UNIVERSAL KZB EQUATIONS: THE ELLIPTIC CASE

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To Yuri Ivanovich Manin on his 70th birthday

Abstract. We define a universal version of the Knizhnik-Zamolodchikov-Bernard (KZB) connection in genus 1. This is a flat connection over a principal bundle on the moduli space of elliptic curves with marked points. It restricts to a flat connection on configuration spaces of points on elliptic curves, which can be used for proving the formality of the pure braid groups on genus 1 surfaces. We study the monodromy of this connection and show that it gives rise to a relation between the KZ associator and a generating series for iterated integrals of Eisenstein forms. We show that the universal KZB connection realizes as the usual KZB connection for simple Lie algebras, and that in the $\mathfrak{sl}_n$ case this realization factors through the Cherednik algebras. This leads us to define a functor from the category of equivariant D-modules on $\mathfrak{sl}_n$ to that of modules over the Cherednik algebra, and to compute the character of irreducible equivariant D-modules over $\mathfrak{sl}_n$ which are supported on the nilpotent cone.

Contents

Introduction

The KZ system was introduced in [KZ] as a system of equations satisfied by correlation functions in conformal field theory. It was then realized that this system has a universal version ([Dr3]). The monodromy of this system leads to representations of the braid groups, which can be used for proving the formality of the configuration spaces of $\mathbb{C}$, i.e., the fact that the fundamental groups of these spaces are formal (i.e., their Lie algebras are isomorphic with their associated graded, which is the holonomy Lie algebra and thus has an explicit presentation). This fact was first proved in the framework of minimal model theory ([Su, Ko]). These results gave rise to Drinfeld's theory of associators and quasi-Hopf algebras ([Dr2, Dr3]); one of the purposes of this work was to give an algebraic construction of the formality isomorphisms, and indeed one of its by-products is the fact that these isomorphisms can be defined over $\mathbb{Q}$.

In the case of configuration spaces over surfaces of genus ≥ 1, similar Lie algebra isomorphisms were constructed by Bezrukavnikov ([Bez]), using results of Kriz ([Kr]). In this series of papers, we will show that this result can be reproved using a suitable flat connection over configuration spaces. This connection is a universal version of the KZB connection ([Be1, Be2]), which is the higher genus analogue of the KZ connection.

In this paper, we focus on the case of genus 1. We define the universal KZB connection (Section 1), and rederive from there the formality result (Section 2). As in the integrable case of the KZB connection, the universal KZB connection extends from the configuration spaces $\bar{C}(E_\tau, n)/S_n$ to the moduli space $\mathcal{M}_{1,[n]}$ of elliptic curves with n unordered marked points (Section 3). This means that: (a) the connection can be extended to the directions of variation of moduli, and (b) it is modular invariant.

This connection then gives rise to a monodromy morphism $\gamma_n : \Gamma_{1,[n]} \to \mathbf{G}_n \rtimes S_n$, which we analyze in Section 4. The images of most generators can be expressed using the KZ associator, but the image $\tilde{\Theta}$ of the S-transformation expresses using iterated integrals of Eisenstein series. The relations between generators give rise to relations between $\tilde{\Theta}$ and the KZ associator, identities (28). This identity may be viewed as an elliptic analogue of the pentagon identity, as it is a "de Rham" analogue of the relation 6AS in [HLS] (in [Ma], the question was asked of the existence of this kind of identity).

In Section 5, we investigate how to algebraically construct a morphism $\Gamma_{1,[n]} \to \mathbf{G}_n \rtimes S_n$. We show that a morphism $\overline{\mathrm{B}}_{1,n} \to \exp(\hat{\bar{\mathfrak{t}}}_{1,n}) \rtimes S_n$ can be constructed using an associator only (here $\overline{\mathrm{B}}_{1,n}$ is the reduced braid group of n points on the torus). [Dr3] then implies that the formality isomorphism can be defined over $\mathbb{Q}$. In the last part of Section 5, we develop the analogue of the theory of quasitriangular quasibialgebras (QTQBA's), namely elliptic structures over QTQBA's. These structures give rise to representations of $\overline{\mathrm{B}}_{1,n}$, and they can be modified by twist. We hope that in the case of a simple Lie algebra, and using suitable twists, the elliptic structure given in Section 5.4 will give rise to elliptic structures over the quantum group $U_q(\mathfrak{g})$ (where $q \in \mathbb{C}^\times$) or over the Lusztig quantum group (when q is a root of unity), yielding back the representations of $\overline{\mathrm{B}}_{1,n}$ from conformal field theory.

In Section 6, we show that the universal KZB connection indeed specializes to the ordinary KZB connection.

Sections 7-9 are dedicated applications of the ideas of the preceding sections (in particular, Section 6) to representation theory of Cherednik algebras.

More precisely, In Section 7, we construct a homomorphism from the Lie algebra $\bar{\mathfrak{t}}_{1,n} \rtimes \mathfrak{d}$ to the rational Cherednik algebra $H_n(k)$ of type A_{n-1}. This allows us to consider the elliptic KZB connection with values in representations of the rational Cherednik algebra. The monodromy of this connection then gives representations of the true Cherednik algebra (i.e. the double affine Hecke algebra). In particular, this gives a simple way of constructing an isomorphism between the rational Cherednik algebra and the double affine Hecke algebra, with formal deformation parameters.

In Section 8, we consider the special representation V_N of the rational Cherednik algebra $H_n(k)$, $k = N/n$, for which the elliptic KZB connection is the KZB connection for (holomorphic) n-point correlation functions of the WZW model for $\mathrm{SL}_N(\mathbb{C})$ on the elliptic curve, when the marked points are labeled by the vector representation $\mathbb{C}^N$. This representation is realized in the space of equivariant polynomial functions on $\mathfrak{sl}_N$ with values in $(\mathbb{C}^N)^{\otimes n}$, and we show that it is irreducible, and calculate its character.

In Section 9, we generalize the construction of Section 8, by replacing, in the construction of V_N, the space of polynomial functions on $\mathfrak{sl}_N$ with an arbitrary D-module on $\mathfrak{sl}_N$. This gives rise to an exact functor from the category of (equivariant) D-modules on $\mathfrak{sl}_N$ to the category of representations of $H_n(N/n)$. We study this functor in detail. In particular, we show that this functor maps D-modules concentrated on the nilpotent cone to modules from category $\mathcal{O}_-$ of highest weight modules over the Cherednik algebra, and is closely related to the Gan-Ginzburg functor, [GG1]. Using these facts, we show that it maps irreducible D-modules on the nilpotent cone to irreducible representations of the Cherednik algebra, and determine their highest weights. As an application, we compute the decomposition of cuspidal D-modules into irreducible representations of $\mathrm{SL}_N(\mathbb{C})$. Finally, we describe the generalization of the above result to the trigonometric case (which involves D-modules on the group and trigonometric Cherednik algebras), and point out several directions for generalization.

1. Bundles with flat connections on (reduced) configuration spaces

1.1. The Lie algebras $\mathfrak{t}_{1,n}$ and $\bar{\mathfrak{t}}_{1,n}$. Let $n \geq 1$ be an integer and $\mathbf{k}$ be a field of characteristic zero. We define $\mathfrak{t}_{1,n}^{\mathbf{k}}$ as the Lie algebra with generators x_i, y_i $(i = 1, ..., n)$ and t_{ij} $(i \neq j \in \{1, ..., n\})$ and relations

$$t_{ij} = t_{ji}, \quad [t_{ij}, t_{ik} + t_{jk}] = 0, \quad [t_{ij}, t_{kl}] = 0, \tag{1}$$

$$[x_i, y_j] = t_{ij}, \quad [x_i, x_j] = [y_i, y_j] = 0, \quad [x_i, y_i] = -\sum_{j|j\neq i} t_{ij},$$

$$[x_i, t_{jk}] = [y_i, t_{jk}] = 0, \quad [x_i + x_j, t_{ij}] = [y_i + y_j, t_{ij}] = 0.$$

$(i, j, k, l$ are distinct). In this Lie algebra, $\sum_i x_i$ and $\sum_i y_i$ are central; we then define $\bar{\mathfrak{t}}_{1,n}^{\mathbf{k}} := \mathfrak{t}_{1,n}^{\mathbf{k}}/(\sum_i x_i, \sum_i y_i)$. Both $\mathfrak{t}_{1,n}^{\mathbf{k}}$ and $\bar{\mathfrak{t}}_{1,n}^{\mathbf{k}}$ are positively graded, where $\deg(x_i) = \deg(y_i) = 1$.

The symmetric group S_n acts by automorphisms of $\mathfrak{t}_{1,n}^{\mathbf{k}}$ by $\sigma(x_i) := x_{\sigma(i)}$, $\sigma(y_i) := y_{\sigma(i)}$, $\sigma(t_{ij}) := t_{\sigma(i)\sigma(j)}$; this induces an action of S_n by automorphisms of $\bar{\mathfrak{t}}_{1,n}^{\mathbf{k}}$.

We will set $\mathfrak{t}_{1,n} := \mathfrak{t}_{1,n}^{\mathbb{C}}$, $\bar{\mathfrak{t}}_{1,n} := \bar{\mathfrak{t}}_{1,n}^{\mathbb{C}}$ in Sections 1 to 4.

1.2. Bundles with flat connections over $C(E, n)$ and $\bar{C}(E, n)$. Let E be an elliptic curve, $C(E, n)$ be the configuration space $E^n - \{\text{diagonals}\}$ $(n \geq 1)$ and $\bar{C}(E, n) := C(E, n)/E$ be the reduced configuration space. We will define a[1] $\exp(\hat{\bar{\mathfrak{t}}}_{1,n})$-principal bundle with a flat (holomorphic) connection $(\bar{P}_{E,n}, \bar{\nabla}_{E,n}) \to \bar{C}(E, n)$. For this, we define a $\exp(\hat{\mathfrak{t}}_{1,n})$-principal bundle with a flat connection $(P_{E,n}, \nabla_{E,n}) \to C(E, n)$. Its image under the natural morphism $\exp(\hat{\mathfrak{t}}_n) \to \exp(\hat{\bar{\mathfrak{t}}}_n)$ is a $\exp(\hat{\bar{\mathfrak{t}}}_{1,n})$-bundle with connection $(\tilde{P}_{E,n}, \tilde{\nabla}_{E,n}) \to C(E, n)$, and we then prove that $(\tilde{P}_{E,n}, \tilde{\nabla}_{E,n})$ is the pull-back of a pair $(\bar{P}_{E,n}, \bar{\nabla}_{E,n})$ under the canonical projection $C(E, n) \to \bar{C}(E, n)$.

For this, we fix a uniformization $E \simeq E_\tau$, where for $\tau \in \mathfrak{H}$, $\mathfrak{H} := \{\tau \in \mathbb{C}|\Im(\tau) > 0\}$, $E_\tau := \mathbb{C}/\Lambda_\tau$ and $\Lambda_\tau := \mathbb{Z} + \mathbb{Z}\tau$.

We then have $C(E_\tau, n) = (\mathbb{C}^n - \mathrm{Diag}_{n,\tau})/\Lambda_\tau^n$, where $\mathrm{Diag}_{n,\tau} := \{\mathbf{z} = (z_1, ..., z_n) \in \mathbb{C}^n | z_{ij} := z_i - z_j \in \Lambda_\tau \text{ for some } i \neq j\}$. We define $P_{\tau,n}$ as the restriction to $C(E_\tau, n)$ of the bundle over $\mathbb{C}^n/\Lambda_\tau^n$ for which a section on $U \subset \mathbb{C}^n/\Lambda_\tau^n$ is a regular map $f : \pi^{-1}(U) \to \exp(\hat{\mathfrak{t}}_{1,n})$, such that[2] $f(\mathbf{z} + \delta_i) = f(\mathbf{z})$, $f(\mathbf{z} + \tau\delta_i) = e^{-2\pi\,\mathrm{i}\,x_i} f(\mathbf{z})$ (here $\pi : \mathbb{C}^n \to \mathbb{C}^n/\Lambda_\tau^n$ is the canonical projection and δ_i is the ith vector of the canonical basis of $\mathbb{C}^n$).

The bundle $\tilde{P}_{\tau,n} \to C(E_\tau, n)$ derived from $P_{\tau,n}$ is the pull-back of a bundle $\bar{P}_{\tau,n} \to \bar{C}(E_\tau, n)$ since the $e^{-2\pi\,\mathrm{i}\,\bar{x}_i} \in \exp(\hat{\bar{\mathfrak{t}}}_{1,n})$ commute pairwise and their product is 1. Here $x \mapsto \bar{x}$ is the map $\hat{\mathfrak{t}}_{1,n} \to \hat{\bar{\mathfrak{t}}}_{1,n}$.

A flat connection $\nabla_{\tau,n}$ on $P_{\tau,n}$ is then the same as an equivariant flat connection over the trivial bundle over $\mathbb{C}^n - \mathrm{Diag}_{n,\tau}$, i.e., a connection of the form

$$\nabla_{\tau,n} := \mathrm{d} - \sum_{i=1}^{n} K_i(\mathbf{z}|\tau)\,\mathrm{d}\,z_i,$$

where $K_i(-|\tau) : \mathbb{C}^n \to \hat{\mathfrak{t}}_{1,n}$ is holomorphic on $\mathbb{C}^n - \mathrm{Diag}_{n,\tau}$, such that:

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[1] We will denote by $\hat{\mathfrak{g}}$ or $\mathfrak{g}^\wedge$ the degree completion of a positively graded Lie algebra $\mathfrak{g}$.

[2] We set $\mathrm{i} := \sqrt{-1}$, leaving i for indices.

(a) $K_i(\mathbf{z}+\delta_j|\tau) = K_i(\mathbf{z}|\tau)$, $K_i(\mathbf{z}+\tau\delta_j|\tau) = e^{-2\pi\,\mathrm{i}\,\mathrm{ad}(x_j)}(K_i(\mathbf{z}|\tau))$,

(b) $[\partial/\partial z_i - K_i(\mathbf{z}|\tau), \partial/\partial z_j - K_j(\mathbf{z}|\tau)] = 0$ for any i, j.

$\nabla_{\tau,n}$ then induces a flat connection $\tilde{\nabla}_{\tau,n}$ on $\tilde{P}_{\tau,n}$. Then $\tilde{\nabla}_{\tau,n}$ is the pull-back of a (necessarily flat) connection on $\bar{P}_{\tau,n}$ iff:

(c) $\bar{K}_i(\mathbf{z}|\tau) = \bar{K}_i(\mathbf{z}+u(\sum_i \delta_i)|\tau)$ and $\sum_i \bar{K}_i(\mathbf{z}|\tau) = 0$ for $\mathbf{z} \in \mathbb{C}^n - \mathrm{Diag}_{n,\tau}$, $u \in \mathbb{C}$.

In order to define the $K_i(\mathbf{z}|\tau)$, we first recall some facts on theta-functions. There is a unique holomorphic function $\mathbb{C}\times\mathfrak{H} \to \mathbb{C}$, $(z,\tau) \mapsto \theta(z|\tau)$, such that $\{z|\theta(z|\tau)=0\} = \Lambda_\tau$, $\theta(z+1|\tau) = -\theta(z|\tau) = \theta(-z|\tau)$ and $\theta(z+\tau|\tau) = -e^{-\pi\,\mathrm{i}\,\tau}e^{-2\pi\,\mathrm{i}\,z}\theta(z|\tau)$, and $\theta_z(0|\tau) = 1$. We have $\theta(z|\tau+1) = \theta(z|\tau)$, while $\theta(-z/\tau|-1/\tau) = -(1/\tau)e^{(\pi\,\mathrm{i}/\tau)z^2}\theta(z|\tau)$. If $\eta(\tau) = q^{1/24}\prod_{n\geq 1}(1-q^n)$ where $q = e^{2\pi\,\mathrm{i}\,\tau}$, and if we set $\vartheta(z|\tau) := \eta(\tau)^3\theta(z|\tau)$, then $\partial_\tau\vartheta = (1/4\pi\,\mathrm{i})\partial_z^2\vartheta$.

Let us set

$$k(z,x|\tau) := \frac{\theta(z+x|\tau)}{\theta(z|\tau)\theta(x|\tau)} - \frac{1}{x}.$$

When τ is fixed, $k(z,x|\tau)$ belongs to $\mathrm{Hol}(\mathbb{C}-\Lambda_\tau)[[x]]$. Substituting $x = \mathrm{ad}x_i$, we get a linear map $\mathfrak{t}_{1,n} \to (\mathfrak{t}_{1,n}\otimes\mathrm{Hol}(\mathbb{C}-\Lambda_\tau))^\wedge$, and taking the image of t_{ij}, we define

$$K_{ij}(z|\tau) := k(z,\mathrm{ad}\,x_i|\tau)(t_{ij}) = \big(\frac{\theta(z+\mathrm{ad}(x_i)|\tau)}{\theta(z|\tau)}\frac{\mathrm{ad}(x_i)}{\theta(\mathrm{ad}(x_i)|\tau)} - 1\big)(y_j);$$

it is a holomorphic function on $\mathbb{C}-\Lambda_\tau$ with values in $\hat{\mathfrak{t}}_{1,n}$.

Now set $\mathbf{z} := (z_1,\ldots,z_n)$, $z_{ij} := z_i - z_j$ and define

$$K_i(\mathbf{z}|\tau) := -y_i + \sum_{j|j\neq i} K_{ij}(z_{ij}|\tau).$$

Let us check that the $K_i(\mathbf{z}|\tau)$ satisfy condition (c). We have clearly $K_i(\mathbf{z}+u(\sum_i\delta_i)) = K_i(\mathbf{z})$. We have $k(z,x|\tau)+k(-z,-x|\tau) = 0$, so $K_{ij}(z|\tau)+K_{ji}(-z|\tau) = 0$, so that $\sum_i K_i(\mathbf{z}|\tau) = -\sum_i y_i$, which implies $\sum_i \bar{K}_i(\mathbf{z}|\tau) = 0$.

Lemma 1.1. *$K_i(\mathbf{z}+\delta_j|\tau) = K_i(\mathbf{z}|\tau)$ and $K_i(\mathbf{z}+\tau\delta_j|\tau) = e^{-2\pi\,\mathrm{i}\,\mathrm{ad}\,x_j}(K_i(\mathbf{z}|\tau))$, i.e., the $K_i(\mathbf{z}|\tau)$ satisfy condition (a).*

Proof. We have $k(z\pm 1,x|\tau) = k(z,x|\tau)$ so for any j, $K_i(\mathbf{z}+\delta_j|\tau) = K_i(\mathbf{z}|\tau)$. We have $k(z\pm\tau,x|\tau) = e^{\mp 2\pi\,\mathrm{i}\,x}k(z,x|\tau) + (e^{\mp 2\pi\,\mathrm{i}\,x}-1)/x$, so if $j\neq i$, $K_i(\mathbf{z}+\tau\delta_j|\tau) = \sum_{j'\neq i,j}K_{ij'}(z_{ij'}|\tau) + e^{2\pi\,\mathrm{i}\,\mathrm{ad}\,x_i}K_{ij}(z_{ij}|\tau) + \frac{e^{2\pi\,\mathrm{i}\,\mathrm{ad}\,x_i}-1}{\mathrm{ad}\,x_i}(t_{ij}) - y_i$. Then

$$\frac{e^{2\pi\,\mathrm{i}\,\mathrm{ad}\,x_i}-1}{\mathrm{ad}\,x_i}(t_{ij}) = \frac{1-e^{-2\pi\,\mathrm{i}\,\mathrm{ad}\,x_j}}{\mathrm{ad}\,x_j}(t_{ij}) = (1-e^{-2\pi\,\mathrm{i}\,\mathrm{ad}\,x_j})(y_i),$$

$e^{2\pi\,\mathrm{i}\,\mathrm{ad}\,x_i}(K_{ij}(z_{ij}|\tau)) = e^{-2\pi\,\mathrm{i}\,\mathrm{ad}\,x_j}(K_{ij}(z_{ij}|\tau))$ and for $j'\neq i,j$, $K_{ij'}(z_{ij'}|\tau) = e^{-2\pi\,\mathrm{i}\,\mathrm{ad}\,x_j}(K_{ij'}(z_{ij'}|\tau))$, so $K_i(\mathbf{z}+\tau\delta_j|\tau) = e^{-2\pi\,\mathrm{i}\,\mathrm{ad}\,x_j}(K_i(\mathbf{z}|\tau))$.

