Experiments with general cubic surfaces

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Dedicated to Yuri Ivanovich Manin on the occasion of his 70th birthday

Abstract

For general cubic surfaces, we test numerically the conjecture of Manin (in the refined form due to E. Peyre) about the asymptotics of points of bounded height on Fano varieties. We also study the behaviour of the height of the smallest rational point versus the Tamagawa type number introduced by Peyre.

1 Introduction

The arithmetic of cubic surfaces is a fascinating subject. To a large extent, it was initiated by the work of Yu. I. Manin, particularly by his fundamental and influential book on *Cubic Forms* [Ma].

In this article, we study the distribution of rational points on general cubic surfaces over $\mathbb{Q}$. The main problems are

- Existence of $\mathbb{Q}$-rational points,
- Asymptotics of $\mathbb{Q}$-rational points,
- The height of the smallest point.

1.1. ***Existence of rational points.*** —— Let V be an algebraic variety defined over $\mathbb{Q}$. Recall that the *Hasse principle* is said to hold for V if

$$V(\mathbb{Q}) = \emptyset \iff \exists\, \nu \in \mathrm{Val}(\mathbb{Q})\colon V(\mathbb{Q}_\nu) = \emptyset\,.$$

For quadrics in $\mathbf{P}^n_{\mathbb{Q}}$, the Hasse principle holds by the famous Theorem of Hasse-Minkowski. It is, however, well-known that for smooth cubic surfaces over $\mathbb{Q}$ the Hasse principle does not hold, in general. This is explained by the Brauer-Manin obstruction (See section 2 for details).

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1.2. ***Asymptotics of rational points.*** —— On the asymptotics of rational points of bounded height, there is the following famous conjecture due to Yu. I. Manin [FMT].

1.3. Conjecture (Manin)**.** —— *Let V be an arbitrary Fano variety over $\mathbb{Q}$ and* H *be an anticanonical height on V. Then, there exist a dense, Zariski open subset $V^\circ \subseteq V$ and a constant C such that*

$$(*) \qquad \#\{x \in V^\circ(\mathbb{Q}) \mid \mathrm{H}(x) < B\} \sim C B \log^{\operatorname{rk}\operatorname{Pic}(V)-1} B$$

for $B \to \infty$.

Motivated by results obtained by the classical circle method, E. Peyre refined Manin's conjecture by a conjectural value for the leading coefficient C.

More precisely, let V be a smooth hypersurface in $\mathbf{P}^{d+1}_{\mathbb{Q}}$ given by an equation $f = 0$. Assume that $\operatorname{rk}\operatorname{Pic}(V) = 1$ and suppose there is no Brauer-Manin obstruction on V. Then, Peyre's constant is equal to the Tamagawa type number τ given by $\tau := \prod_{p \in \mathbb{P} \cup \{\infty\}} \tau_p$ where

$$\tau_p = \Big(1 - \frac{1}{p}\Big) \cdot \lim_{n \to \infty} \frac{\#V(\mathbb{Z}/p^n\mathbb{Z})}{p^{dn}}$$

for p finite and

$$\tau_\infty = \frac{1}{2} \int\limits_{\substack{x \in [-1,1]^{d+2} \\ f(x)=0}} \frac{1}{\|(\operatorname{grad} f)(x))\|_2} \, dS \, .$$

Here, dS denotes the usual hypersurface measure on the cone $C_V(\mathbb{R})$, considered as a hypersurface in $\mathbb{R}^{d+2}$.

Conjecture 1.3 is established [Bi] for smooth complete intersections of multidegree $d_1, \ldots, d_n$ in the case that the dimension of V is very large compared to $d_1, \ldots, d_n$. Further, it is proven for projective spaces and quadrics. Finally, there are a number of further special cases in which Manin's conjecture is known to be true. (See, e.g., [Pe, sec. 4].)

Recently, numerical evidence for Conjecture 1.3 has been presented in the case of the threefolds $V^e_{a,b}$ given by $ax^e = by^e + z^e + v^e + w^e$ in $\mathbf{P}^4_{\mathbb{Q}}$ for $e = 3$ and 4 [EJ1].

1.4. ***The smallest point.*** —— It would be desirable to have an a-priori upper bound for the height of the smallest $\mathbb{Q}$-rational point on V as this would allow to effectively decide whether $V(\mathbb{Q}) \neq \emptyset$ or not.

