GRAPH COMPLEXES WITH LOOPS AND WHEELS

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To Yuri Ivanovich Manin on his 70th birthday

§1. Introduction

The first instances of graph complexes have been introduced in the theory of operads and props which have found recently lots of applications in algebra, topology and geometry. Another set of examples has been introduced by Kontsevich [Ko1, Ko2] as a way to expose highly non-trivial interrelations between certain infinite dimensional Lie algebras and topological objects, including moduli spaces of curves, invariants of odd dimensional manifolds, and the group of outer automorphisms of a free group.

Motivated by the problem of deformation quantization we introduce and study directed graph complexes with oriented loops and wheels. We show that universal quantizations of Poisson structures can be understood as morphisms of dg props, $Q : \mathsf{P}\langle\mathcal{D}\rangle \rightarrow \mathsf{Lie^1B}_\infty^{\circlearrowright}$, where

– $\mathsf{P}\langle\mathcal{D}\rangle$ is the dg free prop whose representations in a vector space V describe Maurer-Cartan elements of the Hochschild dg Lie algebra on $\mathcal{O}_V := \widehat{\odot^\bullet} V^*$ (see §2.7 for a precise definition), and

– $\mathsf{Lie^1B}_\infty^{\circlearrowright}$ is the *wheeled* completion of the minimal resolution, $\mathsf{Lie^1B}_\infty$, of the prop, $\mathsf{Lie^1B}$, of Lie 1-bialgebras; it is defined explicitly in §2.6 and is proven to have the property that its representations in a *finite-dimensional* vector space V correspond to Maurer-Cartan elements in the Schouten Lie algebra $\wedge^\bullet \mathcal{T}_V$, where $\mathcal{T}_V := \mathrm{Der}\mathcal{O}_V$.

In the theory of props one is most interested in those directed graph complexes which contain *no* loops and wheels. A major advance in understanding the cohomology groups of such complexes was recently accomplished in [Ko2, MaVo, Va] using key ideas of $\frac{1}{2}$prop and Koszul duality. In particular, these authors were able to compute cohomologies of directed versions (without loops and wheels though) of Kontsevich's ribbon graph complex and the "commutative" graph complex, and show that they both are acyclic almost everywhere. This paper is an attempt to extend some of the results of [Ko2, MaVo, Va] to a more difficult situation when the directed graphs are allowed to contain loops and wheels (i.e. directed paths which begin and end at the same vertex). In this case the answer differs markedly from the unwheeled case: we prove, for example, that while the cohomology of the wheeled extension, $\mathsf{Lie}_\infty^{\circlearrowright}$, of the operad of Lie_∞-algebras remains acyclic almost everywhere (see Theorem 4.1.1 for a precise formula for $\mathsf{H}^\bullet(\mathsf{Lie}_\infty^{\circlearrowright})$), the cohomology of the wheeled extension of the operad Ass_∞ gets more complicated. Both these complexes describe irreducible summands of directed "commutative" and, respectively, ribbon graph complexes with the restriction on absence of wheels dropped.

The wheeled complex $\mathsf{Lie}_\infty^{\circlearrowright}$ is a subcomplex of the above mentioned graph complex $\mathsf{Lie^1B}_\infty^{\circlearrowright}$, which plays a central role in deformation quantizations of Poisson structures. Using Theorem 4.1.1 on $\mathsf{H}^\bullet(\mathsf{Lie}_\infty^{\circlearrowright})$ we show in §4.2 that a subcomplex of $\mathsf{Lie^1B}_\infty^{\circlearrowright}$ which is spanned by graphs with

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at most genus 1 wheeles is also acyclic almost everywhere. However this acyclicity breaks for graphs with higher genus wheels: we find an explicit cohomology class with 3 wheels in §4.2.4 which proves that the natural epimorphism, $\mathsf{Lie^1B}_\infty \to \mathsf{Lie^1B}$, *fails* to stay quasi-isomorphism when extended to the wheeled completions, $\mathsf{Lie^1B}_\infty^{\circlearrowright} \to \mathsf{Lie^1B}^{\circlearrowright}$.

This paper grew up from the project on props and quantizations which was launched in 2004 with an attempt to create a prop profile of deformation quantization of Poisson structures (and continued on with [MMS] on minimal wheeled resolutions of main classical operads and with [Me3] on propic proof of a formality theorem associated with quantizations of Lie bialgebras). It is organized as follows. In §2 we remind some basic facts about props and graph complexes and describe a universal construction which associates dg props to a class of sheaves of dg Lie algebras on smooth formal manifolds, and apply that construction to the sheaf of polyvector fields and the sheaf of polydifferential operators creating thereby associated dg props $\mathsf{Lie^1B}_\infty$ and, respectively, $\mathsf{P}\langle\mathcal{D}\rangle$. In §3 we develop new methods for computing cohomology of directed graph complexes with wheels, and prove several theorems on cohomology of wheeled completions of minimal resolutions of dioperads. In §4 we apply these methods and results to compute cohomology of several concrete graph complexes. In §5 we use ideas of cyclic homology to construct a cyclic multicomplex computing cohomology of wheeled completions of dg operads.

A few words about our notations. The cardinality of a finite set I is denoted by $|I|$. The degree of a homogeneous element, a, of a graded vector space is denoted by $|a|$ (this should never lead to a confusion). $\mathbb{S}_n$ stands for the group of all bijections, $[n] \to [n]$, where $[n]$ denotes (here and everywhere) the set $\{1, 2, \dots, n\}$. The set of positive integers is denoted by $\mathbb{N}^*$. If $V = \oplus_{i\in\mathbb{Z}} V^i$ is a graded vector space, then $V[k]$ is a graded vector space with $V[k]^i := V^{i+k}$.

We work throughout over the field k of characteristic 0.

§2. Dg props versus sheaves of dg Lie algebras

2.1. Props. An $\mathbb{S}$-bimodule, E, is, by definition, a collection of graded vector spaces, $\{E(m,n)\}_{m,n\geq 0}$, equipped with a left action of the group $\mathbb{S}_m$ and with a right action of $\mathbb{S}_n$ which commute with each other. For any graded vector space M the collection, $\mathsf{End}\langle M\rangle = \{\mathsf{End}\langle M\rangle(m,n) := \mathrm{Hom}(M^{\otimes n}, M^{\otimes m})\}_{m,n\geq 0}$, is naturally an $\mathbb{S}$-bimodule. A *morphism* of $\mathbb{S}$-bimodules, $\phi : E_1 \to E_2$, is a collection of equivariant linear maps, $\{\phi(m,n) : E_1(m,n) \to E_2(m,n)\}_{m,n\geq 0}$. A morphism $\phi : E \to \mathsf{End}\langle M\rangle$ is called a *representation* of an $\mathbb{S}$-bimodule E in a graded vector space M.

There are two natural associative binary operations on the $\mathbb{S}$-bimodule $\mathsf{End}\langle M\rangle$,

$$\bigotimes : \mathsf{End}\langle M\rangle(m_1,n_1) \otimes \mathsf{End}\langle M\rangle(m_2,n_2) \longrightarrow \mathsf{End}\langle M\rangle(m_1+m_2, n_1+n_2),$$

$$\circ : \mathsf{End}\langle M\rangle(p,m) \otimes \mathsf{End}\langle M\rangle(m,n) \longrightarrow \mathsf{End}\langle M\rangle(p,n),$$

and a distinguished element, the identity map $\mathbf{1} \in \mathsf{End}\langle M\rangle(1,1)$.

Axioms of prop ("*pro*ducts and *p*ermutations") are modelled on the properties of $(\bigotimes, \circ, \mathbf{1})$ in $\mathsf{End}\langle M\rangle$ (see [Mc]).

2.1.1. Definition. A *prop*, E, is an $\mathbb{S}$-bimodule, $E = \{E(m,n)\}_{m,n\geq 0}$, equipped with the following data,

- a linear map called *horizontal composition*,

$$\begin{array}{rccc} \bigotimes: & E(m_1,n_1)\otimes E(m_2,n_2) & \longrightarrow & E(m_1+m_2,n_1+n_2)\\ & \mathfrak{e}_1\otimes\mathfrak{e}_2 & \longrightarrow & \mathfrak{e}_1\bigotimes\mathfrak{e}_2 \end{array}$$

  such that $(\mathfrak{e}_1\bigotimes\mathfrak{e}_2)\bigotimes\mathfrak{e}_3=\mathfrak{e}_1\bigotimes(\mathfrak{e}_2\bigotimes\mathfrak{e}_3)$ and $\mathfrak{e}_1\bigotimes\mathfrak{e}_2=(-1)^{|\mathfrak{e}_1||\mathfrak{e}_1|}\sigma_{m_1,m_2}(\mathfrak{e}_2\bigotimes\mathfrak{e}_1)\sigma_{n_2,n_1}$ where σ_{m_1,m_2} is the following permutation in $\mathbb{S}_{m_1+m_2}$,

$$\left(\begin{array}{ccccccccc} 1 & , & \ldots & , & m_2 & , & m_2+1 & , & \ldots & , & m_2+m_1\\ 1+m_1 & , & \ldots & , & m_2+m_1 & , & 1 & , & \ldots & , & m_1 \end{array}\right);$$

- a linear map called *vertical composition*,

$$\begin{array}{rccc} \circ: & E(p,m)\otimes E(m,n) & \longrightarrow & E(p,n)\\ & \mathfrak{e}_1\otimes\mathfrak{e}_2 & \longrightarrow & \mathfrak{e}_1\circ\mathfrak{e}_2 \end{array}$$

  such that $(\mathfrak{e}_1\circ\mathfrak{e}_2)\circ\mathfrak{e}_3=\mathfrak{e}_1\circ(\mathfrak{e}_2\circ\mathfrak{e}_3)$ whenever both sides are defined;
- an algebra morphism, $i_n: k[\mathbb{S}_n]\to(E(n,n),\circ)$, such that (i) for any $\sigma_1\in\mathbb{S}_{n_1}$, $\sigma_2\in\mathbb{S}_{n_2}$ one has $i_{n_1+n_2}(\sigma_1\times\sigma_2)=i_{n_1}(\sigma_1)\bigotimes i_{n_2}(\sigma_2)$, and (ii) for any $\mathfrak{e}\in E(m,n)$ one has $\mathbf{1}^{\otimes m}\circ\mathfrak{e}=\mathfrak{e}\circ\mathbf{1}^{\otimes n}=\mathfrak{e}$ where $\mathbf{1}:=i_1(\mathrm{Id})$.

A morphism of props, $\phi: E_1\to E_2$, is a morphism of the associated $\mathbb{S}$-bimodules which respects, in the obvious sense, all the prop data.

A *differential* in a prop E is a collection of degree 1 linear maps, $\{\delta: E(m,n)\to E(m,n)\}_{m,n\geq 0}$, such that $\delta^2=0$ and

$$\begin{aligned} \delta(\mathfrak{e}_1\bigotimes\mathfrak{e}_2) &= (\delta\mathfrak{e}_1)\bigotimes\mathfrak{e}_2+(-1)^{|\mathfrak{e}_1|}\mathfrak{e}_1\bigotimes\delta\mathfrak{e}_2,\\ \delta(\mathfrak{e}_3\circ\mathfrak{e}_4) &= (\delta\mathfrak{e}_3)\circ\mathfrak{e}_4+(-1)^{|\mathfrak{e}_3|}\mathfrak{e}_3\circ\delta\mathfrak{e}_4, \end{aligned}$$

for any $\mathfrak{e}_1,\mathfrak{e}_2\in E$ and any $\mathfrak{e}_3,\mathfrak{e}_4\in E$ such that $\mathfrak{e}_3\circ\mathfrak{e}_4$ makes sense. Note that $d\mathbf{1}=0$.

For any dg vector space (M,d) the associated prop $\mathsf{End}\langle M\rangle$ has a canonically induced differential which we always denote by the same symbol d.

A *representation* of a dg prop (E,δ) in a dg vector space (M,d) is, by definition, a morphism of props, $\phi: E\to\mathsf{End}\langle M\rangle$, which commutes with differentials, $\phi\circ\delta=d\circ\phi$. (Here and elsewhere $\circ$ stands for the composition of maps; it will always be clear from the context whether $\circ$ stands for the composition of maps or for the vertical composition in props.)

2.1.2. Remark. If $\psi:(E_1,\delta)\to(E_2,\delta)$ is a morphism of dg props, and $\phi:(E_2,\delta)\to(\mathsf{End}\langle M\rangle,d)$ is a representation of E_2, then the composition, $\phi\circ\psi$, is a representation of E_1. Thus representations can be "pulled back".

2.1.3. Free props. Let $\mathfrak{G}^{\uparrow}(m,n)$, $m,n\geq 0$, be the space of *directed* (m,n)*-graphs*, G, that is, connected 1-dimensional CW complexes satisfying the following conditions:

(i) each edge (that is, 1-dimensional cell) is equipped with a direction;
(ii) if we split the set of all vertices (that is, 0-dimensional cells) which have exactly one adjacent edge into a disjoint union, $V_{in}\sqcup V_{out}$,
  with V_{in} being the subset of vertices with the adjacent edge directed from the vertex,
  and V_{out} the subset of vertices with the adjacent edge directed towards the vertex,
  then $|V_{in}|\geq n$ and $|V_{out}|\geq m$;
(iii) precisely n of vertices from V_{in} are labelled by $\{1,\ldots,n\}$ and are called *inputs*;
(iv) precisely m of vertices from V_{out} are labelled by $\{1,\ldots,m\}$ and are called *outputs*;

(v) there are no oriented *wheels*, i.e. directed paths which begin and end at the same vertex; in particular, there are no *loops* (oriented wheels consisting of one internal edge). Put another way, directed edges generate a continuous flow on the graph which we always assume in our pictures to go from bottom to the top.

Note that $G \in \mathfrak{G}^{\uparrow}(m,n)$ may not be connected. Vertices in the complement,

$$v(G) := \overline{inputs \sqcup outputs},$$

are called *internal vertices*. For each internal vertex v we denote by $In(v)$ (resp., by $Out(v)$) the set of those adjacent half-edges whose orientation is directed towards (resp., from) the vertex. Input (resp., output) vertices together with adjacent edges are called *input* (resp., *output*) *legs*. The graph with one internal vertex, n input legs and m output legs is called the (m,n)-*corolla*.

We set $\mathfrak{G}^{\uparrow} := \sqcup_{m,n}\mathfrak{G}^{\uparrow}(m,n)$.

The *free* prop, $\mathsf{P}\langle E\rangle$, generated by an $\mathbb{S}$-module, $E = \{E(m,n)\}_{m,n\geq 0}$, is defined by (see, e.g., [MaVo, Va])

$$\mathsf{P}\langle E\rangle(m,n) := \bigoplus_{G\in\mathfrak{G}^{\uparrow}(m,n)} \left(\bigotimes_{v\in v(G)} E(Out(v),In(v))\right)_{AutG}$$

where

- $E(Out(v),In(v)) := \mathrm{Bij}([m],Out(v)) \times_{\mathbb{S}_m} E(m,n) \times_{\mathbb{S}_n} \mathrm{Bij}(In(v),[n])$ with Bij standing for the set of bijections,
- $Aut(G)$ stands for the automorphism group of the graph G.

An element of the summand above, $G\langle E\rangle := \left(\bigotimes_{v\in v(G)} E(Out(v),In(v))\right)_{AutG}$, is often called a *graph G with internal vertices decorated by elements of E*, or just a *decorated graph*.

A differential, δ, in a free prop $\mathsf{P}\langle E\rangle$ is completely determined by its values,

$$\delta : E(Out(v),In(v)) \longrightarrow \mathsf{P}\langle E\rangle(|Out(v)|,|In(v)|),$$

on decorated corollas (whose unique internal vertex is denoted by v).

Prop structure on an $\mathbb{S}$-bimodule $E = \{E(m,n)\}_{m,n\geq 0}$ provides us, for any graph $G \in \mathfrak{G}^{\uparrow}(m,n)$, with a well-defined *evaluation* morphism of $\mathbb{S}$-bimodules,

$$\mathsf{ev} : G\langle E\rangle \longrightarrow E(m,n).$$

In particular, if a decorated graph $C \in \mathsf{P}\langle E\rangle$ is built from two corollas, $C_1 \in \mathfrak{G}(m_1,n_1)$ and $C_2 \in \mathfrak{G}(m_2,n_2)$ by gluing jth output leg of C_2 with ith input leg of C_1, and if the vertices of these corollas are decorated, respectively, by elements $a \in E(m_1,n_1)$ and $b \in E(m_2,n_2)$, then we reserve a special notation,

$$a\ {}_i\circ_j\ b := \mathsf{e}v(C) \in E(m_1+m_2-1, n_1+n_2-1),$$

for the resulting evaluation map.

2.1.4. Completions. Any free prop $\mathsf{P}\langle E\rangle$ is naturally a direct sum, $\mathsf{P}\langle E\rangle = \oplus_{n\geq 0}\mathsf{P}_n\langle E\rangle$, of subspaces spanned by the number of vertices of underlying graphs. Each summand $\mathsf{P}_n\langle E\rangle$ has a natural filtration by the genus, g, of the underlying graphs (which is, by definition, equal to the first Betti number of the associated CW complex). Hence each $\mathsf{P}_n\langle E\rangle$ can be completed with respect to this filtration. Similarly, there is a filtration by the number of vertices. We shall always work in this paper with completed with respect to these filtrations free props and hence use the same notation, $\mathsf{P}\langle E\rangle$, and the same name, *free prop*, for the completed version.

2.2. Dioperads and $\frac{1}{2}$props. A *dioperad* is an $\mathbb{S}$-bimodule, $E = \{E(m,n)\}_{\substack{m,n\geq 1\\ m+n\geq 3}}$, equipped with a set of compositions,

$$\{ {}_i\circ_j : E(m_1,n_1)\otimes E(m_2,n_2) \longrightarrow E(m_1+m_2-1, n_1+n_2-1)\}_{\substack{1\leq i\leq n_1\\ 1\leq j\leq m_2}},$$

which satisfy the axioms imitating the properties of the compositions ${}_i\circ_j$ in a generic prop. We refer to [Ga], where this notion was introduced, for a detailed list of these axioms. The *free* dioperad generated by an $\mathbb{S}$-bimodule E is given by,

$$\mathsf{D}\langle E\rangle(m,n) := \bigoplus_{G\in\mathfrak{T}(m,n)} G\langle E\rangle$$

where $\mathfrak{T}(m,n)$ is a subset of $\mathfrak{G}(m,n)$ consisting of connected trees (i.e., connected graphs of genus 0).