Now $K_i(\mathbf{z}+\tau\delta_i|\tau) = -\sum_i y_i - \sum_{j|j\neq i}K_j(\mathbf{z}+\tau\delta_i|\tau) = -\sum_i y_i - e^{-2\pi\,\mathrm{i}\,\mathrm{ad}\,x_i}(\sum_{j|j\neq i}K_j(\mathbf{z}|\tau)) = e^{-2\pi\,\mathrm{i}\,\mathrm{ad}\,x_i}(-\sum_i y_i - \sum_{j|j\neq i}K_j(\mathbf{z}|\tau)) = e^{-2\pi\,\mathrm{i}\,\mathrm{ad}\,x_i}K_i(\mathbf{z}|\tau)$ (the first and last equality follow from the proof of (c), the second equality has just been proved, the third equality follows from the centrality of $\sum_i y_i$). □

Proposition 1.2. *$[\partial/\partial z_i - K_i(\mathbf{z}|\tau), \partial/\partial z_j - K_j(\mathbf{z}|\tau)] = 0$, i.e., the $K_i(\mathbf{z}|\tau)$ satisfy condition (b).*

Proof. For $i\neq j$, let us set $K_{ij} := K_{ij}(z_{ij}|\tau)$. Recall that $K_{ij}+K_{ji} = 0$, therefore if $\partial_i := \partial/\partial z_i$

$$\partial_i K_{ij} - \partial_j K_{ji} = 0, \quad [y_i - K_{ij}, y_j - K_{ji}] = -[K_{ij}, y_i + y_j].$$

Moreover, if i,j,k,l are distinct, then $[K_{ik},K_{jl}]=0$. It follows that if $i\neq j$,

$$[\partial_i - K_i(\mathbf{z}|\tau), \partial_j - K_j(\mathbf{z}|\tau)]$$
$$= [y_i+y_j,K_{ij}] + \sum_{k|k\neq i,j}\big([K_{ik},K_{jk}]+[K_{ij},K_{jk}]+[K_{ij},K_{ik}]+[y_j,K_{ik}]-[y_i,K_{jk}]\big).$$

Let us assume for a while that if $k\notin\{i,j\}$, then

$$-[y_i,K_{jk}]-[y_j,K_{ki}]-[y_k,K_{ij}]+[K_{ji},K_{ki}]+[K_{kj},K_{ij}]+[K_{ik},K_{jk}]=0 \qquad (2)$$

(this is the universal version of the classical dynamical Yang-Baxter equation).

Then (2) implies that

$$[\partial_i - K_i(\mathbf{z}|\tau), \partial_j - K_j(\mathbf{z}|\tau)] = [y_i+y_j,K_{ij}] + \sum_{k|k\neq i,j}[y_k,K_{ij}] = [\sum_k y_k, K_{ij}] = 0$$

(as $\sum_k y_k$ is central), which proves the proposition.

Let us now prove (2). If $f(x)\in\mathbb{C}[[x]]$, then

$$[y_k, f(\mathrm{ad}x_i)(t_{ij})] = \frac{f(\mathrm{ad}x_i)-f(-\mathrm{ad}x_j)}{\mathrm{ad}x_i+\mathrm{ad}x_j}[-t_{ki},t_{ij}],$$
$$[y_i, f(\mathrm{ad}x_j)(t_{jk})] = \frac{f(\mathrm{ad}x_j)-f(-\mathrm{ad}x_k)}{\mathrm{ad}x_j+\mathrm{ad}x_k}[-t_{ij},t_{jk}] = \frac{f(\mathrm{ad}x_j)-f(\mathrm{ad}x_i+\mathrm{ad}x_j)}{-\mathrm{ad}x_i}[-t_{ij},t_{jk}],$$
$$[y_j, f(\mathrm{ad}x_k)(t_{ki})] = \frac{f(\mathrm{ad}x_k)-f(-\mathrm{ad}x_i)}{\mathrm{ad}x_k+\mathrm{ad}x_i}[-t_{jk},t_{ki}] = \frac{f(-\mathrm{ad}x_i-\mathrm{ad}x_j)-f(-\mathrm{ad}x_i)}{-\mathrm{ad}x_j}[-t_{jk},t_{ki}].$$

The first identity is proved as follows:

$$[y_k,(\mathrm{ad}x_i)^n(t_{ij})] = -\sum_{s=0}^{n-1}(\mathrm{ad}x_i)^s(\mathrm{ad}t_{ki})(\mathrm{ad}x_i)^{n-1-s}(t_{ij}) = -\sum_{s=0}^{n-1}(\mathrm{ad}x_i)^s(\mathrm{ad}t_{ki})(-\mathrm{ad}x_j)^{n-1-s}(t_{ij})$$
$$= -\sum_{s=0}^{n-1}(\mathrm{ad}x_i)^s(-\mathrm{ad}x_j)^{n-1-s}(\mathrm{ad}t_{ki})(t_{ij}) = f(\mathrm{ad}x_i,-\mathrm{ad}x_j)([-t_{ki},t_{ij}]),$$

where $f(u,v)=(u^n-v^n)/(u-v)$. The two next identities follow from this one and from the fact that $x_i+x_j+x_k$ commutes with t_{ij},t_{ik},t_{jk}.

Then, if we write $k(z,x)$ instead of $k(z,x|\tau)$, the l.h.s. of (2) is equal to

$$\Big(k(z_{ij},-\mathrm{ad}x_j)k(z_{ik},\mathrm{ad}x_i+\mathrm{ad}x_j) - k(z_{ij},\mathrm{ad}x_i)k(z_{jk},\mathrm{ad}x_i+\mathrm{ad}x_j) + k(z_{ik},\mathrm{ad}x_i)k(z_{jk},\mathrm{ad}x_j)$$
$$+\frac{k(z_{jk},\mathrm{ad}x_j)-k(z_{jk},\mathrm{ad}x_i+\mathrm{ad}x_j)}{\mathrm{ad}x_i} + \frac{k(z_{ik},\mathrm{ad}x_i)-k(z_{ij},\mathrm{ad}x_i+\mathrm{ad}x_j)}{\mathrm{ad}x_j}$$
$$-\frac{k(z_{ij},\mathrm{ad}x_i)-k(z_{ij},-\mathrm{ad}x_j)}{\mathrm{ad}x_i+\mathrm{ad}x_j}\Big)[t_{ij},t_{ik}].$$

So (2) follows from the identity

$$k(z,-v)k(z',u+v)-k(z,u)k(z'-z,u+v)+k(z',u)k(z'-z,v)$$
$$+\frac{k(z'-z,v)-k(z'-z,u+v)}{u}+\frac{k(z',u)-k(z',u+v)}{v}-\frac{k(z,u)-k(z,-v)}{u+v}=0,$$

where u,v are formal variables, which is a consequence of the theta-functions identity

$$\big(k(z,-v)-\frac{1}{v}\big)\big(k(z',u+v)+\frac{1}{u+v}\big)-\big(k(z,u)+\frac{1}{u}\big)\big(k(z'-z,u+v)+\frac{1}{u+v}\big)$$
$$+\big(k(z',u)+\frac{1}{u}\big)\big(k(z'-z,v)+\frac{1}{v}\big)=0. \qquad (3)$$

□

We have therefore proved:

Theorem 1.3. *$(P_{\tau,n},\nabla_{\tau,n})$ is a flat connection on $C(E_\tau,n)$, and the induced flat connection $(\tilde P_{\tau,n},\tilde\nabla_{\tau,n})$ is the pull-back of a unique flat connection $(\bar P_{\tau,n},\bar\nabla_{\tau,n})$ on $\bar C(E_\tau,n)$.*

1.3. **Bundles with flat connections on $C(E,n)/S_n$ and $\bar C(E,n)/S_n$.** The group S_n acts freely by automorphisms of $C(E,n)$ by $\sigma(z_1,...,z_n):=(z_{\sigma^{-1}(1)},...,z_{\sigma^{-1}(n)})$. This descends to a free action of S_n on $\bar C(E,n)$. We set $C(E,[n]):=C(E,n)/S_n$, $\bar C(E,[n]):=\bar C(E,n)/S_n$.

We will show that $(P_{\tau,n},\nabla_{\tau,n})$ induces a bundle with flat connection $(P_{\tau,[n]},\nabla_{\tau,[n]})$ on $C(E_\tau,[n])$ with group $\exp(\hat{\mathfrak t}_{1,n})\rtimes S_n$, and similarly $(\bar P_{\tau,n},\bar\nabla_{\tau,n})$ induces $(\bar P_{\tau,[n]},\bar\nabla_{\tau,[n]})$ on $\bar C(E_\tau,[n])$ with group $\exp(\hat{\bar{\mathfrak t}}_{1,n})\rtimes S_n$.

We define $P_{\tau,[n]}\to C(E_\tau,[n])$ by the condition that a section of $U\subset C(E_\tau,[n])$ is a regular map $\pi^{-1}(U)\to\exp(\hat{\mathfrak t}_{1,n})\rtimes S_n$, satisfying again $f(\mathbf z+\delta_i)=f(\mathbf z)$, $f(\mathbf z+\tau\delta_i)=e^{-2\pi\,\mathrm i\,x_i}f(\mathbf z)$ and the additional requirement $f(\sigma\mathbf z)=\sigma f(\mathbf z)$ (where $\tilde\pi:\mathbb C^n-\mathrm{Diag}_{\tau,n}\to C(E_\tau,[n])$ is the canonical projection). It is clear that $\nabla_{\tau,n}$ is S_n-invariant, which implies that it defines a flat connection $\nabla_{\tau,[n]}$ on $C(E_\tau,[n])$.

The bundle $\bar P(E_\tau,[n])\to\bar C(E_\tau,[n])$ is defined by the additional requirement $f(\mathbf z+u(\sum_i\delta_i))=f(\mathbf z)$ and $\bar\nabla_{\tau,n}$ then induces a flat connection $\bar\nabla_{\tau,[n]}$ on $\bar C(E_\tau,[n])$.

2. Formality of pure braid groups on the torus

2.1. **Reminders on Malcev Lie algebras.** Let $\mathbf k$ be a field of characteristic 0 and let $\mathfrak g$ be a pronilpotent $\mathbf k$-Lie algebra. Set $\mathfrak g^1=\mathfrak g$, $\mathfrak g^{k+1}=[\mathfrak g,\mathfrak g^k]$; then $\mathfrak g=\mathfrak g^1\supset\mathfrak g^2...$ is a decreasing filtration of $\mathfrak g$. The associated graded Lie algebra is $\mathrm{gr}(\mathfrak g):=\oplus_{k\geq1}\mathfrak g^k/\mathfrak g^{k+1}$; we also consider its completion $\hat{\mathrm{gr}}(\mathfrak g):=\hat\oplus_{k\geq1}\mathfrak g^k/\mathfrak g^{k+1}$ (here $\hat\oplus$ is the direct product). We say that $\mathfrak g$ is formal iff there exists an isomorphism of filtered Lie algebras $\mathfrak g\simeq\hat{\mathrm{gr}}(\mathfrak g)$, whose associated graded morphism is the identity. We will use the following fact: if $\mathfrak g$ is a pronilpotent Lie algebra, $\mathfrak t$ is a positively graded Lie algebra, and there exists an isomorphism $\mathfrak g\simeq\hat{\mathfrak t}$ of filtered Lie algebras, then $\mathfrak g$ is formal, and the associated graded morphism $\mathrm{gr}(\mathfrak g)\to\mathfrak t$ is an isomorphism of graded Lie algebras.

If Γ is a finitely generated group, there exists a unique pair $(\Gamma(\mathbf k),i_\Gamma)$ of a prounipotent algebraic group $\Gamma(\mathbf k)$ and a group morphism $i_\Gamma:\Gamma\to\Gamma(\mathbf k)$, which is initial in the category of all pairs (U,j), where U is prounipotent $\mathbf k$-algebraic group and $j:\Gamma\to U$ is a group morphism.

We denote by $\mathrm{Lie}(\Gamma)_{\mathbf k}$ the Lie algebra of $\Gamma(\mathbf k)$. Then we have $\Gamma(\mathbf k)=\exp(\mathrm{Lie}(\Gamma)_{\mathbf k})$; $\mathrm{Lie}(\Gamma)_{\mathbf k}$ is a pronilpotent Lie algebra. We have $\mathrm{Lie}(\Gamma)_{\mathbf k}=\mathrm{Lie}(\Gamma)_{\mathbb Q}\otimes\mathbf k$. We say that Γ is formal iff $\mathrm{Lie}(\Gamma)_{\mathbb C}$ is formal (one can show that this implies that $\mathrm{Lie}(\Gamma)_{\mathbb Q}$ is formal).

When Γ is presented by generators $g_1,...,g_n$ and relations $R_i(g_1,...,g_n)$ $(i=1,...,p)$, $\mathrm{Lie}(\Gamma)_{\mathbb Q}$ is the quotient of the topologically free Lie algebra $\hat{\mathfrak f}_n$ generated by $\gamma_1,...,\gamma_n$ by the topological ideal generated by $\log(R_i(e^{\gamma_1},...,e^{\gamma_n}))$ $(i=1,...,p)$.

The decreasing filtration of $\hat{\mathfrak f}_n$ is $\hat{\mathfrak f}_n=(\hat{\mathfrak f}_n)^1\supset(\hat{\mathfrak f}_n)^2\supset...$, where $(\hat{\mathfrak f}_n)^k$ is the part of $\hat{\mathfrak f}_n$ of degree $\geq k$ in the generators $\gamma_1,...,\gamma_n$. The image of this filtration by the projection is map is the decreasing filtration $\mathrm{Lie}(\Gamma)_{\mathbb Q}=\mathrm{Lie}(\Gamma)^1_{\mathbb Q}\supset\mathrm{Lie}(\Gamma)^2_{\mathbb Q}\supset...$ of $\mathrm{Lie}(\Gamma)_{\mathbb Q}$.

2.2. **Presentation of $\mathrm{PB}_{1,n}$.** For $\tau\in\mathfrak H$, let $U_\tau\subset\mathbb C^n-\mathrm{Diag}_{n,\tau}$ be the open subset of all $\mathbf z=(z_1,...,z_n)$, of the form $z_i=a_i+\tau b_i$, where $0<a_1<...<a_n<1$ and $0<b_1<...<b_n<1$. If $\mathbf z_0=(z_1^0,...,z_n^0)\in U_\tau$, its image $\overline{\mathbf z}_0$ in E_τ^n actually belongs to the configuration space $C(E_\tau,n)$.

The pure braid group of n points on the torus $\mathrm{PB}_{1,n}$ may be viewed as $\mathrm{PB}_{1,n}=\pi_1(C(E_\tau,n),\overline{\mathbf z}_0)$. Denote by $X_i,Y_i\in\mathrm{PB}_{1,n}$ the classes of the projection of the paths $[0,1]\ni t\mapsto\mathbf z_0-t\delta_i$ and $[0,1]\ni t\mapsto\mathbf z_0-t\tau\delta_i$.

Set $A_i:=X_i...X_n$, $B_i:=Y_i...Y_n$ for $i=1,...,n$. According to [Bi1], A_i,B_i $(i=1,...,n)$ generate $\mathrm{PB}_{1,n}$ and a presentation of $\mathrm{PB}_{1,n}$ is, in terms of these generators:

$$(A_i,A_j)=(B_i,B_j)=1\ (\text{any }\ i,j),\quad (A_1,B_j)=(B_1,A_j)=1\ (\text{any }\ j),$$

$$(B_k, A_kA_j^{-1}) = (B_kB_j^{-1}, A_k) = C_{jk}\ (j \le k), \quad (A_i, C_{jk}) = (B_i, C_{jk}) = 1\ (i \le j \le k),$$

where $(g, h) = ghg^{-1}h^{-1}$.

2.3. **Alternative presentations of $\mathfrak{t}_{1,n}$.** We now give two variants of the defining presentation of $\mathfrak{t}_{1,n}$. Presentation (A) below is the original presentation in [Bez], and presentation (B) will be suited to the comparison with the above presentation of $\mathrm{PB}_{1,n}$.

Lemma 2.1. $\mathfrak{t}_{1,n}$ *admits the following presentations:*

(A) generators are x_i, y_i $(i = 1, ..., n)$, relations are $[x_i, y_j] = [x_j, y_i]$ $(i \neq j)$, $[x_i, x_j] = [y_i, y_j] = 0$ (any i, j), $[\sum_j x_j, y_i] = [\sum_j y_j, x_i] = 0$ (any i), $[x_i, [x_j, y_k]] = [y_i, [y_j, x_k]] = 0$ (i, j, k are distinct);

(B) generators are a_i, b_i $(i = 1, ..., n)$, relations are $[a_i, a_j] = [b_i, b_j] = 0$ (any i, j), $[a_1, b_j] = [b_1, a_j] = 0$ (any j), $[a_j, b_k] = [a_k, b_j]$ (any i, j), $[a_i, c_{jk}] = [b_i, c_{jk}] = 0$ $(i \le j \le k)$, where $c_{jk} = [b_k, a_k - a_j]$.

The isomorphism of presentations (A) and (B) is $a_i = \sum_{j=i}^n x_j$, $b_i = \sum_{j=i}^n y_j$.

Proof. Let us prove that the initial relations for x_i, y_i, t_{ij} imply the relations (A) for x_i, y_i. Let us assume the initial relations. If $i \neq j$, since $[x_i, y_j] = t_{ij}$ and $t_{ij} = t_{ji}$, we get $[x_i, y_j] = [x_j, y_i]$. The relations $[x_i, x_j] = [y_i, y_j] = 0$ (any i, j) are contained in the initial relations. For any i, since $[x_i, y_i] = -\sum_{j|j\neq i} t_{ij}$ and $[x_j, y_i] = t_{ji} = t_{ij}$ $(j \neq i)$, we get $[\sum_j x_j, y_i] = 0$. Similarly, $[\sum_j y_j, x_i] = 0$ (for any i). If i, j, k are distinct, since $[x_j, y_k] = t_{jk}$ and $[x_i, t_{jk}] = 0$, we get $[x_i, [x_j, y_k]] = 0$ and similarly we prove $[x_i, [y_j, x_k]] = 0$.

Let us now prove that the relations (A) for x_i, y_i imply the initial relations for x_i, y_i and $t_{ij} := [x_i, y_j]$ $(i \neq j)$. Assume the relations (A). If $i \neq j$, since $[x_i, y_j] = [x_j, y_i]$, we have $t_{ij} = t_{ji}$. The relation $t_{ij} = [x_i, y_j]$ $(i \neq j)$ is clear and $[x_i, x_j] = [y_i, y_j] = 0$ (any i, j) are already in relations (A). Since for any i, $[\sum_j x_j, y_i] = 0$, we get $[x_i, y_i] = -\sum_{j|j\neq i}[x_j, y_i] = -\sum_{j|j\neq i} t_{ji} = -\sum_{j|j\neq i} t_{ij}$. If i, j, k are distinct, the relations $[x_i, [x_j, y_k]] = [y_i, [y_j, x_k]] = 0$ imply $[x_i, t_{jk}] = [y_i, t_{jk}] = 0$. If $i \neq j$, since $[\sum_k x_k, x_i] = [\sum_k x_k, y_j] = 0$, we get $[\sum_k x_k, t_{ij}] = 0$ and $[x_k, t_{ij}] = 0$ for $k \notin \{i, j\}$ then implies $[x_i + x_j, t_{ij}] = 0$. One proves similarly $[y_i + y_j, t_{ij}] = 0$. We have already shown that $[x_i, t_{kl}] = [y_j, t_{kl}] = 0$ for i, j, k, l distinct, which implies $[[x_i, y_j], t_{kl}] = 0$, i.e., $[t_{ij}, t_{kl}] = 0$. If i, j, k are distinct, we have shown that $[t_{ij}, y_k] = 0$ and $[t_{ij}, x_i + x_j] = 0$, which implies $[t_{ij}, [x_i + x_j, y_k]] = 0$, i.e., $[t_{ij}, t_{ik} + t_{jk}] = 0$.