When V is a conic, Legendre's theorem on zeroes of ternary quadratic forms yields an effective bound for the smallest point. For quadrics of arbitrary dimension, the same is true by an observation due to J. W. S. Cassels [Ca]. Further, there is a theorem of C. L. Siegel [Si, Satz 1] which provides a generalization to hypersurfaces defined by norm equations. This certainly includes some special cubic surfaces but, in general, no theoretical upper bound is known for the height of the smallest $\mathbb{Q}$-rational point on a cubic surface.

If one had an error term [S-D] for $(*)$ uniform over all cubic surfaces V of Picard rank 1 then this would imply that the height $\mathrm{m}(V)$ of the smallest $\mathbb{Q}$-rational point is always less than $\frac{C}{\tau(V)^\alpha}$ for certain constants $\alpha > 1$ and $C > 0$.

1.5. Remarks. —— i) The investigations on quartic threefolds made in [EJ2] indicate that one might have even $\mathrm{m}(V) < \frac{C(\varepsilon)}{\tau(V)^{1+\varepsilon}}$ for any $\varepsilon > 0$.

ii) Assuming equidistribution, one would expect that the height of the smallest $\mathbb{Q}$-rational point on V should be even $\sim\frac{1}{\tau(V)}$. An inequality of the form $\mathrm{m}(V) < \frac{C}{\tau(V)}$ is, however, known to be wrong, in general [EJ2].

1.6. *The results.* —— We consider two families of cubic surfaces which are produced by a random number generator. For each of these surfaces, we do the following.

i) We verify that the Galois group acting on the 27 lines is equal to $W(E_6)$.

ii) We compute E. Peyre's constant $\tau(V)$.

iii) Up to a certain bound for the anticanonical height, we count all $\mathbb{Q}$-rational points on the surface V.

Thereby, we establish the Hasse principle for each of the surfaces considered. Further, we test numerically the conjecture of Manin, in the refined form due to E. Peyre, on the asymptotics of points of bounded height. Finally, we study the behaviour of the height of the smallest $\mathbb{Q}$-rational point versus E. Peyre's constant. This means, we test the estimates formulated in 1.4 and Remark 1.5.i).

2 Background

2.1. *27 lines.* —— Recall that a non-singular cubic surface defined over $\overline{\mathbb{Q}}$ contains exactly 27 lines. The symmetries of the configuration of the 27 lines respecting the intersection pairing are given by the Weyl group $W(E_6)$ [Ma, Theorem 23.9.ii].

2.2. Fact. —— *Let V be a smooth cubic surface defined over $\mathbb{Q}$ and let K be the field of definition of the 27 lines on V. Then K is a Galois extension of $\mathbb{Q}$. The Galois group $\mathrm{Gal}(K/\mathbb{Q})$ is a subgroup of $W(E_6)$.*

2.3. Remarks. —— i) $W(E_6)$ contains a subgroup U of index two which is isomorphic to the simple group of order 25 920. It is of Lie type $B_2(\mathbb{F}_3)$, i.e. $U \cong \Omega_5(\mathbb{F}_3) \subset \mathrm{SO}_5(\mathbb{F}_3)$.

ii) The operation of $W(E_6)$ on the 27 lines gives rise to a transitive permutation representation $\iota\colon W(E_6) \to S_{27}$. It turns out that the image of ι is contained in the alternating group A_{27}. We will call an element $\sigma \in W(E_6)$ *even* if $\sigma \in U$ and *odd,* otherwise. This should not be confused with the sign of $\iota(\sigma) \in S_{27}$ which is always even.

2.4. *The Brauer-Manin obstruction.* — For Fano varieties, all known obstructions against the Hasse principle are explained by the following observation.

