbb{Z})$. Thus, the minimal number $l(A_S)$ of generators of A_S is one. Thus, $l(A_S) \le$ rk $S - 2$. By Theorem 1.14.4 in [10], a primitive embedding of $T(X)$ into the cohomology lattice of K3 (which is an even unimodular lattice of signature $(3, 19)$) is then unique, up to isomorphisms. It follows that the isomorphism between transcendental periods of X and Y can be extended to an isomorphism between periods of X and Y. By Global Torelli Theorem for K3 surfaces [15], the K3 surfaces X and Y are isomorphic. (These considerations are now standard.)

Let $H \in N \subset S$ be a primitive sublattice with rk $N = 2$. Since $H \cdot S = H \cdot H\mathbb{Z} = 2 \cdot 5 \cdot 13\mathbb{Z}$, it follows that $H \cdot N = 2 \cdot 5 \cdot 13\mathbb{Z}$, and the invariant $\gamma = 2 \cdot 5 \cdot 13$ is the same for any sublattice $N \subset S$ containing H. By Theorem 4.5, then $M_{X'}(r, H, s)$ is not isomorphic to X' for any general K3 surface (X', H) with $N(X') = N$.

This finishes the proof. $\qquad\square$

Similar arguments can be used to prove the following general statement for $\rho(X) \ge 12$ which shows that there are many cases when $Y = M_X(r, H, s) \cong X$ which don't follow from divisorial conditions on moduli. Its first statement is well-known (e. g. see Proposition 2.2.1 in [1]).

Theorem 5.2. *Let (X, H) be a polarized K3 surface with $\rho(X) \geq 12$, and for $r, s \geq 1$ let (r, H, s) be a primitive isotropic Mukai vector on X, i. e. $H^2 = 2rs$ and $(c, d) = 1$. Assume that $\widetilde{H} \cdot N(X) = \gamma\mathbb{Z}$.*

Then $Y = M_X(r, H, s) \cong X$ if $(\gamma, c) = 1$ (Mukai necessary condition).

On the other hand, if (4.24) satisfies, the isomorphism $Y = M_X(r, H, s) \cong X$ does not follow from any divisorial condition on moduli of polarized K3 surfaces. I. e. for any primitive 2-dimensional sublattice $H \in N \subset N(X)$, there exists a polarized K3 surface (X', H) with $N(X') = N$ such that $Y' = M_{X'}(r, H, s)$ is not isomorphic to X'.

Proof. Since $\rho(X) \geq 12$, then rk $T(X) \leq 22 - 12 = 10$ and $l(A_{T(X)}) \leq$ rk $T(X) = 10$. Since $N(X)$ and $T(X)$ are orthogonal complements to one another in the unimodular lattice $H^2(X, \mathbb{Z})$, it follows that $A_{N(X)} \cong A_{T(X)}$ and $l(A_{N(X)}) \leq 10 \leq$ rk $N(X) - 2$. By Theorem 1.14.4 in [10], a primitive embedding of $T(X)$ into the cohomology lattice of K3 is then unique up to isomorphisms. Like in the proof of Theorem 5.1, it follows that $Y \cong X$.

Let us prove the second statement. Since $H \cdot N(X) = \gamma\mathbb{Z}$ and $H \in N \subset N(X)$, it follows that $H \cdot N(X') = \gamma(N)\mathbb{Z}$ where $\gamma|\gamma(N)$. If $(c, \gamma(N)) > 1$, then Y' is not isomorphic to X because $[T(Y') : T(X')] = (c, \gamma(N)) > 1$ by Mukai's result (4.2). Assume $(c, \gamma(N)) = 1$. Obviously, (4.24) for γ implies (4.24) for $\gamma(N)$. Let X' be a general K3 surface with $N(X') = N$. By Theorem 4.4, then $Y' = M_{X'}(r, H, s)$ is not isomorphic to X'.

This finishes the proof. $\square$

Theorems 5.1 and 5.2 can be unified to the following the most general (known) statement: when $Y = M_X(r, H, s) \cong X$ for any primitive isotropic Mukai vector on X satisfying Mukai's necessary condition.