Another and less obvious reduction of the notion of prop was introduced by Kontsevich in [Ko2] and studied in detail in [MaVo]: a $\frac{1}{2}$*prop* is an $\mathbb{S}$-bimodule, $E = \{E(m,n)\}_{\substack{m,n\geq 1\\ m+n\geq 3}}$, equipped with two sets of compositions,

$$\{ {}_1\circ_j : E(m_1,1)\otimes E(m_2,n_2) \longrightarrow E(m_1+m_2-1, n_2)\}_{1\leq j\leq m_2}$$

and

$$\{ {}_i\circ_1 : E(m_1,n_1)\otimes E(1,n_2) \longrightarrow E(m_1+m_2-1, n_2)\}_{1\leq i\leq n_1}$$

satisfying the axioms which imitate the properties of the compositions ${}_1\circ_j$ and ${}_i\circ_1$ in a generic dioperad. The *free* $\frac{1}{2}$prop generated by an $\mathbb{S}$-bimodule E is given by,

$$\frac{1}{2}\mathsf{P}\langle E\rangle(m,n) := \bigoplus_{G\in\frac{1}{2}\mathfrak{T}(m,n)} G\langle E\rangle,$$

where $\frac{1}{2}\mathfrak{T}(m,n)$ is a subset of $\mathfrak{T}(m,n)$ consisting of those directed trees which, for each pair of internal vertices, (v_1,v_2), connected by an edge directed from v_1 to v_2 have either $|Out(v_1)| = 1$ or/and $|In(v_2)| = 1$. Such trees have at most one vertex v with $|Out(v)|\geq 2$ and $|In(v)|\geq 2$.

Axioms of dioperad (resp., $\frac{1}{2}$prop) structure on an $\mathbb{S}$-bimodule E ensure that there is a well-defined evaluation map,

$$\mathsf{ev} : G\langle E\rangle \longrightarrow E(m,n),$$

for each $G\in\mathfrak{T}(m,n)$ (resp., $G\in\frac{1}{2}\mathfrak{T}(m,n)$).

2.2.1. Free resolutions. A *free resolution* of a dg prop P is, by definition, a dg free prop, $(\mathsf{P}\langle E\rangle,\delta)$, generated by some $\mathbb{S}$-bimodule E together with a morphism of dg props, $\alpha : (\mathsf{P}\langle E\rangle,\delta)\rightarrow P$, which is a homology isomorphism.

If the differential δ in $\mathsf{P}\langle E\rangle$ is decomposable (with respect to prop's vertical and /or horizontal compositions), then $\alpha : (\mathsf{P}\langle E\rangle,\delta)\rightarrow P$ is called a *minimal model* of P.

Similarly one defines free resolutions and minimal models, $(\mathsf{D}\langle E\rangle,\delta)\rightarrow P$ and $(\frac{1}{2}\mathsf{P}\langle E\rangle,\delta)\rightarrow P$, of dioperads and $\frac{1}{2}$props .

2.2.2. Forgetful functors and their adjoints. There is an obvious chain of forgetful functors, $\mathsf{Prop} \longrightarrow \mathsf{Diop} \longrightarrow \frac{1}{2}\mathsf{Prop}$. Let

$$\Omega_{\frac{1}{2}\mathsf{P}\to\mathsf{D}} : \frac{1}{2}\mathsf{Prop} \longrightarrow \mathsf{Diop}, \quad \Omega_{\mathsf{D}\to\mathsf{P}} : \mathsf{Diop} \longrightarrow \mathsf{Prop}, \quad \Omega_{\frac{1}{2}\mathsf{P}\to\mathsf{P}} : \frac{1}{2}\mathsf{Prop} \longrightarrow \mathsf{Prop},$$

be the associated adjoints. The main motivation behind introducing the notion of $\frac{1}{2}$prop is a very useful fact that the functor $\Omega_{\frac{1}{2}\mathsf{P}\to\mathsf{P}}$ is exact [Ko2, MaVo], i.e., it commutes with the cohomology functor. Which in turn is due to the fact that, for any $\frac{1}{2}$prop P, there exists a kind of PBW lemma which represents $\Omega_{\frac{1}{2}\mathsf{P}\to\mathsf{P}}\langle P\rangle$ as a vector space *freely* generated by a family decorated graphs,

$$\Omega_{\frac{1}{2}\mathsf{P}\to\mathsf{P}}\langle P\rangle(m,n) := \bigoplus_{G\in\overline{\mathfrak{G}}(m,n)} G\langle E\rangle,$$

where $\overline{\mathfrak{G}}(m,n)$ is a subset of $\mathfrak{G}(m,n)$ consisting of so called *reduced* graphs, G, which satisfy the following defining property [MaVo]: for each pair of internal vertices, (v_1, v_2), of G which are connected by a single edge directed from v_1 to v_2 one has $|Out(v_1)| \geq 2$ and $|In(v_2)| \geq 2$. The prop structure on $\Omega_{\frac{1}{2}\mathsf{P}\to\mathsf{P}}\langle P\rangle$ is given by

(i) horizontal compositions := disjoint unions of decorated graphs,
(ii) vertical compositions := graftings followed by $\frac{1}{2}$prop compositions of all those pairs of vertices (v_1, v_2) which are connected by a single edge directed from v_1 to v_2 and have either $|Out(v_1)| = 1$ or/and $|In(v_2)| = 1$ (if there are any).

2.3. Graph complexes with wheels. Let $\mathfrak{G}^{\circlearrowright}(m,n)$ be the set of all directed (m,n)-graphs which satisfy conditions 2.1.3(i)-(iv), and set $\mathfrak{G}^{\circlearrowright} := \sqcup_{m,n}\mathfrak{G}(m,n)$. A vertex (resp., edge or half-edge) of a graph $G \in \mathfrak{G}^{\circlearrowright}$ which belongs to an oriented wheel is called a *cyclic* vertex (resp., edge or half-edge).

Note that for each internal vertex of $G \in \mathfrak{G}^{\circlearrowright}(m,n)$ there is still a well defined separation of adjacent half-edges into input and output ones, as well as a well defined separation of legs into input and output ones.

For any $\mathbb{S}$-bimodule $E = \{E(m,n)\}_{m,n\geq 0}$, we define an $\mathbb{S}$-bimodule,

$$\mathsf{P}^{\circlearrowright}\langle E\rangle(m,n) := \bigoplus_{G\in\mathfrak{G}(m,n)} \left(\bigotimes_{v\in v(G)} E(Out(v), In(v))\right)_{AutG},$$

and notice that $\mathsf{P}^{\circlearrowright}\langle E\rangle$ has a natural prop structure with respect to disjoint union and grafting of graphs. Clearly, this prop contains the free prop $\mathsf{P}\langle E\rangle$ as a natural sub-prop.

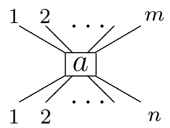
A *derivation* in $\mathsf{P}^{\circlearrowright}\langle E\rangle$ is, by definition, a collection of linear maps, $\delta : \mathsf{P}^{\circlearrowright}\langle E\rangle(m,n) \to \mathsf{P}^{\circlearrowright}\langle E\rangle(m,n)$ such that, for any $G \in \mathfrak{G}$ and any element of $\mathsf{P}^{\circlearrowright}\langle E\rangle(m,n)$ of the form,

$$\mathfrak{e} = \underset{\text{orderings of v(G)}}{\mathsf{coequalizer}} \left(\mathfrak{e}_1 \otimes \mathfrak{e}_2 \otimes \ldots \otimes \mathfrak{e}_{|v(G)|}\right), \quad \mathfrak{e}_k \in E(Out(v_k), In(v_k)) \text{ for } 1 \leq k \leq |v(G)|,$$

one has

$$\delta\mathfrak{e} = \underset{\text{orderings of v(G)}}{\mathsf{coequalizer}} \left(\sum_{k=1}^{|v(G)|} (-1)^{|\mathfrak{e}_1|+\ldots+|\mathfrak{e}_{k-1}|}\mathfrak{e}_1 \otimes \ldots \otimes \delta\mathfrak{e}_k \otimes \ldots \otimes \mathfrak{e}_{|v(G)|}\right).$$

Put another way, a graph derivation is completely determined by its values on decorated corollas,



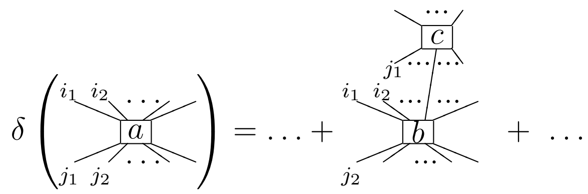
$$a \in E(m,n),$$

that is, by linear maps,

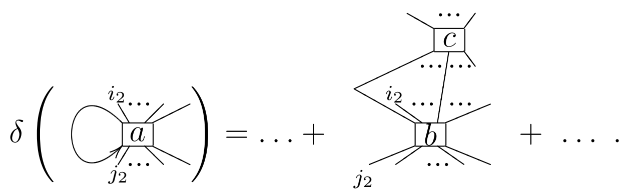
$$\delta : E(m,n) \longrightarrow \mathsf{P}^{\circlearrowright}\langle E\rangle(m,n).$$

A *differential* in $\mathsf{P}^{\circlearrowright}\langle E\rangle$ is, by definition, a degree 1 derivation δ satisfying the condition $\delta^2 = 0$.

2.3.1. Remark. If $(\mathsf{P}\langle E\rangle, \delta)$ is a free dg prop generated by an $\mathbb{S}$-bimodule E, then δ extends naturally to a differential on $\mathsf{P}^{\circlearrowright}\langle E\rangle$ which we denote by the same symbol δ. It is worth pointing out that such an induced differential may *not* preserve the number of oriented wheels. For example, if δ applied to an element $a \in E(m,n)$ (which we identify with the a-decorated (m,n)-corolla) contains a summand of the form,



then the value of δ on the graph obtained from this corolla by gluing output i_1 with input j_1 into a loop,



contains a term with no oriented wheels at all. Thus propic differential can, in general, *decrease* the number of wheels. Notice in this connection that if δ is induced on $\mathsf{P}^{\circlearrowright}\langle E\rangle$ from the minimal model of a $\frac{1}{2}$prop, then such summands are impossible and hence the differential preserves the number of wheels.

Vector spaces $\mathsf{P}\langle E\rangle$ and $\mathsf{P}^{\circlearrowright}\langle E\rangle$ have a natural positive gradation,

$$\mathsf{P}\langle E\rangle = \bigoplus_{k\geq 1} \mathsf{P}_k\langle E\rangle, \quad \mathsf{P}^{\circlearrowright}\langle E\rangle = \bigoplus_{k\geq 1} \mathsf{P}^{\circlearrowright}\langle E\rangle,$$

by the number, k, of internal vertices of underlying graphs. In particular, $\mathsf{P}_1\langle E\rangle(m,n)$ is spanned by decorated (m,n)-corollas and can be identified with $E(m,n)$.

2.3.2. Representations of $\mathsf{P}^{\circlearrowright}\langle E\rangle$. Any representation, $\phi : E \to \mathrm{End}\langle M\rangle$, of an $\mathbb{S}$-bimodule E in a finite dimensional vector space M can be naturally extended to representations of props, $\mathsf{P}\langle E\rangle \to \mathrm{End}\langle M\rangle$ and $\mathsf{P}^{\circlearrowright}\langle E\rangle \to \mathrm{End}\langle M\rangle$. In the latter case decorated graphs with oriented wheels are mapped into appropriate traces.

2.3.3. Remark. Prop structure on an $\mathbb{S}$-bimodule $E = \{E(m,n)\}$ can be defined as a family of evaluation linear maps,

$$\mu_G : G\langle E\rangle \longrightarrow E(m,n), \quad \forall\, G \in \mathfrak{G}^{\uparrow},$$

satisfying certain associativity axiom (cf. §2.1.3). Analogously, one can define a *wheeled prop struture* on E as a family of linear maps,

$$\mu_G : G\langle E\rangle \longrightarrow E(m,n), \quad \forall\, G \in \mathfrak{G}^{\circlearrowright},$$

such that

(i) $\mu_{(m,n)-corolla} = \mathrm{Id}$,
(ii) $\mu_G = \mu_{G/H} \circ \mu_H$ for every subgraph $H \in \mathfrak{G}^{\circlearrowright}$ of G, where G/H is obtained from G by collapsing to the single vertex every connected component of H, and $\mu_H : G\langle E\rangle \to G/H\langle E\rangle$ is the evaluation map on the subgraph H and identity on its complement.

Claim. For every finite-dimensional vector space M the associated endomorphism prop $\mathrm{End}\langle M\rangle$ *has a natural structure of wheeled prop.*

The notion of representation of $\mathsf{P}^{\circlearrowright}\langle E\rangle$ in a finite dimensional vector space M introduced above is just a morphism of *wheeled* props, $\mathsf{P}^{\circlearrowright}\langle E\rangle \to \mathrm{End}\langle M\rangle$. We shall discuss these issues in detail elsewhere as in the present paper we are most interested in computing cohomology of *free* dg wheeled props $(\mathsf{P}^{\circlearrowright}\langle E\rangle, \delta)$, where the composition maps μ_G are tautological.

2.4. Formal graded manifolds. For a finite-dimensional vector space M we denote by $\mathcal{M}$ the associated formal graded manifold. The distinguished point of the latter is always denoted by $*$. The structure sheaf, $\mathcal{O}_{\mathcal{M}}$, is (non-canonically) isomorphic to the completed graded symmetric tensor algebra, $\hat{\odot} M^*$. A choice of a particular isomorphism, $\phi : \mathcal{O}_{\mathcal{M}} \to \hat{\odot} M^*$, is called a choice of a local coordinate system on $\mathcal{M}$. If $\{e_\alpha\}_{\alpha\in I}$ is a basis in M and $\{t^\alpha\}_{\alpha\in I}$ the associated dual basis in M^*, then one may identify $\mathcal{O}_{\mathcal{M}}$ with the graded commutative formal power series ring $\mathbb{R}[[t^\alpha]]$.

Free modules over the ring $\mathcal{O}_{\mathcal{M}}$ are called locally free sheaves (=vector bundles) on $\mathcal{M}$. The $\mathcal{O}_{\mathcal{M}}$-module, $\mathcal{T}_{\mathcal{M}}$, of derivations of $\mathcal{O}_{\mathcal{M}}$ is called the tangent sheaf on $\mathcal{M}$. Its dual, $\Omega^1_{\mathcal{M}}$, is called the cotangent sheaf. One can form their (graded skewsymmetric) tensor products such as the sheaf of polyvector fields, $\wedge^\bullet \mathcal{T}_{\mathcal{M}}$, and the sheaf of differential forms, $\Omega^\bullet_{\mathcal{M}} = \wedge^\bullet \Omega^1_{\mathcal{M}}$. The first sheaf is naturally a sheaf of Lie algebras on $\mathcal{M}$ with respect to the Schouten bracket.

One can also define a sheaf of polydifferential operators, $\mathcal{D}_{\mathcal{M}} \subset \oplus_{i\geq 0}\mathrm{Hom}_{\mathbb{R}}(\mathcal{O}_{\mathcal{M}}^{\otimes i}, \mathcal{O}_{\mathcal{M}})$. The latter is naturally a sheaf of dg Lie algebras on $\mathcal{M}$ with respect to the Hochschild differential, d_H, and brackets, $[\ ,\ ]_H$.

2.5. Geometry ⇒ graph complexes. We shall sketch here a simple trick which associates to a sheaf of dg Lie algebras, $\mathcal{G}_{\mathcal{M}}$, over a smooth graded formal manifold $\mathcal{M}$ a dg prop, $\mathsf{P}^{\circlearrowright}\langle E_{\mathcal{G}}\rangle$, generated by a certain $\mathbb{S}$-bimodule $E_{\mathcal{G}}$.

We assume that

(i) $\mathcal{G}_{\mathcal{M}}$ is built from direct sums and tensor products of (any order) jets of the sheaves $\mathcal{T}_{\mathcal{M}}^{\otimes\bullet} \otimes {\Omega^1_{\mathcal{M}}}^{\otimes\bullet}$ and their duals (thus $\mathcal{G}_{\mathcal{M}}$ can be defined for *any* formal smooth manifold $\mathcal{M}$, i.e., its definition does not depend on the dimension of $\mathcal{M}$),
(ii) the differential and the Lie bracket in $\mathcal{G}_M$ can be represented, in a local coordinate system, by polydifferential operators and natural contractions between duals.

The motivating examples are $\wedge^\bullet\mathcal{T}_{\mathcal{M}}$, $\mathcal{D}_{\mathcal{M}}$ and the sheaf of Nijenhuis dg Lie algebras on $\mathcal{M}$ (see [Me2]).

By assumption (i), a choice of a local coordinate system on $\mathcal{M}$, identifies $\mathcal{G}_{\mathcal{M}}$ with a subspace in

$$\bigoplus_{p,m\geq 0} \hat{\odot}^{\bullet} M^* \otimes \mathrm{Hom}(M^{\otimes p}, M^{\otimes m}) = \prod_{p,q,m\geq 0} \mathrm{Hom}(\odot^p M \otimes M^{\otimes q}, M^{\otimes m}) \subset \prod_{m,n\geq 0} \mathrm{Hom}(M^{\otimes n}\ M^{\otimes m}).$$

Let Γ be a degree 1 element in $\mathcal{G}_{\mathcal{M}}$. Denote by $\Gamma^m_{p,q}$ the bit of Γ which lies in $\mathrm{Hom}(\odot^p M \otimes M^{\otimes q}, M^{\otimes m})$ and set $\Gamma^m_n := \oplus_{p+q=n}\Gamma^m_{p,q} \in \mathrm{Hom}(M^{\otimes n}\ M^{\otimes m})$.

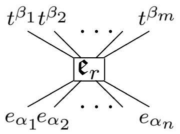
There exists a uniquely defined finite-dimensional $\mathbb{S}$-bimodule, $E_{\mathcal{G}} = \{E_{\mathcal{G}}(m,n)\}_{m,n\geq 0}$, whose representations in the vector space M are in one-to-one correspondence with Taylor components, $\Gamma^m_n \in \mathrm{Hom}(M^{\otimes n}\ M^{\otimes m})$, of a degree 1 element Γ in $\mathcal{G}_{\mathcal{M}}$. Set $\mathsf{P}^{\circlearrowright}\langle\mathcal{G}\rangle := \mathsf{P}^{\circlearrowright}\langle E_{\mathcal{G}}\rangle$ (see Sect. 2.3).

Next we employ the dg Lie algebra structure in $\mathcal{G}_{\mathcal{M}}$ to introduce a differential, δ, in $\mathsf{P}^{\circlearrowright}\langle\mathcal{G}\rangle$. The latter is completely determined by its restriction to the subspace of $\mathsf{P}^{\circlearrowright}_1\langle\mathcal{G}\rangle$ spanned by decorated corollas (without attached loops).