Let us prove that the relations (A) for x_i, y_i imply relations (B) for $a_i := \sum_{j=i}^n x_j$, $b_i := \sum_{j=i}^n y_j$. Summing up the relations $[x_{i'}, x_{j'}] = [y_{i'}, y_{j'}] = 0$ and $[x_{i'}, y_{j'}] = [x_{j'}, y_{i'}]$ for $i' = i, ..., n$ and $j' = j, ..., n$, we get $[a_i, a_j] = [b_i, b_j] = 0$ and $[a_i, b_j] = [a_j, b_i]$ (for any i, j). Summing up $[\sum_j x_j, y_{i'}] = [\sum_j y_j, x_{i'}] = 0$ for $i' = i, ..., n$, we get $[a_1, b_i] = [a_i, b_1] = 0$ (for any i). Finally, $c_{jk} = \sum_{\alpha=j}^{k-1}\sum_{\beta=k}^n t_{\alpha\beta}$ (in terms of the initial presentation) so the relations $[x_{i'}, t_{\alpha\beta}] = 0$ for $i' \neq \alpha, \beta$ and $[x_\alpha + x_\beta, t_{\alpha\beta}] = 0$ imply $[a_i, c_{jk}] = 0$ for $i \le j \le k$. Similarly, one shows $[b_i, c_{jk}] = 0$ for $i \le j \le k$.

Let us prove that the relations (B) for a_i, b_i imply relations (A) for $x_i := a_i - a_{i+1}$, $y_i := b_i - b_{i+1}$ (with the convention $a_{n+1} = b_{n+1} = 0$). As before, $[a_i, a_j] = [b_i, b_j] = 0$, $[a_i, b_j] = [a_j, b_i]$ imply $[x_i, x_j] = [y_i, y_j] = 0$, $[x_i, y_j] = [x_j, y_i]$ (for any i, j). We set $t_{ij} := [x_i, y_j]$ for $i \neq j$, then we have $t_{ij} = t_{ji}$. We have for $j < k$, $t_{jk} = c_{jk} - c_{j,k+1} - c_{j+1,k} + c_{j+1,k+1}$ (we set $c_{i,n+1} := 0$), so $[a_i, c_{jk}] = 0$ implies $[\sum_{i'=i}^n x_{i'}, t_{jk}] = 0$ for $i \le j < k$. When $i < j < k$, the difference between this relation and its analogue of $(i+1, j, k)$ gives $[x_i, t_{jk}] = 0$ for $i < j < k$. This can be rewritten $[x_i, [x_j, y_k]] = 0$ and since $[x_i, x_j] = 0$, we get $[x_j, [x_i, y_k]] = 0$, so $[x_j, t_{ik}] = 0$ and by changing indices, $[x_i, t_{jk}] = 0$ for $j < i < k$. Rewriting again $[x_i, t_{jk}] = 0$ for $i < j < k$ as $[x_i, [y_j, x_k]] = 0$ and using $[x_i, x_k] = 0$, we get $[x_k, [x_i, y_j]] = 0$. i.e., $[x_k, t_{ij}] = 0$, which we rewrite $[x_i, t_{jk}] = 0$ for $j < k < i$. Finally, $[x_i, t_{jk}] = 0$ for $j < k$ and $i \notin \{j, k\}$, which implies $[x_i, t_{jk}] = 0$ for i, j, k different. One proves similarly $[y_i, t_{jk}] = 0$ for i, j, k different. ∎

2.4. **The formality of** $\mathrm{PB}_{1,n}$**.** The flat connection $\mathrm{d}-\sum_{i=1}^n K_i(\mathbf{z}|\tau)\,\mathrm{d}\,z_i$ gives rise to a monodromy representation $\mu_{\mathbf{z}_0,\tau}:\mathrm{PB}_{1,n}=\pi_1(C,\overline{\mathbf{z}}_0)\to\exp(\hat{\mathfrak{t}}_{1,n})$, which factors through a morphism $\mu_{\mathbf{z}_0,\tau}(\mathbb{C}):\mathrm{PB}_{1,n}(\mathbb{C})\to\exp(\hat{\mathfrak{t}}_{1,n})$. Let $\mathrm{Lie}(\mu_{\mathbf{z}_0,\tau}):\mathrm{Lie}(\mathrm{PB}_{1,n})_{\mathbb{C}}\to\hat{\mathfrak{t}}_{1,n}$ be the corresponding morphism between pronilpotent Lie algebras.

Proposition 2.2. $\mathrm{Lie}(\mu_{\mathbf{z}_0,\tau})$ *is an isomorphism of filtered Lie algebras, so that* $\mathrm{PB}_{1,n}$ *is formal.*

Proof. As we have seen, $\mathrm{Lie}(\mathrm{PB}_{1,n})_{\mathbb{C}}$ (denoted $\mathrm{Lie}(\mathrm{PB}_{1,n})$ in this proof) is the quotient of the topologically free Lie algebra generated by α_i,β_i $(i=1,...,n)$ by the topological ideal generated by $[\alpha_i,\alpha_j]$, $[\beta_i,\beta_j]$, $[\alpha_1,\beta_j]$, $[\beta_1,\alpha_j]$, $\log(e^{\beta_k},e^{\alpha_k-\alpha_j})-\log(e^{\beta_k-\beta_j},e^{\alpha_k})$, $[\alpha_i,\gamma_{jk}]$, $[\beta_i,\gamma_{jk}]$ where $\gamma_{jk}=\log(e^{\beta_k},e^{\alpha_k-\alpha_j})$.

This presentation and the above presentation (B) of $\mathfrak{t}_{1,n}$ imply that there is a morphism of graded Lie algebras $p_n:\mathfrak{t}_{1,n}\to\mathrm{gr}\,\mathrm{Lie}(\mathrm{PB}_{1,n})$ defined by $a_i\mapsto[\alpha_i]$, $b_i\mapsto[\beta_i]$, where $\alpha\mapsto[\alpha]$ is the projection map $\mathrm{Lie}(\mathrm{PB}_{1,n})\to\mathrm{gr}_1\,\mathrm{Lie}(\mathrm{PB}_{1,n})$.

p_n is surjective because $\mathrm{gr}\,\mathrm{Lie}\,\Gamma$ is generated in degree 1 (as the associated graded of any quotient of a topologically free Lie algebra).

There is a unique derivation $\tilde\Delta_0\in\mathrm{Der}(\mathfrak{t}_{1,n})$, such that $\tilde\Delta_0(x_i)=y_i$ and $\tilde\Delta_0(y_i)=0$. This derivation gives rise to a one-parameter group of automorphisms of $\mathrm{Der}(\mathfrak{t}_{1,n})$, defined by $\exp(s\tilde\Delta_0)(x_i):=x_i+sy_i$, $\exp(s\tilde\Delta_0)(y_i)=y_i$.

$\mathrm{Lie}(\mu_{\mathbf{z}_0,\tau})$ induces a morphism $\mathrm{gr}\,\mathrm{Lie}(\mu_{\mathbf{z}_0,\tau}):\mathrm{gr}\,\mathrm{Lie}(\mathrm{PB}_{1,n})\to\mathfrak{t}_{1,n}$. We will now prove that

$$\mathrm{gr}\,\mathrm{Lie}(\mu_{\mathbf{z}_0,\tau})\circ p_n=\exp(-\frac{\tau}{2\pi\,\mathrm{i}}\tilde\Delta_0)\circ w,\tag{4}$$

where w is the automorphism of $\mathfrak{t}_{1,n}$ defined by $w(a_i)=-b_i$, $w(b_i)=2\pi\,\mathrm{i}\,a_i$.

$\mu_{\mathbf{z}_0,\tau}$ is defined as follows. Let $F_{\mathbf{z}_0}(\mathbf{z})$ be the solution of $(\partial/\partial z_i)F_{\mathbf{z}_0}(\mathbf{z})=K_i(\mathbf{z}|\tau)F_{\mathbf{z}_0}(\mathbf{z})$, $F_{\mathbf{z}_0}(\mathbf{z}_0)=1$ on U_τ; let $H_\tau:=\{\mathbf{z}=(z_1,...,z_n)|z_i=a_i+\tau b_i,0<a_1<...<a_n<1\}$ and $V_\tau:=\{\mathbf{z}=(z_1,...,z_n)|z_i=a_i+\tau b_i,0<b_1<...<b_n<1\}$; let $F^H_{\mathbf{z}_0}$ and $F^V_{\mathbf{z}_0}$ be the analytic prolongations of $F_{\mathbf{z}_0}$ to H_τ and V_τ; then

$$F^H_{\mathbf{z}_0}(\mathbf{z}+\delta_i)=F^H_{\mathbf{z}_0}(\mathbf{z})\mu_{\mathbf{z}_0,\tau}(X_i),\quad e^{2\pi\,\mathrm{i}\,x_i}F^V_{\mathbf{z}_0}(\mathbf{z}+\tau\delta_i)=F^V_{\mathbf{z}_0}(\mathbf{z})\mu_{\mathbf{z}_0,\tau}(Y_i).$$

We have $\log F_{\mathbf{z}_0}(\mathbf{z})=-\sum_i(z_i-z_i^0)y_i+$ terms of degree ≥ 2, where $\mathfrak{t}_{1,n}$ is graded by $\deg(x_i)=\deg(y_i)=1$, which implies that $\log\mu_{\mathbf{z}_0,\tau}(X_i)=-y_i+$ terms of degree ≥2, $\log\mu_{\mathbf{z}_0,\tau}(Y_i)=2\pi\,\mathrm{i}\,x_i-\tau y_i+$ terms of degree ≥ 2. Therefore $\mathrm{Lie}(\mu_{\mathbf{z}_0,\tau})(\alpha_i)=\log\mu_{\mathbf{z}_0,\tau}(A_i)=-b_i+$ terms of degree ≥2, $\mathrm{Lie}(\mu_{\mathbf{z}_0,\tau})(\beta_i)=\log\mu_{\mathbf{z}_0,\tau}(B_i)=2\pi\,\mathrm{i}\,a_i-\tau b_i+$ terms of degree ≥ 2. So $\mathrm{gr}\,\mathrm{Lie}(\mu_{\mathbf{z}_0,\tau})([\alpha_i])=-b_i$, $\mathrm{gr}\,\mathrm{Lie}(\mu_{\mathbf{z}_0,\tau})([\beta_i])=2\pi\,\mathrm{i}\,a_i-\tau b_i$.

It follows that $\mathrm{gr}\,\mathrm{Lie}(\mu_{\mathbf{z}_0,\tau})\circ p_n$ is the endomorphism $a_i\mapsto -b_i$, $b_i\mapsto 2\pi\,\mathrm{i}\,a_i-\tau b_i$ of $\mathfrak{t}_{1,n}$, which is the automorphism $\exp(-\frac{\tau}{2\pi\,\mathrm{i}}\tilde\Delta_0)\circ w$; this proves (4).

Since we already proved that p_n is surjective, it follows that $\mathrm{gr}\,\mathrm{Lie}(\mu_{\mathbf{z}_0,\tau})$ and p_n are both isomorphisms. As $\mathrm{Lie}(\mathrm{PB}_{1,n})$ and $\hat{\mathfrak{t}}_{1,n}$ are both complete and separated, $\mathrm{Lie}(\mu_{\mathbf{z}_0,\tau})$ is bijective, and since it is a morphism, it is an isomorphism of filtered Lie algebras. □

2.5. **The formality of** $\overline{\mathrm{PB}}_{1,n}$**.** Let $\mathbf{z}_0\in U_\tau$ and $[\mathbf{z}_0]\in\bar C(E_\tau,n)$ be its image. We set $\overline{\mathrm{PB}}_{1,n}:=\pi_1(\bar C(E_\tau,n),[\mathbf{z}_0])$. Then $\overline{\mathrm{PB}}_{1,n}$ is the quotient of $\mathrm{PB}_{1,n}$ by its central subgroup (isomorphic to $\mathbb{Z}^2$) generated by A_1 and B_1. We have $\mu_{\mathbf{z}_0,\tau}(A_1)=e^{-\sum_i y_i}$ and $\mu_{\mathbf{z}_0,\tau}(B_1)=e^{2\pi\,\mathrm{i}\sum_i x_i-\tau\sum_i y_i}$, so $\mathrm{Lie}(\mu_{\mathbf{z}_0,\tau})(\alpha_1)=-a_1$, $\mathrm{Lie}(\mu_{\mathbf{z}_0,\tau})(\beta_1)=2\pi\,\mathrm{i}\,a_1-\tau b_1$, which implies that $\mathrm{Lie}(\mu_{\mathbf{z}_0,\tau})$ induces an isomorphism between $\mathrm{Lie}(\overline{\mathrm{PB}}_{1,n})_{\mathbb{C}}$ and $\bar{\mathfrak{t}}_{1,n}$. In particular, $\overline{\mathrm{PB}}_{1,n}$ is formal.

Remark 2.3. Let $\mathrm{Diag}_n:=\{(\mathbf{z},\tau)\in\mathbb{C}^n\times\mathfrak{H}|\mathbf{z}\in\mathrm{Diag}_{n,\tau}\}$ and let $U\subset(\mathbb{C}^n\times\mathfrak{H})-\mathrm{Diag}_n$ be the set of all $(\mathbf{z},\tau)$ such that $\mathbf{z}\in U_\tau$. Each element of U gives rise to a Lie algebra isomorphism $\mu_{\mathbf{z},\tau}:\mathrm{Lie}(\mathrm{PB}_{1,n})\simeq\hat{\mathfrak{t}}_{1,n}$. For an infinitesimal $(\mathrm{d}\,\mathbf{z},\mathrm{d}\,\tau)$, the composition $\mu_{\mathbf{z}+\mathrm{d}\,\mathbf{z},\tau+\mathrm{d}\,\tau}\circ\mu^{-1}_{\mathbf{z},\tau}$ is then an infinitesimal automorphism of $\hat{\mathfrak{t}}_{1,n}$. This defines a flat connection over U with values in the trivial Lie algebra bundle with Lie algebra $\mathrm{Der}(\hat{\mathfrak{t}}_{1,n})$. When $\mathrm{d}\,\tau = 0$, the infinitesimal automorphism has the form $\exp(\sum_i K_i(\mathbf{z}|\tau)\,\mathrm{d}\,z_i)$, so the connection has the form $\mathrm{d} - \sum_i \mathrm{ad}(K_i(\mathbf{z}|\tau))\,\mathrm{d}\,z_i - \tilde\Delta(\mathbf{z}|\tau)\,\mathrm{d}\,\tau$, where $\tilde\Delta : U \to \mathrm{Der}(\hat{\mathfrak{t}}_{1,n})$ is a meromorphic map with poles at Diag_n. In the next section, we determine a map $\Delta : (\mathbb{C}^n \times \mathfrak{H}) - \mathrm{Diag}_n \to \mathrm{Der}(\hat{\mathfrak{t}}_{1,n})$ with the same flatness properties as $\tilde\Delta(\mathbf{z}|\tau)$.

2.6. **The isomorphisms** $\mathrm{B}_{1,n}(\mathbb{C}) \simeq \exp(\hat{\mathfrak{t}}_{1,n}) \rtimes S_n$, $\overline{\mathrm{B}}_{1,n}(\mathbb{C}) \simeq \exp(\hat{\bar{\mathfrak{t}}}_{1,n}) \rtimes S_n$. Let $\mathbf{z}_0$ be as above; we define $\mathrm{B}_{1,n} := \pi_1(C(E_\tau, [n]), [\mathbf{z}_0])$ and $\overline{\mathrm{B}}_{1,n} := \pi_1(\bar{C}(E_\tau, [n]), [\overline{\mathbf{z}}_0])$, where $x \mapsto [x]$ is the canonical projection $C(E_\tau, n) \to C(E_\tau, [n])$ or $\bar{C}(E_\tau, n) \to \bar{C}(E_\tau, [n])$.

We have an exact sequence $1 \to \mathrm{PB}_{1,n} \to \mathrm{B}_{1,n} \to S_n \to 1$, We then define groups $\mathrm{B}_{1,n}(\mathbb{C})$ fitting in an exact sequence $1 \to \mathrm{PB}_{1,n}(\mathbb{C}) \to \mathrm{B}_{1,n}(\mathbb{C}) \to S_n \to 1$ as follows: the morphism $\mathrm{B}_{1,n} \to \mathrm{Aut}(\mathrm{PB}_{1,n})$ extends to $\mathrm{B}_{1,n} \to \mathrm{Aut}(\mathrm{PB}_{1,n}(\mathbb{C}))$; we then construct the semidirect product $\mathrm{PB}_{1,n}(\mathbb{C}) \rtimes \mathrm{B}_{1,n}$; then $\mathrm{PB}_{1,n}$ embeds diagonally as a normal subgroup of this semidirect product, and $\mathrm{B}_{1,n}(\mathbb{C})$ is defined as the quotient $(\mathrm{PB}_{1,n}(\mathbb{C}) \rtimes \mathrm{B}_{1,n})/\,\mathrm{PB}_{1,n}$.

The monodromy of $\nabla_{\tau,[n]}$ then gives rise to a group morphism $\mathrm{B}_{1,n} \to \exp(\hat{\mathfrak{t}}_{1,n}) \rtimes S_n$, which factors through $\mathrm{B}_{1,n}(\mathbb{C}) \to \exp(\hat{\mathfrak{t}}_{1,n}) \rtimes S_n$. Since this map commutes with the natural morphisms to S_n and using the isomorphism $\mathrm{PB}_{1,n}(\mathbb{C}) \simeq \exp(\hat{\mathfrak{t}}_{1,n})$, we obtain that $\mathrm{B}_{1,n}(\mathbb{C}) \to \exp(\hat{\mathfrak{t}}_{1,n}) \rtimes S_n$ is an isomorphism.

Similarly, starting from the exact sequence $1 \to \overline{\mathrm{PB}}_{1,n} \to \overline{\mathrm{B}}_{1,n} \to S_n \to 1$ one defines a group $\overline{\mathrm{B}}_{1,n}(\mathbb{C})$ fitting in an exact sequence $1 \to \overline{\mathrm{PB}}_{1,n} \to \overline{\mathrm{B}}_{1,n}(\mathbb{C}) \to S_n \to 1$ together with an isomorphism $\overline{\mathrm{B}}_{1,n}(\mathbb{C}) \to \exp(\hat{\bar{\mathfrak{t}}}_{1,n}) \rtimes S_n$.

3. Bundles with flat connection on $\mathcal{M}_{1,n}$ and $\mathcal{M}_{1,[n]}$

We first define Lie algebras of derivations of $\bar{\mathfrak{t}}_{1,n}$ and a related group $\mathbf{G}_n$. We then define a principal $\mathbf{G}_n$-bundle with flat connection of $\mathcal{M}_{1,n}$ and a principal $\mathbf{G}_n \rtimes S_n$-bundle with flat connection on the moduli space $\mathcal{M}_{1,[n]}$ of elliptic curves with n unordered marked points.

3.1. **Derivations of the Lie algebras $\mathfrak{t}_{1,n}$ and $\bar{\mathfrak{t}}_{1,n}$ and associated groups.** Let $\mathfrak{d}$ be the Lie algebra with generators Δ_0, d, X and δ_{2m} $(m \geq 1)$, and relations:

$$[d, X] = 2X, \quad [d, \Delta_0] = -2\Delta_0, \quad [X, \Delta_0] = d,$$

$$[\delta_{2m}, X] = 0, \quad [d, \delta_{2m}] = 2m\delta_{2m}, \quad \mathrm{ad}(\Delta_0)^{2m+1}(\delta_{2m}) = 0.$$

Proposition 3.1. *We have a Lie algebra morphism $\mathfrak{d} \to \mathrm{Der}(\mathfrak{t}_{1,n})$, denoted by $\xi \mapsto \tilde\xi$, such that*

$$\tilde d(x_i) = x_i, \tilde d(y_i) = -y_i, \tilde d(t_{ij}) = 0, \quad \tilde X(x_i) = 0, \tilde X(y_i) = x_i, \tilde X(t_{ij}) = 0,$$

$$\tilde\Delta_0(x_i) = y_i, \tilde\Delta_0(y_i) = 0, \tilde\Delta_0(t_{ij}) = 0,$$

$$\tilde\delta_{2m}(x_i) = 0, \tilde\delta_{2m}(t_{ij}) = [t_{ij}, (\mathrm{ad}\,x_i)^{2m}(t_{ij})], \tilde\delta_{2m}(y_i) = \sum_{j|j\neq i} \frac{1}{2} \sum_{p+q=2m-1} [(\mathrm{ad}\,x_i)^p(t_{ij}), (-\,\mathrm{ad}\,x_i)^q(t_{ij})].$$

This induces a Lie algebra morphism $\mathfrak{d} \to \mathrm{Der}(\bar{\mathfrak{t}}_{1,n})$.