2.5. Observation (Manin)**.** — *Let V be a non-singular variety over $\mathbb{Q}$. Choose an element $\alpha \in \mathrm{Br}(V)$ [Ma, Definition 41.3]. Then, any $\mathbb{Q}$-rational point $x \in V(\mathbb{Q})$ gives rise to an adelic point $(x_\nu)_\nu \in V(\mathbf{A}_\mathbb{Q})$ satisfying the non-trivial condition*

$$\sum_{\nu \in \mathrm{Val}(\mathbb{Q})} \mathrm{inv}(\alpha|_{x_\nu}) = 0\,.$$

Here, $\mathrm{inv}\colon \mathrm{Br}(\mathbb{Q}_\nu) \to \mathbb{Q}/\mathbb{Z}$ *(respectively* $\mathrm{inv}\colon \mathrm{Br}(\mathbb{R}) \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}$*) denotes the canonical isomorphism.*

$\mathrm{inv}(\alpha|_{x_\nu})$ depends continuously on $x_\nu \in V(\mathbb{Q}_\nu)$. Further, Yu. I. Manin proved [Ma, Corollary 44.2.5] that, for each non-singular variety V over $\mathbb{Q}$, there exists a finite set $S \subset \mathrm{Val}(\mathbb{Q})$ such that $\mathrm{inv}(\alpha|_{x_\nu}) = 0$ for every $\alpha \in \mathrm{Br}(V)$, $\nu \notin S$, and $x_\nu \in V(\mathbb{Q}_\nu)$. This implies that the Brauer-Manin obstruction, if present, is an obstruction against the principle of weak approximation.

Denote by $\pi\colon V \to \mathrm{Spec}(\mathbb{Q})$ the structural map. It is obvious that altering $\alpha \in \mathrm{Br}(V)$ by some Brauer class $\pi^*\rho$ for $\rho \in \mathrm{Br}(\mathbb{Q})$ does not change the obstruction defined by α. By consequence, it is only the factor group $\mathrm{Br}(V)/\pi^*\mathrm{Br}(\mathbb{Q})$ which is relevant for the Brauer-Manin obstruction. The latter is canonically isomorphic to $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(V_{\overline{\mathbb{Q}}}))$ [Ma, Lemma 43.1.1]. In particular, if $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(V_{\overline{\mathbb{Q}}})) = 0$ then there is no Brauer-Manin obstruction on V.

For a smooth cubic surface V, the geometric Picard group $\mathrm{Pic}(V_{\overline{\mathbb{Q}}})$ is generated by the classes of the 27 lines on $V_{\overline{\mathbb{Q}}}$. Its first cohomology group can be described in terms of the Galois action on these lines. Indeed, there is a canonical isomorphism [Ma, Proposition 31.3]

$$(+) \qquad H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(V_{\overline{\mathbb{Q}}})) \cong \mathrm{Hom}((NF \cap F_0)/NF_0, \mathbb{Q}/\mathbb{Z}).$$

Here, $F \subset \mathrm{Div}(V_{\overline{\mathbb{Q}}})$ is the group generated by the 27 lines, $F_0 \subset F$ denotes the subgroup of principal divisors, and N is the norm map under the operation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})/H$, H being the stabilizer of F.

2.6. Remark. — Consider the particular case when the Galois group acts transitively on the 27 lines. Then, (+) shows that $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(V_{\overline{\mathbb{Q}}})) = 0$. In particular, there is no Brauer-Manin obstruction in this case.

It is expected that the Hasse principle holds for all cubic surfaces such that $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(V_{\overline{\mathbb{Q}}})) = 0$. (See [CS, Conjecture C].)

3 Computation of the Galois group

Let V be a smooth cubic surface defined over $\mathbb{Q}$ and let K be the field of definition of the 27 lines on V. By Fact 2.2, $K/\mathbb{Q}$ is a Galois extension and the Galois group $G := \mathrm{Gal}(K/\mathbb{Q})$ is a subgroup of $W(E_6)$. For general cubic surfaces, G is actually equal to $W(E_6)$. To verify this for particular examples, the following lemma is useful.

3.1. Lemma. — *Let $H \subseteq W(E_6)$ be a subgroup which acts transitively on the 27 lines and contains an element of order five. Then, either H is the subgroup $U \subset W(E_6)$ of index two or $H = W(E_6)$.*

Proof. $H \cap U$ still acts transitively on the 27 lines and still contains an element of order five. Thus, we may suppose $H \subseteq U$.

Assume that $H \subsetneq U$. Denote by k the index of H in U. The natural action of U on the set of cosets U/H yields a permutation representation $i\colon U \to S_k$. As U is simple, i is necessarily injective. In particular, since $\#U \nmid 8!$, we see that $k > 8$. Let us consider the stabilizer $H' \subset H$ of one of the lines. As H acts transitively, it follows that $\#H' = \frac{\#H}{27} = \frac{\#U}{27 \cdot k} = \frac{960}{k}$. We distinguish two cases.