First we replace the Taylor coefficients, Γ^m_n, of the section Γ by the decorated (m,n)-corollas

- with the unique internal vertex decorated by a basis element, $\{\mathfrak{e}_r\}_{r\in J}$, of $E_{\mathcal{G}}(m,n)$,
- with input legs labeled by basis elements, $\{e_\alpha\}$, of the vector space M and output legs labeled by the elements of the dual basis, $\{t^\beta\}$.

Next we consider a formal linear combination,

$$\overline{\Gamma}^m_n = \sum_r \sum_{\substack{\alpha_1,\ldots,\alpha_n\\ \beta_1,\ldots,\beta_m}}$$ 

$$t^{\alpha_1}\otimes\ldots\otimes t^{\alpha_n}\otimes e_{\beta_1}\otimes\ldots\otimes e_{\beta_m}\ \in \mathsf{P}^{\circlearrowright}_1\langle\mathcal{G}\rangle\otimes\mathrm{Hom}(M^{\otimes n}, M^{\otimes m}).$$

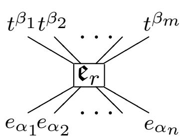
This expression is essentially a component of the Taylor decomposition of Γ,

$$\Gamma^m_n = \sum_{\substack{\alpha_1\ldots\alpha_n\\ \beta_1\ldots\beta_m}} \Gamma^{\beta_1\ldots\beta_m}_{\alpha_1\ldots\alpha_n} t^{\alpha_1}\otimes\ldots\otimes t^{\alpha_n}\otimes e_{\beta_1}\otimes\ldots\otimes e_{\beta_m},$$

in which the numerical coefficient $\Gamma^{\beta_1\ldots\beta_m}_{\alpha_1\ldots\alpha_n}$ is substituted by a decorated labeled graph. More precisely, the interrelation between $\overline{\Gamma} = \oplus_{m,n\geq 0}\overline{\Gamma}^m_n$ and $\Gamma = \oplus_{m,n\geq 0}\Gamma^m_n \in \mathcal{G}_{\mathcal{M}}$ can be described as follows: a choice of any particular representation of the $\mathbb{S}$-bimodule $E_{\mathcal{G}}$,

$$\phi : \left\{E_{\mathcal{G}}(m,n) \to \mathrm{Hom}(M^{\otimes n}, M^{\otimes m})\right\}_{m,n\geq 0},$$

defines an element $\Gamma = \phi(\overline{\Gamma}) \in \mathcal{G}_{\mathcal{M}}$ which is obtained from $\overline{\Gamma}$ by replacing each graph,



by the value, $\overset{r}{\Gamma}{}^{\beta_1\ldots\beta_m}_{\alpha_1\ldots\alpha_n} \in \mathbb{R}$, of $\phi(\mathfrak{e}_r\} \in \mathrm{Hom}(M^{\otimes n}, M^{\otimes m})$ on the basis vector $e_{\alpha_1}\otimes\ldots\otimes e_{\alpha_n}\otimes t^{\beta_1}\otimes\ldots\otimes t^{\beta_m}$ (so that $\Gamma^{\beta_1\ldots\beta_m}_{\alpha_1\ldots\alpha_n} = \sum_r \overset{r}{\Gamma}{}^{\beta_1\ldots\beta_m}_{\alpha_1\ldots\alpha_n}$) .

In a similar way one can define an element,

$$\overline{[\cdots[[d\Gamma,\Gamma],\Gamma]\cdots]} \in \mathsf{P}^{\circlearrowright}_n\langle\mathcal{G}\rangle\otimes\mathrm{Hom}(M^{\otimes\bullet}, M^{\otimes\bullet})$$

for any Lie word,

$$[\cdots[[d\Gamma,\Gamma],\Gamma]\ldots],$$

built from Γ, $d\Gamma$ and $n-1$ Lie brackets. In particular, there are uniquely defined elements,

$$\overline{d\Gamma} \in \mathsf{P}^{\circlearrowright}_1\langle \mathcal{G}\rangle \otimes \mathrm{Hom}(M^{\otimes\bullet}, M^{\otimes\bullet}), \quad \overline{\frac{1}{2}[\Gamma,\Gamma]} \in \mathsf{P}^{\circlearrowright}_2\langle \mathcal{G}\rangle \otimes \mathrm{Hom}(M^{\otimes\bullet}, M^{\otimes\bullet}),$$

whose values, $\phi(\overline{d\Gamma})$ and $\phi(\overline{\frac{1}{2}[\Gamma,\Gamma]})$, for any particular choice of representation ϕ of the $\mathbb{S}$-bimodule $E_{\mathcal{G}}$, coincide respectively with $d\Gamma$ and $\frac{1}{2}[\Gamma,\Gamma]$.

Finally one defines a differential δ in the graded space $\mathsf{P}^{\circlearrowright}\langle \mathcal{G}\rangle$ by setting

$$\delta\overline{\Gamma} = \overline{d\Gamma} + \overline{\frac{1}{2}[\Gamma,\Gamma]}, \qquad (\star\star)$$

i.e. by equating the graph coefficients of both sides. That $\delta^2 = 0$ is clear from the following calculation,

$$\begin{aligned}
\delta^2\overline{\Gamma} &= \delta\left(\overline{d\Gamma} + \overline{\frac{1}{2}[\Gamma,\Gamma]}\right)\\
&= \delta\overline{d\Gamma} + \overline{[d\Gamma + \frac{1}{2}[\Gamma,\Gamma],\Gamma]}\\
&= -\overline{d(d\Gamma + \frac{1}{2}[\Gamma,\Gamma])} + \overline{[d\Gamma,\Gamma]} + \frac{1}{2}\overline{[[\Gamma,\Gamma],\Gamma]}\\
&= -\overline{[d\Gamma,\Gamma]} + \overline{[d\Gamma,\Gamma]}\\
&= 0,
\end{aligned}$$

where we used both the axioms of dg Lie algebra in $\mathcal{G}_{\mathcal{M}}$ and the axioms of the differential in $\mathsf{P}^{\circlearrowright}\langle \mathcal{G}\rangle$. This completes the construction of $(\mathsf{P}^{\circlearrowright}\langle \mathcal{G}\rangle, \delta)$[1].

2.5.1. Remarks. (i) If the differential and Lie brackets in $\mathcal{G}_{\mathcal{M}}$ contain no traces, then the expression $\overline{d\Gamma} + \overline{\frac{1}{2}[\Gamma,\Gamma]}$ does not contain graphs with oriented wheels. Hence formula $(\star\star)$ can be used to introduce a differential in the free prop, $\mathsf{P}\langle \mathcal{G}\rangle$, generated by the $\mathbb{S}$-bimodule $E_{\mathcal{G}}$.

(ii) If the differential and Lie brackets in $\mathcal{G}_{\mathcal{M}}$ contain no traces and are given by first order differential operators, then the expression $\overline{d\Gamma} + \overline{\frac{1}{2}[\Gamma,\Gamma]}$ is a tree. Therefore formula $(\star\star)$ can be used to introduce a differential in the free dioperad, $\mathsf{D}\langle \mathcal{G}\rangle$.

2.5.2. Remark. The above trick also works for sheaves,

$$\left(\mathcal{G}_{\mathcal{M}}, \mu_n : \wedge^n\mathcal{G}_{\mathcal{M}} \to \mathcal{G}_{\mathcal{M}}[2-n], n = 1,2,\ldots\right),$$

of L_∞ algebras over $\mathcal{M}$. The differential in $\mathsf{P}^{\circlearrowright}\langle \mathcal{G}\rangle$ (or in $\mathsf{P}\langle \mathcal{G}\rangle$, if appropriate) is defined by,

$$\delta\overline{\Gamma} = \sum_{n=1}^{\infty}\frac{1}{n!}\overline{\mu_n(\Gamma,\ldots,\Gamma)}.$$

The term $\overline{\mu_n(\Gamma,\ldots,\Gamma)}$ corresponds to decorated graphs with n internal vertices.

[1]As a first approximation to the propic translation of *non*-flat geometries (Yang-Mills, Riemann, etc.) one might consider the following version of the “trick”: in addition to generic element $\Gamma \in \mathcal{G}_{\mathcal{M}}$ of degree 1 take into consideration (probably, *non* generic) element of degree 2, $F \in \mathcal{G}_{\mathcal{M}}$, extend appropriately the $\mathbb{S}$-bimodule $E_{\mathcal{G}}$ to accommodate the associated “curvature” F-corollas, and then (attempt to) define the differential δ in $\mathsf{P}^{\circlearrowright}\langle E_{\mathcal{G}}\rangle$ by equating graph coefficients in the expressions, $\delta\overline{\Gamma} = \overline{F} + \overline{d\Gamma} + \overline{\frac{1}{2}[\Gamma,\Gamma]}$ and $\delta\overline{F} = \overline{dF} + \overline{[\Gamma,F]}$.

2.5.3. Remark. Any sheaf of dg Lie subalgebras, $\mathcal{G}'_{\mathcal{M}} \subset \mathcal{G}_{\mathcal{M}}$, defines a dg prop, $(\mathsf{P}^{\circlearrowright}\langle \mathcal{G}' \rangle, \delta)$, which is a quotient of $(\mathsf{P}^{\circlearrowright}\langle \mathcal{G} \rangle, \delta)$ by the ideal generated by decorated graphs lying in the complement, $\mathsf{P}^{\circlearrowright}\langle \mathcal{G} \rangle \setminus \mathsf{P}^{\circlearrowright}\langle \mathcal{G}' \rangle$. Similar observation holds true for $\mathsf{P}\langle \mathcal{G} \rangle$ and $\mathsf{P}\langle \mathcal{G}' \rangle$ (if they are defined).

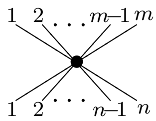
2.6. Example (polyvector fields). Let us consider the sheaf of polyvector fields, $\wedge^\bullet \mathcal{T}_{\mathcal{M}} := \sum_{i\geq 0} \wedge^i \mathcal{T}_{\mathcal{M}}[1-i]$, equipped with the Schouten Lie bracket, $[\ ,\ ]_S$, and vanishing differential. A degree one section, Γ, of $\wedge^\bullet \mathcal{T}_{\mathcal{M}}$ decomposes into a direct sum, $\oplus_{i\geq 0}\Gamma_i$, with $\Gamma_i \in \wedge^i \mathcal{T}_{\mathcal{M}}$ having degree $2-i$ with respect to the grading of the underlying manifold. In a local coordinate system Γ can be represented as a Taylor series,

$$\Gamma = \sum_{m,n\geq 0} \sum_{\substack{\alpha_1\ldots\alpha_n \\ \beta_1\ldots\beta_m}} \Gamma^{\beta_1\ldots\beta_m}_{\alpha_1\ldots\alpha_n} (e_{\beta_1} \wedge \ldots \wedge e_{\beta_m}) \otimes (t^{\alpha_1} \odot \ldots \odot t^{\alpha_n}).$$

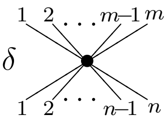
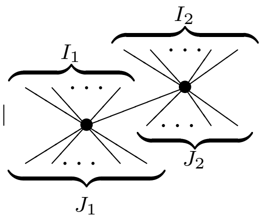
As $\Gamma^{\beta_1\ldots\beta_m}_{\alpha_1\ldots\alpha_n} = \Gamma^{[\beta_1\ldots\beta_m]}_{(\alpha_1\ldots\alpha_n)}$ has degree $2-m$, we conclude that the associated $\mathbb{S}$-bimodule $E_{\wedge^\bullet \mathcal{T}}$ is given by

$$E_{\wedge^\bullet \mathcal{T}}(m,n) = \mathbf{sgn_m} \otimes \mathbf{1_n}[m-2], \quad m,n \geq 0,$$

where $\mathbf{sgn_m}$ stands for the one dimensional sign representation of Σ_m and $\mathbf{1_n}$ stands for the trivial one-dimensional representation of Σ_n. Then a generator of $\mathsf{P}\langle \wedge^\bullet \mathcal{T} \rangle$ can be represented by the directed planar (m,n)-corolla,



with skew-symmetric outgoing legs and symmetric ingoing legs. The formula $(\star\star)$ in Sect. 2.5 gives the following explicit expression for the induced differential, δ, in $\mathsf{P}\langle \wedge^\bullet \mathcal{T} \rangle$,

δ  $= \sum_{\substack{I_1 \sqcup I_2 = (1,\ldots,m) \\ J_1 \sqcup J_2 = (1,\ldots,n) \\ |I_1|\geq 0, |I_2|\geq 1 \\ |J_1|\geq 1, |J_2|\geq 0}} (-1)^{\sigma(I_1\sqcup I_2)+|I_1||I_2|}$ 

where $\sigma(I_1 \sqcup I_2)$ is the sign of the shuffle $I_1 \sqcup I_2 = (1,\ldots,m)$.

2.6.1. Proposition. *There is a one-to-one correspondence between representations,*

$$\phi : (\mathsf{P}\langle \wedge^\bullet \mathcal{T} \rangle, \delta) \longrightarrow (\mathsf{End}\langle M \rangle, d),$$

of $(\mathsf{P}\langle \wedge^\bullet \mathcal{T} \rangle, \delta)$ *in a dg vector space* (M,d) *and Maurer-Cartan elements,* γ, *in* $\wedge^\bullet \mathcal{T}_{\mathcal{M}}$, *that is, degree one elements satisfying the equation,* $[\gamma,\gamma]_S = 0$.

Proof. Let ϕ be a representation. Images of the above (m,n)-corollas under ϕ provide us with a collection of linear maps, $\Gamma^m_n : \odot^n M \to \wedge^m M[2-m]$ which we assemble, as in Sect. 2.5, into a section, $\Gamma = \sum_{m,n} \Gamma^m_n$, of $\wedge^\bullet \mathcal{T}_{\mathcal{M}}$.

The differential d in M can be interpreted as a linear (in the coordinates $\{t^\alpha\}$) degree one section of $\mathcal{T}_{\mathcal{M}}$ which we denote by the same symbol.

Finally, the commutativity of ϕ with the differentials implies

$$[-d+\Gamma, -d+\Gamma]_S = 0.$$

Thus setting $\gamma = -d+\Gamma$ one gets a Maurer-Cartan element in $\wedge^\bullet \mathcal{T}_{\mathcal{M}}$.

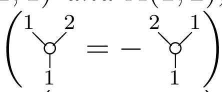
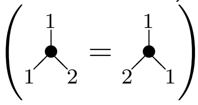
Reversely, if γ is a Maurer-Cartan element in $\wedge^\bullet \mathcal{T}_\mathcal{M}$, then decomposing the sum $d+\gamma$ into a collection of its Taylor series components as in Sect. 2.5, one gets a representation ϕ. $\Box$

Let $\wedge_0^\bullet \mathcal{T}_\mathcal{M} = \sum_{i\geq 1} \wedge_0^i \mathcal{T}_\mathcal{M}[1-i]$ be a sheaf of Lie subalgebras of $\wedge^\bullet \mathcal{T}_\mathcal{M}$ consisting of those elements which vanish at the distinguished point $* \in \mathcal{M}$, have no $\wedge^0\mathcal{T}_\mathcal{M}[2]$-component, and whose $\wedge^1\mathcal{T}_\mathcal{M}[1]$-component is at least quadratic in the coordinates $\{t^\alpha\}$. The associated dg free prop, $\mathsf{P}\langle \wedge_0^\bullet \mathcal{T}\rangle$, is generated by (m,n)-corollas with $m,n\geq 1$, $m+n\geq 1$, and has a surprisingly small cohomology, a fact which is of key importance for our proof of the deformation quantization theorem.

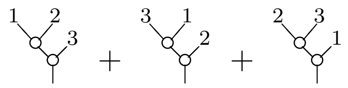
2.6.2. Theorem. *The cohomology of* $(\mathsf{P}\langle \wedge_0^\bullet \mathcal{T}\rangle, \delta)$ *is equal to a quadratic prop,* $\mathsf{Lie^1B}$*, which is a quotient,*

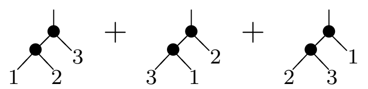
$$\mathsf{Lie^1B} = \frac{\mathsf{P}\langle A\rangle}{\mathsf{Ideal} < R >}\ ,$$

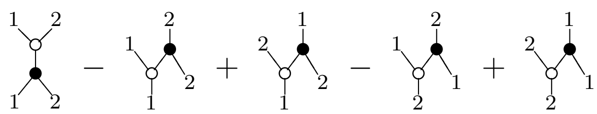
of the free prop generated by the following $\mathbb{S}$*-bimodule* A*,*

- *all* $A(m,n)$ *vanish except* $A(2,1)$ *and* $A(1,2)$*,*
- $A(2,1) := \mathbf{sgn_2}\otimes \mathbf{1_1} =$ *span* 
- $A(1,2) := \mathbf{1_1}\otimes \mathbf{1_2}[-1] =$ *span* 

modulo the ideal generated by the following relations, R*,*

R_1 :  $\in \mathsf{P}\langle A\rangle(3,1)$

R_2 :  $\in \mathsf{P}\langle A\rangle(1,3)$

R_3 :  $\in \mathsf{P}\langle A\rangle(2,2)$.

Proof. The cohomology of $(\mathsf{P}\langle \wedge_0^\bullet \mathcal{T}\rangle, \delta)$ can not be computed directly. At the dioperadic level the theorem was established in [Me1]. That this result extends to the level of props can be easily shown using either ideas of perturbations of 1/2props and path filtrations developed in [Ko2, MaVo] or the idea of Koszul duality for props developed in [Va]. One can argue, for example, as follows: for any $f\in \mathsf{P}\langle \wedge_0^\bullet \mathcal{T}\rangle$, define the natural number,

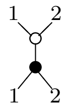
$$|f| := \begin{array}{l}\text{number of directed paths in the graph } f\\ \text{which connect input legs with output ones.}\end{array}$$

and notice that the differential δ preserves the filtration,

$$F_p\mathsf{P}\langle \wedge_0^\bullet \mathcal{T}\rangle := \{\text{span } f\in \mathsf{P}\langle \wedge_0^\bullet \mathcal{T}\rangle :\ |f|\leq p\}\,.$$

The associated spectral sequence, $\{E_r\mathsf{P}\langle \wedge_0^\bullet \mathcal{T}\rangle, \delta_r\}_{r\geq 0}$, is exhaustive and bounded below so that it converges to the cohomology of $(\mathsf{P}\langle \wedge_0^\bullet \mathcal{T}\rangle, \delta)$.