Proof. The fact that $\tilde\Delta_0, \tilde d, \tilde X$ are derivations and commute according to the Lie bracket of $\mathfrak{sl}_2$ is clear.

Let us prove that $\tilde\delta_{2m}$ is a derivation. We have $\tilde\delta_{2m}(t_{ij}) = [t_{ij}, \sum_{i<j}(\mathrm{ad}x_i)^{2m}(t_{ij})]$, which implies that $\tilde\delta_{2m}$ preserves the infinitesimal pure braid identities. It clearly preserves the relations $[x_i, x_j] = 0$, $[x_i, y_j] = t_{ij}$, $[x_k, t_{ij}] = 0$, $[x_i + x_j, t_{ij}] = 0$.

Let us prove that $\tilde{\delta}_{2m}$ preserves the relation $[y_k, t_{ij}] = 0$, i.e., that $[\tilde{\delta}_\varphi(y_k), t_{ij}] + [y_k, \tilde{\delta}_\varphi(t_{ij})] = 0$.

$$[\tilde{\delta}_{2m}(y_k), t_{ij}] = \frac{1}{2} \sum_{p+q=2m-1} (-1)^q [[(\mathrm{ad}x_k)^p(t_{ki}), (\mathrm{ad}x_k)^q(t_{ki})] + [(\mathrm{ad}x_k)^p(t_{kj}), (\mathrm{ad}x_k)^q(t_{kj})], t_{ij}]$$

$$= \frac{1}{2} \sum_{p+q=2m-1} (-1)^{q+1} [[(\mathrm{ad}x_k)^p(t_{ki}), (\mathrm{ad}x_k)^q(t_{kj})] + [(\mathrm{ad}x_k)^p(t_{kj}), (\mathrm{ad}x_k)^q(t_{ki})], t_{ij}]$$

$$= \sum_{p+q=2m-1} (-1)^{q+1} [[(\mathrm{ad}x_k)^p(t_{ki}), (\mathrm{ad}x_k)^q(t_{kj})], t_{ij}] = [t_{ij}, \sum_{p+q=2m-1} (-1)^p (\mathrm{ad}x_i)^p (\mathrm{ad}x_j)^q ([t_{ki}, t_{kj}])].$$

On the other hand, $[y_k, \tilde{\delta}_{2m}(t_{ij})] = [y_k, [t_{ij}, (\mathrm{ad}x_i)^{2m}(t_{ij})]] = [t_{ij}, [y_k, (\mathrm{ad}x_i)^{2m}(t_{ij})]]$. Now

$$[y_k, (\mathrm{ad}x_i)^{2m}(t_{ij})] = -\sum_{\alpha+\beta=2m-1} (\mathrm{ad}x_i)^\alpha \big([t_{ki}, (\mathrm{ad}x_i)^\beta(t_{ij})]\big)$$

$$= -\sum_{\alpha+\beta=2m-1} (\mathrm{ad}x_i)^\alpha [t_{ki}, (-\mathrm{ad}x_j)^\beta(t_{ij})] = -\sum_{\alpha+\beta=2m-1} (\mathrm{ad}x_i)^\alpha (-\mathrm{ad}x_j)^\beta ([t_{ki}, t_{ij}])$$

$$= \sum_{p+q=2m-1} (-1)^{p+1} (\mathrm{ad}x_i)^p (\mathrm{ad}x_j)^q ([t_{ki}, t_{kj}]).$$

Hence $[\tilde{\delta}_{2m}(y_k), t_{ij}] + [y_k, \tilde{\delta}_{2m}(t_{ij})] = 0$.

Let us prove that $\tilde{\delta}_{2m}$ preserves the relation $[y_i, y_j] = 0$, i.e., that $[\tilde{\delta}_{2m}(y_i), y_j] + [y_i, \tilde{\delta}_{2m}(y_j)] = 0$.

We have

$$[y_i, \tilde{\delta}_{2m}(y_j)] = \frac{1}{2}[y_i, \sum_{p+q=2m-1} (-1)^q [(\mathrm{ad}x_j)^p(t_{ji}), (\mathrm{ad}x_j)^q(t_{ji})]]$$
$$+ \frac{1}{2} \sum_{k \neq i,j} [y_i, \sum_{p+q=2m-1} (-1)^q [(\mathrm{ad}x_j)^p(t_{jk}), (\mathrm{ad}x_j)^q(t_{jk})]].$$

Now

$$\frac{1}{2}[y_i, \sum_{p+q=2m-1} (-1)^q [(\mathrm{ad}x_j)^p(t_{ji}), (\mathrm{ad}x_j)^q(t_{ji})]] - (i \leftrightarrow j) \tag{5}$$
$$= -\frac{1}{2}[y_i + y_j, \sum_{p+q=2m-1} (-1)^q [(\mathrm{ad}x_i)^p(t_{ij}), (\mathrm{ad}x_i)^q(t_{ij})]]$$
$$= \sum_{p+q=2m-1} (-1)^{q+1} [[y_i + y_j, (\mathrm{ad}x_i)^p(t_{ij})], (\mathrm{ad}x_i)^q(t_{ij})].$$

A computation similar to the above computation of $[y_k, (\mathrm{ad}x_i)^{2m}(t_{ij})]$ yields

$$[y_i + y_j, (\mathrm{ad}x_i)^p(t_{ij})] = (-1)^p \sum_{\alpha+\beta=p-1} [(\mathrm{ad}x_k)^\alpha(t_{ik}), (\mathrm{ad}x_j)^\beta(t_{jk})],$$

so

$$(5) = \sum_{\alpha+\beta+\gamma=2m-2} [(\mathrm{ad}x_i)^\alpha(t_{ij}), [(\mathrm{ad}x_k)^\beta(t_{ik}), (\mathrm{ad}x_j)^\gamma(t_{jk})]].$$

If now $k \neq i, j$, then

$$[y_i, \frac{1}{2} \sum_{p+q=2m-1} (-1)^q [(\mathrm{ad}x_j)^p(t_{jk}), (\mathrm{ad}x_j)^q(t_{jk})]] = \sum_{p+q=2m-1} (-1)^q [[y_i, (\mathrm{ad}x_j)^p(t_{jk})], (\mathrm{ad}x_j)^q(t_{jk})].$$

As we have seen,

$$[y_j, (\mathrm{ad}x_i)^p(t_{ik})] = (-1)^p \sum_{\alpha+\beta=p-1} (-\mathrm{ad}x_i)^\alpha(\mathrm{ad}x_k)^\beta[t_{ij}, t_{ik}]$$
$$= (-1)^{p+1} \sum_{\alpha+\beta=p-1} [(-\mathrm{ad}x_i)^\alpha(t_{ij}), (\mathrm{ad}x_k)^\beta(t_{jk})]$$

So we get

$$[y_i, \frac{1}{2} \sum_{p+q=2m-1} (-1)^q[(\mathrm{ad}x_j)^p(t_{jk}), (\mathrm{ad}x_j)^q(t_{jk})]]$$
$$= \sum_{\alpha+\beta+\gamma=2m-2} [[(\mathrm{ad}x_i)^\alpha(t_{ij}), (\mathrm{ad}x_k)^\beta(t_{ik})], (\mathrm{ad}x_j)^\gamma(t_{jk})]$$

therefore

$$[y_i, \frac{1}{2} \sum_{p+q=2m-1} (-1)^q[(\mathrm{ad}x_j)^p(t_{jk}), (\mathrm{ad}x_j)^q(t_{jk})]] - (i \leftrightarrow j)$$
$$= \sum_{\alpha+\beta+\gamma=2m-2} [(\mathrm{ad}x_i)^\alpha(t_{ij}), [(\mathrm{ad}x_k)^\beta(t_{ik}), (\mathrm{ad}x_j)^\gamma(t_{jk})]].$$

Therefore $[y_i, \tilde\delta_{2m}(y_j)] + [\tilde\delta_{2m}(y_i), y_j] = 0$.

Since $\tilde\delta_{2m}(\sum_i x_i) = \tilde\delta_{2m}(\sum_i y_i) = 0$ and $\sum_i x_i$ and $\sum_i y_i$ are central, $\tilde\delta_{2m}$ preserves the relations $[\sum_i x_i, y_j] = 0$ and $[\sum_k x_k, t_{ij}] = [\sum_k y_k, t_{ij}] = 0$. It follows that $\tilde\delta_{2m}$ preserves the relations $[x_i + x_j, t_{ij}] = [y_i + y_j, t_{ij}] = 0$ and $[x_i, y_i] = -\sum_{j|j\neq i} t_{ij}$. All this proves that $\tilde\delta_{2m}$ is a derivation.

Let us show that $\mathrm{ad}(\tilde\Delta_0)^{2m+1}(\tilde\delta_{2m}) = 0$ for $m \geq 1$. We have

$$\mathrm{ad}(\tilde\Delta_0)^{2m+1}(\tilde\delta_{2m})(x_i) = -(2m+1)\tilde\Delta_0^{2m} \circ \tilde\delta_{2m} \circ \tilde\Delta_0(x_i) = -(2m+1)\tilde\Delta_0^{2m} \circ \tilde\delta_{2m}(y_i)$$
$$= -(2m+1)\tilde\Delta_0^{2m}(\sum_{j|j\neq i} \frac{1}{2} \sum_{p+q=2m-1} [(\mathrm{ad}\, x_i)^p(t_{ij}), (-\mathrm{ad}\, x_i)^q(t_{ij})]) = 0;$$

the last part of this computation implies that $\mathrm{ad}(\tilde\Delta_0)^{2m+1}(\tilde\delta_{2m})(y_i) = 0$, therefore $\mathrm{ad}(\tilde\Delta_0)^{2m+1}(\tilde\delta_{2m}) = 0$.

We have clearly $[\tilde X, \tilde\delta_{2m}] = 0$ and $[\tilde d, \tilde\delta_{2m}] = 2m\tilde\delta_{2m}$. It follows that we have a Lie algebra morphism $\mathfrak{d} \to \mathrm{Der}(\mathfrak{t}_{1,n})$. Since $\tilde d, \tilde\Delta_0, \tilde X$ and $\tilde\delta_{2m}$ all map $\mathbb{C}(\sum_i x_i) \oplus \mathbb{C}(\sum_i y_i)$ to itself, this induces a Lie algebra morphism $\mathfrak{d} \to \mathrm{Der}(\bar{\mathfrak{t}}_{1,n})$. □

Let e, f, h be the standard basis of $\mathfrak{sl}_2$. Then we have a Lie algebra morphism $\mathfrak{d} \to \mathfrak{sl}_2$, defined by $\delta_{2n} \mapsto 0$, $d \mapsto h$, $X \mapsto e$, $\Delta_0 \mapsto f$. We denote by $\mathfrak{d}_+ \subset \mathfrak{d}$ its kernel.

Since the morphism $\mathfrak{d} \to \mathfrak{sl}_2$ has a section (given by $e, f, h \mapsto X, \Delta_0, d$), we have a semidirect product decomposition $\mathfrak{d} = \mathfrak{d}_+ \rtimes \mathfrak{sl}_2$.

We then have

$$\bar{\mathfrak{t}}_{1,n} \rtimes \mathfrak{d} = (\bar{\mathfrak{t}}_{1,n} \rtimes \mathfrak{d}_+) \rtimes \mathfrak{sl}_2.$$

Lemma 3.2. *$\bar{\mathfrak{t}}_{1,n} \rtimes \mathfrak{d}_+$ is positively graded.*

Proof. We define compatible $\mathbb{Z}^2$-gradings of $\mathfrak{d}$ and $\bar{\mathfrak{t}}_{1,n}$ by $\deg(\Delta_0) = (-1, 1)$, $\deg(d) = (0, 0)$, $\deg(X) = (1, -1)$, $\deg(\delta_{2m}) = (2m+1, 1)$, $\deg(x_i) = (1, 0)$, $\deg(y_i) = (0, 1)$, $\deg(t_{ij}) = (1, 1)$.

We define the support of $\mathfrak{d}$ (resp., $\bar{\mathfrak{t}}_{1,n}$) as the subset of $\mathbb{Z}^2$ of indices for which the corresponding component of $\mathfrak{d}$ (resp., $\bar{\mathfrak{t}}_{1,n}$) is nonzero.

Since the $\bar x_i$ on one hand, the $\bar y_i$ on the other hand generate abelian Lie subalgebras of $\bar{\mathfrak{t}}_{1,n}$, the support of $\bar{\mathfrak{t}}_{1,n}$ is contained in $\mathbb{N}^2_{>0} \cup \{(1, 0), (0, 1)\}$.

On the other hand, $\mathfrak{d}_+$ is generated by the $\mathrm{ad}(\Delta_0)^p(\delta_{2m})$, which all have degrees in $\mathbb{N}^2_{>0}$. It follows that the support of $\mathfrak{d}_+$ is contained in $\mathbb{N}^2_{>0}$.

Therefore the support of $\bar{\mathfrak{t}}_{1,n} \rtimes \mathfrak{d}_+$ is contained in $\mathbb{N}^2_{>0} \cup \{(1,0),(0,1)\}$, so this Lie algebra is positively graded. $\square$

Lemma 3.3. *$\bar{\mathfrak{t}}_{1,n} \rtimes \mathfrak{d}_+$ is a sum of finite dimensional $\mathfrak{sl}_2$-modules; $\mathfrak{d}_+$ is a sum of irreducible odd dimensional $\mathfrak{sl}_2$-modules.*

Proof. A generating space for $\bar{\mathfrak{t}}_{1,n}$ is $\sum_i (\mathbb{C}\bar{x}_i \oplus \mathbb{C}\bar{y}_i)$, which is a sum of finite dimensional $\mathfrak{sl}_2$-modules, so $\bar{\mathfrak{t}}_{1,n}$ is a sum of finite dimensional $\mathfrak{sl}_2$-modules.

A generating space for $\mathfrak{d}_+$ is the sum over $m \geq 1$ of its $\mathfrak{sl}_2$-submodules generated by the δ_{2m}, which are zero or irreducible odd dimensional, therefore $\mathfrak{d}_+$ is a sum of odd dimensional $\mathfrak{sl}_2$-modules. (In fact, the $\mathfrak{sl}_2$-submodule generated by δ_{2m} is nonzero, as it follows from the construction of the above morphism $\mathfrak{d}_+ \to \mathrm{Der}(\bar{\mathfrak{t}}_{1,n})$ that $\delta_{2m} \neq 0$.) $\square$

It follows that $\bar{\mathfrak{t}}_{1,n}$, $\bar{\mathfrak{d}}_+$ and $\bar{\mathfrak{t}}_{1,n} \rtimes \mathfrak{d}_+$ integrate to $\mathrm{SL}_2(\mathbb{C})$-modules (while $\bar{\mathfrak{d}}_+$ even integrates to a $\mathrm{PSL}_2(\mathbb{C})$-module).

We can form in particular the semidirect products

$$\mathbf{G}_n := \exp((\bar{\mathfrak{t}}_{1,n} \rtimes \mathfrak{d}_+)^\wedge) \rtimes \mathrm{SL}_2(\mathbb{C})$$

and $\exp(\hat{\mathfrak{d}}_+) \rtimes \mathrm{PSL}_2(\mathbb{C})$; we have morphisms $\mathbf{G}_n \to \exp(\hat{\mathfrak{d}}_+) \rtimes \mathrm{PSL}_2(\mathbb{C})$ (this is a 2-covering if $n = 1$ since $\bar{\mathfrak{t}}_{1,1} = 0$).

Observe that the action of S_n by automorphisms of $\bar{\mathfrak{t}}_{1,n}$ extends to an action on $\bar{\mathfrak{t}}_{1,n} \rtimes \mathfrak{d}$, where the action on $\mathfrak{d}$ is trivial. This gives rise to an action of S_n by automorphisms of $\mathbf{G}_n$.

3.2. **Bundle with flat connection on $\mathcal{M}_{1,n}$.** The semidirect product $((\mathbb{Z}^n)^2 \times \mathbb{C}) \rtimes \mathrm{SL}_2(\mathbb{Z})$ acts on $(\mathbb{C}^n \times \mathfrak{H}) - \mathrm{Diag}_n$ by $(\mathbf{n}, \mathbf{m}, u) * (\mathbf{z}, \tau) := (\mathbf{n} + \tau\mathbf{m} + u(\sum_i \delta_i), \tau)$ for $(\mathbf{n}, \mathbf{m}, u) \in (\mathbb{Z}^n)^2 \times \mathbb{C}$ and $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) * (\mathbf{z}, \tau) := (\frac{\mathbf{z}}{\gamma\tau+\delta}, \frac{\alpha\tau+\beta}{\gamma\tau+\delta})$ for $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ (here $\mathrm{Diag}_n := \{(\mathbf{z}, \tau) \in \mathbb{C}^n \times \mathfrak{H} |$ for some $i \neq j, z_{ij} \in \Lambda_\tau\}$). The quotient then identifies with the moduli space $\mathcal{M}_{1,n}$ of elliptic curves with n marked points.

Set $\mathbf{G}_n := \exp((\bar{\mathfrak{t}}_{1,n} \rtimes \mathfrak{d}_+)^\wedge) \rtimes \mathrm{SL}_2(\mathbb{C})$. We will define a principal $\mathbf{G}_n$-bundle with flat connection $(\mathcal{P}_n, \nabla_{\mathcal{P}_n})$ over $\mathcal{M}_{1,n}$.

For $u \in \mathbb{C}^\times$, $u^d := \left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C}) \subset \mathbf{G}_n$ and for $v \in \mathbb{C}$, $e^{vX} := \left(\begin{smallmatrix} 1 & v \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C}) \subset \mathbf{G}_n$. Since $[X, \bar{x}_i] = 0$, we consistently set $\exp(aX + \sum_i b_i \bar{x}_i) := \exp(aX)\exp(\sum_i b_i \bar{x}_i)$.

Proposition 3.4. *There exists a unique principal $\mathbf{G}_n$-bundle $\mathcal{P}_n$ over $\mathcal{M}_{1,n}$, such that a section of $U \subset \mathcal{M}_{1,n}$ is a function $f : \pi^{-1}(U) \to \mathbf{G}_n$ (where $\pi : (\mathbb{C}^n \times \mathfrak{H}) - \mathrm{Diag}_n \to \mathcal{M}_{1,n}$ is the canonical projection), such that $f(\mathbf{z} + \delta_i|\tau) = f(\mathbf{z} + u(\sum_i \delta_i)|\tau) = f(\mathbf{z}|\tau)$, $f(\mathbf{z} + \tau\delta_i|\tau) = e^{-2\pi\mathrm{i}\,\bar{x}_i} f(\mathbf{z}|\tau)$, $f(\mathbf{z}|\tau+1) = f(\mathbf{z}|\tau)$ and $f(\frac{\mathbf{z}}{\tau}| - \frac{1}{\tau}) = \tau^d \exp(\frac{2\pi\mathrm{i}}{\tau}(\sum_i z_i\bar{x}_i + X)) f(\mathbf{z}|\tau)$.*

Proof. Let $c_{\tilde{g}} : \mathbb{C}^n \times \mathfrak{H} \to \mathbf{G}_n$ be a family of holomorphic functions (where $\tilde{g} \in ((\mathbb{Z}^n)^2 \times \mathbb{C}) \rtimes \mathrm{SL}_2(\mathbb{Z})$) satisfying the cocycle condition $c_{\tilde{g}\tilde{g}'}(\mathbf{z}|\tau) = c_{\tilde{g}}(\tilde{g}' * (\mathbf{z}|\tau)) c_{\tilde{g}'}(\mathbf{z}|\tau)$. Then there exists a unique principal $\mathbf{G}_n$-bundle over $\mathcal{M}_{1,n}$ such that a section of $U \subset \mathcal{M}_{1,n}$ is a function $f : \pi^{-1}(U) \to \mathbf{G}_n$ such that $f(\tilde{g} * (\mathbf{z}|\tau)) = c_{\tilde{g}}(\mathbf{z}|\tau) f(\mathbf{z}|\tau)$.