Theorem 5.3. *Let X be a K3 surface, the Picard lattice $N(X)$ is unique in its genus, and the natural homomorphism*

$$O(N(X)) \to O(q_{N(X)})$$

is surjective where $q_{N(X)}$ is the discriminant quadratic form of $N(X)$. Equivalently, any isomorphism of the transcendental periods of X and another K3 surface can be extended to an isomorphisms of periods of X and the other K3 surface.

Then for any primitive isotropic Mukai vector $v = (r, H, s)$ on X such that $(c, \gamma) = 1$ (Mukai necessary condition), one has $Y = M_X(r, H, s) \cong X$.

On the other hand, if X is general, i. e. $\mathrm{Aut}(T(X), H^{2,0}(X)) = \pm 1$, and (4.24) satisfies, then the isomorphism $Y = M_X(r, H, s) \cong X$ does not follow from any divisorial condition on moduli of polarized K3 surfaces (X, H). I. e. for any primitive 2-dimensional sublattice $H \in N \subset N(X)$, there exists a polarized K3 surface (X', H) with $N(X') = N$ such that $Y' = M_{X'}(r, H, s)$ is not isomorphic to X'.

The results of Section 4 and these results suggest the following general concepts.

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Let $r \in \mathbb{N}$ and $s \in \mathbb{Z}$. We formally put $H^2 = 2rs$ and introduce $c = (r,s)$ and $a = r/c$, $b = s/c$. Let $d \in \mathbb{N}$, $(d,c) = 1$ and $d^2 | ab$. We call

$$(r, H, s), \ H^2 = 2rs, \ d \tag{5.1}$$

the *type of a primitive isotropic Mukai vector of K3.* Clearly, *a Mukai vector of the type* (5.1) *on a K3 surface* X is just an element $H \in N(X)$ such that $H^2 = 2rs$ and $\widetilde{H} = H/d$ is primitive. Like above, we introduce $d_a = (d, a)$, $d_b = (d, b)$ and put $a_1 = a/d_a^2$, $b_1 = b/d_b^2$. Then $\widetilde{H}^2 = 2a_1 b_1 c^2$.

Let N be an even lattice which can be primitively embedded into a Picard lattice of some algebraic K3 surface (equivalently, there exists a Kählerian K3 surface with this Picard lattice). This is equivalent for N to be either negative definite, or semi-negative definite with 1-dimensional kernel, or hyperbolic (i. e. N has the signature $(1, \rho - 1)$), and to have a primitive embedding into an even unimodular lattice of the signature $(3, 19)$. Further, we call N as an *abstract K3 Picard lattice* (or just a K3 Picard lattice). Let $H \in N$. We call $H \in N$ as a *polarized (abstract) K3 Picard lattice,* in spite of H^2 can be non-positive. We consider such pairs up to natural isomorphisms. Another polarized K3 Picard lattice $H' \in N'$ is called isomorphic to $H \in N$ if there exists an isomorphism $f : N \cong N'$ of lattices such that $f(H) = H'$.

Definition 5.1. Let us fix a type (5.1) of a primitive isotropic Mukai vector of K3. A polarized K3 Picard lattice $H \in N$ is called *critical for correspondences of a K3 surface with itself via moduli of sheaves for the type* (5.1) *of Mukai vector* (further we abbreviate this as $H \in N$ *is a critical polarized K3 Picard lattice for the type* (5.1)) if $H^2 = 2rs$ and $\widetilde{H} = H/d \in N$ is primitive; moreover the conditions (a) and (b) below satisfy:

(a) for any K3 surface X such that $H \in N \subset N(X)$ is a primitive sublattice, one has $Y = M_X(r, H, s) \cong X$.

(b) the condition (a) above is not valid if one replaces $H \in N$ by $H \in N_1$ for any primitive sublattice $H \in N_1 \subset N$ of N of strictly smaller rank rk $N_1 <$ rk N.

In ([14], Theorem 2.3.3), for a polarized K3 Picard lattice $H \in N$, the criterion is given for a general (and then any) K3 surface with $H \in N = N(X)$ to have $Y = M_X(r, H, s) \cong X$. Moreover, by the specialization principle (Lemma 2.1.1 in [14]), if this criterion is satisfied, then $Y = M_X(r, H, s) \cong X$ for any K3 surface X such that $H \in N \subset N(X)$ is a primitive sublattice. Thus, for the problem of describing in terms of Picard lattices, of all K3 surfaces X such that $Y = M_X(r, H, s) \cong X$, the main problem is as follows.