By Koszulness of the operad Lie and exactness of the functor $\Omega_{\frac{1}{2}\mathsf{P}\to\mathsf{P}}$, the zeroth term of the spectral sequence, $(E_0\mathsf{P}\langle\wedge^\bullet_0\mathcal{T}\rangle,\delta_0)$, is precisely the minimal resolution of a quadratic prop, $\mathsf{Lie}^1\mathsf{B}'$, generated by the $\mathbb{S}$-bimodule A modulo the ideal generated by relations R_1, R_2 and the following one,

R'_3 : 

$= 0.$

As the differential δ vanishes on the generators of A, this spectral sequence degenerates at the first term, $(E_1\mathsf{P}\langle\wedge^\bullet_0\mathcal{T}\rangle, d_1=0)$, implying the isomorphism

$$\bigoplus_{p\geq 1}\frac{F_{p+1}H(\mathsf{P}\langle\wedge^\bullet_0\mathcal{T}\rangle,\delta)}{F_pH(\mathsf{P}\langle\wedge^\bullet_0\mathcal{T}\rangle,\delta)}=\mathsf{Lie}^1\mathsf{B}'.$$

There is a natural surjective morphism of dg props, $p:(\mathsf{P}\langle\wedge^\bullet_0\mathcal{T}\rangle,\delta)\to\mathsf{Lie}^1\mathsf{B}$. Define the dg prop (X,δ) via an exact sequence,

$$0\longrightarrow(X,\delta)\stackrel{i}{\longrightarrow}(\mathsf{P}\langle\wedge^\bullet_0\mathcal{T}\rangle,\delta)\stackrel{p}{\longrightarrow}(\mathsf{Lie}^1\mathsf{B},0)\longrightarrow 0.$$

The filtration on $(\mathsf{P}\langle\wedge^\bullet_0\mathcal{T}\rangle,\delta)$ induces filtrations on sub- and quotient complexes,

$$0\longrightarrow(F_pX,\delta)\stackrel{i}{\longrightarrow}(F_p\mathsf{P}\langle\wedge^\bullet_0\mathcal{T}\rangle,\delta)\stackrel{p}{\longrightarrow}(F_p\mathsf{Lie}^1\mathsf{B},0)\longrightarrow 0,$$

and hence an exact sequence of 0th terms of the associated spectral sequences,

$$0\longrightarrow(E_0X,\delta_0)\stackrel{i_0}{\longrightarrow}(E_0\mathsf{P}\langle\wedge^\bullet_0\mathcal{T}\rangle,\delta_0)\stackrel{p_0}{\longrightarrow}(\bigoplus_{p\geq 1}\frac{F_{p+1}\mathsf{Lie}^1\mathsf{B}}{F_p\mathsf{Lie}^1\mathsf{B}},0)\longrightarrow 0.$$

By the above observation,

$$E_1\mathsf{P}\langle\wedge^\bullet_0\mathcal{T}\rangle=\bigoplus_{p\geq 1}\frac{F_{p+1}H(\mathsf{P}\langle\wedge^\bullet_0\mathcal{T}\rangle,\delta)}{F_pH(\mathsf{P}\langle\wedge^\bullet_0\mathcal{T}\rangle,\delta)}=\mathsf{Lie}^1\mathsf{B}'.$$

On the other hand, it is not hard to check that

$$\bigoplus_{p\geq 1}\frac{F_{p+1}\mathsf{Lie}^1\mathsf{B}}{F_p\mathsf{Lie}^1\mathsf{B}}=\mathsf{Lie}^1\mathsf{B}'.$$

Thus the map p_0 is a quasi-isomorphism implying vanishing of E_1X and hence acyclicity of (X,δ). Thus the projection map p is a quasi-isomorphism. □

2.6.3. Corollary. The dg prop $(\mathsf{P}\langle\wedge^\bullet_0\mathcal{T}\rangle,\delta)$ is a minimal model of the prop $\mathsf{Lie}^1\mathsf{B}$: the natural morphism of dg props,

$$p:(\mathsf{P}\langle\wedge^\bullet_0\mathcal{T}\rangle,\delta)\longrightarrow(\mathsf{Lie}^1\mathsf{B},\text{vanishing differential}).$$

which sends to zero all generators of $\mathsf{P}\langle\wedge^\bullet_0\mathcal{T}\rangle$ except those in $A(2,1)$ and $A(1,2)$, is a quasi-isomorphism. Hence we can and shall re-denote $\mathsf{P}\langle\wedge^\bullet_0\mathcal{T}\rangle$ as $\mathsf{Lie}^1\mathsf{B}_\infty$.

2.7. Example (polydifferential operators). Let us consider the sheaf of dg Lie algebras,

$$\mathcal{D}_{\mathcal{M}}\subset\bigoplus_{k\geq 0}\mathrm{Hom}(\mathcal{O}_{\mathcal{M}}^{\otimes k},\mathcal{O}_{\mathcal{M}})[1-k],$$

consisting of polydifferential operators on $\mathcal{O}_{\mathcal{M}}$ which, for $k\geq 1$, vanish on every element $f_1\otimes\ldots\otimes f_k\in\mathcal{O}_{\mathcal{M}}^{\otimes k}$ with at least one function, f_i, $i=1,\ldots,k$, constant. A degree one section,

$\Gamma \in \mathcal{D}_{\mathcal{M}}$, decomposes into a sum, $\sum_{k\geq 0}\Gamma_k$, with $\Gamma_k \in \mathrm{Hom}_{2-k}(\mathcal{O}_{\mathcal{M}}^{\otimes k}, \mathcal{O}_{\mathcal{M}})$. In a local coordinate system, $(t^\alpha, \partial/\partial t^\alpha \simeq e_\alpha)$ on $\mathcal{M}$, Γ can be represented as a Taylor series,

$$\Gamma = \sum_{k\geq 0}\sum_{I_1,\ldots,I_k,J} \Gamma_J^{I_1,\ldots,I_k} e_{I_1}\otimes\ldots e_{I_k}\otimes t^J,$$

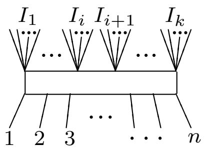
where for each fixed k and $|J|$ only a finite number of coefficients $\Gamma_J^{I_1,\ldots,I_k}$ is non-zero. The summation runs over multi-indices, $I := \alpha_1\alpha_2\ldots\alpha_{|I|}$, and $e_I := e_{\alpha_1}\odot\ldots\odot e_{\alpha_{|I|}}$, $t^I := t^{\alpha_1}\odot\ldots t^{\alpha_{|I|}}$. Hence the associated $\mathbb{S}$-bimodule $E_{\mathcal{D}}$ is given by

$$E_{\mathcal{D}}(m,n) = \mathbf{E}(m)\otimes \mathbf{1_n}, \quad m,n\geq 0,$$

where

$$\begin{aligned} \mathbf{E}(0) &:= \mathbb{R}[-2],\\ \mathbf{E}(m\geq 1) &:= \bigoplus_{k\geq 1}\bigoplus_{\substack{[m]=I_1\sqcup\ldots\sqcup I_k\\ |I_1|,\ldots,|I_k|\geq 1}} \mathrm{Ind}^{\mathbb{S}_m}_{\mathbb{S}_{|I_1|}\times\ldots\times\mathbb{S}_{|I_k|}} \mathbf{1}_{|I_1|}\otimes\ldots\otimes\mathbf{1}_{|I_k|}[k-2] \end{aligned}$$

The basis of $\mathsf{P}_1\langle\mathcal{D}\rangle(m,n)$ can be represented by directed planar corollas of the form,



$$\simeq \quad \Gamma_J^{I_1,\ldots,I_k},$$

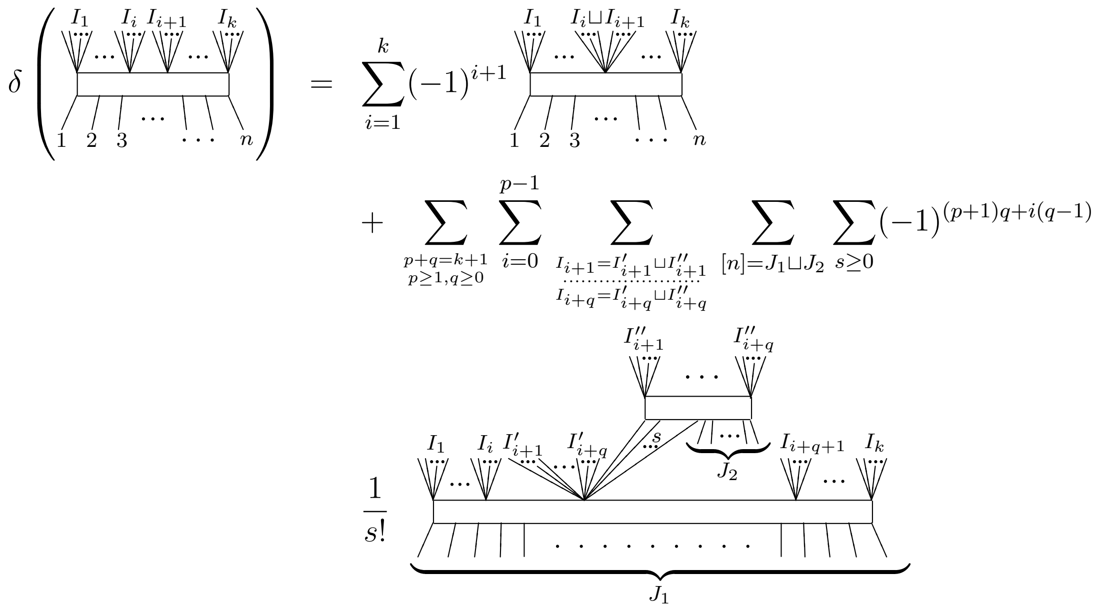
where

- the input legs are labeled by the set $[n] := \{1,2,\ldots,n\}$ and are symmetric (so that it does not matter how labels from $[n]$ are distributed over them),
- the output legs (if there are any) are labeled by the set $[m]$ partitioned into k disjoint non-empty subsets,

$$[m] = I_1\sqcup\ldots\sqcup I_i\sqcup I_{i+1}\sqcup\ldots\sqcup I_k,$$

  and legs in each I_i-bunch are symmetric (so that it does not matter how labels from the set I_i are distributed over legs in I_ith bunch).

The $\mathbb{Z}$-grading in $\mathsf{P}\langle\mathcal{D}\rangle$ is defined by associating degree $2-k$ to such a corolla. The formula $(\star\star)$ in Sect. 2.5 provides us with the following explicit expression for the differential, δ, in $\mathsf{P}\langle\mathcal{D}\rangle$,



where the first sum comes from the Hochschild differential d_{H} and the second sum comes from the Hochschild brackets $[\ ,\ ]_{\mathrm{H}}$. The s-summation in the latter runs over the number, s, of edges connecting the two internal vertices. As s can be zero, the r.h.s. above contains disconnected graphs (more precisely, disjoint unions of two corollas).

2.7.1. Proposition. *There is a one-to-one correspondence between representations,*

$$\phi : (\mathsf{P}\langle\mathcal{D}\rangle, \delta) \longrightarrow (\mathsf{End}\langle M\rangle, d),$$

of $(\mathsf{P}\langle\mathcal{D}\rangle, \delta)$ *in a dg vector space* (M, d) *and Maurer-Cartan elements,* γ*, in* $\mathcal{D}_{\mathcal{M}}$*, that is, degree one elements satisfying the equation,* $d_{\mathrm{H}}\gamma + \frac{1}{2}[\gamma,\gamma]_{\mathrm{H}} = 0$*.*

Proof is similar to the proof of Proposition 2.6.1.

2.8. Remark. Kontsevich's formality map [Ko1] can be interpreted as a morphism of dg props,

$$F_\infty : (\mathsf{P}\langle\mathcal{D}\rangle, \delta) \longrightarrow (\mathsf{P}\langle\wedge^\bullet\mathcal{T}\rangle^{\circlearrowright}, \delta).$$

Vice versa, any morphism of the above dg props gives rise to a *universal* formality map in the sense of [Ko1]. Note that the dg prop of polyvector fields, $\mathsf{P}\langle\wedge^\bullet\mathcal{T}\rangle$, appears above in the wheel extended form, $\mathsf{P}\langle\wedge^\bullet\mathcal{T}\rangle^{\circlearrowright}$. This is not accidental: it is not hard to (by quantizing, e.g. a pair consisting of a linear Poisson structure and quadratic homological vector field) there does *not* exist a morphism between ordinary (i.e. unwheeled) dg props, $(\mathsf{P}\langle\mathcal{D}\rangle, \delta) \longrightarrow (\mathsf{P}\langle\wedge^\bullet\mathcal{T}\rangle, \delta)$, satisfying the quasi-classical limit. Thus wheeled completions of classical dg props are absolutely necessary at least from the point of view of applications to geometric problems.

We shall next investigate how wheeled completion of directed graph complexes affects their cohomology.

§3. Directed graph complexes with loops and wheels

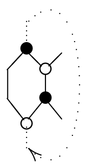
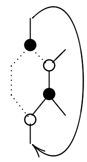
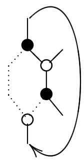
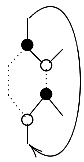
3.1. $\mathfrak{G}^{\circlearrowright}$ **versus** $\mathfrak{G}^{\uparrow}$**.** One of the most effective methods for computing cohomology of dg free props (that is, $\mathfrak{G}^{\uparrow}$-graph complexes) is based on the idea of interpreting the differential as a perturbation of its $\frac{1}{2}$propic part which, in this $\mathfrak{G}^{\uparrow}$-case, can often be singled out by the path filtration [Ko2, MaVo]. However, one can *not* apply this idea directly to graphs with wheels — it is shown below that a filtration which singles out the $\frac{1}{2}$propic part of the differential does *not* exist in general even for dioperadic differentials. Put another way, if one takes a $\mathfrak{G}^{\uparrow}$-graph complex, $(\mathsf{P}\langle E\rangle, \delta)$, enlarges it by adding decorated graphs with wheels while keeping the original differential δ unchanged, then one ends up in a very different situation in which the idea of $\frac{1}{2}$props is no longer directly applicable.

3.2. Graphs with back-in-time edges. Here we suggest the following trick to solve the problem: we further enlarge our set of graphs with wheels, $\mathfrak{G}^{\circlearrowright} \rightsquigarrow \mathfrak{G}^{+}$, by putting a mark on one (and only one) of the edges in each wheel, and then study the natural "forgetful" surjection, $\mathfrak{G}^{+} \to \mathfrak{G}^{\circlearrowright}$. The point is that $\mathfrak{G}^{+}$-graph complexes again admit a filtration which singles out the $\frac{1}{2}$prop part of the differential and hence their cohomology are often easily computable.

More precisely, let $\mathfrak{G}^{+}(m,n)$ be the set of all directed (m,n)-graphs G which satisfy conditions 2.1.3(i)-(iv), and the following one:

(v) every oriented wheel in G (if any) has one and only one of its internal edges marked (say, dashed) and called *back-in-time* edge.

For example,

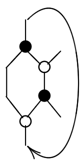
 ,  ,  and 

are four different graphs in $\mathfrak{G}^{+}(1,1)$.

Clearly, we have a natural surjection,

$$u : \mathfrak{G}^{+}(m,n) \longrightarrow \mathfrak{G}^{\circlearrowright}(m,n),$$

which forgets the markings. For example, the four graphs above are mapped under u into the same graph,

 $\in \mathfrak{G}^{\circlearrowright}(1,1)$,

and, in fact, span its pre-image under u.

3.3. Graph complexes. Let $E = \{E(m,n)\}_{m,n\geq 1, m+n\geq 3}$ be an $\mathbb{S}$-bimodule and let $(\mathsf{P}\langle E\rangle, \delta)$ be a dg free prop on E with the differential δ which preserves connectedness and genus, that is, δ applied to any decorated (m,n)-corolla creates a connected (m,n)-tree. Such a differential can be called *dioperadic*, and from now on we restrict ourselves to dioperadic differentials only. This restriction is not that dramatic: every dg free prop with a non-dioperadic but connected[2] differential always admits a filtration which singles out its dioperadic part [MaVo]. Thus the

[2]A differential δ in $\mathsf{P}\langle E\rangle$ is called *connected* if it preserves the filtration of $\mathsf{P}\langle E\rangle$ by the number of connected (in the topological sense) components.

technique we develop here in §3 can, in principle, be applied to a wheeled extension of any dg free prop with a connected differential.

We enlarge the $\mathfrak{G}^{\uparrow}$-graph complex $(\mathsf{P}\langle E\rangle, \delta)$ in two ways,

$$\mathsf{P}^{\circlearrowright}\langle E\rangle(m,n) := \bigoplus_{G\in\mathfrak{G}^{\circlearrowright}(m,n)} \left(\bigotimes_{v\in v(G)} E(Out(v), In(v))\right)_{AutG},$$

$$\mathsf{P}^{+}\langle E\rangle(m,n) := \bigoplus_{G\in\mathfrak{G}^{+}(m,n)} \left(\bigotimes_{v\in v(G)} E(Out(v), In(v))\right)_{AutG},$$

and notice that both $\mathsf{P}^{\circlearrowright}\langle E\rangle := \{\mathsf{P}^{\circlearrowright}\langle E\rangle(m,n)\}$ and $\mathsf{P}^{+}\langle E\rangle := \{\mathsf{P}^{\circlearrowright}\langle E\rangle(m,n)\}$ have a natural structure of dg prop with respect to disjoint unions, grafting of graphs, and the original differential δ. Clearly, they contain $(\mathsf{P}\langle E\rangle, \delta)$ as a dg subprop. There is a natural morphism of dg props,

$$u : (\mathsf{P}^{+}\langle E\rangle, \delta) \longrightarrow (\mathsf{P}^{\circlearrowright}\langle E\rangle, \delta),$$

which forgets the markings. Let $(\mathsf{L}^{+}\langle E\rangle, \delta) := \mathsf{Ker}\, u$ and denote the natural inclusion $\mathsf{L}^{+}\langle E\rangle \subset \mathsf{P}^{+}\langle E\rangle$ by i.