First case: $k > 16$. Then, $k \geq 20$ and $\#H' \leq 48$. This implies that the 5-Sylow subgroup is normal in H'. Its conjugate by some $\sigma \in H$ therefore depends only on $\overline{\sigma} \in H/H'$. By consequence, the number n of 5-Sylow subgroups in H is a divisor of $\#H/\#H' = 27$. Sylow's congruence $n \equiv 1 \pmod 5$ yields that $n = 1$.

Let $H_5 \subset H$ be the 5-Sylow subgroup. Then, $\iota(H_5) \subset S_{27}$ is generated by a product of disjoint 5-cycles leaving at least two lines fixed. It is, therefore, not normal in the transitive group $\iota(H)$. This is a contradiction.

Second case: $9 \leq k \leq 16$. We have $k \,|\, 960$. On the other hand, the assumption $5 \,|\, \#H$ implies $5 \nmid k$. This shows, there are only two possibilities, $k = 12$ and $k = 16$. As, in U, there is no subgroup of index eight or less, $H \subset U$ must be a maximal subgroup. In particular, the permutation representation $i\colon U \to S_k$ is primitive.

Primitive permutation representations of degree up to 20 have been classified already in the late 19th century. It is well known that no group of order 25 920 allows a primitive permutation representation of degree 12 or 16 [Si, Table 1]. □

3.2. Remark. — The subgroups of the simple group U have been completely classified by L. E. Dickson [Di] in 1904. It would not be complicated to deduce the lemma from Dickson's list.

Let the smooth cubic surface V be given by a homogeneous equation $f = 0$ with integral coefficients. We want to compute the Galois group G.

An affine part of a general line ℓ can be described by four coefficients a, b, c, d via the parametrization $\ell\colon t \mapsto (1 : t : a + bt : c + dt)$. ℓ is contained in S if and only if it intersects S in at least four points. This implies that

$$f(\ell(0)) = f(\ell(1)) = f(\ell(2)) = f(\ell(3)) = 0$$

is a system of equations for a, b, c, d which encodes that ℓ is contained in S.

By a Gröbner base calculation in `SINGULAR`, we compute a univariate polynomial g of minimal degree belonging to the ideal generated by the equations. If g is of degree 27 then the splitting field of g is equal to the field K of definition of the 27 lines on V. We then use van der Waerden's criterion [PZ, Proposition 2.9.35]. More precisely, our algorithm works as follows.

3.3. Algorithm. —— Given the equation $f = 0$ of a smooth cubic surface, this algorithm verifies $G = W(E_6)$.

i) Compute a univariate polynomial $0 \neq g \in \mathbb{Z}[d]$ of minimal degree such that

$$g \in (f(\ell(0)), f(\ell(1)), f(\ell(2)), f(\ell(3))) \subset \mathbb{Q}[a, b, c, d].$$

If g is not of degree 27 then terminate with an error message. In this case, the coordinate system for the lines is not sufficiently general. If we are erroneously given a singular cubic surface then the algorithm will fail at this point.

ii) Factor g modulo all primes below a given limit. Ignore the primes dividing the leading coefficient of g.

iii) If one of the factors is multiple then go to the next prime immediately. Otherwise, check whether the decomposition type corresponds to one of the cases listed below,

$$A := \{(9,9,9)\}, \qquad B := \{(1,1,5,5,5,5,5),(2,5,5,5,10)\},$$
$$C := \{(1,4,4,6,12),(2,5,5,5,10),(1,2,8,8,8)\}.$$

iv) If each of the cases occurred for at least one of the primes then output the message "The Galois group is equal to $W(E_6)$." and terminate.

Otherwise, output "Can not prove that the Galois group is equal to $W(E_6)$."