Problem 5.1. *For a given type of a primitive isotropic Mukai vector* (5.1) *of K3, describe all* **critical polarized K3 Picard lattices** $H \in N$ *(for the problem of correspondences of a K3 surface with itself via moduli of sheaves).*

Now we have the following examples of solution of this problem.

By (3.9), or Corollary 3.2, or Remark 4.1, we have classification of critical polarized K3 Picard lattices of the rank one.

Example 5.1. For the type (r, H, s), $H^2 = 2rs$, d where $c = 1$ and either $a_1 = 1$ or $b_1 = \pm 1$, we obtain that $N = \mathbb{Z}\widetilde{H}$ where $\widetilde{H}^2 = 2a_1 b_1$ gives all critical polarized K3 Picard lattices $H = d\widetilde{H} \in N$ of the rank one.

Example 5.2. For the type of Mukai vector which is different from Example 5.1, classification of the critical polarized K3 Picard lattices of the rank 2 is given by equivalent Theorems 4.1, 4.2 or 4.3.

Example 5.3. For the Mukai vector of the type $(5, H, 13)$ with $H^2 = 2 \cdot 5 \cdot 13$ and $d = 1$, the polarized Picard lattice $H \in S$ of Theorem 5.1 is critical of the rank rk S=3, by Theorem 5.1. Obviously, there are plenty of similar examples. It would be very interesting and non-trivial to find all critical polarized K3 Picard lattices $H \in S$ of the rank 3.

Example 5.4. By Theorem 5.2, we should expect that there exist critical polarized K3 Picard lattices of the rank more than 3. On the other hand, the same Theorem 5.2 gives that the rank of a critical polarized K3 Picard lattice is less or equal to 12.

We have

Theorem 5.4. *For any type (r, H, s), $H^2 = 2rs$ and d of a primitive isotropic Mukai vector of K3, the rank of a critical polarized K3 Picard lattice $H \in N$ is not more than 12: we have rk $N \leq 12$.*

Proof. Let $H \in N$ be a critical polarized K3 Picard lattice of this type and rk $N \geq 13$. Let us take any primitive sublattice $H \in N' \subset N$ of the rk $N' = 12$ such that $\widetilde{H} \cdot N' = \widetilde{H} \cdot N$. Obviously, it does exist. Let X be an algebraic K3 surface such that $H \in N' \subset N(X)$. Then rk $N(X) \geq 12$ and $Y = M_X(r, H, s) \cong X$ by Theorem 5.2.

Then the condition (b) of Definition 5.1 is not satisfied, and we get a contradiction. Thus, rk $N \leq 12$.

This finishes the proof. $\square$

It would be very interesting to give an exact estimate for the rank of critical polarized K3 Picard lattices.

Problem 5.2. *For a given type (5.1) of a primitive isotropic Mukai vector of K3, give the exact estimate of the rank rk N of a critical polarized K3 Picard lattices $H \in N$ of this type (for the problem of isomorphisms of K3 surfaces with itself).*

Now we don't know the answer to this problem for any type (5.1) different from Example 5.1.

6 Compositions of correspondences of a K3 surface with itself via moduli of sheaves. The general Problem of classification of correspondences of a K3 surface with itself via moduli of sheaves

Here we want to interpret the results above in terms of the action of correspondences on 2-dimensional cohomology lattice of a K3 surface. Moreover, we try to formulate the general problem of classification of correspondences of a K3 surface with itself via moduli of sheaves.

Let $v = (r, H, s)$ be a primitive isotropic Mukai vector on a K3 surface X and $Y = M_X(r, H, s)$. We denote by π_X and π_Y the corresponding projections of $X \times Y$ to X and Y respectively.