3.4. Fact. There is a short exact sequence of graph complexes,

$$0 \longrightarrow (\mathsf{L}^{+}\langle E\rangle, \delta) \xrightarrow{i} (\mathsf{P}^{+}\langle E\rangle, \delta) \xrightarrow{u} (\mathsf{P}^{\circlearrowright}\langle E\rangle, \delta) \longrightarrow 0,$$

where u is the map which forgets markings. Thus, if the natural inclusion of complexes, $i : (\mathsf{L}^{+}\langle E\rangle, \delta) \to (\mathsf{P}^{+}\langle E\rangle, \delta)$ induces a monomorphism in cohomology, $[i] : H(\mathsf{L}^{+}\langle E\rangle, \delta) \to H(\mathsf{P}^{+}\langle E\rangle, \delta)$, then

$$H(\mathsf{P}^{\circlearrowright}\langle E\rangle, \delta) = \frac{H(\mathsf{P}^{+}\langle E\rangle, \delta)}{H(\mathsf{L}^{+}\langle E\rangle, \delta)}.$$

Put another way, if $[i]$ is a monomorphism, then $H(\mathsf{P}^{\circlearrowright}\langle E\rangle, \delta)$ is obtained from $H(\mathsf{P}^{+}\langle E\rangle, \delta)$ simply by forgetting the markings.

3.5. Functors which adjoin wheels. We are interested in this paper in dioperads, D, which are either free, $\mathsf{D}\langle E\rangle$, on an $\mathbb{S}$-bimodule $E = \{E(m,n)\}_{m,n\geq 1, m+n\geq 3}$, or are naturally represented as quotients of free dioperads,

$$D = \frac{\mathsf{D}\langle E\rangle}{< I >},$$

modulo the ideals are generated by some relations $I \subset \mathsf{D}\langle E\rangle$. Then the free prop, $\Omega_{\mathsf{D}\to P}\langle E\rangle$, generated by D is simply the quotient of the free prop, $\mathsf{P}\langle E\rangle$,

$$\Omega_{\mathsf{D}\to\mathsf{P}}\langle D\rangle := \frac{\mathsf{P}\langle E\rangle}{< I >},$$

by the ideal generated by the same relations I. Now we define two other props[3],

$$\Omega_{\mathsf{D}\to\mathsf{P}^{\circlearrowright}}\langle D\rangle := \frac{\mathsf{P}^{\circlearrowright}\langle E\rangle}{< I >^{\circlearrowright}}, \qquad \Omega_{\mathsf{D}\to\mathsf{P}^{+}}\langle D\rangle := \frac{\mathsf{P}^{+}\langle E\rangle}{< I >^{+}},$$

where $< I >^{\circlearrowright}$ (resp., $< I >^{+}$) is the subspace of those graphs G in $\mathsf{P}^{\circlearrowright}\langle E\rangle$ (resp., in $\mathsf{P}^{+}\langle E\rangle$) which satisfy the following condition: there exists a (possibly empty) set of cyclic edges whose breaking up into two legs produces a graph lying in the ideal $< I >$ which defines the prop $\Omega_{\mathsf{D}\to P}\langle D\rangle$.

[3]These props are particular examples of *wheeled props* which will be discussed in detail elsewhere.

Analogously one defines functors $\Omega_{\frac{1}{2}\mathsf{P}\to\mathsf{P}^{\circlearrowright}}$ and $\Omega_{\frac{1}{2}\mathsf{P}\to\mathsf{P}^{+}}$.

From now on we abbreviate notations as follows,

$$D^{\uparrow} := \Omega_{\mathsf{D}\to\mathsf{P}}\langle D\rangle, \quad D^{+} := \Omega_{\mathsf{D}\to\mathsf{P}^{+}}\langle D\rangle, \quad D^{\circlearrowright} := \Omega_{\mathsf{D}\to\mathsf{P}^{\circlearrowright}}\langle D\rangle,$$

for values of the above defined functors on dioperads, and, respectively

$$D_0^{\uparrow} := \Omega_{\frac{1}{2}\mathsf{P}\to\mathsf{P}}\langle D_0\rangle, \quad D_0^{+} := \Omega_{\frac{1}{2}\mathsf{P}\to\mathsf{P}^{\circlearrowright}}\langle D_0\rangle, \quad D_0^{\circlearrowright} := \Omega_{\frac{1}{2}\mathsf{P}\to\mathsf{P}^{\circlearrowright}}\langle D_0\rangle.$$

for their values on $\frac{1}{2}$props.

3.5.1. Facts. **(i)** If D is a dg dioperad, then both $D^{\circlearrowright}$ and D^{+} are naturally dg props. **(ii)** If D is an operad, then both $D^{\circlearrowright}$ and D^{+} may contain at most one wheel.

3.5.2. Proposition. *Any finite-dimensional representation of the dioperad D lifts to a representation of its wheeled prop extension $D^{\circlearrowright}$.*

Proof. If $\phi : D \to \mathrm{End}\langle M\rangle$ is a representation, then we first extend it to a representation, $\phi^{\circlearrowright}$, of $\mathsf{P}^{\circlearrowright}\langle E\rangle \subset$ as in Sect. 2.3.2 and then notice that $\phi^{\circlearrowright}(f) = 0$ for any $f \in < I >^{\circlearrowright}$. □

3.5.3. Definition. Let D be a Koszul dioperad with $(D_\infty, \delta) \to (D, 0)$ being its minimal resolution. The dioperad D is called *stably Koszul* if the associated morphism of the wheeled completions,

$$(D_\infty^{\circlearrowright}, \delta) \longrightarrow (D^{\circlearrowright}, 0)$$

remains a quasi-isomorphism.

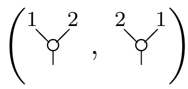
3.5.4. Example. The notion of stable Koszulness is non-trivial. Just adding oriented wheels to a minimal resolution of a Koszul operad while keeping the differential unchanged may alter the cohomology group of the resulting graph complex as the following example shows.

Claim. *The operad,* Ass*, of associative algebras is not stably Koszul.*

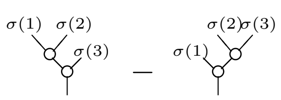
Proof. The operad Ass can be represented a quotient,

$$\mathsf{Ass} = \frac{\mathsf{Oper}\langle E\rangle}{\mathsf{Ideal} < R >},$$

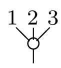
of the free operad, $\mathsf{Oper}\langle E\rangle$, generated by the following $\mathbb{S}$-module E,

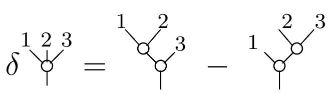
$E(n) := \begin{cases} k[\mathbb{S}_2] = \text{span} \end{cases}$  for $n = 2$; 0 otherwise,

modulo the ideal generated by the following relations,

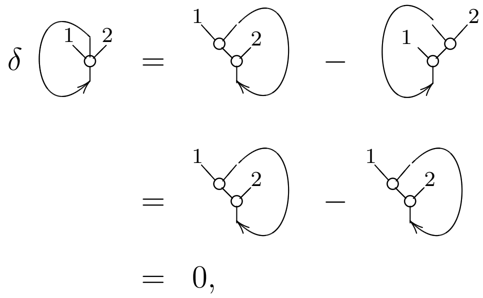


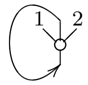
$$= 0, \qquad \forall \sigma \in \mathbb{S}_3.$$

Hence the minimal resolution, $(\mathsf{Ass}_\infty, \delta)$ of Ass contains a degree -1 corolla  such that

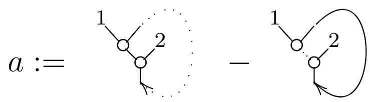


Therefore, in its wheeled extension, $(\mathsf{Ass}_\infty^{\circlearrowright}, \delta)$, one has



implying existence of a non-trivial cohomology class,  in $\mathsf{H}(\mathsf{A}ss_\infty^{\circlearrowright}, \delta)$ which does *not* belong to $\mathsf{Ass}^{\circlearrowright}$. Thus $\mathsf{A}ss$ is Koszul, but *not* stably Koszul. □

It is instructive to see explicitly how the map $[i] : \mathsf{H}(\mathsf{L}^+\langle\mathsf{Ass}_\infty\rangle, \delta) \to \mathsf{H}(\mathsf{P}^+\langle\mathsf{A}ss_\infty\rangle, \delta)$ fails to be a monomorphism. As $\mathsf{L}^+\langle\mathsf{Ass}_\infty\rangle$ does not contain loops, the element



defines a non-trivial cohomology class, $[a]$, in $\mathsf{H}(\mathsf{L}^+\langle\mathsf{Ass}_\infty\rangle, \delta)$, whose image, $[i]([a])$, in $\mathsf{H}(\mathsf{P}^+\langle\mathsf{Ass}_\infty\rangle, \delta)$ vanishes.

3.6. Koszul substitution laws. Let $P = \{P(n)\}_{n\geq 1}$ and $Q = \{Q(n)\}_{n\geq 1}$ be two quadratic Koszul operads generated,

$$P := \frac{\mathsf{P}\langle E_P(2)\rangle}{< I_P >}, \qquad Q := \frac{\mathsf{P}\langle E_Q(2)\rangle}{< I_Q >},$$

by $\mathbb{S}_2$-modules $E_P(2)$, and, respectively, $E_Q(2)$.

One can canonically associate [MaVo] to such a pair the $\frac{1}{2}$prop, $P \diamond Q^\dagger$, with

$$P \diamond Q^\dagger(m,n) = \begin{cases} P(n) & \textit{for } m = 1, n \geq 2, \\ Q(m) & \textit{for } n = 1, m \geq 2, \\ 0 & \textit{otherwise}, \end{cases}$$

and the $\frac{1}{2}$prop compositions,

$$\left\{ {}_1\circ_j : P \diamond Q^\dagger(m_1, 1) \otimes P \diamond Q^\dagger(m_2, n_2) \longrightarrow P \diamond Q^\dagger(m_1 + m_2 - 1, n_2)\right\}_{1\leq j\leq m_2}$$

being zero for $n_2 \geq 2$ and coinciding with the operadic composition in Q for $n_2 = 1$, and

$$\left\{ {}_i\circ_1 : P \diamond Q^\dagger(m_1, n_1) \otimes P \diamond Q^\dagger(1, n_2) \longrightarrow P \diamond Q^\dagger(m_1 + m_2 - 1, n_2)\right\}_{1\leq i\leq n_1}$$

being zero for $m_1 \geq 2$ and otherwise coinciding with the operadic composition in P for $m_1 = 1$.

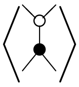
Let $D_0 = \Omega_{\frac{1}{2}\mathsf{P}\to\mathsf{D}}\langle P \diamond Q^\dagger\rangle$ be the associated free dioperad, $D_0^!$ its Koszul dual dioperad, and $(D_{0\infty} := \mathbf{D}D_0^!, \delta_0)$ the associated cobar construction [Ga]. As D_0 is Koszul [Ga, MaVo], the latter provides us with the dioperadic minimal model of D_0. By exactness of $\Omega_{\frac{1}{2}\mathsf{P}\to\mathsf{P}}$, the dg free prop, $(D_{0\infty}^\uparrow := \Omega_{\mathsf{D}\to\mathsf{P}}\langle D_{0\infty}\rangle, \delta_0)$, is the minimal model of the prop $D_0^\uparrow := \Omega_{\mathsf{D}\to\mathsf{P}}\langle D_0\rangle \simeq \Omega_{\frac{1}{2}\mathsf{P}\to\mathsf{P}}\langle P \diamond Q^\dagger\rangle$.

3.6.1. Remark. The prop $D_0^{\uparrow}$ can be equivalently defined as the quotient,

$$\frac{P * Q^{\dagger}}{I_0}$$

where $P * Q^{\dagger}$ is the free product of props associated to operads P and[4] $Q^{\dagger}$, and the ideal I_0 is generated by graphs of the form,

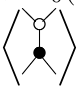
$$I_0 = \mathsf{span}\left\langle \quad \right\rangle \simeq D_0(2,1) \otimes D_0(1,2) = E_Q(2) \otimes E_P(2)$$

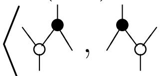


with white vertex decorated by elements of $E_Q(2)$ and black vertex decorated by elements of $E_P(2)$.

Let us consider a morphism of $\mathbb{S}_2$-bimodules,

$$\begin{array}{rccc} \lambda: & D_0(2,1) \otimes D_0(1,2) & \longrightarrow & D_0(2,2) \\ & \mathsf{span}\left\langle \quad \right\rangle & \longrightarrow & \mathsf{span}\left\langle \quad , \quad \right\rangle \end{array}$$





and define [MaVo] the dioperad, D_λ, as the quotient of the free dioperad generated by the two spaces of binary operations, $D_0(2,1) = E_Q(2)$ and $D_0(1,2) = E_P(2)$, modulo the ideal generated by relations in P, relations in Q as well as the followings ones,

$$I_\lambda = \mathsf{span}\{f - \lambda f : \forall f \in D_0(2,1) \otimes D_0(1,2)\}.$$

Note that in notations of § 3.6.1 the associated prop, $D_\lambda^{\uparrow} := \Omega_{\mathsf{D}\to P}\langle D_\lambda\rangle$, is just the quotient, $P * Q^{\dagger}/I_\lambda$.

The substitution law λ is called *Koszul*, if D_λ is isomorphic to D_0 as an $\mathbb{S}$-bimodule. Which implies that D_λ is Koszul [Ga]. Koszul duality technique provides $\mathbf{D}D_0^! \simeq \mathbf{D}D_\lambda^!$ with a perturbed differential δ_λ such that $(\mathbf{D}D_0^!, \delta_\lambda)$ is the minimal model, $(D_{\lambda\infty}, \delta_\lambda)$, of the dioperad D_λ.

3.7. Theorem [MaVo, Va] . *The dg free prop $D_{\lambda\infty}^{\uparrow} := \Omega_{\mathsf{D}\to P}\langle D_{\lambda\infty}\rangle$ is the minimal model of the prop $D_\lambda^{\uparrow}$, i.e. the natural morphism*

$$(D_{\lambda\infty}^{\uparrow}, \delta_\lambda) \longrightarrow (D_\lambda^{\uparrow}, 0),$$

which sends to zero all vertices of $D_{\lambda\infty}^{\uparrow}$ except binary ones decorated by elements of $E_P(2)$ and $E_Q(2)$, is a quasi-isomorphism.

Proof. The main point is that

$$F_p := \left\{ \mathrm{span}\langle f \in D_{\lambda\infty}^{\uparrow}\rangle \ : \ \begin{array}{l} \text{number of directed paths in the graph } f \\ \text{which connect input legs with output ones} \end{array} \ \leq p \right\}.$$

defines a filtration of the complex $D_{\lambda\infty}^{\uparrow}$. The associated spectral sequence, $\{E_r, d_r\}_{r\geq 0}$, is exhaustive and bounded below so that it converges to the cohomology of $(D_{\lambda\infty}^{\uparrow}, \delta_\lambda)$.

The zeroth term of this spectral sequence is isomorphic to $(D_{0\infty}^{\uparrow}, \delta_0)$ and hence, by Koszulness of the dioperad D_0 and exactness of the functor $\Omega_{\frac{1}{2}\mathsf{P}\to\mathsf{P}}$, has the cohomology, E_1, isomorphic to $D_0^{\uparrow}$. Which, by Koszulness of D_λ, is isomorphic to $D_\lambda^{\uparrow}$ as an $\mathbb{S}$-bimodule. As $\{d_r = 0\}_{r\geq 1}$, the result follows along the same lines as in the second part of the proof of Theorem 2.6.2. □

[4]the symbol † stands for the functor on props, $P = \{P(m,n)\} \to P^{\dagger} = \{P^{\dagger}(m,n)\}$ which reverses "time flow", i.e. $P^{\dagger}(m,n) := P(n,m)$.

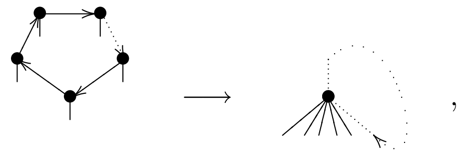
3.8. Cohomology of graph complexes with marked wheels. In this section we analyze the functor $\Omega_{\frac{1}{2}\mathsf{P}\to\mathsf{P}^+}$. The following statement is one of the motivations for its introduction (it does *not* hold true for the "unmarked" version $\Omega_{\frac{1}{2}\mathsf{P}\to\mathsf{P}^{\circlearrowright}}$).

3.8.1. Theorem. *The functor $\Omega_{\frac{1}{2}\mathsf{P}\to\mathsf{P}^+}$ is exact.*

Proof. Let T be an arbitrary dg $\frac{1}{2}$prop. The main point is that we can use $\frac{1}{2}$prop compositions and presence of marks on cyclic edges to represent $\Omega_{\frac{1}{2}\mathsf{P}\to\mathsf{P}^+}\langle T\rangle$ as a vector space *freely* generated by a family of decorated graphs,

$$\Omega_{\frac{1}{2}P\to\mathsf{P}^+}\langle T\rangle(m,n) = \bigoplus_{G\in\overline{\mathfrak{G}^+}(m,n)} G\langle P\rangle,$$

where $\overline{\mathfrak{G}^+}(m,n)$ is a subset of $\mathfrak{G}^+(m,n)$ consisting of so called *reduced* graphs, G, which satisfy the following defining property: for each pair of internal vertices, (v_1,v_2), of G which are connected by an unmarked edge directed from v_1 to v_2 one has $|Out(v_1)|\geq 2$ and $|In(v_2)|\geq 2$. Put another way, given an arbitrary T-decorated graph with wheels, one can perform $\frac{1}{2}$prop compositions ("contractions") along all unmarked internal edges (v_1,v_2) which do not satisfy the above conditions. The result is a reduced decorated graph (with wheels). Which is uniquely defined by the original one. Notice that marks are vital for this contraction procedure, e.g.



to be well-defined.

Then we have

$$\begin{aligned}
H^*\left(\Omega_{\frac{1}{2}\mathsf{P}\to\mathsf{P}^+}\langle T\rangle(m,n)\right) &= H^*\left(\bigoplus_{G\in\bar{\mathfrak{G}}^+(m,n)}\left(\bigotimes_{v\in v(G)} T(Out(v),In(v))\right)_{AutG}\right)\\
&= \bigoplus_{G\in\bar{\mathfrak{G}}^+(m,n)} H^*\left(\bigotimes_{v\in v(G)} T(Out(v),In(v))\right)_{AutG} \quad \text{by Maschke's theorem}\\
&= \bigoplus_{G\in\bar{\mathfrak{G}}^+(m,n)}\left(\bigotimes_{v\in v(G)} H^*(T)(Out(v),In(v))\right)_{AutG} \quad \text{by Künneth formula}\\
&= \Omega_{\frac{1}{2}\mathsf{P}\to\mathsf{P}^+}\langle H^*(T)\rangle(m,n).
\end{aligned}$$

In the second line we used the fact that the group $AutG$ is finite. □

Another motivation for introducing graph complexes with *marked* wheels is that they admit a filtration which singles out the $\frac{1}{2}$propic part of the differential. A fact which we heavily use in the proof of the following

3.8.2. Theorem. *Let D_λ be a Koszul dioperad with Koszul substitution law and let $(D_{\lambda\infty},\,\delta)$ be its minimal resolution. The natural morphism of graph complexes,*

$$(D^+_{\lambda\infty},\,\delta_\lambda)\longrightarrow(D^+_\lambda,0)$$

is a quasi-isomorphism.