We will now prove that there is a unique cocycle such that $c_{(u,0,0)} = c_{(0,\delta_i,0)} = 1$, $c_{(0,0,\delta_i)} = e^{-2\pi\mathrm{i}\,\bar{x}_i}$, $c_S = 1$ and $c_T(\mathbf{z}|\tau) = \tau^d \exp(\frac{2\pi\mathrm{i}}{\tau}(\sum_i z_i\bar{x}_i + X))$, where $S = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Such a cocycle is the same as a family of functions $c_g : \mathbb{C}^n \times \mathfrak{H} \to \mathbf{G}_n$ (where $g \in \mathrm{SL}_2(\mathbb{Z})$), satisfying the cocycle conditions $c_{gg'}(\mathbf{z}|\tau) = c_g(g' * (\mathbf{z}|\tau)) c_{g'}(\mathbf{z}|\tau)$ for $g, g' \in \mathrm{SL}_2(\mathbb{Z})$, and $c_g(\mathbf{z} + \delta_i|\tau) = e^{2\pi\mathrm{i}\,\gamma\bar{x}_i} c_g(\mathbf{z}|\tau)$, $c_g(\mathbf{z} + \tau\delta_i|\tau) = e^{-2\pi\mathrm{i}\,\delta\bar{x}_i} c_g(\mathbf{z}|\tau) e^{2\pi\mathrm{i}\,\bar{x}_i}$ and $c_g(\mathbf{z} + u(\sum_i \delta_i)|\tau) = c_g(\mathbf{z}|\tau)$ for $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

Lemma 3.5. *There exists a unique family of functions $c_g : \mathbb{C}^n \times \mathfrak{H} \to \mathbf{G}_n$ such that $c_{gg'}(\mathbf{z}|\tau) = c_g(g' * (\mathbf{z}|\tau)) c_{g'}(\mathbf{z}|\tau)$ for $g, g' \in \mathrm{SL}_2(\mathbb{Z})$, with*

$$c_S(\mathbf{z}|\tau) = 1, \quad c_T(\mathbf{z}|\tau) = \tau^d e^{(2\pi\mathrm{i}/\tau)(\sum_j z_j\bar{x}_j + X)}.$$

Proof. $\mathrm{SL}_2(\mathbb{Z})$ is the group generated by $\tilde S$, $\tilde T$ and relations $\tilde T^4 = 1$, $(\tilde S\tilde T)^3 = \tilde T^2$, $\tilde S\tilde T^2 = \tilde T^2\tilde S$. Let $\langle \tilde S, \tilde T\rangle$ be the free group with generators $\tilde S, \tilde T$; then there is a unique family of maps $c_{\tilde g} : \mathbb{C}^n \times \mathfrak{H} \to \mathbf{G}_n$, $\tilde g \in \langle \tilde S, \tilde T\rangle$ satisfying the cocycle conditions (w.r.t. the action of $\langle \tilde S, \tilde T\rangle$ on $\mathbb{C}^n \times \mathfrak{H}$ through its quotient $\mathrm{SL}_2(\mathbb{Z})$) and $c_{\tilde S} = c_S$, $c_{\tilde T} = c_T$. It remains to show that $c_{\tilde T^4} = 1$, $c_{(\tilde S\tilde T)^3} = c_{\tilde T^2}$ and $c_{\tilde S\tilde T^2} = c_{\tilde T^2\tilde S}$.

For this, we show that $c_{\tilde T^2}(\mathbf{z}|\tau) = (-1)^d$. We have $c_{\tilde T^2}(\mathbf{z}|\tau) = c_T(\mathbf{z}/\tau| - 1/\tau)c_T(\mathbf{z}|\tau) = (-\tau)^{-d}\exp(-2\pi\,\mathrm{i}\,\tau(\sum_j (z_j/\tau)\bar x_j + X))\tau^d \exp(\frac{2\pi\,\mathrm{i}}{\tau}(\sum_j z_j\bar x_j + X)) = (-1)^d$ since $\tau^d X\tau^{-d} = \tau^2 X$, $\tau^d\bar x_i\tau^{-d} = \tau\bar x_i$.

Since $((-1)^d)^2 = 1^d = 1$, we get $c_{\tilde T^4} = 1$. Since $c_{\tilde S}$ and $c_{\tilde T^2}$ are both constant and commute, we also get $c_{\tilde S\tilde T^2} = c_{\tilde T^2\tilde S}$.

We finally have $c_{\tilde S\tilde T}(\mathbf{z}|\tau) = c_T(\mathbf{z}|\tau)$ while $\tilde S\tilde T = \begin{pmatrix} 1 & -1\\ 1 & 0\end{pmatrix}$, $(\tilde S\tilde T)^2 = \begin{pmatrix} 0 & -1\\ 1 & -1\end{pmatrix}$ so

$$c_{(\tilde S\tilde T)^3}(\mathbf{z}|\tau) = c_T(\frac{\mathbf{z}}{\tau-1}|\frac{1}{1-\tau})c_T(\frac{\mathbf{z}}{\tau}|\frac{\tau-1}{\tau})c_T(\mathbf{z}|\tau) = (\frac{1}{1-\tau})^d\exp(-2\pi\,\mathrm{i}\sum z_j\bar x_j + 2\pi\,\mathrm{i}(1-\tau)X)$$
$$(\frac{\tau-1}{\tau})^d\exp(\frac{2\pi\,\mathrm{i}}{\tau-1}\sum_j z_j\bar x_j + 2\pi\,\mathrm{i}\,\frac{\tau}{\tau-1}X)\tau^d\exp(\frac{2\pi\,\mathrm{i}}{\tau}(\sum_j z_j\bar x_j + X))$$
$$= (-1)^d\exp(\frac{2\pi\,\mathrm{i}}{1-\tau}(\sum_j z_j\bar x_j + X))\exp(\frac{2\pi\,\mathrm{i}}{\tau(\tau-1)}(\sum_j z_j\bar x_j + X))\exp(\frac{2\pi\,\mathrm{i}}{\tau}(\sum_j z_j\bar x_j + X)) = (-1)^d,$$

so $c_{(\tilde S\tilde T)^3} = c_{\tilde T^2}$. □

End of proof of Proposition 3.4. We now check that the maps c_g satisfy the remaining conditions, i.e., $c(\mathbf{z} + u(\sum_i\delta_i)|\tau) = c_g(\mathbf{z}|\tau)$, $c_g(\mathbf{z} + \delta_i|\tau) = e^{2\pi\,\mathrm{i}\,\gamma\bar x_i}c_g(\mathbf{z}|\tau)$, $c_g(\mathbf{z} + \tau\delta_i|\tau) = e^{-2\pi\,\mathrm{i}\,\delta\bar x_i}c_g(\mathbf{z}|\tau)e^{2\pi\,\mathrm{i}\,\bar x_i}$. The cocycle identity $c_{gg'}(\mathbf{z}|\tau) = c_g(g' * (\mathbf{z}|\tau))c_{g'}(\mathbf{z}|\tau)$ implies that it suffices to prove these identities for $g = S$ and $g = T$. They are trivially satisfied if $g = S$. When $g = T$, the first identity follows from $\sum_i \bar x_i = 0$, the third identity follows from the fact that $(X, \bar x_1, ..., \bar x_n)$ is a commutative family, the second identity follows from the same fact together with $\tau^d\bar x_i\tau^{-d} = \tau\bar x_i$. □

Set
$$g(z, x|\tau) := \frac{\theta(z + x|\tau)}{\theta(z|\tau)\theta(x|\tau)}\big(\frac{\theta'}{\theta}(z + x|\tau) - \frac{\theta'}{\theta}(x|\tau)\big) + \frac{1}{x^2} = k_x(z, x|\tau),$$
(we set $f'(z|\tau) := (\partial/\partial z)f(z|\tau)$).

We have $g(z, x|\tau) \in \mathrm{Hol}((\mathbb{C}\times\mathfrak{H}) - \mathrm{Diag}_1)[[x]]$, therefore $g(z, \mathrm{ad}\,\bar x_i|\tau)$ is a linear map $\bar{\mathfrak{t}}_{1,n} \to (\mathrm{Hol}((\mathbb{C}\times\mathfrak{H}) - \mathrm{Diag}_1)\otimes\bar{\mathfrak{t}}_{1,n})^\wedge$, so $g(z, \mathrm{ad}\,\bar x_i|\tau)(\bar t_{ij}) \in (\mathrm{Hol}((\mathbb{C}\times\mathfrak{H}) - \mathrm{Diag}_1)\otimes\bar{\mathfrak{t}}_{1,n})^\wedge$. Therefore
$$g(\mathbf{z}|\tau) := \sum_{i<j} g(z_{ij}, \mathrm{ad}\,\bar x_i|\tau)(\bar t_{ij})$$
is a meromorphic function $\mathbb{C}^n \times \mathfrak{H} \to \hat{\bar{\mathfrak{t}}}_{1,n}$ with only poles at Diag_n.

We set
$$\bar\Delta(\mathbf{z}|\tau) := -\frac{1}{2\pi\,\mathrm{i}}\Delta_0 - \frac{1}{2\pi\,\mathrm{i}}\sum_{n\geq 1} a_{2n}E_{2n+2}(\tau)\delta_{2n} + \frac{1}{2\pi\,\mathrm{i}}g(\mathbf{z}|\tau),$$
where $a_{2n} = -(2n+1)B_{2n+2}(2\,\mathrm{i}\,\pi)^{2n+2}/(2n+2)!$ and B_n are the Bernoulli numbers given by $x/(e^x - 1) = \sum_{r\geq 0}(B_r/r!)x^r$. This is a meromorphic function $\mathbb{C}^n \times \mathfrak{H} \to (\bar{\mathfrak{t}}_{1,n} \rtimes \mathfrak{d}_+)^\wedge \rtimes \mathfrak{n}_+ \subset \mathrm{Lie}(\mathbf{G}_{1,n})$ (where $\mathfrak{n}_+ = \mathbb{C}\Delta_0 \subset \mathfrak{sl}_2$) with only poles at Diag_n.

For $\psi(x) = \sum_{n\geq 1} b_{2n}x^{2n}$, we set $\delta_\psi := \sum_{n\geq 1} b_{2n}\delta_{2n}$, $\Delta_\psi := \Delta_0 + \sum_{n\geq 1} b_{2n}\delta_{2n}$. If we set
$$\varphi(x|\tau) = -x^{-2} - (\theta'/\theta)'(x|\tau) + (x^{-2} + (\theta'/\theta)'(x|\tau))_{|x=0} = g(0, 0|\tau) - g(0, x|\tau),$$
then $\varphi(x|\tau) = \sum_{n\geq 1} a_{2n}E_{2n+2}(\tau)x^{2n}$, so that
$$\bar\Delta(\mathbf{z}|\tau) = -\frac{1}{2\pi\,\mathrm{i}}\Delta_{\varphi(*|\tau)} + \frac{1}{2\pi\,\mathrm{i}}g(\mathbf{z}|\tau).$$

Theorem 3.6. *There is a unique flat connection $\nabla_{\mathcal{P}_n}$ on $\mathcal{P}_n$, whose pull-back to $(\mathbb{C}^n \times \mathfrak{H}) - \mathrm{Diag}_n$ is the connection*

$$\mathrm{d} - \bar{\Delta}(\mathbf{z}|\tau)\,\mathrm{d}\,\tau - \sum_i \bar{K}_i(\mathbf{z}|\tau)\,\mathrm{d}\,z_i$$

on the trivial $\mathbf{G}_n$-bundle.

Proof. We should check that the connection $\mathrm{d} - \bar{\Delta}(\mathbf{z}|\tau)\,\mathrm{d}\,\tau - \sum_i \bar{K}_i(\mathbf{z}|\tau)\,\mathrm{d}\,z_i$ is equivariant and flat, which is expressed as follows (taking into account that we already checked the equivariance and flatness of $\mathrm{d} - \sum_i \bar{K}_i(\mathbf{z}|\tau)\,\mathrm{d}\,z_i$ for any τ):

(equivariance) for $g = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

$$\frac{1}{\gamma\tau+\delta}\bar{K}_i(\frac{\mathbf{z}}{\gamma\tau+\delta}|\frac{\alpha\tau+\beta}{\gamma\tau+\delta}) = \mathrm{Ad}(c_g(\mathbf{z}|\tau))(\bar{K}_i(\mathbf{z}|\tau)) + [(\partial/\partial z_i)c_g(\mathbf{z}|\tau)]c_g(\mathbf{z}|\tau)^{-1}, \tag{6}$$

$$\bar{\Delta}(\mathbf{z}+\delta_i|\tau) = \bar{\Delta}(\mathbf{z}+u(\sum_i \delta_i)|\tau) = \bar{\Delta}(\mathbf{z}|\tau), \quad \bar{\Delta}(\mathbf{z}+\tau\delta_i|\tau) = e^{-2\pi\,\mathrm{i}\,\mathrm{ad}\,x_i}(\bar{\Delta}(\mathbf{z}|\tau) - \bar{K}_i(\mathbf{z}|\tau)), \tag{7}$$

$$\begin{aligned}\frac{1}{(\gamma\tau+\delta)^2}\bar{\Delta}(\frac{\mathbf{z}}{\gamma\tau+\delta}|\frac{\alpha\tau+\beta}{\gamma\tau+\delta}) &= \mathrm{Ad}(c_g(\mathbf{z}|\tau))(\bar{\Delta}(\mathbf{z}|\tau)) + \frac{\gamma}{\gamma z+\delta}\sum_{i=1}^n z_i\,\mathrm{Ad}(c_g(\mathbf{z}|\tau))(\bar{K}_i(\mathbf{z}|\tau))\\ &+ [(\frac{\partial}{\partial\tau} + \frac{\gamma}{\gamma\tau+\delta}\sum_{i=1}^n z_i\frac{\partial}{\partial z_i})c_g(\mathbf{z}|\tau)]c_g(\mathbf{z}|\tau)^{-1},\end{aligned} \tag{8}$$

(flatness) $[\partial/\partial\tau - \bar{\Delta}(\mathbf{z}|\tau), \partial/\partial z_i - \bar{K}_i(\mathbf{z}|\tau)] = 0$.

Let us now check the equivariance identity (6) for $\bar{K}_i(\mathbf{z}|\tau)$. The cocycle identity $c_{gg'}(\mathbf{z}|\tau) = c_g(g' * (\mathbf{z}|\tau))c_{g'}(\mathbf{z}|\tau)$ implies that it suffices to check it when $g = S$ and $g = T$. When $g = S$, this is the identity $\bar{K}_i(\mathbf{z}|\tau+1) = \bar{K}_i(\mathbf{z}|\tau)$, which follows from the identity $\theta(z|\tau+1) = \theta(z|\tau)$. When $g = T$, we have to check the identity

$$\frac{1}{\tau}\bar{K}_i(\frac{\mathbf{z}}{\tau}|-\frac{1}{\tau}) = \mathrm{Ad}(\tau^d e^{\frac{2\pi\,\mathrm{i}}{\tau}(\sum_i z_i\bar{x}_i + X)})(\bar{K}_i(\mathbf{z}|\tau)) + 2\pi\,\mathrm{i}\,\bar{x}_i. \tag{9}$$

We have

$$\begin{aligned}&2\pi\,\mathrm{i}\,\bar{x}_i - \mathrm{Ad}(e^{2\pi\,\mathrm{i}(\sum_i z_i\bar{x}_i + X)})(\bar{y}_i/\tau)\\ &= -\mathrm{Ad}(e^{2\pi\,\mathrm{i}(\sum_i z_i\bar{x}_i)})(\bar{y}_i/\tau) \quad (\text{as } \mathrm{Ad}(e^{2\pi\,\mathrm{i}\,\tau X})(\bar{y}_i/\tau) = \bar{y}_i/\tau + 2\pi\,\mathrm{i}\,\bar{x}_i)\\ &= -\frac{\bar{y}_i}{\tau} - \frac{e^{2\pi\,\mathrm{i}\,\mathrm{ad}(\sum_k z_k\bar{x}_k)} - 1}{\mathrm{ad}(\sum_k z_k\bar{x}_k)}([\sum_j z_j\bar{x}_j, \frac{\bar{y}_i}{\tau}]) = -\frac{\bar{y}_i}{\tau} - \frac{e^{2\pi\,\mathrm{i}\,\mathrm{ad}(\sum_k z_k\bar{x}_k)} - 1}{\mathrm{ad}(\sum_k z_k\bar{x}_k)}(\sum_{j|j\neq i}\frac{z_{ji}}{\tau}\bar{t}_{ij})\\ &= -\frac{\bar{y}_i}{\tau} - \sum_{j|j\neq i}\frac{e^{2\pi\,\mathrm{i}\,\mathrm{ad}(\sum_k z_k\bar{x}_k)} - 1}{\mathrm{ad}(\sum_k z_k\bar{x}_k)}(\frac{z_{ji}}{\tau}\bar{t}_{ij}) = -\frac{\bar{y}_i}{\tau} - \sum_{j|j\neq i}\frac{e^{2\pi\,\mathrm{i}\,\mathrm{ad}(z_{ij}\bar{x}_i)} - 1}{\mathrm{ad}(z_{ij}\bar{x}_i)}(\frac{z_{ji}}{\tau}\bar{t}_{ij})\\ &= -\frac{\bar{y}_i}{\tau} + \sum_{j|j\neq i}\frac{e^{2\pi\,\mathrm{i}\,\mathrm{ad}(z_{ij}\bar{x}_i)} - 1}{\mathrm{ad}(\bar{x}_i)}(\frac{\bar{t}_{ij}}{\tau}),\end{aligned}$$

therefore

$$\frac{1}{\tau}(\sum_j \frac{e^{2\pi\,\mathrm{i}\,z_{ij}\,\mathrm{ad}\,\bar{x}_i} - 1}{\mathrm{ad}\,\bar{x}_i}(\bar{t}_{ij}) - \bar{y}_i) = -\mathrm{Ad}(\tau^d e^{\frac{2\pi\,\mathrm{i}}{\tau}(\sum_i z_i\bar{x}_i + X)})(\bar{y}_i) + 2\pi\,\mathrm{i}\,\bar{x}_i. \tag{10}$$

We have $\theta(z/\tau|-1/\tau) = (1/\tau)e^{(\pi\,\mathrm{i}/\tau)z^2}\theta(z|\tau)$, therefore

$$\frac{1}{\tau}k(\frac{z}{\tau}, x|-\frac{1}{\tau}) = e^{2\pi\,\mathrm{i}\,zx}k(z, \tau x|\tau) + \frac{e^{2\pi\,\mathrm{i}\,zx} - 1}{x\tau}. \tag{11}$$

Substituting $(z,x)=(z_{ij},\mathrm{ad}\,\bar x_i)$ $(j\neq i)$, applying to $\bar t_{ij}$, summing over j and adding up identity (10), we get

$$\frac{1}{\tau}\Big(\sum_{j|j\neq i}k\big(\frac{z_{ij}}{\tau},\mathrm{ad}\,\bar x_i|-\frac{1}{\tau}\big)(\bar t_{ij})-\bar y_i\Big)$$
$$=\sum_{j|j\neq i}e^{2\pi\mathrm{i}\,z_{ij}\,\mathrm{ad}\,\bar x_i}k(z_{ij},\tau\,\mathrm{ad}\,\bar x_i|\tau)(\bar t_{ij})-\mathrm{Ad}(\tau^d e^{\frac{2\pi\mathrm{i}}{\tau}(\sum_i z_i\bar x_i+X)})(\bar y_i)+2\pi\mathrm{i}\,\bar x_i.$$

Since $e^{2\pi\mathrm{i}\,z_{ij}\,\mathrm{ad}\,\bar x_i}k(z_{ij},\tau\,\mathrm{ad}\,\bar x_i|\tau)(\bar t_{ij})=\mathrm{Ad}(\tau^d e^{(2\pi\mathrm{i}/\tau)(\sum_i z_i\bar x_i+X)})\big(k(z_{ij},\mathrm{ad}\,\bar x_i)(\bar t_{ij})\big)$, this implies (9). This ends the proof of (6).