By Mukai ([5], Theorem 1.5), the corresponding quasi-universal sheaf $\mathcal{E}$ on $X \times Y$ and defined by this sheaf algebraic cycle

$$Z_{\mathcal{E}} = \left(\pi_X{}^* \sqrt{td_X} \right) \cdot ch(\mathcal{E}) \cdot \left(\pi_Y{}^* \sqrt{td_Y} \right) / \sigma(\mathcal{E}) \tag{6.1}$$

give the isomorphism of the full cohomology groups and the corresponding Hodge structures

$$f_{Z_{\mathcal{E}}} : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q}), \quad t \mapsto \pi_{Y*}(Z_{\mathcal{E}} \cdot \pi_X{}^* t). \tag{6.2}$$

Moreover, according to Mukai, it defines the isomorphism of lattices (or isometry)

$$f_{Z_{\mathcal{E}}} : v^{\perp} \to H^4(Y, \mathbb{Z}) \oplus H^2(Y, \mathbb{Z})$$

where $f_{Z_{\mathcal{E}}}(v) = w \in H^4(Y, \mathbb{Z})$ is the fundamental cocycle, and the orthogonal complement $v^{\perp}$ is taken in the Mukai lattice $\widetilde{H}(X, \mathbb{Z})$. It follows Mukai's formula (3.4) which we have used in Sect. 3.

In particular, taking the composition of $f_{Z_{\mathcal{E}}}$ with the projection $\pi : H^4(Y, \mathbb{Z}) \oplus H^2(Y, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$, we obtain the embedding of lattices

$$f_{Z_{\mathcal{E}}} : H^{\perp}_{H^2(X, \mathbb{Z})} \to H^2(Y, \mathbb{Z})$$

which can be extended to the isometry

$$\widetilde{f}_{Z_{\mathcal{E}}} : H^2(X, \mathbb{Q}) \to H^2(Y, \mathbb{Q}) \tag{6.3}$$

of the quadratic forms over $\mathbb{Q}$ by Witt's Theorem.

If $H^2 = 0$, this extension is unique.

If $H^2 \neq 0$, there are two such extensions different by ± 1 on $\mathbb{Z}H$. Let us agree to take

$$\widetilde{f}_{Z_{\mathcal{E}}}(\widetilde{H}) = ch \tag{6.4}$$

where h is defined in (3.5), and we use Proposition 3.2 which relates periods of X and Y.

The defined Hodge isometry (6.3) can be considered as a little change of the Mukai's algebraic cycle (6.1) to get an isometry in second cohomologies. Clearly, it is also defined by some algebraic cycle because changes the Mukai isomorphism (6.2) only in the algebraic part.

By Proposition 3.2, we obtain that the isometry $\widetilde{f}_{\mathcal{E}}$ is given by the embeddings

$$\widetilde{H}^{\perp} \subset h^{\perp} = [\widetilde{H}^{\perp}, 2a_1 b_1 ct^*(\widetilde{H})], \ \mathbb{Z}\widetilde{H} \subset \mathbb{Z}h, \ \widetilde{H} = ch, H^{2,0}(X) = H^{2,0}(Y). \tag{6.5}$$

This identifies quadratic forms $H^2(X, \mathbb{Q}) = H^2(Y, \mathbb{Q})$ over $\mathbb{Q}$, and the lattices $H^2(X, \mathbb{Z})$, $H^2(Y, \mathbb{Z})$ as its two sublattices.

Let

$$O(H^2(X, \mathbb{Q}))_0 = \{f \in O(H^2(X, \mathbb{Q})) \mid f | T(X) = \pm 1\} \cong$$

$$\cong O(N(X) \otimes \mathbb{Q}) \times \{\pm 1_{T(X)}\},$$

and

$$O(H^2(X, \mathbb{Z}))_0 = O(H^2(X, \mathbb{Z})) \cap O(H^2(X, \mathbb{Q}))_0.$$

By Global Torelli Theorem for K3 surfaces [15], we immediately obtain:

Proposition 6.1. *If a K3 surface X is general for its Picard lattice, then $Y = M_X(r, H, s) \cong X$ if and only if there exists an automorphism $\phi(r, H, s) \in O(H^2(X, \mathbb{Q})_0)$ such that $\phi(H^2(X, \mathbb{Z})) = H^2(Y, \mathbb{Z})$.*

If $Y \cong X$, then we can give the definition.

Definition 6.1. If $Y = M_X(r, H, s) \cong X$ and X is general for its Picard lattice, then the isomorphism of Proposition 6.1

$$\phi(r, H, s) \mod O(H^2(X, \mathbb{Z}))_0 \in O(H^2(X, \mathbb{Q}))_0 / O(H^2(X, \mathbb{Z}))_0$$

is called the *action on $H^2(X, \mathbb{Q})$* of the correspondence of a general (for its Picard lattice) K3 surface X with itself via moduli of sheaves $Y = M_X(r, H, s)$ on X with the primitive isotropic Mukai vector $v = (r, H, s)$.