3.4. Remarks. —— i) The cases above are functioning as follows.

a) Case B shows that the order of the Galois group is divisible by five.

b) Cases A and B together guarantee that g is irreducible. Therefore, by Lemma 3.1, A and B prove that G contains the index two subgroup $U \subset W(E_6)$.

c) Case C is a selection of the most frequent odd conjugacy classes in $W(E_6)$.

ii) One could replace cases B and C by their common element $(2, 5, 5, 5, 10)$. This would lead to a simpler but less efficient algorithm.

iii) Actually, a decomposition type as considered in step iii) does not always represent a single conjugacy class in $W(E_6)$. Two elements $\iota(\sigma)$, $\iota(\sigma') \in S_{27}$ might be conjugate in S_{27} via a permutation $\tau \notin \iota(W(E_6))$.

For example, as is easily seen using `GAP`, the decomposition type $(3, 6, 6, 6, 6)$ falls into three conjugacy classes two of which are even and one is odd (cf. Remark 2.3.ii)). However, all the decomposition types searched for in Algorithm 3.3 do represent single conjugacy classes.

iv) Since we expect $G = W(E_6)$, we can estimate the probability of each case by the Čebotarev density theorem. Case A has a probability of $\frac{1}{9}$. This is the lowest value among the three cases.

v) As we do not use the factors explicitly, it is enough to compute their degrees and to check that each of them occurs with multiplicity one. This means, we only have to compute $\gcd(\overline{g}(X), \overline{g}'(X))$ and $\gcd(\overline{g}(X), X^{p^d} - X)$ in $\mathbb{F}_p[X]$ for $d = 1, \ldots, 13$ [Co, Algorithm 3.4.2 and 3.4.3].

4 Computation of Peyre's constant

4.1. *The Euler product.* —— We want to compute the product over all τ_p. For a finite place p, we have

$$\tau_p = \left(1 - \frac{1}{p}\right) \cdot \lim_{n \to \infty} \frac{V(\mathbb{Z}/p^n\mathbb{Z})}{p^{2n}} .$$

If the reduction $V_{\mathbb{F}_p}$ is smooth then the sequence under the limit is constant by virtue of Hensel's Lemma. Otherwise, it becomes stationary after finitely many steps.

We approximate the infinite product over all the τ_p by the finite product taken over all primes less than 100. Numerical experiments show that the primes between 100 and 300 do not lead to a significant change.

4.2. *The factor at the infinite place.* —— We want to compute

$$\tau_\infty = \frac{1}{2} \int\limits_R \frac{1}{\|\operatorname{grad} f\|_2} \, dS$$

where the domain of integration is given by

$$R = \{(x, y, z, w) \in [-1, 1]^4 \mid f(x, y, z, w) = 0\} .$$

Here, dS denotes the usual hypersurface measure on R, considered as a hypersurface in $\mathbb{R}^4$. Thus, τ_∞ is given by a three-dimensional integral.

Since f is a homogeneous polynomial, we may reduce to an integral over the boundary of R which is a two-dimensional domain. In our particular case, we have $\deg f = 3$. Then, a direct computation leads to

$$\begin{aligned}\tau_\infty = & \int\limits_{R_0} \frac{1}{\|(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial w})\|_2} \, dA + \int\limits_{R_1} \frac{1}{\|(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial w})\|_2} \, dA \\ & + \int\limits_{R_2} \frac{1}{\|(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial w})\|_2} \, dA + \int\limits_{R_3} \frac{1}{\|(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})\|_2} \, dA\end{aligned}$$

where the domains of integration are

$$R_i = \{(x_0, x_1, x_2, x_3) \in [-1, 1]^4 \mid x_i = 1 \text{ and } f(x_0, x_1, x_2, x_3) = 0\}.$$

dA denotes the two-dimensional hypersurface measure on R_i, considered as a hypersurface in $\mathbb{R}^3$.

We therefore have to integrate a smooth function over a compact part of a smooth two-dimensional submanifold in $\mathbb{R}^3$. To do this, we approximate the domain of integration by a triangular mesh.

4.3. Algorithm (Generating a triangular mesh**).** — i) The domain of integration is the intersection of a manifold with a cube. We split this cube into eight smaller cubes and iterate this procedure four times, recursively.

During recursion, we exclude those cubes which obviously do not intersect the manifold. To do this, we estimate $\|\operatorname{grad} f\|_2$ on each cube.

ii) Then, each resulting cube is split into six simplices.

iii) For each edge of each simplex which intersects the manifold, we approximate the point of intersection. This leads to a mesh consisting of one or two triangles per simplex.