Let us now check the shift identities (7) in $\bar\Delta(\mathbf{z}|\tau)$. The first part is immediate; let us check the last identity. We have $k(z+\tau,x|\tau)=e^{-2\pi\mathrm{i}\,x}g(z,x|\tau)+(e^{-2\pi\mathrm{i}\,x}-1)/x$, therefore $g(z+\tau,x|\tau)=e^{-2\pi\mathrm{i}\,x}g(z,x|\tau)-2\pi\mathrm{i}\,e^{-2\pi\mathrm{i}\,x}k(z,x|\tau)+\frac{1}{x}(\frac{1-e^{-2\pi\mathrm{i}\,x}}{x}-2\pi\mathrm{i}\,e^{-2\pi\mathrm{i}\,x})$. Substituting $(z,x)=(z_{ij},\mathrm{ad}\,\bar x_i)$ $(j\neq i)$, applying to $\bar t_{ij}$, summing up and adding up $\sum_{k,l|k,l\neq j}g(z_{kl},\mathrm{ad}\,\bar x_k|\tau)(\bar t_{kl})$, we get

$$g(\mathbf{z}+\tau\delta_i|\tau)$$
$$=e^{-2\pi\mathrm{i}\,\mathrm{ad}\,\bar x_i}(g(\mathbf{z}|\tau))-2\pi\mathrm{i}\,e^{-2\pi\mathrm{i}\,\mathrm{ad}\,\bar x_i}(\bar K_i(\mathbf{z}|\tau)+\bar y_i)+\sum_{j|j\neq i}\frac{1}{\mathrm{ad}\,\bar x_i}\Big(\frac{1-e^{-2\pi\mathrm{i}\,\mathrm{ad}\,\bar x_i}}{\mathrm{ad}\,\bar x_i}-2\pi\mathrm{i}\,e^{-2\pi\mathrm{i}\,\mathrm{ad}\,\bar x_i}\Big)(\bar t_{ij})$$
$$=e^{-2\pi\mathrm{i}\,\mathrm{ad}\,\bar x_i}(g(\mathbf{z}|\tau))-2\pi\mathrm{i}\,e^{-2\pi\mathrm{i}\,\mathrm{ad}\,\bar x_i}(\bar K_i(\mathbf{z}|\tau)+\bar y_i)-\Big(\frac{1-e^{-2\pi\mathrm{i}\,\mathrm{ad}\,\bar x_i}}{\mathrm{ad}\,\bar x_i}-2\pi\mathrm{i}\,e^{-2\pi\mathrm{i}\,\mathrm{ad}\,\bar x_i}\Big)(\bar y_i)$$
$$=e^{-2\pi\mathrm{i}\,\mathrm{ad}\,\bar x_i}(g(\mathbf{z}|\tau))-2\pi\mathrm{i}\,e^{-2\pi\mathrm{i}\,\mathrm{ad}\,\bar x_i}(\bar K_i(\mathbf{z}|\tau))-\frac{1-e^{-2\pi\mathrm{i}\,\mathrm{ad}\,\bar x_i}}{\mathrm{ad}\,\bar x_i}(\bar y_i);$$

on the other hand, we have $e^{-2\pi\mathrm{i}\,\mathrm{ad}\,\bar x_i}(\Delta_0)=\Delta_0+\frac{1-e^{-2\pi\mathrm{i}\,\mathrm{ad}\,\bar x_i}}{\mathrm{ad}\,\bar x_i}(\bar y_i)$ (as $[\Delta_0,\bar x_i]=\bar y_i$), therefore $g(\mathbf{z}+\delta_i|\tau)-\Delta_0=e^{-2\pi\mathrm{i}\,\mathrm{ad}\,\bar x_i}(g(\mathbf{z}|\tau)-\Delta_0-2\pi\mathrm{i}\,\bar K_i(\mathbf{z}|\tau))$. Since the δ_{2n} commute with $\bar x_i$, we get $\bar\Delta(\mathbf{z}+\tau\delta_i|\tau)=e^{-2\pi\mathrm{i}\,\mathrm{ad}\,\bar x_i}(\bar\Delta(\mathbf{z}|\tau)-\bar K_i(\mathbf{z}|\tau))$, as wanted.

Let us now check the equivariance identities (8) for $\bar\Delta(\mathbf{z}|\tau)$. As above, the cocycle identities imply that it suffices to check (8) for $g=S,T$. When $g=S$, this identity follows from $\sum_i\bar K_i(\mathbf{z}|\tau)=0$. When $g=T$, it is written

$$\frac{1}{\tau^2}\bar\Delta\big(\frac{\mathbf{z}}{\tau}|-\frac{1}{\tau}\big)=\mathrm{Ad}(c_T(\mathbf{z}|\tau))\Big(\bar\Delta(\mathbf{z}|\tau)+\frac{1}{\tau}\sum_i z_i\bar K_i(\mathbf{z}|\tau)\Big)+\frac{d}{\tau}-2\pi\mathrm{i}\,X. \qquad (12)$$

The modularity identity (11) for $k(z,x|\tau)$ implies that

$$\frac{1}{\tau^2}g\big(\frac{z}{\tau},x|-\frac{1}{\tau}\big)=e^{2\pi\mathrm{i}\,zx}g(z,\tau x|\tau)+\frac{2\pi\mathrm{i}\,z}{\tau}e^{2\pi\mathrm{i}\,zx}k(z,\tau x|\tau)+\frac{1-e^{2\pi\mathrm{i}\,zx}}{\tau^2x^2}+\frac{2\pi\mathrm{i}\,z}{\tau^2}\frac{e^{2\pi\mathrm{i}\,zx}}{x}.$$

This implies

$$\frac{1}{\tau^2}\sum_{i<j}g\big(\frac{z_{ij}}{\tau},\mathrm{ad}\,\bar x_i|-\frac{1}{\tau}\big)(\bar t_{ij})=\sum_{i<j}e^{2\pi\mathrm{i}\,z_{ij}\,\mathrm{ad}\,\bar x_i}g(z_{ij},\tau\,\mathrm{ad}\,\bar x_i|\tau)(\bar t_{ij})$$
$$+\sum_{i<j}\frac{2\pi\mathrm{i}}{\tau}z_{ij}e^{2\pi\mathrm{i}\,z_{ij}\,\mathrm{ad}\,\bar x_i}k(z_{ij},\tau\,\mathrm{ad}\,\bar x_i|\tau)(\bar t_{ij})+\sum_{i<j}\Big(\frac{1-e^{2\pi\mathrm{i}\,z_{ij}\,\mathrm{ad}\,\bar x_i}}{\tau^2(\mathrm{ad}\,\bar x_i)^2}+\frac{2\pi\mathrm{i}\,z_{ij}}{\tau^2}\frac{e^{2\pi\mathrm{i}\,z_{ij}\,\mathrm{ad}\,\bar x_i}}{\mathrm{ad}\,\bar x_i}\Big)(\bar t_{ij}).$$

We compute as above

$$\sum_{i<j}e^{2\pi\mathrm{i}\,z_{ij}\,\mathrm{ad}\,\bar x_i}g(z_{ij},\tau\,\mathrm{ad}\,\bar x_i|\tau)(\bar t_{ij})=\mathrm{Ad}(\tau^d e^{\frac{2\pi i}{\tau}(\sum_i z_i\bar x_i+X)})(g(\mathbf{z}|\tau)),$$

$$\sum_{i<j}\frac{2\pi\mathrm{i}}{\tau}z_{ij}e^{2\pi\mathrm{i}\,z_{ij}\,\mathrm{ad}\,\bar x_i}k(z_{ij},\tau\,\mathrm{ad}\,\bar x_i|\tau)(\bar t_{ij})=\sum_i\frac{2\pi\mathrm{i}}{\tau}z_i\Big(\sum_{j|j\neq i}e^{2\pi\mathrm{i}\,z_{ij}\,\mathrm{ad}\,\bar x_i}k(z_{ij},\tau\,\mathrm{ad}\,\bar x_i|\tau)(\bar t_{ij})\Big)$$

(using $k(z,x|\tau)+k(-z,-x|\tau)=0$) and

$$\sum_{i<j} e^{2\pi\mathrm{i}\,z_{ij}\,\mathrm{ad}\,\bar{x}_i}k(z_{ij},\tau\,\mathrm{ad}\,\bar{x}_i|\tau)(\bar{t}_{ij}) = \mathrm{Ad}(\tau^d e^{\frac{2\pi i}{\tau}(\sum_i z_i\bar{x}_i+X)})(\bar{K}_i(\mathbf{z}|\tau)+\bar{y}_i).$$

Therefore

$$\frac{1}{\tau^2}g(\frac{\mathbf{z}}{\tau}|-\frac{1}{\tau}) = \mathrm{Ad}(c_T(\mathbf{z}|\tau))\Big(g(\mathbf{z}|\tau)+\frac{2\pi\mathrm{i}}{\tau}\sum_i z_i\bar{K}_i(\mathbf{z}|\tau)+\frac{2\pi\mathrm{i}}{\tau}\sum_i z_i\bar{y}_i\Big)$$
$$+\sum_{i<j}(\frac{1-e^{2\pi\mathrm{i}\,z_{ij}\,\mathrm{ad}\,\bar{x}_i}}{\tau^2(\mathrm{ad}\,\bar{x}_i)^2}+\frac{2\pi\mathrm{i}\,z_{ij}}{\tau^2}\frac{e^{2\pi\mathrm{i}\,z_{ij}\,\mathrm{ad}\,\bar{x}_i}}{\mathrm{ad}\,\bar{x}_i})(\bar{t}_{ij}),$$

which implies

$$\frac{1}{\tau^2}\bar{\Delta}(\frac{\mathbf{z}}{\tau}|-\frac{1}{\tau}) = \mathrm{Ad}(c_T(\mathbf{z}|\tau))\big(\bar{\Delta}(\mathbf{z}|\tau)+\frac{1}{\tau}\sum_i \bar{K}_i(\mathbf{z}|\tau)\big)$$
$$+\mathrm{Ad}(c_T(\mathbf{z}|\tau))(\frac{1}{\tau}\sum_i z_i\bar{y}_i)+\frac{1}{2\pi\mathrm{i}}\sum_{i<j}(\frac{1-e^{2\pi\mathrm{i}\,z_{ij}\,\mathrm{ad}\,\bar{x}_i}}{\tau^2(\mathrm{ad}\,\bar{x}_i)^2}+\frac{2\pi\mathrm{i}\,z_{ij}}{\tau^2}\frac{e^{2\pi\mathrm{i}\,z_{ij}\,\mathrm{ad}\,\bar{x}_i}}{\mathrm{ad}\,\bar{x}_i})(\bar{t}_{ij})$$
$$+\frac{1}{2\pi\mathrm{i}}\big(\mathrm{Ad}(c_T(\mathbf{z}|\tau))(\Delta_{\varphi(*|\tau)})-\frac{1}{\tau^2}\Delta_{\varphi(*|-1/\tau)}\big).$$

To prove (12), it then suffices to prove

$$\mathrm{Ad}(c_T(\mathbf{z}|\tau))(\frac{1}{\tau}\sum_i z_i\bar{y}_i)+\frac{1}{2\pi\mathrm{i}}\sum_{i<j}(\frac{1-e^{2\pi\mathrm{i}\,z_{ij}\,\mathrm{ad}\,\bar{x}_i}}{\tau^2(\mathrm{ad}\,\bar{x}_i)^2}+\frac{2\pi\mathrm{i}\,z_{ij}}{\tau^2}\frac{e^{2\pi\mathrm{i}\,z_{ij}\,\mathrm{ad}\,\bar{x}_i}}{\mathrm{ad}\,\bar{x}_i})(\bar{t}_{ij})$$
$$+\frac{1}{2\pi\mathrm{i}}\big(\mathrm{Ad}(c_T(\mathbf{z}|\tau))(\Delta_{\varphi(*|\tau)})-\frac{1}{\tau^2}\Delta_{\varphi(*|-1/\tau)}\big)=\frac{d}{\tau}-2\pi\mathrm{i}\,X. \tag{13}$$

We compute

$$\mathrm{Ad}(c_T(\mathbf{z}|\tau))(\frac{1}{\tau}\sum_i z_i\bar{y}_i)=\frac{1}{\tau^2}\sum_i z_i\bar{y}_i+\frac{2\pi\mathrm{i}}{\tau}\sum_i z_i\bar{x}_i+\sum_{i<j}(-\frac{1}{\tau^2})z_{ij}\frac{e^{2\pi\mathrm{i}\,z_{ij}\,\mathrm{ad}\,\bar{x}_i}-1}{\mathrm{ad}\,\bar{x}_i}(\bar{t}_{ij}).$$

We also have $\mathrm{Ad}(c_T(\mathbf{z}|\tau))(E_{2n+2}(\tau)\delta_{2n})=\frac{1}{\tau^2}E_{2n+2}(-\frac{1}{\tau})\delta_{2n}$ since $[\delta_{2n},\bar{x}_i]=[\delta_{2n},X]=0$ and $[d,\delta_{2n}]=2n\delta_{2n}$, and since $E_{2n+2}(-1/\tau)=\tau^{2n+2}E_{2n+2}(\tau)$. This implies

$$\mathrm{Ad}(c_T(\mathbf{z}|\tau))(\delta_{\varphi(*|\tau)})=\delta_{\varphi(*|-1/\tau)}.$$

We now compute $\mathrm{Ad}(c_T(\mathbf{z}|\tau))(\Delta_0)-(1/\tau^2)\Delta_0$. We have $\mathrm{Ad}(c_T(\mathbf{z}|\tau))(\Delta_0)=\mathrm{Ad}(e^{2\pi\mathrm{i}\sum_i z_i\bar{x}_i})\circ\mathrm{Ad}(\tau^d e^{(2\pi\mathrm{i}/\tau)X})(\Delta_0)$, and $\mathrm{Ad}(\tau^d e^{(2\pi\mathrm{i}/\tau)X})(\Delta_0)=(1/\tau^2)\Delta_0+(2\pi\mathrm{i}/\tau)d-(2\pi\mathrm{i})^2X$. Now $\mathrm{Ad}(e^{2\pi\mathrm{i}\sum_i z_i\bar{x}_i})(X)=X$, $\mathrm{Ad}(e^{2\pi\mathrm{i}\sum_i z_i\bar{x}_i})(d)=d-2\pi\mathrm{i}\sum_i z_i\bar{x}_i$. We now compute

$$\mathrm{Ad}(e^{2\pi\mathrm{i}\sum_i z_i\bar{x}_i})(\Delta_0)=\Delta_0+\frac{e^{2\pi\mathrm{i}\sum_i z_i\,\mathrm{ad}\,\bar{x}_i}-1}{2\pi\mathrm{i}\,\mathrm{ad}(\sum_i z_i\bar{x}_i)}([2\pi\mathrm{i}\sum_i z_i\bar{x}_i,\Delta_0])$$
$$=\Delta_0-\frac{e^{2\pi\mathrm{i}\sum_i z_i\,\mathrm{ad}\,\bar{x}_i}-1}{\mathrm{ad}(\sum_i z_i\bar{x}_i)}(\sum_i z_i\bar{y}_i)=\Delta_0-\sum_i\frac{e^{2\pi\mathrm{i}\sum_{j|j\neq i}z_{ji}\,\mathrm{ad}\,\bar{x}_j}-1}{\mathrm{ad}(\sum_{j|j\neq i}z_{ji}\bar{x}_j)}(z_i\bar{y}_i)$$
$$=\Delta_0-\sum_i\Big(2\pi\mathrm{i}\,z_i\bar{y}_i+\frac{1}{\mathrm{ad}(\sum_{j|j\neq i}z_{ji}\bar{x}_j)}(\frac{e^{2\pi\mathrm{i}\sum_{j|j\neq i}z_{ji}\,\mathrm{ad}\,\bar{x}_j}-1}{\mathrm{ad}(\sum_{j|j\neq i}z_{ji}\bar{x}_j)}-2\pi\mathrm{i})([\sum_{j|j\neq i}z_{ji}\bar{x}_j,z_i\bar{y}_i])\Big)$$
$$=\Delta_0-\sum_i 2\pi\mathrm{i}\,z_i\bar{y}_i-\sum_{i\neq j}\Big(\frac{1}{\mathrm{ad}(\bar{x}_j)}(\frac{e^{2\pi\mathrm{i}\,z_{ji}\,\mathrm{ad}\,\bar{x}_j}-1}{\mathrm{ad}(z_{ji}\bar{x}_j)}-2\pi\mathrm{i})(z_i\bar{t}_{ij})\Big);$$

the last sum decomposes as

$$\sum_{i<j}\frac{1}{\mathrm{ad}(\bar x_j)}(\frac{e^{2\pi\,\mathrm{i}\,z_{ji}\,\mathrm{ad}\,\bar x_j}-1}{\mathrm{ad}(z_{ji}\bar x_j)}-2\pi\,\mathrm{i})(z_i\bar t_{ij})+\sum_{i>j}\frac{1}{\mathrm{ad}(\bar x_j)}(\frac{e^{2\pi\,\mathrm{i}\,z_{ji}\,\mathrm{ad}\,\bar x_j}-1}{\mathrm{ad}(z_{ji}\bar x_j)}-2\pi\,\mathrm{i})(z_i\bar t_{ij})$$
$$=\sum_{i<j}\frac{1}{\mathrm{ad}(\bar x_j)}(\frac{e^{2\pi\,\mathrm{i}\,z_{ji}\,\mathrm{ad}\,\bar x_j}-1}{\mathrm{ad}(z_{ji}\bar x_j)}-2\pi\,\mathrm{i})(z_i\bar t_{ij})+\frac{1}{\mathrm{ad}(\bar x_i)}(\frac{e^{2\pi\,\mathrm{i}\,z_{ij}\,\mathrm{ad}\,\bar x_i}-1}{\mathrm{ad}(z_{ij}\bar x_i)}-2\pi\,\mathrm{i})(z_j\bar t_{ij})$$
$$=\sum_{i<j}\frac{1}{\mathrm{ad}(\bar x_i)}(\frac{e^{2\pi\,\mathrm{i}\,z_{ij}\,\mathrm{ad}\,\bar x_i}-1}{\mathrm{ad}(z_{ij}\bar x_i)}-2\pi\,\mathrm{i})(z_{ji}\bar t_{ij}),$$

so

$$\mathrm{Ad}(e^{2\pi\,\mathrm{i}\sum_i z_i\bar x_i})(\Delta_0)=\Delta_0-2\pi\,\mathrm{i}\sum_i z_i\bar y_i-\sum_{i<j}\frac{1}{\mathrm{ad}(\bar x_i)}(\frac{e^{2\pi\,\mathrm{i}\,z_{ij}\,\mathrm{ad}\,\bar x_i}-1}{\mathrm{ad}(z_{ij}\bar x_i)}-2\pi\,\mathrm{i})(z_{ji}\bar t_{ij}),$$

and finally

$$\mathrm{Ad}(c_T(\mathbf{z}|\tau))(\Delta_{\varphi(*|\tau)})-\frac{1}{\tau^2}\Delta_{\varphi(*|-1/\tau)}$$
$$=-\frac{2\pi\,\mathrm{i}}{\tau^2}\sum_i z_i\bar y_i-\frac{1}{\tau^2}\sum_{i<j}\frac{1}{\mathrm{ad}(\bar x_i)}(\frac{e^{2\pi\,\mathrm{i}\,z_{ij}\,\mathrm{ad}\,\bar x_i}-1}{\mathrm{ad}(z_{ij}\bar x_i)}-2\pi\,\mathrm{i})(z_{ji}\bar t_{ij})+\frac{2\pi\,\mathrm{i}}{\tau}(d-2\pi\,\mathrm{i}\sum_i z_i\bar x_i)-(2\pi\,\mathrm{i})^2X,$$

which implies (13). This proves (12) and therefore (8).