By Global Torelli Theorem for K3 surfaces [15], the group $O(H^2(X, \mathbb{Z}))_0$ mod ± 1 can be considered as generated by correspondences defined by graphs of automorphisms of X and by the reflections $s_\delta : x \mapsto x + (x \cdot \delta)\delta$, $x \in H^2(X, \mathbb{Z})$, in elements $\delta \in N(X)$ with $\delta^2 = -2$. By Riemann–Roch Theorem for K3 surfaces, $\pm \delta$ contains an effective curve E. If $\Delta \subset X \times X$ is the diagonal, the effective 2-dimensional algebraic cycle $\Delta + E \times E \subset X \times X$ acts as the reflection s_δ in $H^2(X, \mathbb{Z})$ (the author knows this from Mukai [8]). Thus, considering actions of correspondences modulo $O(H^2(X, \mathbb{Z}))_0 \mod \pm 1$ is very natural.

Let us consider Tyurin's isomorphism (3.9) defined by the Mukai vector $v = (\pm H^2/2, H, \pm 1)$ where $H \in N(X)$ has $H^2 \neq 0$ and $\pm H^2 > 0$. Assume that

$\widetilde{H} = H/d$ is primitive. Then $M_X(\pm H^2/2, H, \pm 1) \cong M_X(\pm \widetilde{H}^2/2 \ \widetilde{H}, \pm 1)$, and we can assume that $\widetilde{H}$ is primitive.

Then $c = 1$, $a_1 = \pm \widetilde{H}^2/2$ and $b_1 = \pm 1$, $m(a_1, b_1) \equiv -1 \mod 2a_1 b_1$, $h = \widetilde{H}$. We have

$$H^2(X, \mathbb{Z}) = [\mathbb{Z}\widetilde{H}, \widetilde{H}^\perp, \widetilde{H} + t^*(\widetilde{H})],$$

$$H^2(Y, \mathbb{Z}) = [\mathbb{Z}\widetilde{H}, \widetilde{H}^\perp, \widetilde{H} - t^*(\widetilde{H})].$$

Then the reflection $s_{\widetilde{H}}$ with respect to $\widetilde{H}$,

$$s_{\widetilde{H}}(x) = x - \frac{2(x \cdot \widetilde{H})\widetilde{H}}{\widetilde{H}^2}, \quad x \in H^2(X, \mathbb{Q}),$$

belongs to $O(H^2(X, \mathbb{Q}))_0$, and $s_{\widetilde{H}}(H^2(X, \mathbb{Z})) = H^2(Y, \mathbb{Z})$. Moreover, the reflections s_H and $s_{\widetilde{H}}$ coincide.

Thus, we obtain

Proposition 6.2. *For a K3 surface X and $H \in N(X)$ with $\pm H^2 > 0$, the Tyurin isomorphism*

$$M_X(\pm H^2, H, \pm 1) \cong X$$

defines the correspondence of X with itself with the action

$$s_H \mod O(H^2(X, \mathbb{Z})_0)$$

where s_H is the reflection in the element H.

By classical and well-known results, their compositions generate the full group $O(H^2(X, \mathbb{Q}))_0 \mod \pm 1$.

6.1 The general problem of classification of correspondences of a K3 surface with itself via moduli of sheaves

We will need some notations. For a sublattice $N \subset N(X)$, we introduce

$$O(N \otimes \mathbb{Q})_0 = \{f \in O(H^2(X, \mathbb{Q})) \mid f|N^\perp_{H^2(X, \mathbb{Z})} = \pm 1\}$$

and

$$O(N)_0 = O(H^2(X, \mathbb{Z})) \cap O(N \otimes \mathbb{Q})_0.$$

Let X be a general (for its Picard lattice) K3 surface X and $N(X)$ its Picard lattice. The problem of classification of correspondences of X with itself via moduli of sheaves consists of the following problems (from the author's point of view now):

(1) Find all primitive isotropic Mukai vectors (r, H, s) on X such that $Y = M_X(r, H, s) \cong X$.

(2) For a primitive isotropic Mukai vector (r, H, s) from (1), find all critical polarized Picard sublattices $H \in N(r, H, s) \subset N(X)$.