The next step is to compute the contribution of each triangle to the integral. For this, we use some adaption of the midpoint rule. We approximate the integrand by its value at the barycenter C of the triangle. Note that this point usually lies outside R_i. Algorithm 4.3 guarantees only that the three vertices of each facet are contained in R_i. We correct by an additional factor, the cosine of the angle between the normal vector of the triangle and the gradient vector $\operatorname{grad} f$ at the center C.

4.4. Remark. — It is not a priori clear that these correctional factors converge to 1 when the triangles become arbitrarily small. H. A. Schwarz's cylindrical surface [Sch] constitutes a famous counterexample.

5 Numerical Data

5.1. The surfaces studied. — A general cubic surface is described by twenty coefficients. With current technology, it is impossible to study all cubic surfaces with coefficients below a given bound. Therefore, we decided to work with coefficient vectors provided by a random number generator. Our first sample consists of 20 000 surfaces with randomly chosen coefficients in the interval $[0 \dots 50]$. The second sample consists of 20 000 surfaces with coefficients randomly chosen in the interval $[-100 \dots 100]$.

Using Algorithm 3.3 we proved that, for each of the surfaces studied, the full Galois group $W(E_6)$ acts on the 27 lines. The largest prime used was 457. This means that all our examples are general from the Galois point of view. By consequence, their Picard ranks are equal to 1. Further, according to Remark 2.6, the Brauer-Manin obstruction is not present on any of the surfaces considered.

We counted all $\mathbb{Q}$-rational points of height less than 250 on the surfaces of the first sample. It turns out that, on two of these surfaces, there are no $\mathbb{Q}$-rational points occurring as the equation is unsolvable in $\mathbb{Q}_p$ for some small p. In this situation, Manin's conjecture is true, trivially. On each of the remaining surfaces, we found at least one $\mathbb{Q}$-rational point. 228 examples contained less than ten points. On the other hand, 1213 examples contained at least one hundred $\mathbb{Q}$-rational points. The largest number of points found was 335.

For the second sample, the search bound was 500. Again, on two of these surfaces, there are no $\mathbb{Q}$-rational points occurring as the equation is unsolvable in a certain $\mathbb{Q}_p$. There were 202 examples containing between one and nine points. 1857 examples contained at least one hundred $\mathbb{Q}$-rational points. The largest number of points found was 349.

Furthermore, we computed an approximation of Peyre's constant for each surface.

5.2. *The density results.* —— For each of the surfaces considered we calculated the quotient

$$\#\{\text{ points of height} < B \text{ found }\} \,/\, \#\{\text{ points of height} < B \text{ expected }\}.$$

Let us visualize the distribution of the quotients by two histograms.

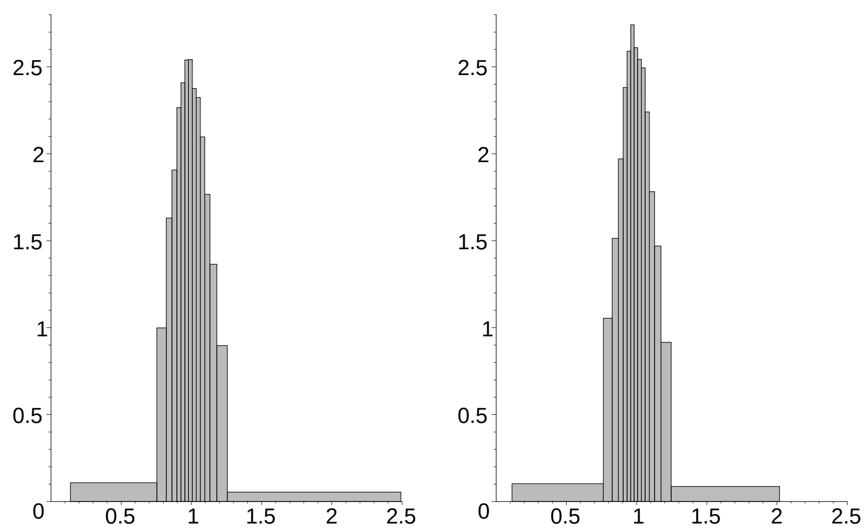

Figure 1: Distribution of the quotients for the first and the second sample.