We then prove that flatness identity $[\partial/\partial\tau-\bar\Delta(\mathbf{z}|\tau),\partial/\partial z_i-\bar K_i(\mathbf{z}|\tau)]=0$. For this, we will prove that $(\partial/\partial\tau)\bar K_i(\mathbf{z}|\tau)=(\partial/\partial\tau)\bar\Delta(\mathbf{z}|\tau)$, and that $[\bar\Delta(\mathbf{z}|\tau),\bar K_i(\mathbf{z}|\tau)]=0$.

Let us first prove

$$(\partial/\partial\tau)\bar K_i(\mathbf{z}|\tau)=(\partial/\partial z_i)\bar\Delta(\mathbf{z}|\tau). \tag{14}$$

We have $(\partial/\partial\tau)\bar K_i(\mathbf{z}|\tau)=\sum_{j|j\neq i}(\partial_\tau k)(z_{ij},\mathrm{ad}\,\bar x_i|\tau)(\bar t_{ij})$ and $(\partial/\partial z_i)\bar\Delta(\mathbf{z}|\tau)=(2\pi\,\mathrm{i})^{-1}\sum_{j|j\neq i}$ $(\partial_z g)(z_{ij},\mathrm{ad}\,\bar x_i)(\bar t_{ij})$ (where $\partial_\tau:=\partial/\partial\tau$, $\partial_z=\partial/\partial z$) so it suffices to prove the identity $(\partial_\tau k)(z,x|\tau)=(2\pi\,\mathrm{i})^{-1}(\partial_z g)(z,x|\tau)$, i.e., $(\partial_\tau k)(z,x|\tau)=(2\pi\,\mathrm{i})^{-1}(\partial_z\partial_x k)(z,x|\tau)$. In this identity, $k(z,x|\tau)$ may be replaced by $\tilde k(z,x|\tau):=k(z,x|\tau)+1/x=\theta(z+x|\tau)/(\theta(z|\tau)\theta(x|\tau))$. Dividing by $\tilde k(z,x|\tau)$, the wanted identity is rewritten as

$$2\pi\,\mathrm{i}\,\big(\frac{\partial_\tau\theta}{\theta}(z+x|\tau)-\frac{\partial_\tau\theta}{\theta}(z|\tau)-\frac{\partial_\tau\theta}{\theta}(x|\tau)\big)=(\frac{\theta'}{\theta})'(z+x|\tau)+\big(\frac{\theta'}{\theta}(z+x|\tau)-\frac{\theta'}{\theta}(z|\tau)\big)\big(\frac{\theta'}{\theta}(z+x|\tau)-\frac{\theta'}{\theta}(x|\tau)\big)$$

(recall that $f'(z|\tau)=\partial_z f(z|\tau)$), or taking into account the heat equation $4\pi\,\mathrm{i}(\partial_\tau\theta/\theta)(z|\tau)=(\theta''/\theta)(z|\tau)-12\pi\,\mathrm{i}(\partial_\tau\eta/\eta)(\tau)$, as follows

$$\begin{aligned}&2\Big(\frac{\theta'}{\theta}(z|\tau)\frac{\theta'}{\theta}(x|\tau)-\frac{\theta'}{\theta}(x|\tau)\frac{\theta'}{\theta}(z+x|\tau)-\frac{\theta'}{\theta}(z|\tau)\frac{\theta'}{\theta}(z+x|\tau)\Big)\\&+\frac{\theta''}{\theta}(z|\tau)+\frac{\theta''}{\theta}(x|\tau)+\frac{\theta''}{\theta}(z+x|\tau)-12\pi\,\mathrm{i}\,\frac{\partial_\tau\eta}{\eta}(\tau)=0\end{aligned} \tag{15}$$

Let us prove (15). Denote its l.h.s. by $F(z,x|\tau)$. Since $\theta(z|\tau)$ is odd w.r.t. z, $F(z,x|\tau)$ is invariant under the permutation of $z,x,-z-x$. The identities $(\theta'/\theta)(z+\tau|\tau)=(\theta'/\theta)(z|\tau)-2\pi\,\mathrm{i}$ and $(\theta''/\theta)(z+\tau|\tau)=(\theta''/\theta)(z|\tau)-4\pi\,\mathrm{i}(\theta'/\theta)(z|\tau)+(2\pi\,\mathrm{i})^2$ imply that $F(z,x|\tau)$ is elliptic in z,x (w.r.t. the lattice Λ_τ). The possible poles of $F(z,x|\tau)$ as a function of z are simple at $z=0$ and $z=-x$ (mod Λ_τ), but one checks that $F(z,x|\tau)$ is regular at these points, so it is constant in z. By the $\mathfrak{S}_3$-symmetry, it is also constant in x, hence it is a function of τ only: $F(z,x|\tau)=F(\tau)$.

To compute this function, we compute $F(z,0|\tau)=[-2(\theta'/\theta)'-2(\theta'/\theta)^2+2\theta''/\theta](z|\tau)+(\theta''/\theta)(0|\tau)-12\pi\,\mathrm{i}(\partial_\tau\eta/\theta)(\tau)$, hence $F(\tau)=(\theta''/\theta)(0|\tau)-12\pi\,\mathrm{i}(\partial_\tau\eta/\eta)(\tau)$. The above heat equation then implies that $F(\tau)=4\pi\,\mathrm{i}(\partial_\tau\theta/\theta)(0|\tau)$. Now $\theta'(0|\tau)=1$ implies that $\theta(z|\tau)$ has the expansion $\theta(z|\tau)=z+\sum_{n\geq 2}a_n(\tau)z^n$ as $z\to 0$, which implies $(\partial_\tau\theta/\theta)(0|\tau)=0$. So $F(\tau)=0$, which implies (15) and therefore (14).

We now prove

$$[\bar\Delta(\mathbf{z}|\tau),\bar K_i(\mathbf{z}|\tau)]=0. \tag{16}$$

Since τ is constant in what follows, we will write $k(z,x)$, $g(z,x)$, φ instead of $k(z,x|\tau)$, $g(z,x|\tau)$, $\varphi(*|\tau)$. For $i\neq j$, let us set $g_{ij}:=g(z_{ij},\operatorname{ad}\bar x_i)(\bar t_{ij})$. Since $g(z,x|\tau)=g(-z,-x|\tau)$, we have $g_{ij}=g_{ji}$. Recall that $\bar K_{ij}=k(z_{ij},\operatorname{ad}\bar x_i)(t_{ij})$.

We have

$$\begin{aligned}2\pi\,\mathrm{i}[\bar\Delta(\mathbf{z}|\tau),\bar K_i(\mathbf{z}|\tau)]&=[-\Delta_\varphi+\sum_{i,j|i<j}g_{ij},-\bar y_i+\sum_{j|j\neq i}\bar K_{ij}] \qquad (17)\\ &=[\Delta_\varphi,\bar y_i]+\sum_{j|j\neq i}\Big(-[\Delta_\varphi,\bar K_{ij}]+[\bar y_i,g_{ij}]+[g_{ij},\bar K_{ij}]\Big)\\ &+\sum_{j,k|j\neq i,k\neq i,j<k}\big([\bar y_i,g_{jk}]+[g_{ik}+g_{jk},\bar K_{ij}]+[g_{ij}+g_{jk},\bar K_{ik}]\big).\end{aligned}$$

One computes

$$[\Delta_\varphi,\bar y_i]=\sum_\alpha[f_\alpha(\operatorname{ad}\bar x_i)(\bar t_{ij}),g_\alpha(-\operatorname{ad}\bar x_i)(\bar t_{ij})],\quad\text{where }\sum_\alpha f_\alpha(u)g_\alpha(v)=\frac12\frac{\varphi(u)-\varphi(v)}{u-v}. \tag{18}$$

If $f(x)\in\mathbb{C}[[x]]$, then

$$\begin{aligned}&[\Delta_0,f(\operatorname{ad}\bar x_i)(\bar t_{ij})]-[\bar y_i,f'(\operatorname{ad}\bar x_i)(\bar t_{ij})]=\sum_\alpha[h_\alpha(\operatorname{ad}\bar x_i)(\bar t_{ij}),k_\alpha(\operatorname{ad}\bar x_i)(\bar t_{ij})]\\ &+\sum_{k|k\neq i,j}\frac{f(\operatorname{ad}\bar x_i)-f(-\operatorname{ad}\bar x_j)-f'(-\operatorname{ad}\bar x_j)(\operatorname{ad}\bar x_i+\operatorname{ad}\bar x_j)}{(\operatorname{ad}\bar x_i+\operatorname{ad}\bar x_j)^2}([\bar t_{ij},\bar t_{ik}]),\end{aligned}$$

where

$$\sum_\alpha h_\alpha(u)k_\alpha(v)=\frac12\Big(\frac1{v^2}\big(f(u+v)-f(u)-vf'(u)\big)-\frac1{u^2}\big(f(u+v)-f(v)-uf'(v)\big)\Big).$$

Since $g(z,x)=k_x(z,x)$, we get

$$\begin{aligned}&-[\Delta_0,\bar K_{ij}]+[\bar y_i,g_{ij}]=-\sum_\alpha[f^{ij}_\alpha(\operatorname{ad}\bar x_i)(\bar t_{ij}),g^{ij}_\alpha(\operatorname{ad}\bar x_i)(\bar t_{ij})] \qquad (19)\\ &+\sum_{k|k\neq i,j}\frac{k(z_{ij},\operatorname{ad}\bar x_i)-k(z_{ij},-\operatorname{ad}\bar x_j)-(\operatorname{ad}\bar x_i+\operatorname{ad}\bar x_j)k_x(z_{ij},-\operatorname{ad}\bar x_j)}{(\operatorname{ad}\bar x_i+\operatorname{ad}\bar x_j)^2}([\bar t_{ij},\bar t_{jk}]),\end{aligned}$$

where

$$\sum_\alpha f^{ij}_\alpha(u)g^{ij}_\alpha(v)=\frac12\Big(\frac1{v^2}\big(k(z_{ij},u+v)-k(z_{ij},u)-vk_x(z_{ij},u)\big)-\frac1{u^2}\big(k(z_{ij},u+v)-k(z_{ij},v)-uk_x(z_{ij},v)\big)\Big).$$

For $f(x)\in\mathbb{C}[[x]]$, we have

$$[\delta_\varphi,f(\operatorname{ad}\bar x_i)(\bar t_{ij})]=\sum_\alpha[l_\alpha(\operatorname{ad}\bar x_i)(\bar t_{ij}),m_\alpha(\operatorname{ad}\bar x_i)(\bar t_{ij})],\quad\text{where}\sum_\alpha l_\alpha(u)m_\alpha(v)=f(u+v)\varphi(v),$$

therefore

$$-[\delta_\varphi,\bar K_{ij}]=-\sum_\alpha[l^{ij}_\alpha(\operatorname{ad}\bar x_i)(\bar t_{ij}),m^{ij}_\alpha(\operatorname{ad}\bar x_i)(\bar t_{ij})],\quad\text{where }\sum_\alpha l^{ij}_\alpha(u)m^{ij}_\alpha(v)=k(z_{ij},u+v)\varphi(v). \tag{20}$$

For $j,k\neq i$ and $j<k$, we have

$$[\bar y_i,g_{jk}]+[g_{ik}+g_{jk},\bar K_{ij}]+[g_{ij}+g_{jk},\bar K_{ik}]=[\bar y_i,g_{jk}]-[g_{ki},\bar K_{ji}]-[g_{ji},\bar K_{ki}]+[g_{jk},\bar K_{ij}]+[g_{jk},\bar K_{ik}],$$

and since for any $f(x) \in \mathbb{C}[[x]]$, $[\bar y_i, f(\mathrm{ad}\,\bar x_i)(\bar t_{jk})] = -\frac{f(\mathrm{ad}\,\bar x_j)-f(-\mathrm{ad}\,\bar x_k)}{\mathrm{ad}\,\bar x_j+\mathrm{ad}\,\bar x_k}([\bar t_{ij},\bar t_{jk}])$, we get

$$[\bar y_i, g_{jk}] + [g_{ik}+g_{jk},\bar K_{ij}] + [g_{ij}+g_{jk},\bar K_{ik}] \tag{21}$$
$$= \Big(-\frac{g(z_{jk},\mathrm{ad}\,\bar x_j)-g(z_{jk},-\mathrm{ad}\,\bar x_k)}{\mathrm{ad}\,\bar x_j+\mathrm{ad}\,\bar x_k} - g(z_{ki},\mathrm{ad}\,\bar x_k)k(z_{ji},\mathrm{ad}\,\bar x_j) + g(z_{ji},\mathrm{ad}\,\bar x_j)k(z_{ki},\mathrm{ad}\,\bar x_k)$$
$$- g(z_{kj},\mathrm{ad}\,\bar x_k)k(z_{ij},\mathrm{ad}\,\bar x_i) + g(z_{jk},\mathrm{ad}\,\bar x_j)k(z_{ik},\mathrm{ad}\,\bar x_i)\Big)([\bar t_{ij},\bar t_{jk}]).$$

Summing up (18), (19), (20) and (21), (17) gives

$$2\pi\,\mathrm{i}[\bar\Delta(\mathbf{z}|\tau),\bar K_i(\mathbf{z}|\tau)]$$
$$= \sum_{j|j\neq i}\sum_\alpha [F_\alpha^{ij}(\mathrm{ad}\,\bar x_i)(\bar t_{ij}), G_\alpha^{ij}(\mathrm{ad}\,\bar x_i)(\bar t_{ij})] + \sum_{j,k|j\neq i,k\neq i} H(z_{ij},z_{ik},-\mathrm{ad}\,\bar x_j,-\mathrm{ad}\,\bar x_k)([t_{ij},t_{jk}]),$$

where $\sum_\alpha F_\alpha^{ij}(u)G_\alpha^{ij}(v) = L(z_{ij},u,v)$,

$$L(z,u,v) = \frac12\frac{\varphi(u)-\varphi(v)}{u+v} + \frac12 k(z,u+v)(\varphi(u)-\varphi(v)) + \frac12(g(z,u)k(z,v)-k(z,u)g(z,v))$$
$$-\frac12\Big(\frac{1}{v^2}\big(k(z,u+v)-k(z,u)-vk_x(z,u)\big) - \frac{1}{u^2}\big(k(z,u+v)-k(z,v)-uk_x(z,v)\big)\Big)$$

and

$$H(z,z',u,v) = \frac{1}{v^2}\big(k(z,u+v)-k(z,u)-vk_x(z,u)\big) - \frac{1}{u^2}\big(k(z',u+v)-k(z',v)-uk_x(z',v)\big)$$
$$+\frac{1}{u+v}\big(g(z'-z,-u)-g(z'-z,v)\big) - g(-z',-v)k(-z,-u) + g(-z,-u)k(-z',-v)$$
$$-g(z-z',-v)k(z,u+v) + g(z'-z,-u)k(z',u+v).$$

Explicit computation shows that $H(z,z',u,v)=0$, which implies that $L(z,u,v)=0$ since $L(z,u,v) = -\frac12 H(z,z,u,v)$. This proves (16). □

Remark 3.7. Define $\Delta(\mathbf{z}|\tau)$ by the same formula as $\bar\Delta(\mathbf{z}|\tau)$, replacing $\bar x_i, \bar y_i$ by x_i, y_i. Then $\mathrm{d} - \Delta(\mathbf{z}|\tau)\,\mathrm{d}\,\tau - \sum_i K_i(\mathbf{z}|\tau)\,\mathrm{d}\,z_i$ is flat. This can be interpreted as follows.

Let $N_+ \subset \mathrm{SL}_2(\mathbb{C})$ be the connected subgroup with Lie algebra $\mathbb{C}\Delta_0$. Set $\tilde{\mathbf{N}}_n := \exp((\mathfrak{t}_{1,n}\rtimes\mathfrak{d}_+)^\wedge)\rtimes N_+$, $\mathbf{N}_n := \exp((\bar{\mathfrak{t}}_{1,n}\rtimes\mathfrak{d}_+)^\wedge)\rtimes N_+$ and $\tilde{\mathbf{G}}_n := \exp((\mathfrak{t}_{1,n}\rtimes\mathfrak{d}_+)^\wedge)\rtimes\mathrm{SL}_2(\mathbb{C})$. Then we have a diagram of groups

$$\begin{array}{ccc} \tilde{\mathbf{N}}_n & \to & \mathbf{N}_n \\ \downarrow & & \downarrow \\ \tilde{\mathbf{G}}_n & \to & \mathbf{G}_n \end{array}$$

The trivial $\mathbf{N}_n$-bundle on $(\mathfrak{H}\times\mathbb{C}^n)-\mathrm{Diag}_n$ with flat connection $\mathrm{d}-\bar\Delta(\mathbf{z}|\tau)\,\mathrm{d}\,\tau - \sum_i \bar K_i(\mathbf{z}|\tau)\,\mathrm{d}\,z_i$ admits a reduction to $\tilde{\mathbf{N}}_n$, where the bundle is again trivial and the connection is $\mathrm{d}-\Delta(\mathbf{z}|\tau)\,\mathrm{d}\,\tau - \sum_i K_i(\mathbf{z}|\tau)\,\mathrm{d}\,z_i$.

$((\mathbb{Z}^2)^2\times\mathbb{C})\rtimes\mathrm{SL}_2(\mathbb{Z})$ contains the subgroups $(\mathbb{Z}^n)^2$, $(\mathbb{Z}^n)^2\times\mathbb{C}$, $(\mathbb{Z}^n)^2\rtimes\mathrm{SL}_2(\mathbb{Z})$. We denote the corresponding quotients of $(\mathbb{C}^n\times\mathfrak{H})-\mathrm{Diag}_n$ by $C(n)$, $\bar C(n)$, $\tilde{\mathcal{M}}_{1,n}$. These fit in the diagram

$$\begin{array}{ccc} \tilde C(n) & \to & C(n) \\ \downarrow & & \downarrow \\ \tilde{\mathcal{M}}_{1,n} & \to & \mathcal{M}_{1,n} \end{array}$$

The pair $(\mathcal{P}_n,\nabla_{\mathcal{P}_n})$ can be pulled back to $\mathbf{G}_n$-bundles over these covers of $\mathcal{M}_{1,n}$. These pull-backs admit G-structures, where G is the corresponding group in the above diagram of groups.

We have natural projections $C(n)\to\mathfrak{H}$, $\bar C(n)\to\mathfrak{H}$. The fibers of $\tau\in\mathfrak{H}$ are respectively $C(E_\tau,n)$ and $\bar C(E_\tau,n)$. The pair $(\mathcal{P}_n,\nabla_n)$ can be pulled back to $C(E_\tau,n)$ and $\bar C(E_\tau,n)$;

these pull-backs admit G-structures, where $G = \exp(\mathfrak{t}_{1,n})$ and $\exp(\bar{\mathfrak{t}}_{1,n})$, which coincide with $(P_{n,\tau}, \nabla_{n,\tau})$ and $(\bar{P}_{n,\tau}, \bar{\nabla}_{n,\tau})$.

3.3. **Bundle with flat connection over $\mathcal{M}_{1,[n]}$.** The semidirect product $((\mathbb{Z}^n)^2 \times \mathbb{C}) \rtimes (\mathrm{SL}_2(\mathbb{C}) \times S_n)$ acts on $(\mathbb{C}^n \times \mathfrak{H}) - \mathrm{Diag}_n$ as follows: the action of $((\mathbb{Z}^n)^2 \times \mathbb{C}) \rtimes \mathrm{SL}_2(\mathbb{C})$ is as above and the action of S_n is $\sigma * (z_1, ..., z_n, \tau) := (z_{\sigma^{-1}(1)}, ..., z_{\sigma^{-1}(n)}, \tau)$. The quotient then identifies with $\mathcal{M}_{1,[n]}$.

We will define a principal $\mathbf{G}_n \rtimes S_n$-bundle with a flat connection $(\mathcal{P}_{[n]}, \nabla_{\mathcal{P}_{[n]}})$ over $\mathcal{M}_{1,[n]}$.