For each of them, the corresponding action of the correspondence $\phi(r,H,s)$ from Definition 6.1 can be taken from $O(N(r,H,s) \otimes \mathbb{Q})_0$. We denote it as $\phi_{N(r,H,s)}$, and it looks like a reflection with respect to $N(r,H,s)$. For two critical polarized Picard sublattices $H \in N(r,H,s)$ and $H \in N'(r,H,s)$, the automorphisms $\phi_{N(r,H,s)}$ and $\phi_{N'(r,H,s)}$ are different by an automorphism from $O(H^2(X,\mathbb{Z}))_0$.

(3) The structures (1) and (2) are important because for two primitive isotropic Mukai vectors (r,H,s) and (r',H',s') from (1) and two their critical polarized Picard sublattices $H \in N(r,H,s)$ and $H' \in N(r',H',s')$, the isomorphism between $M_X(r,H,s)$ and $M_X(r',H',s')$ which is defined by

$$\phi_{N(r',H',s')}\phi_{N(r,H,s)}^{-1},$$

comes from K3 surfaces with the Picard sublattice $N(r,H,s) + N(r',H',s') \subset N(X)$, and it can be considered as a natural isomorphism between these moduli.

(4) All these generators $\phi_{N(r,H,s)} \mod O(N(r,H,s))_0$ can be considered as natural generators for correspondences of X with itself via moduli of sheaves, together with automorphisms of X and reflections s_δ, $\delta \in N(X)$ and $\delta^2 = -2$. They and their relations are the natural subject to study.

For $\rho(X) = 1$, 2, problems (1)—(4) are solved. See Sections 3 and 4. Results of Sect. 5 show that these problems are very non-trivial for $\rho(X) \geq 3$.

As an example, let us take a general K3 surface X with the Picard lattice $N(X) = S$ of Theorem 5.1 of the rank three (or any other Picard lattice of the rank three which satisfies Theorem 5.3). Let $v = (r,H,s)$ be a primitive isotropic Mukai vector on X. Then $Y = M_X(r,H,s) \cong X$ if and only if $(\gamma,c) = 1$ where $\widetilde{H} \cdot S = \gamma\mathbb{Z}$. Moreover, we have three cases:

(a) If $c = 1$ and either $a_1 = 1$ or $b_1 = \pm 1$ (Tyurin's case), then the critical sublattice is $N(v) = \mathbb{Z}\widetilde{H}$, it has the rank one and is unique. The corresponding $\phi_{N(v)} = s_H \mod O(H^2(X,\mathbb{Z}))_0$.

(b) If $v = (r,H,s)$ is different from (a), but the critical sublattice $N(v)$ has the rank two (the divisorial case), then all critical sublattices $N(v)$ are generated by $\widetilde{H}$ and $\widetilde{h}_1 \in [\widetilde{H}, a_1 c N(X)]$ with $\widetilde{h}_1^2 = \pm 2a_1 c$ or $\widetilde{h}_1 \in [\widetilde{H}, b_1 c N(X)]$ with $\widetilde{h}_1^2 = \pm 2b_1 c$. (See theorems of Sect. 4). All these $N(v)$ give automorphisms $\phi_{N(v)}$ which are different by elements from $O(H^2(X,\mathbb{Z}))_0$.

(c) If $v = (r,H,s)$ is different from (a) and (b), then the critical sublattice $N(v) = N(X)$ has the rank three. These cases really happen by Theorem 5.1. We obtain $\phi_{N(v)} \mod O(H^2(X,\mathbb{Z}))_0$.

Any two v_1, v_2 satisfying one of these conditions (a), (b) or (c), and corresponding their critical sublattices $N(v_1)$, $N(v_2)$ generate natural isomorphisms $\phi_{N(v_2)}\phi_{N(v_1)}^{-1}$ between corresponding moduli of sheaves over X (all of them are isomorphic to X) which are specializations of the corresponding isomorphisms from the Picard sublattice $N(v_1) + N(v_2) \subset N(X)$.

A reader can see that our general idea is that a very complicated structure of correspondences of a general (for its Picard lattice) K3 surface X with itself via moduli of sheaves is hidden inside of the abstract Picard lattice $N(X)$, and we try to recover this structure. This should lead to some non-trivial constructions related to the abstract Picard lattice $N(X)$ and more closely relate it to the geometry of the K3 surface.