Table 1: Parameters of the distribution

	first sample	second sample
mean	0.999 04	1.000 23
standard deviation	0.172 92	0.165 55

5.3. ***The results for the smallest point.*** —— For each of the surfaces in our samples, we determined the height $\mathrm{m}(V)$ of its smallest point. Let us visualize the distribution of these values versus the Tamagawa type number.

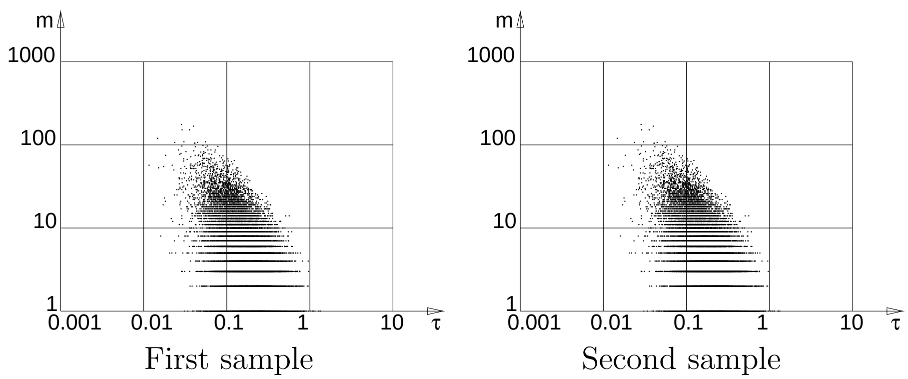

Figure 2: The smallest height of a rational point versus the Tamagawa number.

5.4. ***Conclusion.*** —— Our experiments suggest that, for general cubic surfaces V over $\mathbb{Q}$, the following assertions hold.

i) There are no obstructions against the Hasse principle.

ii) Manin's conjecture is true in the form refined by E. Peyre.

Further, it is apparent from the diagrams that the experiment agrees with the expectation for the heights of the smallest points formulated in 1.4. For both samples, the slope of a line tangent to the top right of the scatter plot, is near (-1). This indicates that, as formulated in Remark 1.5.i), even the estimate $\mathrm{m}(V) < \frac{C(\varepsilon)}{\tau(V)^{1+\varepsilon}}$ should be true for any $\varepsilon > 0$.

6 A concrete example

Let us conclude the article by some results on the particular cubic surface V given by

$$x^3 + 2xy^2 + 11y^3 + 3xz^2 + 5y^2w + 7zw^2 = 0.$$

This example is not one of the surfaces produced by the random number generator. Our intention is just to present the output of our algorithms in a specific (and not too artificial) example and, most notably, to show the intermediate results of Algorithm 3.3.

The first step of that algorithm works well on V, i.e. the polynomial g is indeed of degree 27. Its coefficients become rather large. The absolutely largest one is that at d^{13}. It is equal to 38 300 982 629 255 010. The leading coefficient of g is exactly $5^3 \cdot 7^{12}$.

Algorithm 3.3 finds case A at $p = 373$. The common decomposition type $(2, 5, 5, 5, 10)$ of the cases B and C occurs at $p = 19$, 31, 59, 61, 191, 199, and 223. Consequently, it is the full $W(E_6)$ which acts as the Galois group on the 27 lines on V.

6.1. Remark. —— The first explicit example of a smooth cubic surface over $\mathbb{Q}$ admitting the property that the Galois group acting on the 27 lines is equal to $W(E_6)$ has been constructed by T. Ekedahl [Ek, Corollary 2.2].

V has bad reduction at $p = 2$, 3, 7, 23, and 22 359 013 270 232 677. As approximations of the Euler product, we get

$$\prod_{p<100} \tau_p = 0{,}689\,380 \quad \text{and} \quad \prod_{p<300} \tau_p = 0{,}729\,750.$$

For the factor at the infinite place, we find using five recursions

$$\tau_\infty = 1{,}7657.$$

Altogether, E. Peyre's constant is approximately $\tau \approx 1{,}289$.

There are 345 $\mathbb{Q}$-rational points on V of height less than 250 and 693 $\mathbb{Q}$-rational points of height less than 500. The smallest points are $(0 : 0 : 1 : 0)$ and $(0 : 0 : 0 : 1)$, the smallest non-obvious point is $(1 : 2 : (-3) : (-2))$.

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