Proof. Consider first a filtration of the complex $(D^+_{\lambda\infty},\, \delta_\lambda)$ by the number of marked edges, and let $(D^+_{\lambda\infty},\, b)$ denote 0th term of the associated spectral sequence (which, as we shall show below, degenerates at the 1st term).

To any decorated graph $f \in D^+_{\lambda\infty} = \Omega_{\mathsf{D}\to\mathsf{P}^+}\langle D_{\lambda\infty}\rangle$ one can associate a graph without wheels, $\overline{f} \in \Omega_{\mathsf{D}\to P}\langle D_{\lambda\infty}\rangle$, by breaking every marked cyclic edge into two legs (one of which is input and another one is output). Let $|\overline{f}|$ be the number of directed paths in the graph $\overline{f}$ which connect input legs with output ones. Then

$$F_p := \left\{ f \in D^+_{\lambda\infty} \ : \ |\overline{f}| \leq p \right\}.$$

defines a filtration of the complex $(D^+_{\lambda\infty},\, b)$.

The zeroth term of the spectral sequence, $\{E_r, d_r\}_{r\geq 0}$, associated to this filtration is isomorphic to $(\Omega_{\frac{1}{2}\mathsf{P}\to\mathsf{P}^+}\langle D^0_\infty\rangle, \delta_0)$ and hence, by Theorem 3.8.1, has the cohomology, E_1, equal to $D^+_0 := \Omega_{\frac{1}{2}\mathsf{P}\to\mathsf{P}^+}\langle D_0\rangle$. Which, by Koszulness of D_λ, is isomorphic as a vector space to D^+_λ. As differentials of all higher terms of both our spectral sequences vanish, the result follows. □

3.8.3. Remark. In the proof of Theorem 3.8.2 the $\frac{1}{2}$propic part, δ_0, of the differential δ_λ was singled out in two steps: first we introduced a filtration by the number of marked edges, and then a filtration by the number of paths, $|\bar{f}|$, in the unwheeled graphs $\overline{f}$. As the following lemma shows, one can do it in one step. Let $w(f)$ stand for the number of marked edges in a decorated graph $f \in D^+_{\lambda\infty}$.

3.8.4. Lemma *The sequence of vector spaces spaces, $p \in \mathbb{N}$,*

$$\mathsf{F}_p := \left\{ span\langle f \in D^+_{\lambda\infty}\rangle \ : \ ||f|| := 3^{w(f)}|\overline{f}| \leq p \right\},$$

defines a filtration of the complex $(D^+_{\lambda\infty}, \delta_\lambda)$ whose spectral sequence has 0-th term isomorphic to $(D^+_{0\infty}, \delta_0)$.

Proof. It is enough to show that for any graph f in $D^+_{\lambda\infty}$ with $w(f) \neq 0$ one has, $||\delta_\lambda f|| \leq ||f||$.

We can, in general, split $\delta_\lambda f$ into two groups of summands,

$$\delta_\lambda f = \sum_{a\in I_1} g_a + \sum_{b\in I_2} g_b$$

where $w(g_a) = w(f)$, $\forall a \in I_1$, and $w(g_b) = w(f) - p_b$ for some $p_b \geq 1$ and all $b \in I_2$.

For any $a \in I_1$,

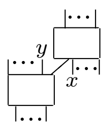
$$||g_a|| = 3^{w(f)}|\overline{g_a}| \leq 3^{w(f)}|\overline{f}| = ||f||.$$

So it remains to check the inequality $||g_b|| \leq ||f||$, $\forall b \in I_2$.

We can also split $\delta_\lambda\overline{f}$ into two groups of summands,

$$\delta_\lambda\overline{f} = \sum_{a\in I_1} h_a + \sum_{b\in I_2} h_b$$

where $\{h_b\}_{b\in I_2}$ is the set of all those summands which contain two-vertex subgraphs of the form,



with contain half-edges of the type x and y corresponding to broken wheeled paths in f. Every graph g_b is obtained from the corresponding h_b by gluing some number of connected to y output legs with the same number of connected to x input legs into new internal non-cyclic edges. This gluing operation creates p_b new paths connecting some internal vertices in $\overline{h_b}$, and hence may increase the total number of paths in $\overline{h_b}$, but no more than by the factor of p_b+1, i.e. $|\overline{g_b}| \leq (p_b+1)|h_b|$, $\forall b \in I_2$.

Finally, we have

$$||g_b|| = 3^{w(f)-p_b}|\overline{g_b}| \leq 3^{w(f)-p_b}(p_b+1)|h_b| < 3^{w(f)}|\overline{f}| = ||f||, \quad \forall b \in I_2.$$

The part of the differential δ_λ which preserves the filtration must in fact preserve both the number of marked edges, $w(f)$, and the number of paths, $|\overline{f}|$, for any decorated graph f. Hence this is precisely δ_0. □

3.9. Graph complexes with unmarked wheels built on $\frac{1}{2}$props. Let $\left(T = \frac{1}{2}\mathsf{P}\langle E\rangle/<I>, \delta\right)$ be a dg $\frac{1}{2}$prop. In §3.5 we defined its wheeled extension,

$$\left(T^{\circlearrowright} := \frac{\mathsf{P}^{\circlearrowright}\langle E\rangle}{<I>^{\circlearrowright}}, \delta\right).$$

Now we specify a dg subprop, $\Omega_{\mathsf{no}-oper}\langle T\rangle \subset T^{\circlearrowright}$, whose cohomology is easy to compute.

3.9.1. Definition. Let $E = \{E(m,n)\}_{m,n\geq 1, m+n\geq 3}$ be an $\mathbb{S}$-bimodule, and $\mathsf{P}^{\circlearrowright}\langle E\rangle$ the associated prop of decorated graphs with wheels. We say that a wheel W in a graph $G \in \mathsf{P}^{\circlearrowright}\langle E\rangle$ is *operadic* if all its cyclic vertices are decorated either by elements of $\{E(1,n)\}_{n\geq 2}$ only, or by elements $E(n,1)_{n\geq 2}$ only. Vertices of operadic wheels are called *operadic cyclic vertices.* Notice that operadic wheels can be of geometric genus 1 only.

Let $\mathsf{P}^{\circlearrowright}{}_{\mathsf{no}-oper}\langle E\rangle$ be the subspace of $\mathsf{P}^{\circlearrowright}\langle E\rangle$ consisting of graphs with no operadic wheels, and let

$$\Omega_{\mathsf{no}-oper}\langle T\rangle = \frac{\mathsf{P}^{\circlearrowright}{}_{\mathsf{no}-oper}\langle E\rangle}{<I>^{\circlearrowright}},$$

be the associated dg sub-prop of $(T^{\circlearrowright}, \delta)$.

Clearly, $\Omega_{\mathsf{no}-oper}$ is a functor from the category of dg $\frac{1}{2}$props to the category of dg props. It is worth pointing out that this functor can *not* be extended to dg dioperads as differential can, in general, create operadic wheels from non-operadic ones.

3.9.2. Theorem. *The functor* $\Omega_{\mathsf{no}-oper}$ *is exact.*

Proof. Let (T,δ) be an arbitrary dg $\frac{1}{2}$prop. Every wheel in $\Omega_{\mathsf{no}-oper}\langle T\rangle$ contains at least one cyclic edge along which $\frac{1}{2}$prop composition in T is not possible. This fact allows one to non-ambiguously perform such compositions along all those cyclic and non-cyclic edges at which such a composition makes sense, and hence represent $\Omega_{\mathsf{no}-oper}\langle T\rangle$ as a vector space *freely* generated by a family of decorated graphs,

$$\Omega_{\mathsf{no}-oper}\langle T\rangle(m,n) = \bigoplus_{G\in\overline{\mathfrak{G}^{\circlearrowright}}(m,n)} G\langle T\rangle$$

where $\overline{\mathfrak{G}^{\circlearrowright}}(m,n)$ is a subset of $\mathfrak{G}^{\circlearrowright}(m,n)$ consisting of *reduced* graphs, G, which satisfy the following defining properties: (i) for each pair of internal vertices, (v_1,v_2), of G which are connected by an edge directed from v_1 to v_2 one has $|Out(v_1)| \geq 2$ and $|In(v_2)| \geq 2$; (ii) there are no operadic wheels in G. The rest of the proof is exactly the same as in §3.8.1. □

Let P and Q be Koszul operads and let D_0 be the associated Koszul dioperad (defined in §3.6) whose minimal resolution is denoted by $(D_{0\infty}, \delta_0)$.

3.9.3. Corollary. $H(\Omega_{\text{no}-oper}\langle D_{0\infty}\rangle, \delta_0) = \Omega_{\frac{1}{2}\mathsf{P}\to P}\langle D_0\rangle$.

Proof. By Theorem 3.9.2,

$$H(\Omega_{\text{no}-oper}\langle D_{0\infty}\rangle, \delta_0) = \Omega_{\text{no}-oper}\langle H(D_{0\infty}, \delta_0)\rangle = \Omega_{\text{no}-oper}\langle D_0\rangle.$$

But the latter space can not have graphs with wheels as any such a wheel would contain at least one "non-reduced" internal cyclic edge corresponding to composition,

$$\circ_{1,1} : D_0(m, 1) \otimes D_0(1, n) \longrightarrow D_0(m, n),$$

which is zero by the definition of D_0 (see §3.6). □

3.10. Theorem. *For any Koszul operads P and Q,*

(i) the natural morphism of graph complexes,

$$(D^{\circlearrowright}_{0\infty}, \delta_0) \longrightarrow (D^{\circlearrowright}_0, 0)$$

is a quasi-isomorphism if and only if the operads P and Q are stably Koszul;

(ii) there is, in general, an isomorphism of $\mathbb{S}$-bimodules,

$$H(D^{\circlearrowright}_{0\infty}, \delta_0) = \frac{H(P^{\circlearrowright}_\infty) * H(Q^{\circlearrowright}_\infty)^\dagger}{I_0}$$

where $H(P^{\circlearrowright}_\infty)$ and $H(Q^{\circlearrowright}_\infty)$ are cohomologies of the wheeled completions of the minimal resolutions of the operads P and Q, $*$ stands for the free product of PROPs, and the ideal I_0 is defined in § 3.6.1.

Proof. (i) The necessity of the condition is obvious. Let us prove its sufficiency.

Let P and Q be stably Koszul operads so that the natural morphisms,

$$(P^{\circlearrowright}_\infty, \delta_P) \to P^{\circlearrowright} \quad \text{and} \quad (Q^{\circlearrowright}_\infty, \delta_Q) \to Q^{\circlearrowright},$$

are quasi-isomorphisms, where (P_∞, δ_P) and (Q_∞, δ_Q) are minimal resolutions of P and Q respectively.

Consider a filtration of the complex $(D^{\circlearrowright}_{0\infty}, \delta_0)$,

$$F_p := \{\text{span}\langle f \in D^{\circlearrowright}_{0\infty}\rangle : \ |f|_2 - |f|_1 \ \leq p\},$$

where

- $|f|_1$ is the number of cyclic vertices in f which belong to operadic wheels;
- $|f|_2$ is the number of non-cyclic half-edges attached to cyclic vertices in f which belong to operadic wheels.

Note that $|f|_2 - |f|_1 \geq 0$. Let $\{E_r, d_r\}_{r\geq 0}$ be the associated spectral sequence. The differential d_0 in E_0 is given by its values on the vertices as follows:

(a) on every non-cyclic vertex and on every cyclic vertex which does *not* belong to an operadic wheel one has $d_0 = \delta_0$;
(b) on every cyclic vertex which belongs to an operadic wheel one has $d_0 = 0$.

Hence modulo the action of finite groups (which we can ignore by Maschke theorem) the complex (E_0, d_0) is isomorphic to the complex $(\Omega_{no-oper}\langle D_{0\infty}\rangle, \delta_0)$, tensored with a trivial complex (i.e. one with vanishing differential). By Corollary 3.9.3 and Künneth formula we obtain,

$$E_1 = H(E_0, d_0) = W_1/h(W_2)$$

where

- W_1 is the subspace of $\mathsf{P}^{\circlearrowright}\langle E_P \oplus E_Q^\dagger\rangle$ consisting of graphs whose wheels (if any) are operadic; here the $\mathbb{S}$-bimodule $E_P \oplus E_Q^\dagger$ is given by

$$(E_P \oplus E_Q^\dagger)(m,n) = \begin{cases} E_P(2), \text{the space of generators of } P, & \text{if } m=1, n=2 \\ E_Q(2), \text{the space of generators of } Q, & \text{if } m=2, n=1 \\ 0, & \text{otherwise;} \end{cases}$$

- W_2 is the subspace of $\mathsf{P}^{\circlearrowright}\langle E_P \oplus E_Q^\dagger \oplus I_P \oplus I_Q^\dagger\rangle$ consisting of graphs, G, whose wheels (if any) are operadic and satisfy the following condition: the elements of I_P and $I_Q^\dagger$ are used to decorate at least one non-cyclic vertex in G. Here I_P and $I_Q^\dagger$ are $\mathbb{S}$-bimodules of relations of the quadratic operads P and $Q^\dagger$ respectively.
- the map $h : W_2 \to W_1$ is defined to be the identity on vertices decorated by elements of $E_P \oplus E_Q^\dagger$, and the tautological (in the obvious sense) morphism on vertices decorated by elements of I_P and $I_Q^\dagger$.

To understand all the remaining terms $\{E_r, d_r\}_{r\geq 1}$ of the spectral sequence we step aside and contemplate for a moment a purely operadic graph complex with wheels, say, $(P_\infty^{\circlearrowright}, \delta_P)$.

The complex $(P_\infty^{\circlearrowright}, \delta_P)$ is naturally a subcomplex of $(D_{0\infty}^{\circlearrowright}, \delta_0)$. Let

$$F_p := \{\text{span}\langle f \in P_\infty^{\circlearrowright}\rangle : \ |f|_2 - |f|_1 \ \leq p\},$$

be the induced filtration, and let $\{E_r^P, d_r^P\}_{r\geq 0}$ be the associated spectral sequence. Then $E_1^P = H(E_0^P, d_0^P)$ is a subcomplex of E_1.

The main point is that, modulo the action of finite groups, the spectral sequence $\{E_r, d_r\}_{r\geq 1}$ is isomorphic to the tensor product of spectral sequences of the form $\{E_r^P, d_r^P\}_{r\geq 1}$ and $\{E_r^Q, d_r^Q\}_{r\geq 1}$. By assumption, the latter converge to $P^{\circlearrowright}$ and $Q^{\circlearrowright}$ respectively. Which implies the result.

(ii) The argument is exactly the same as in (i) except the very last paragraph: the spectral sequences of the form $\{E_r^P, d_r^P\}_{r\geq 1}$ and $\{E_r^Q, d_r^Q\}_{r\geq 1}$ converge, respectively, to $H(P_\infty^{\circlearrowright})$ and to $H(Q_\infty^{\circlearrowright})$ (rather than to $P^{\circlearrowright}$ and $Q^{\circlearrowright}$). □

3.11. Operadic wheeled extension. Let D_λ be a dioperad and $D_{\lambda\infty}$ be its minimal resolution. Let $D_{\lambda\infty}^{\circlearrowright\!\!\to}$ be a dg subprop of $D_{\lambda\infty}^{\circlearrowright}$ spanned by graphs with at most operadic wheels (see §3.9.1). Similarly one defines a subprop, $D_\lambda^{\circlearrowright\!\!\to}$, of $D_\lambda^{\circlearrowright}$.

3.11.1. Theorem. *For any Koszul operads P and Q and any Koszul substitution law λ,*

(i) the natural morphism of graph complexes,

$$(D_{\lambda\infty}^{\circlearrowright\!\!\to}, \delta_\lambda) \longrightarrow D_\lambda^{\circlearrowright\!\!\to},$$

is a quasi-isomorphism if and only if the operads P and Q are both stably Koszul.

(ii) there is, in general, an isomorphism of $\mathbb{S}$-bimodules,

$$H(D^{\looparrowright}_{\lambda\infty},\, \delta_\lambda) = H(D^{\looparrowright}_{0\infty},\, \delta_0) = \frac{H(P^{\circlearrowright}_\infty) * H(Q^{\circlearrowright}_\infty)^\dagger}{I_0}.$$

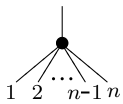
where $H(P^{\circlearrowright}_\infty)$ and $H(Q^{\circlearrowright}_\infty)$ are cohomologies of the wheeled completions of the minimal resolutions of the operads P and Q.

Proof. Use spectral sequence of a filtration, $\{F_p\}$, defined similar to the one introduced in the proof of Theorem 3.10. We omit full details as they are analogous to §3.10. □

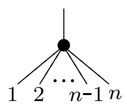
In the next section we apply some of the above results to compute cohomology of several concrete graph complexes with wheels.

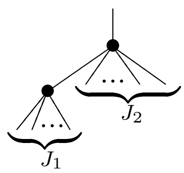
§4. Examples

4.1. Wheeled operad of strongly homotopy Lie algebras. Let $(\mathsf{Lie}_\infty, \delta)$ be the minimal resolution of the operad, Lie, of Lie algebras. It can be identified with the subcomplex of $(\mathsf{Lie}^1\mathsf{B}_\infty, \delta)$ spanned by connected trees built on degree one $(1, n)$-corollas, $n \geq 2$,



with the differential given by

δ 

$= \sum_{[n]=J_1\sqcup J_2,\ |J_1|\geq 2, |J_2|\geq 1}$ 

Let $\mathsf{Lie}^{\circlearrowright}_\infty$ and $\mathsf{Lie}^{\circlearrowright}$ and wheeled extensions of Lie_∞ and, respectively, Lie (see §3.5 for precise definitions).

4.1.1. Theorem. *The operad,* Lie*, of Lie algebras is stably Koszul, i.e.* $\mathsf{H}(\mathsf{Lie}^{\circlearrowright}_\infty) = \mathsf{Lie}^{\circlearrowright}$.