References

[1] C. Madonna and V.V. Nikulin, *On a classical correspondence between K3 surfaces*, Proc. Steklov Inst. of Math. **241** (2003), 120–153; (see also math.AG/0206158).

[2] C. Madonna and V.V. Nikulin, *On a classical correspondence between K3 surfaces II*, in: Clay Mathematics Proceedings, Vol. 3 (Strings and Geometry), Douglas M., Gauntlett J., Gross M. editors, 2004, pp.285–300; (see also math.AG/0304415).

[3] C. Madonna and V.V. Nikulin, *Explicit correspondences of a K3 surface with itself*, Izvestiya. Mathematics, (2007) (to appear); (see also math.AG/0605362, 0606239, 0606289).

[4] Sh. Mukai, *Symplectic structure of the moduli space of sheaves on an Abelian or K3 surface*, Inv. Math. **77** (1984), 101–116.

[5] Sh. Mukai, *On the moduli space of bundles on K3 surfaces I*, in: Vector bundles on algebraic varieties, Tata Inst. Fund. Res. Studies in Math. no. **11** (1987), 341–413.

[6] Sh. Mukai, *Duality of polarized K3 surfaces*, Hulek K. (ed.) New trends in algebraic geometry. Selected papers presented at the Euro conference, Warwick, UK, July 1996, Cambridge University Press. London Math. Soc. Lect. Notes Ser. **264**, Cambridge, 1999, pp. 311–326.

[7] Sh. Mukai, *Vector bundles on a K3 surface*, Proc. ICM 2002 in Beijing, Vol. 3, pp. 495–502.

[8] Sh. Mukai, *Cycles on product of two K3 surfaces*, Lecture in the University of Liverpool, February 2002.

[9] V.V. Nikulin, *Finite automorphism groups of Kählerian surfaces of type K3*, Trans. Moscow Math. Soc. **38** (1980), 71–135.

[10] V.V. Nikulin *Integral symmetric bilinear forms and some of their geometric applications*, Math. USSR Izv. **14** (1980), no. 1, 103–167.

[11] V.V. Nikulin *On the quotient groups of the automorphism groups of hyperbolic forms by the subgroups generated by 2-reflections, Algebraic-geometric applications*, J. Soviet Math., **22** (1983), 1401–1476.

[12] V.V. Nikulin, *On correspondences between K3 surfaces*, Math. USSR Izv. **30** (1988), no.2, 375–383.

[13] V.V. Nikulin, *On correspondences of a K3 surface with itself. I*, Proc. Steklov Inst. Math. **246** (2004), 204–226 (see also math.AG/0307355).

[14] V.V. Nikulin, *On Correspondences of a K3 surfaces with itself. II*, Contemporary mathematics (2006), to appear (see also math.AG/0309348, (2003), 62 pages).

[15] I.I. Pjatetskiĭ-Šapiro and I.R. Šafarevich, *A Torelli theorem for algebraic surfaces of type K3*, Math. USSR Izv. **5** (1971), no. 3, 547–588.

[16] B. Saint-Donat *Projective models of K–3 surfaces*, Amer. J. of Mathem. **96** (1974), no. 4, 602–639.

[17] I.R. Shafarevich (ed.) *Algebraic surfaces*, Proc. Stekov Inst. Math. **75** (1965).

[18] A.N. Tyurin, *Cycles, curves and vector bundles on algebraic surfaces*, Duke Math. J. **54** (1987), no. 1, 1–26.

[19] A.N. Tyurin, *Special 0-cycles on a polarized K3 surface*, Math. USSR Izv. **30** (1988), no. 1, 123–143.

[20] A.N. Tyurin, *Symplectic structures on the varieties of moduli of vector bundles on algebraic surfaces with $p_g > 0$*, Math. USSR Izv. **33** (1989), no. 1, 139–177.

[21] K. Yoshioka, *Irreducibility of moduli spaces of vector bundles on K3 surfaces*, Preprint math.AG/9907001, 21 pages.

[22] K. Yoshioka, *Some examples of Mukai's reflections on K3 surfaces*, J. reine angew. Math. **515**, (1999), 97–123 (see also math.AG/9902105)

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