Proof. We shall show that the natural morphism of dg props,

$$(\mathsf{Lie}^{\circlearrowright}_\infty, \delta) \longrightarrow (\mathsf{Lie}^{\circlearrowright}, 0)$$

is a quasi-isomorphism. Consider a surjection of graph complexes (cf. Sect. 3.4),

$$u : (\mathsf{Lie}^{+}_\infty, \delta) \longrightarrow (\mathsf{Lie}^{\circlearrowright}_\infty, \delta)$$

where Lie^{+}_∞ is the marked extension of $\mathsf{Lie}^{\circlearrowright}_\infty$, i.e. the one in which one cyclic edge in every wheel is marked. This surjection respects the filtrations,

$$F_p\mathsf{Lie}^{+}_\infty := \left\{\mathrm{span}\langle f \in \mathsf{Lie}^{+}_\infty\rangle : \text{ total number of cyclic vertices in } f \geq p\right\},$$

$$F_p\mathsf{Lie}^{\circlearrowright}_\infty := \left\{\mathrm{span}\langle f \in \mathsf{Lie}^{\circlearrowright}_\infty\rangle : \text{ total number of cyclic vertices in } f \geq p\right\},$$

and hence induces a morphism of the associated 0-th terms of the spectral sequences,

$$u_0 : (\mathsf{E}^{+}_0, \partial_0) \longrightarrow (\mathsf{E}^{\circlearrowright}_0, \partial_0).$$

The point is that the (pro-)cyclic group acting on $(\mathsf{E}^{+}_0, \partial_0)$ by shifting the marked edge one step further along orientation commutes with the differential ∂_0 so that u_0 is nothing but the projection to the coinvariants with respect to this action. As we work over a field of characteristic 0 coinvariants can be identified with invariants in $(\mathsf{E}_0^+, \partial_0)$. Hence we get, by Maschke theorem,

$$\mathsf{H}(\mathsf{E}_0^{\circlearrowright}, \partial_0) = \text{cyclic invariants in } \mathsf{H}(\mathsf{E}_0^+, \partial_0).$$

The next step is to compute the cohomology of the complex $(\mathsf{E}_0^+, \partial_0)$. Consider its filtration,

$$\mathcal{F}_p := \left\{ \text{span}\langle f \in \mathsf{E}_0^+ \rangle : \begin{array}{l} \text{total number of non-cyclic input} \\ \text{edges at cyclic vertices in } f \end{array} \leq p \right\},$$

and let $\{\mathcal{E}_r, \delta_r\}_{r\geq 0}$ be the associated spectral sequence. We shall show below that the latter degenerates at the second term (so that $\mathcal{E}_2 \simeq \mathsf{H}(\mathsf{E}_0^+, \partial_0)$). The differential δ_0 in $\mathcal{E}_0$ is given by its values on the vertices as follows:

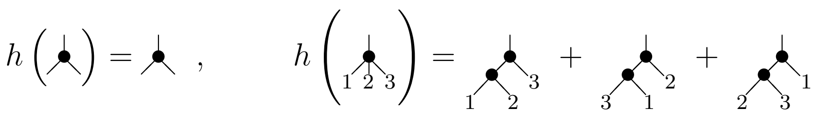
(i) on every non-cyclic vertex one has $\delta_0 = \delta$, the differential in Lie_∞;
(ii) on every cyclic vertex $\delta_0 = 0$.

Hence the complex $(\mathcal{E}_0, \delta_0)$ is isomorphic to the direct sum of tensor products of complexes $(\mathsf{Lie}_\infty, \delta)$. By Künneth theorem, we get,

$$\mathcal{E}_1 = V_1 / h(V_2),$$

where

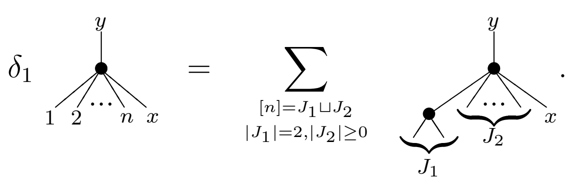
- V_1 is the subspace of Lie_∞^+ consisting of all those graphs whose every non-cyclic vertex is ;
- V_2 is the subspace of Lie_∞^+ whose every non-cyclic vertex is either  or  with the number of vertices of the latter type ≥ 1 ;
- the map $h : V_2 \to V_1$ is given on non-cyclic vertices by



and on all cyclic vertices h is set to be the identity.

The differential δ_1 in $\mathcal{E}_1$ is given by its values on vertices as follows:

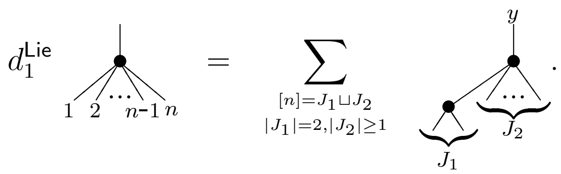
(i) on every non-cyclic vertex one has $\delta_1 = 0$;
(ii) on every cyclic $(1, n+1)$-vertex with cyclic half-edges denoted by x and y, one has



To compute the cohomology of $(\mathcal{E}_1, \delta_1)$ let us step aside and compute the cohomology of the minimal resolution, $(\mathsf{Lie}_\infty, \delta)$ (which we, of course, already know to be equal to Lie), in a slightly unusual way:

$$F_p^{\mathsf{Lie}} := \{\text{span}\langle f \in \mathsf{Lie}_\infty \rangle : \text{number of edges attached to the root vertex of } f \leq p\}$$

is clearly a filtration of the complex $(\mathsf{Lie}_\infty, \delta)$. Let $\{E_r^{\mathsf{Lie}}, d_r^{\mathsf{Lie}}\}_{r\geq 0}$ be the associated spectral sequence. The cohomology classes of $E_1^{\mathsf{Lie}} = \mathsf{H}(E_0^{\mathsf{Lie}}, d_0^{\mathsf{Lie}})$ resemble elements of $\mathcal{E}_1$: they are trees whose root vertex may have any number of edges while all other vertices are binary, $\curlywedge$. The differential d_1^{Lie} is non-trivial only on the root vertex on which it is given by,



The cohomology of $(E_1^{\mathsf{Lie}}, d_1^{\mathsf{Lie}})$ is equal to the operad of Lie algebras. The differential d_1^{Lie} is identical to the differential δ_1 above except for the term corresponding to $|J_2| = 0$. Thus let us define another complex, $(E_1^{\mathsf{Lie}+}, d_1^{\mathsf{Lie}+})$, by adding to E_1^{Lie} trees whose root vertex is a degree -1 corolla ϕ while all other vertices are binary $\curlywedge$. The differential $d_1^{\mathsf{Lie}+}$ is defined on root $(1, n)$-corollas with $n \geq 2$ by formally the same formula as for d_1^{Lie} except that the summation range is extended to include the term with $|I_1| = 0$. We also set $d_1^{\mathsf{Lie}+}\phi = 0$.

Claim. The cohomology of the complex $(E_1^{\mathsf{Lie}+}, d_1^{\mathsf{Lie}+})$ *is a one dimensional vector space spanned by* ϕ .

Proof of the claim. Consider the 2-step filtration, $F_0 \subset F_1$ of the complex $(E_1^{\mathsf{Lie}+}, d_1^{\mathsf{Lie}+})$ by the number of ϕ . The zero-th term of the associated spectral sequence is isomorphic to the direct sum of the complexes,

$$(E_1^{\mathsf{Lie}}, d_1^{\mathsf{Lie}}) \oplus (E_1^{\mathsf{Lie}}[1], d_1^{\mathsf{Lie}}) \oplus (\mathrm{span}\langle\phi\rangle, 0)$$

so that the next term of the spectral sequence is

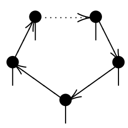
$$\mathsf{Lie} \oplus \mathsf{Lie}[1] \oplus \langle\phi\rangle$$

with the differential being zero on $\mathsf{Lie}[1] \oplus \langle\phi\rangle$ and the the natural isomorphism,

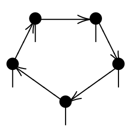
$$\mathsf{Lie} \longrightarrow \mathsf{Lie}[1]$$

on the remaining summand. Hence the claim follows.

The point of the above Claim is that the graph complex $(\mathcal{E}_1, \delta_1)$ is isomorphic to the tensor product of a trivial complex with complexes of the form $(E_1^{\mathsf{Lie}+}, d_1^{\mathsf{Lie}+})$. Which immediately implies that $\mathcal{E}_2 = \mathcal{E}_\infty \simeq H(E_0^+, \partial_0)$ is the direct sum of Lie and the vector space spanned by marked wheels of the type,



whose every vertex is cyclic. Hence the cohomology group $\mathsf{H}(\mathsf{E}_0^{\circlearrowright}, \partial_0) = \mathsf{E}_1^{\circlearrowright}$ we started with is equal to the direct sum of Lie and the space, Z, spanned by unmarked wheels of the type,



whose every vertex is cyclic. As every vertex is binary, the induced differential ∂_1 on $\mathsf{E}_1^{\circlearrowright}$ vanishes, the spectral sequence by the number of cyclic vertices we began with degenerates, and we conclude that this direct sum, $\mathsf{Lie} \oplus Z$, is isomorphic to the required cohomology group $\mathsf{H}(\mathsf{Lie}_\infty^{\circlearrowright}, d)$.

Finally one checks using Jacobi identities that every element of $\mathsf{Lie}^{\circlearrowright}$ containing a wheel can be *uniquely* represented as a linear combination of graphs from Z implying

$$\mathsf{Lie}^{\circlearrowright} \simeq \mathsf{Lie} \oplus Z \simeq \mathsf{H}(\mathsf{Lie}^{\circlearrowright}_{\infty}, d)$$

and completing the proof. □

4.2. Wheeeled prop of polyvector fields. Let $\mathsf{Lie^1B}$ be the prop of Lie 1-bialgebras and $(\mathsf{Lie^1B}_{\infty}, \delta)$ its minimal resolution (see §2.6.3). We denote their wheeled extensions by $\mathsf{Lie^1B}^{\circlearrowright}$ and $(\mathsf{Lie^1B}^{\circlearrowright}_{\infty}, \delta)$ respectively (see §3.5), and their operadic wheeled extensions by $\mathsf{Lie^1B}^{\looparrowright}$ and $(\mathsf{Lie^1B}^{\looparrowright}_{\infty}, \delta)$ (see §3.11). By Theorems 3.11.1 and 4.1.1, we have

4.2.1. Proposition. *The natural epimorphism of dg props,*

$$(\mathsf{Lie^1B}^{\looparrowright}_{\infty}, \delta) \longrightarrow (\mathsf{Lie^1B}^{\looparrowright}, 0)$$

is a quasi-isomorphism.

We shall study next a subcomplex (*not* a subprop!) of the complex $(\mathsf{Lie^1B}^{\circlearrowright}_{\infty}, \delta)$ which is spanned by directed graphs with at most one wheel, i.e. with at most one closed path which begins and ends at the same vertex. We denote this subcomplex by $\mathsf{Lie^1B}^{\circ}_{\infty}$. Similarly we define a subspace $\mathsf{Lie^1B}^{\circ} \subset \mathsf{Lie^1B}^{\circlearrowright}$ spanned by equivalence classes of graphs with at most one wheel.

4.2.2. Theorem. $\mathsf{H}(\mathsf{Lie^1B}^{\circ}_{\infty}, \delta) = \mathsf{Lie^1B}^{\circ}$.

Proof. **(a)** Consider a two step filtration, $F_0 \subset F_1 := \mathsf{Lie^1B}^{\circ}_{\infty}$ of the complex $(\mathsf{Lie^1B}^{\circ}_{\infty}, \delta)$, with $F_0 := \mathsf{Lie^1B}_{\infty}$ being the subspace spanned by graphs with no wheels. We shall show below that the cohomology of the associated direct sum complex,

$$F_0 \bigoplus \frac{F_1}{F_0},$$

is equal to $\mathsf{Lie^1B} \bigoplus \frac{\mathsf{Lie^1B}^{\circ}}{\mathsf{Lie^1B}}$. In fact, the equality $\mathsf{H}(F_0) = \mathsf{Lie^1B}$ is obvious so that it is enough to show below that the cohomology of the complex, $\mathsf{C} := \frac{F_1}{F_0}$, is equal to $\frac{\mathsf{Lie^1B}^{\circ}}{\mathsf{Lie^1B}}$.

(b) Consider a filtration of the complex (C, δ),

$$\mathsf{F}_p\mathsf{C} := \{\mathrm{span}\langle f \in \mathsf{C}\rangle : \text{ number of cyclic vertices in } f \geq p\},$$

and a similar filtration,

$$\mathsf{F}_p\mathsf{C}^+ := \left\{\mathrm{span}\langle f \in \mathsf{C}^+\rangle : \text{ number of cyclic vertices in } f \geq p\right\},$$

of the marked version of C. Let $\{\mathsf{E}_r, \partial_r\}_{r\geq 0}$ and $\{\mathsf{E}^+_r, \partial_r\}_{r\geq 0}$ be the associated spectral sequences. There is a natural surjection of *complexes*,

$$u_0 : (\mathsf{E}^+_0, \partial_0) \longrightarrow (\mathsf{E}^{\circlearrowright}_0, \partial_0).$$

It is easy to see that the differential ∂_0 in E^+_0 commutes with the action of the (pro-)cyclic group on $(\mathsf{E}^+_0, \delta_0)$ by shifting the marked edge one step further along orientation so that u_0 is nothing but the projection to the coinvariants with respect to this action. As we work over a field of characteristic 0, we get by Maschke theorem,

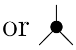
$$\mathsf{H}(\mathsf{E}^{\circlearrowright}_0, \partial_0) = \text{cyclic invariants in } \mathsf{H}(\mathsf{E}^+_0, \partial_0).$$

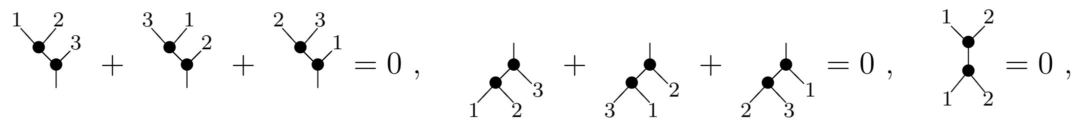
so that we can at this stage work with the complex $(\mathsf{E}^+_0, \partial_0)$. Consider a filtration of the latter,

$$\mathcal{F}_p := \left\{\mathrm{span}\langle f \in \mathsf{E}^+_0\rangle : \begin{array}{l}\text{total number of non-cyclic input}\\ \text{edges at cyclic vertices in } f\end{array} \leq p\right\},$$

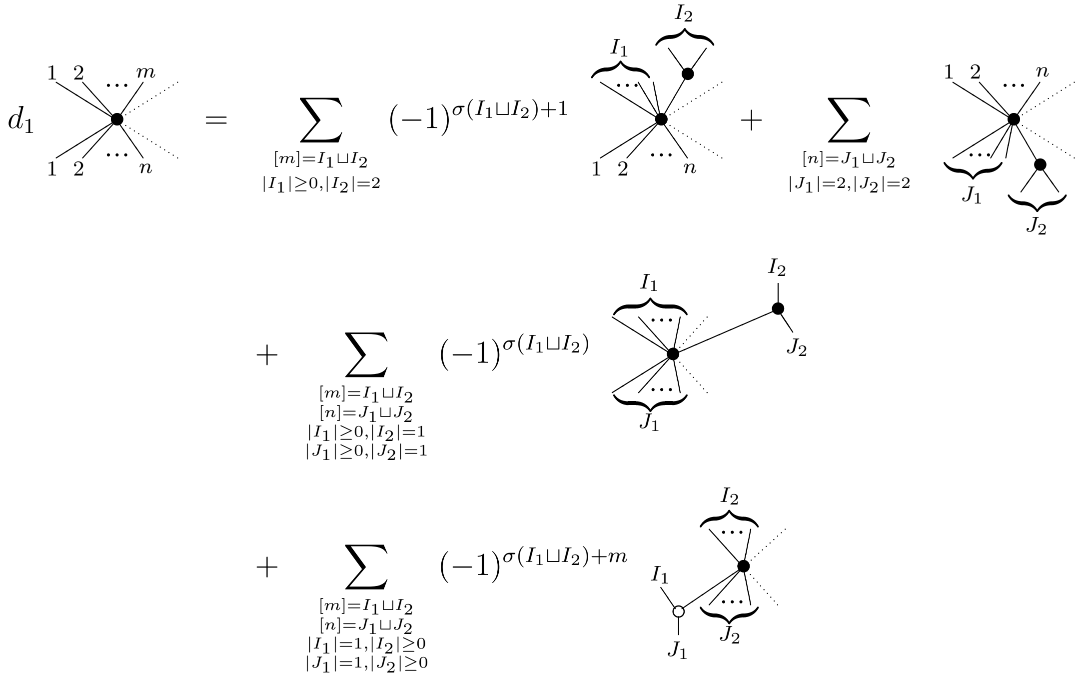
and let $\{\mathcal{E}_r, d_r\}_{r\geq 0}$ be the associated spectral sequence. The differential δ_0 in $\mathcal{E}_0$ is given by its values on the vertices as follows:

(i) on every non-cyclic vertex one has $d_0 = \delta_\lambda$, the differential in $\mathsf{Lie^1B}_\infty$;
(ii) on every cyclic vertex $d_0 = 0$.

Hence the complex $(\mathcal{E}_0, d_0)$ is isomorphic to the direct sum of tensor products of complexes $(\mathsf{Lie}_\infty, \delta)$ with trivial complexes. By Künneth theorem, we conclude that $\mathcal{E}_1 = \mathsf{H}(\mathcal{E}_0, d_0)$ can be identified with the quotient of the subspace in C spanned by graphs whose every non-cyclic vertex is ternary, e.g. either  or  , with respect to the equivalence relation generated by the following equations among non-cyclic vertices,

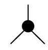
(⋆) 

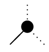
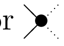
The differential d_1 in $\mathcal{E}_1$ is non-zero only on cyclic vertices,

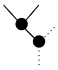
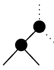
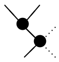
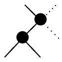


where cyclic half-edges (here and below) are dashed. Then $\mathcal{E}_1$ can be interpreted as a bicomplex, $(\mathcal{E}_1 = \bigoplus_{m,n} \mathcal{E}_1^{m,n}, d_1 = \partial + \bar{\partial})$, with, say, m counting the number of vertices attached to cyclic vertices in "operadic" way (as in the first two summands above), and n counting the number of vertices attached to cyclic vertices in "non-operadic" way (as in the last two summands in the above formula). Note that the assumption that there is only *one* wheel in C is vital for this splitting of the differential d_1 to have sense. The differential ∂ (respectively, $\bar{\partial}$) is equal to the first (respectively, last) two summands in d_1.

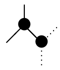
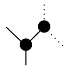
Using Claim in the proof of Theorem 4.1.1 it is not hard check that $\mathsf{H}(\mathcal{E}_1, \partial)$ is isomorphic to the quotient of the subspace of C spanned by graphs whose

— every non-cyclic vertex is ternary, e.g. either  or  ;

— every cyclic vertex is either , or  , or  , with respect to the equivalence relation generated by equations ($\star$) and the following ones,

($\star\star$)  $= 0$,  $= 0$,  $= 0$,  $= 0$.

The differential $\bar{\partial}$ is non-zero only on cyclic vertices of the type  , on which it is given by

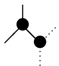
$\bar{\partial}$  $= -$  $-$ 

Hence $\mathsf{A} := \mathsf{H}(\mathsf{H}(\mathcal{E}_1, \partial), \bar{\partial})$ can be identified with the quotient of the subspace of C spanned by graphs whose

— every non-cyclic vertex is ternary, e.g. either  or  ;

— every cyclic vertex is also ternary, e.g. either  or 

with respect to the equivalence relation generated by equations ($\star$), ($\star\star$) and, say, the following one,

 $= 0$.

As all vertices are ternary, all higher differentials in our spectral sequences vanish, and we conclude that

$$\mathsf{H}(\mathsf{C}, \delta) \simeq \mathsf{A},$$

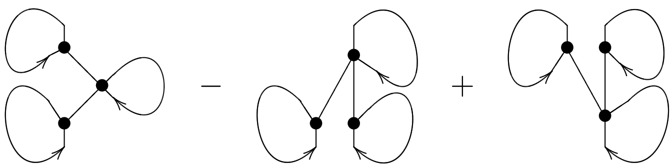
which proves the Theorem. □

4.2.3. Remark. As an independent check of the above arguments one can show using relations $R_1 - R_3$ in §2.6.2 that every element of $\frac{\mathsf{Lie^1B}^{\circ}}{\mathsf{Lie^1B}}$ can indeed be uniquely represented as a linear combinations of graphs from the space A.

4.2.4. Remark. Proposition 4.2.1 and Theorem 4.2.2 can not be extended to the full wheeled prop $\mathsf{Lie^1B}_{\infty}^{\circlearrowright}$, i.e. the natural surjection,

$$\pi : (\mathsf{Lie^1B}_{\infty}^{\circlearrowright}, \delta) \longrightarrow (\mathsf{Lie^1B}, 0),$$

is *not* a quasi-isomorphism. For example, the graph



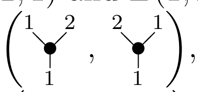
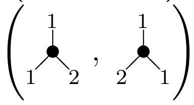
represents a non-trivial cohomology class in $\mathsf{H}^1(\mathsf{Lie^1B}_{\infty}^{\circlearrowright}, \delta)$.

4.3. Wheeled prop of Lie bialgebras. Let LieB be the prop of Lie bialgebras which is generated by the dioperad very similar to $\mathsf{Lie^1B}$ except that both generating Lie and coLie operations are in degree zero. This dioperad is again Koszul with Koszul substitution law so that the analogue of Proposition 4.2.1 holds true for the operadic wheelification, $\mathsf{LieB}_{\infty}^{\circlearrowright}$. In fact, the analogue of Theorem 4.2.2 holds true for $\mathsf{LieB}_{\infty}^{\circ}$.

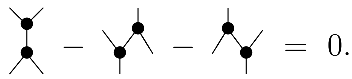
4.5. Prop of infinitesimal bialgebras. Let IB be the dioperad of infinitesimal bialgebras [MaVo] which can be represented as a quotient,

$$\mathsf{IB} = \frac{\mathsf{D}\langle E \rangle}{\mathsf{Ideal} < R >} ,$$

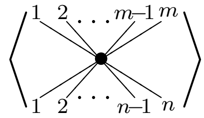
of the free prop generated by the following $\mathbb{S}$-bimodule E,

- all $E(m,n)$ vanish except $E(2,1)$ and $E(1,2)$,
- $E(2,1) := k[\mathbb{S}_2] \otimes \mathbf{1_1} = span$ ,
- $E(1,2) := \mathbf{1_1} \otimes k[\mathbb{S}_2] = span$ ,

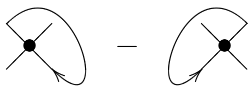
modulo the ideal generated by the associativity conditions for , co-associativity conditions for ,



This is a Koszul dioperad with a Koszul substitution law. Its minimal prop resolution, $(\mathsf{IB}_\infty, \delta)$ is a dg prop freely generated by the $\mathbb{S}$-bimodule $\mathsf{E} = \{\mathsf{E}(m,n)\}_{m,n\geq 1, m+n\geq 3}$, with

$$\mathsf{E}(m,n) := k[\mathbb{S}_m] \otimes k[\mathbb{S}_n][3-m-n] = \text{span}$$  .

By Claim 3.5.4, the analogue of Proposition 4.2.1 does *not* hold true for $\mathsf{IB}_\infty^{\circlearrowright}$. Moreover, it is not hard to check that the graph



represents a non-trivial cohomology class in $\mathsf{H}^{-1}(\mathsf{IB}_\infty^{\circlearrowright})$. Thus neither the analogue of Theorem 4.2.2 holds true for $\mathsf{IB}_\infty^{\circ}$ nor the natural surjection, $\mathsf{IB}_\infty^{\circlearrowright} \to \mathsf{IB}^{\circlearrowright}$, is a quasi-isomorphism. This example is of interest because the wheeled dg prop $\mathsf{IB}_\infty^{\circlearrowright}$ controls the cohomology of a directed version of Kontsevich's ribbon graph complex.

4.5. Wheeled quasi-minimal resolutions. Let P be a graded prop with zero differential admitting a minimal resolution,

$$\pi : (\mathsf{P}_\infty = \mathsf{P}\langle\mathsf{E}\rangle, \delta) \to (\mathsf{P}, 0).$$

We shall use in the following discussion of this pair of props, P_∞ and P, a so called *Tate-Jozefak* grading[5] which, by definition, assignes degree zero to *all* generators of P and hence make P_∞ into a non-positively graded differential prop, $\mathsf{P}_\infty = \bigoplus_{i\leq 0} \mathsf{P}^i_\infty$, with cohomology concentrated in degree zero, $\mathsf{H}^0(\mathsf{P}_\infty, \delta) = \mathsf{P}$. Both props P_∞ and P admit canonically wheeled extensions,

$$\mathsf{P}^{\circlearrowright}_\infty := \bigoplus_{G\in\mathfrak{G}^{\circlearrowright}} G\langle\mathsf{E}\rangle$$
$$\mathsf{P}^{\circlearrowright} := \mathsf{H}^0(\mathsf{P}^{\circlearrowright}_\infty, \delta).$$

However, the natural extension of the epimorphism π,

$$\pi^{\circlearrowright} : (\mathsf{P}^{\circlearrowright}_\infty, \delta) \to (\mathsf{P}, 0).$$

fails in general to stay a quasi-isomorphism.

[5]The Tate-Josefak grading of props $\mathsf{Lie^1B}_\infty$ and $\mathsf{Lie^1B}$, for example, assigns to generating (m,n) corollas degree $3-m-n$.

Note that the dg prop, $(\mathsf{P}_\infty^\circlearrowright, \delta)$, defined above is a *free* prop

$$\mathsf{P}_\infty^\circlearrowright := \bigoplus_{G\in\mathfrak{G}^\uparrow} G\langle \mathsf{E}^\circlearrowright\rangle$$

on the $\mathbb{S}$-bimodule, $\mathsf{E}^\circlearrowright = \{\mathsf{E}(m,n)\}_{m,n\geq 0}$,

$$\mathsf{E}^\circlearrowright(m,n) := \bigoplus_{G\in\mathfrak{G}^\circlearrowright_{indec}(m,n)} G\langle \mathsf{E}\rangle$$

generated by indecomposable (in the propic sense) decorated wheeled graphs. Note that the induced differential is *not* quadratic with respect to the generating set $\mathsf{E}^\circlearrowright$.

4.5.1. Theorem-definition. *There exists a dg free prop, $([\mathsf{P}^\circlearrowright]_\infty, \delta)$, which fits into a commutative diagram of morphisms of props,*

$$\begin{array}{ccc} [\mathsf{P}^\circlearrowright]_\infty & \xrightarrow{\alpha} & \mathsf{P}_\infty^\circlearrowright \\ & \searrow^{qis} & \downarrow{\scriptstyle \pi^\circlearrowright} \\ & & \mathsf{P}^\circlearrowright \end{array}$$

where α is an epimorphism of (nondifferential) props qis a quasi-isomorphism of dg props. The prop $[\mathsf{P}^\circlearrowright]_\infty$ is called a quasi-minimal resolution *of* $\mathsf{P}^\circlearrowright$.

Proof. Let $s_1 : \mathsf{H}^{-1}(\mathsf{P}_\infty^\circlearrowright) \to \mathsf{P}_\infty^\circlearrowright$ be any representation of degree -1 cohomology classes (if there are any) as cycles. Set $\mathsf{E}_1 := \mathsf{H}^{-1}(\mathsf{P}_\infty^\circlearrowright)[1]$ and define a differential graded prop,

$$\mathsf{P}_1 := \mathsf{P}\langle \mathsf{E}^\circlearrowright \oplus \mathsf{E}_1\rangle$$

with the differential δ extended to new generators as $s_1[1]$. By construction, $\mathsf{H}^0(\mathsf{P}_1) = \mathsf{P}^\circlearrowright$, and $\mathsf{H}^{-1}(\mathsf{P}_1) = 0$.

Let $s_2 : \mathsf{H}^{-2}(\mathsf{P}_1) \to \mathsf{P}_1$ be any representation of degree -2 cohomology classes (if there are any) as cycles. Set $\mathsf{E}_2 := \mathsf{H}^{-2}(\mathsf{P}_1)[1]$ and define a differential graded prop,

$$\mathsf{P}_2 := \mathsf{P}\langle \mathsf{E}^\circlearrowright \oplus \mathsf{E}_1 \oplus \mathsf{E}_2\rangle$$

with the differential δ extended to new generators as $s_2[1]$. By construction, $\mathsf{H}^0(\mathsf{P}_2) = \mathsf{P}^\circlearrowright$, and $\mathsf{H}^{-1}(\mathsf{P}_2) = \mathsf{H}^{-2}(\mathsf{P}_2) = 0$.

Continuing by induction we construct a dg free prop, $[\mathsf{P}^\circlearrowright]_\infty := \mathsf{P}\langle \mathsf{E}^\circlearrowright \oplus \mathsf{E}_1 \oplus \mathsf{E}_2 \oplus \mathsf{E}_3 \oplus \ldots\rangle$ with all the cohomology concentrated in Tate-Jozefak degree 0 and equal to $\mathsf{P}^\circlearrowright$. □

4.5.2. Example. The prop $[\mathsf{Ass}^\circlearrowright]_\infty$ has been explicitly described in [MMS] (it is denoted there by $\mathsf{Ass}_\infty^\circlearrowright$).

§5. Wheeled cyclic complex

5.1. Genus 1 wheels. Let $(\mathsf{P}\langle E\rangle, \delta)$ be a dg free prop, and let $(\mathsf{P}^{\circlearrowright}\langle E\rangle, \delta)$ be its wheeled extension. We assume in this section that the differential δ preserves the number of wheels[6]. Then it makes sense to define a subcomplex, $(\mathsf{T}^{\circlearrowright}\langle E\rangle \subset \mathsf{P}^{\circlearrowright}\langle E\rangle$, spanned by graphs with precisely one wheel. In this section we use the ideas of cyclic homology to define a new cyclic bicomplex which computes cohomology of $(\mathsf{T}^{\circlearrowright}\langle E\rangle, \delta)$.

All the above assumptions are satisfied automatically if $(\mathsf{P}\langle E\rangle, \delta)$ is the free dg prop associated with a free dg operad.

We denote by $\mathsf{T}^{+}\langle E\rangle$ the obvious "marked wheel" extension of $\mathsf{T}^{\circlearrowright}\langle E\rangle$ (see §3.2).

5.2. Abbreviated notations for graphs in $\mathsf{T}^{+}\langle E\rangle$. The half-edges attached to any internal vertex of split into, say m, ingoing and, say n, outgoing ones. The differential δ is uniquely determined by its values on such (m,n)-vertices for all possible $m, n \geq 1$. If the vertex is cyclic, then one of its input half-edges is cyclic and one of its output half-edges is also cyclic. In this section we show in pictures only those (half-)edges attached to vertices which are cyclic (unless otherwise is explicitly stated), so that

- $\boxed{e}$ stands for a non-cyclic (m,n)-vertex decorated by an element $e \in E(m,n)$,
- $\overset{|}{\underset{|}{\boxed{e}}}$ is a decorated cyclic (m,n)-vertex with no input or output cyclic half-edges marked,
- $\overset{\vdots}{\underset{|}{\boxed{e}}}$ is a decorated cyclic (m,n)-vertex with the output cyclic half-edge marked,
- $\overset{|}{\underset{\vdots}{\boxed{e}}}$ is a decorated cyclic (m,n)-vertex with the input cyclic half-edge marked.

The differential δ applied to any vertex of the last three types can be uniquely decomposed into the sum of the following three groups of terms,

$$\delta\, \overset{|}{\underset{\vdots}{\boxed{e}}} = \sum_{\alpha\in I_1} \begin{array}{c} | \\ \boxed{e_{a''}} \\ | \\ \boxed{e_{a'}} \\ \vdots \end{array} + \sum_{a\in I_2} \overset{|}{\underset{\vdots}{\boxed{e_{a'}}}} \diagup \boxed{e_{a''}} + \sum_{b\in I_3} \overset{|}{\underset{\vdots}{\boxed{e_{b'}}}} \diagdown \boxed{e_{b''}}$$

where we have shown also non-cyclic *internal* edges in the last two groups of terms. The differential δ applied to $\overset{|}{\underset{|}{\boxed{e}}}$ and $\overset{\vdots}{\underset{|}{\boxed{e}}}$ is given by exactly the same formula except for the presence/position of dashed markings.

5.3. New differential in $\mathsf{T}^{+}\langle E\rangle$. Let us define a new derivation, b, in $\mathsf{T}^{+}\langle E\rangle$, as follows:

- $b\, \boxed{e} := \delta\, \boxed{e}$,
- $b\, \overset{|}{\underset{|}{\boxed{e}}} := \delta\, \overset{|}{\underset{|}{\boxed{e}}}$,
- $b\, \overset{\vdots}{\underset{|}{\boxed{e}}} = \delta\, \overset{\vdots}{\underset{|}{\boxed{e}}}$,
- $b\, \overset{|}{\underset{\vdots}{\boxed{e}}} := \delta\, \overset{|}{\underset{\vdots}{\boxed{e}}} + \sum_{\alpha\in I_1} \begin{array}{c} | \\ \boxed{e_{a''}} \\ \vdots \\ \boxed{e_{a'}} \\ | \end{array}$.

[6]This is not that dramatic loss of generality in the sense there always exits a filtration of $(\mathsf{P}^{\circlearrowright}\langle E\rangle, \delta)$ by the number of wheels whose spectral sequence has zero-th term satisfying our condition on the differential.

5.3.1. Lemma. *The derivation b satisfies $b^2 = 0$, i.e. $(\mathsf{T}^+\langle E\rangle, b)$ is a complex.*

Proof is a straightforward but tedious calculation based solely on the relation $\delta^2 = 0$.

5.4. Action of cyclic groups. The vector space $\mathsf{T}^+\langle E\rangle$ is naturally bigraded,

$$\mathsf{T}^+\langle E\rangle = \sum_{m\geq 0, n\geq 1} \mathsf{T}^+\langle E\rangle_{m,n},$$

where the summand $\mathsf{T}^+\langle E\rangle_{m,n}$ consists of all graphs with m non-cyclic and n cyclic vertices. Note that $\mathsf{T}^+\langle E\rangle_{m,n}$ is naturally a representation space of the cyclic group $\mathbb{Z}_n$ whose generator, t, moves the mark to the next cyclic edge along the orientation. Define also the operator, $N := 1 + t + \ldots + t^n : \mathsf{T}^+\langle E\rangle_{m,n} \to \mathsf{T}^+\langle E\rangle_{m,n}$, which symmetrizes the marked graphs.

5.4.1. Lemma. *$\delta(1-t) = (1-t)b$ and $N\delta = bN$.*

Proof is a straightforward calculation based on the definition of b.

Following the ideas of the theory of cyclic homology (see, e.g., [Lo]) we introduce a 4th quadrant bicomplex,

$$C_{p,q} := C_q, \qquad C_q := \sum_{m+n=q} \mathsf{T}^+\langle E\rangle_{m,n}, \qquad p \leq 0, q \geq 1,$$

with the differentials given by the following diagram,

$$\begin{array}{ccccccccc}
 & & \ldots & & \ldots & & \ldots & & \ldots \\
 & & \uparrow b & & \uparrow \delta & & \uparrow b & & \uparrow \delta \\
\ldots & \xrightarrow{N} & C_4 & \xrightarrow{1-t} & C_4 & \xrightarrow{N} & C_4 & \xrightarrow{1-t} & C_4 \\
 & & \uparrow b & & \uparrow \delta & & \uparrow b & & \uparrow \delta \\
\ldots & \xrightarrow{N} & C_3 & \xrightarrow{1-t} & C_3 & \xrightarrow{N} & C_3 & \xrightarrow{1-t} & C_3 \\
 & & \uparrow b & & \uparrow \delta & & \uparrow b & & \uparrow \delta \\
\ldots & \xrightarrow{N} & C_2 & \xrightarrow{1-t} & C_2 & \xrightarrow{N} & C_2 & \xrightarrow{1-t} & C_2 \\
 & & \uparrow b & & \uparrow \delta & & \uparrow b & & \uparrow \delta \\
\ldots & \xrightarrow{N} & C_1 & \xrightarrow{1-t} & C_1 & \xrightarrow{N} & C_1 & \xrightarrow{1-t} & C_1
\end{array}$$

5.4.2. Theorem. *The cohomology group of the unmarked graph complex, $H(\mathsf{T}^{\circlearrowleft}\langle E\rangle, \delta)$, is equal to the cohomology of the total complex associated with the cyclic bicomplex $C_{\bullet,\bullet}$.*

Proof. The complex $(\mathsf{T}^{\circlearrowleft}\langle E\rangle, \delta)$ can be identified with the cokernel, $C_\bullet/(1-t)$, of the endomorphism $(1-t)$ of the total complex, $C_\bullet$, associated with the bicomplex $C_{\bullet,\bullet}$. As the rows of $C_{\bullet,\bullet}$ are exact [Lo], the claim follows. □

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