Proposition 3.8. *There exists a unique principal $\mathbf{G}_n \rtimes S_n$-bundle $\mathcal{P}_{[n]}$ over $\mathcal{M}_{1,[n]}$, such that a section of $U \subset \mathcal{M}_{1,[n]}$ is a function $f : \tilde{\pi}^{-1}(U) \to \mathbf{G}_n \rtimes S_n$, satisfying the conditions of Proposition 3.4 as well as $f(\sigma\mathbf{z}|\tau) = \sigma f(\mathbf{z}|\tau)$ for $\sigma \in S_n$ (here $\tilde{\pi} : (\mathbb{C}^n \times \mathfrak{H}) - \mathrm{Diag}_n \to \mathcal{M}_{1,[n]}$ is the canonical projection).*

Proof. One checks that $\sigma c_{\tilde{g}}(\mathbf{z}|\tau)\sigma^{-1} = c_{\sigma\tilde{g}\sigma^{-1}}(\sigma^{-1}\mathbf{z})$, where $\tilde{g} \in ((\mathbb{Z}^n)^2 \times \mathbb{C}) \rtimes \mathrm{SL}_2(\mathbb{Z})$, $\sigma \in S_n$. It follows that there is a unique cocycle $c_{(\tilde{g},\sigma)} : \mathbb{C}^n \times \mathfrak{H} \to \mathbf{G}_n \rtimes S_n$ such that $c_{(\tilde{g},1)} = c_{\tilde{g}}$ and $c_{(1,\sigma)}(\mathbf{z}|\tau) = \sigma$. □

Theorem 3.9. *There is a unique flat connection $\nabla_{\mathcal{P}_{[n]}}$ on $\mathcal{P}_{[n]}$, whose pull-back to $(\mathbb{C}^n \times \mathfrak{H}) - \mathrm{Diag}_n$ is the connection $\mathrm{d} - \bar{\Delta}(\mathbf{z}|\tau)\,\mathrm{d}\,\tau - \sum_i \bar{K}_i(\mathbf{z}|\tau)\,\mathrm{d}\,z_i$ on the trivial $\mathbf{G}_n \rtimes S_n$-bundle.*

Proof. Taking into account Theorem 3.6, it remains to show that this connection is S_n-equivariant. We have already mentioned that $\sum_i \bar{K}_i(\mathbf{z}|\tau)\,\mathrm{d}\,z_i$ is equivariant; $\bar{\Delta}(\mathbf{z}|\tau)$ is also checked to be equivariant. □

4. The monodromy morphisms $\Gamma_{1,[n]} \to \mathbf{G}_n \rtimes S_n$

Let $\Gamma_{1,[n]}$ be the mapping class group of genus 1 surfaces with n unordered marked points. It can be viewed as the fundamental group $\pi_1(\mathcal{M}_{1,[n]}, *)$, where $*$ is a base point at infinity which will be specified later. The flat connection on $\mathcal{M}_{1,[n]}$ introduced above gives rise to morphisms $\gamma_n : \Gamma_{1,[n]} \to \mathbf{G}_n \rtimes S_n$, which we now study. This study in divided in two parts: in the first, analytic part, we show that γ_n can be obtained from γ_1 and γ_2, and show that the restriction of γ_n to $\overline{\mathrm{B}}_{1,n}$ can be expressed in terms of the KZ associator only. In the second part, we show that morphisms $\overline{\mathrm{B}}_{1,n} \to \exp(\hat{\bar{\mathfrak{t}}}_{1,n}) \rtimes S_n$ can be constructed algebraically using an arbitrary associator. Finally, we introduce the notion of an elliptic structure over a quasi-bialgebra.

4.1. **The solution $F^{(n)}(\mathbf{z}|\tau)$.** The elliptic KZB system is now

$$(\partial/\partial z_i)F(\mathbf{z}|\tau) = \bar{K}_i(\mathbf{z}|\tau)F(\mathbf{z}|\tau), \quad (\partial/\partial\tau)F(\mathbf{z}|\tau) = \bar{\Delta}(\mathbf{z}|\tau)F(\mathbf{z}|\tau),$$

where $F(\mathbf{z}|\tau)$ is a function $(\mathbb{C}^n \times \mathfrak{H}) - \mathrm{Diag}_n \supset U \to \mathbf{G}_n \rtimes S_n$ invariant under translation by $\mathbb{C}(\sum_i \delta_i)$. Let $D_n := \{(\mathbf{z}, \tau) \in \mathbb{C}^n \times \mathfrak{H} | z_i = a_i + b_i\tau, a_i, b_i \in \mathbb{R}, a_1 < a_2 < ... < a_n < a_1 + 1, b_1 < b_2 < ... < b_n < b_1 + 1\}$. Then $D_n \subset (\mathbb{C}^n \times \mathfrak{H}) - \mathrm{Diag}_n$ is simply connected and invariant under $\mathbb{C}(\sum_i \delta_i)$. A solution of the elliptic KZB system on this domain is then unique, up to right multiplication by a constant. We now determine a particular solution $F^{(n)}(\mathbf{z}|\tau)$.

Let us study the elliptic KZB system in the region $z_{ij} \ll 1$, $\tau \to \mathrm{i}\,\infty$. Then $\bar{K}_i(\mathbf{z}|\tau) = \sum_{j|j\neq i} \bar{t}_{ij}/(z_i - z_j) + O(1)$.

We now compute the expansion of $\bar{\Delta}(\mathbf{z}|\tau)$. The heat equation for ϑ implies the expansion $\vartheta(x|\tau) = \eta(\tau)^3\big(x + 2\pi\,\mathrm{i}\,\partial_\tau \log\eta(\tau)x^3 + O(x^5)\big)$, so $\theta(x|\tau) = x + 2\pi\,\mathrm{i}\,\partial_\tau \log\eta(\tau)x^3 + O(x^5)$, hence

$$g(0, x|\tau) = (\frac{\theta'}{\theta})'(x|\tau) + \frac{1}{x^2} = 4\pi\,\mathrm{i}\,\partial_\tau \log\eta(\tau) + O(x) = -(\pi^2/3)E_2(\tau) + O(x)$$

since $E_2(\tau) = \frac{24}{2\pi\,\mathrm{i}}\partial_\tau \log\eta(\tau)$. We have $g(0,x|\tau) = g(0,0|\tau) - \varphi(x|\tau)$, so

$$g(0,x|\tau) = -\sum_{k\geq 0} a_{2k} x^{2k} E_{2k+2}(\tau),$$

where $a_0 = \pi^2/3$. Then

$$\bar\Delta(\mathbf{z}|\tau) = -\frac{1}{2\pi\,\mathrm{i}}\Big(\Delta_0 + \sum_{k\geq 0} a_{2k}E_{2k+2}(\tau)\big(\delta_{2k} + \sum_{i,j|i<j}(\mathrm{ad}\,\bar x_i)^{2k}(\bar t_{ij})\big)\Big) + o(1)$$

for $z_{ij} \ll 1$ and any $\tau \in \mathfrak{H}$. Since we have an expansion $E_{2k}(\tau) = 1 + \sum_{l>0} a_{kl}e^{2\pi\,\mathrm{i}\,l\tau}$ as $\tau \to \mathrm{i}\infty$, and using Proposition A.3 with $u_n = z_{n1}$, $u_{n-1} = z_{n-1,1}/z_{n1}$,..., $u_2 = z_{21}/z_{31}$, $u_1 = q = e^{2\pi\,\mathrm{i}\,\tau}$, there is a unique solution $F^{(n)}(\mathbf{z}|\tau)$ with the expansion

$$F^{(n)}(\mathbf{z}|\tau) \simeq z_{21}^{\bar t_{12}} z_{31}^{\bar t_{13}+\bar t_{23}} ... z_{n1}^{\bar t_{1n}+...+\bar t_{n-1,n}} \exp\Big(-\frac{\tau}{2\pi\,\mathrm{i}}\Big(\Delta_0 + \sum_{k\geq 0} a_{2k}\big(\delta_{2k} + \sum_{i<j}(\mathrm{ad}\,\bar x_i)^{2k}(\bar t_{ij})\big)\Big)\Big)$$

in the region $z_{21} \ll z_{31} \ll ... \ll z_{n1} \ll 1$, $\tau \to \mathrm{i}\infty$, $(\mathbf{z},\tau) \in D_n$ (here $z_{ij} = z_i - z_j$); here the sign $\simeq$ means that any of the ratios of both sides has the form $1+\sum_{k>0}\sum_{i,a_1,...,a_n} r_k^{i,a_1,...,a_n}(u_1,...,u_n)$, where the second sum is finite with $a_i \geq 0$, $i \in \{1,...,n\}$, $r_k^{i,a_1,...,a_n}(u_1,...,u_n)$ has degree k, and is $O(u_i(\log u_1)^{a_1}...(\log u_n)^{a_n})$.

4.2. **Presentation of $\Gamma_{1,[n]}$.** According to [Bi2], $\Gamma_{1,[n]} = \{\overline{\mathrm{B}}_{1,n} \rtimes \widetilde{\mathrm{SL}_2(\mathbb{Z})}\}/\mathbb{Z}$, where $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ is a central extension $1 \to \mathbb{Z} \to \widetilde{\mathrm{SL}_2(\mathbb{Z})} \to \mathrm{SL}_2(\mathbb{Z}) \to 1$; the action $\alpha : \widetilde{\mathrm{SL}_2(\mathbb{Z})} \to \mathrm{Aut}(\overline{\mathrm{B}}_{1,n})$ is such that for Z the central element $1 \in \mathbb{Z} \subset \widetilde{\mathrm{SL}_2(\mathbb{Z})}$, $\alpha_Z(x) = Z'x(Z')^{-1}$, where Z' is the image of a generator of the center of PB_n (the pure braid group of n points on the plane) under the natural morphism $\mathrm{PB}_n \to \overline{\mathrm{B}}_{1,n}$; $\overline{\mathrm{B}}_{1,n} \rtimes \widetilde{\mathrm{SL}_2(\mathbb{Z})}$ is then $\overline{\mathrm{B}}_{1,n} \times \widetilde{\mathrm{SL}_2(\mathbb{Z})}$ with the product $(p,A)(p',A') = (p\alpha_A(p'), AA')$; this semidirect product is then factored by its central subgroup (isomorphic to $\mathbb{Z}$) generated by $((Z')^{-1}, Z)$.

$\Gamma_{1,[n]}$ is presented explicitly as follows. Generators are σ_i $(i = 1,...,n-1)$, A_i, B_i $(i = 1,...,n)$, C_{jk} $(1 \leq j < k \leq n)$, Θ and Ψ, and relations are:

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\ (i = 1,...,n-2), \quad \sigma_i\sigma_j = \sigma_j\sigma_i\ (1 \leq i < j \leq n),$$
$$\sigma_i^{-1}X_i\sigma_i^{-1} = X_{i+1}, \quad \sigma_iY_i\sigma_i = Y_{i+1}\ (i = 1,...,n-1),$$
$$(\sigma_i, X_j) = (\sigma_i, Y_j) = 1\ (i \in \{1,...,n-1\}, j \in \{1,...,n\}, j \neq i, i+1),$$
$$\sigma_i^2 = C_{i,i+1}C_{i+1,i+2}C_{i,i+2}^{-1}\ (i = 1,...,n-1),$$
$$(A_i, A_j) = (B_i, B_j) = 1\ (\text{any } i,j), \quad A_1 = B_1 = 1,$$
$$(B_k, A_kA_j^{-1}) = (B_kB_j^{-1}, A_k) = C_{jk}\ (1 \leq j < k \leq n),$$
$$(A_i, C_{jk}) = (B_i, C_{jk}) = 1\ (1 \leq i \leq j < k \leq n),$$
$$\Theta A_i\Theta^{-1} = B_i^{-1}, \quad \Theta B_i\Theta^{-1} = B_iA_iB_i^{-1},$$
$$\Psi A_i\Psi^{-1} = A_i, \quad \Psi B_i\Psi^{-1} = B_iA_i, \quad (\Theta, \sigma_i) = (\Psi, \sigma_i) = 1,$$
$$(\Psi, \Theta^2) = 1, \quad (\Theta\Psi)^3 = \Theta^4 = C_{12}...C_{n-1,n}.$$

Here $X_i = A_iA_{i+1}^{-1}$, $Y_i = B_iB_{i+1}^{-1}$ for $i = 1,...,n$ (with the convention $A_{n+1} = B_{n+1} = C_{i,n+1} = 1$). The relations imply

$$C_{jk} = \sigma_{j,j+1...k}...\sigma_{j+n-k,j+n-k+1...n}\sigma_{j,j+1...n-k+j+1}...\sigma_{k-1,k...n},$$

where $\sigma_{i,i+1...j} = \sigma_{j-1}...\sigma_i$. Observe that $C_{12},...,C_{n-1,n}$ commute with each other.

The group $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ is presented by generators Θ, Ψ and Z, and relations: Z is central, $\Theta^4 = (\Theta\Psi)^3 = Z$ and $(\Psi, \Theta^2) = 1$. The morphism $\widetilde{\mathrm{SL}_2(\mathbb{Z})} \to \mathrm{SL}_2(\mathbb{Z})$ is $\Theta \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\Psi \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and the morphism $\Gamma_{1,[n]} \to \mathrm{SL}_2(\mathbb{Z})$ is given by the same formulas and $A_i, B_i, \sigma_i \mapsto 1$.

The elliptic braid group $\overline{\mathrm{B}}_{1,n}$ is the kernel of $\Gamma_{1,[n]} \to \mathrm{SL}_2(\mathbb{Z})$; it has the same presentation as $\Gamma_{1,[n]}$, except for the omission of the generators Θ, Ψ and the relations involving them. The "pure" mapping class group $\Gamma_{1,n}$ is the kernel of $\Gamma_{1,[n]} \to S_n$, $A_i, B_i, C_{jk} \mapsto 1$, $\sigma_i \mapsto \sigma_i$; it has the same presentation as $\Gamma_{1,[n]}$, except for the omission of the σ_i. Finally, recall that $\overline{\mathrm{PB}}_{1,n}$ is the kernel of $\Gamma_{1,[n]} \to \mathrm{SL}_2(\mathbb{Z}) \times S_n$.

Remark 4.1. The extended mapping class group $\tilde{\Gamma}_{1,n}$ of classes of non necessarily orientation-preserving self-homeomorphisms of a surface of type $(1,n)$ fits in a split exact sequence $1 \to \Gamma_{1,n} \to \tilde{\Gamma}_{1,n} \to \mathbb{Z}/2\mathbb{Z} \to 1$; it may be viewed as $\{\overline{\mathrm{PB}}_{1,n} \rtimes \widetilde{\mathrm{GL}_2(\mathbb{Z})}\}/\mathbb{Z}$; it has the same presentation as $\Gamma_{1,n}$ with the additional generator Σ subject to

$$\Sigma^2 = 1, \quad \Sigma\Theta\Sigma^{-1} = \Theta^{-1}, \quad \Sigma\Psi\Sigma^{-1} = \Psi^{-1}, \quad \Sigma A_i \Sigma^{-1} = A_i^{-1}, \quad \Sigma B_i \Sigma^{-1} = A_i B_i A_i^{-1}.$$

4.3. **The monodromy morphisms $\gamma_n : \Gamma_{1,[n]} \to \mathbf{G}_n \rtimes S_n$.** Let $F(\mathbf{z}|\tau)$ be a solution of the elliptic KZB system defined on D_n.

Recall that $D_n := \{(\mathbf{z},\tau) \in \mathbb{C}^n \times \mathfrak{H} | z_i = a_i + b_i\tau, a_i, b_i \in \mathbb{R}, a_1 < a_2 < \ldots < a_n < a_1+1, b_1 < b_2 < \ldots < b_n < b_1 + 1\}$. The domains $H_n := \{(\mathbf{z},\tau) \in \mathbb{C}^n \times \mathfrak{H} | z_i = a_i + b_i\tau, a_i, b_i \in \mathbb{R}, a_1 < a_2 < \ldots < a_n < a_1 + 1\}$ and $D_n := \{(\mathbf{z},\tau) \in \mathbb{C}^n \times \mathfrak{H} | z_i = a_i + b_i\tau, a_i, b_i \in \mathbb{R}, b_1 < b_2 < \ldots < b_n < b_1 + 1\}$ are also simply connected and invariant, and we denote by $F^H(\mathbf{z}|\tau)$ and $F^V(\mathbf{z}|\tau)$ the prolongations of $F(\mathbf{z}|\tau)$ to these domains.

Then $(\mathbf{z},\tau) \mapsto F^H(\mathbf{z} + \sum_{j=i}^n \delta_i|\tau)$ and $(\mathbf{z},\tau) \mapsto e^{2\pi\,\mathrm{i}(\bar{x}_i+\ldots+\bar{x}_n)} F^V(\mathbf{z} + \tau(\sum_{j=i}^n \delta_i)|\tau)$ are solutions of the elliptic KZB system on H_n and D_n respectively. We define $A_i^F, B_i^F \in \mathbf{G}_n$ by

$$F^H(\mathbf{z} + \sum_{j=i}^n \delta_i|\tau) = F^H(\mathbf{z}|\tau)A_i^F, \quad e^{2\pi\,\mathrm{i}(\bar{x}_i+\ldots+\bar{x}_n)} F^V(\mathbf{z} + \tau(\sum_{j=i}^n \delta_i)|\tau) = F^V(\mathbf{z}|\tau)B_i^F.$$

The action of $T^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is $(\mathbf{z},\tau) \mapsto (-\mathbf{z}/\tau, -1/\tau)$; this transformation takes H_n to V_n. Then $(\mathbf{z},\tau) \mapsto c_{T^{-1}}(\mathbf{z}|\tau)^{-1} F^V(-\mathbf{z}/\tau| - 1/\tau)$ is a solution of the elliptic KZB system on H_n (recall that $c_{T^{-1}}(\mathbf{z}|\tau)^{-1} = e^{2\pi\,\mathrm{i}(-\sum_i z_i\bar{x}_i + \tau X)}(-\tau)^d = (-\tau)^d e^{(2\pi\,\mathrm{i}/\tau)(\sum_i z_i\bar{x}_i + X)}$). We define Θ^F by

$$c_{T^{-1}}(\mathbf{z}|\tau)^{-1} F^V(-\mathbf{z}/\tau| - 1/\tau) = F^H(\mathbf{z}|\tau)\Theta^F.$$

The action of $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is $(\mathbf{z},\tau) \mapsto (\mathbf{z},\tau+1)$. This transformation takes H_n to itself. Since $c_S(\mathbf{z}|\tau) = 1$, the function $(\mathbf{z},\tau) \mapsto F^H(\mathbf{z},\tau+1)$ is a solution of the elliptic KZB system on H_n. We define Ψ^F by

$$F^H(\mathbf{z}|\tau+1) = F^H(\mathbf{z}|\tau)\Psi^F.$$

Finally, define σ_i^F by

$$\sigma_i F(\sigma_i^{-1}\mathbf{z}|\tau) = F(\mathbf{z}|\tau)\sigma_i^F,$$

where on the l.h.s. F is extended to the universal cover of $(\mathbb{C}^n \times \mathfrak{H}) - \mathrm{Diag}_n$ (σ_i exchanges z_i and z_{i+1}, z_{i+1} passing to the right of z_i).

Lemma 4.2. *There is a unique morphism $\Gamma_{1,[n]} \to \mathbf{G}_{1,n} \rtimes S_n$, taking X to X^F, where $X = A_i, B_i, \Theta$ or Ψ.*

Proof. This follows from the geometric description of generators of $\Gamma_{1,[n]}$: if $(\mathbf{z}_0,\tau_0) \in D_n$, then A_i is the class of the projection of the path $[0,1] \ni t \mapsto (\mathbf{z}_0 + t\sum_{j=i}^n \delta_j, \tau_0)$, B_i is the class of the projection of $[0,1] \ni t \mapsto (\mathbf{z}_0 + t\tau\sum_{j=i}^n \delta_j, \tau_0)$, Θ is the class of the projection of any path connecting $(\mathbf{z}_0,\tau_0)$ to $(-\mathbf{z}_0/\tau_0, -1/\tau_0)$ contained in H_n, and Ψ is the class of the projection of any path connecting $(\mathbf{z}_0,\tau_0)$ to $(\mathbf{z}_0,\tau_0+1)$ contained in H_n. □

We will denote by $\gamma_n : \Gamma_{1,[n]} \to \mathbf{G}_n \rtimes S_n$ the morphism induced by the solution $F^{(n)}(\mathbf{z}|\tau)$.

4.4. **Expression of $\gamma_n : \Gamma_{1,[n]} \to \mathbf{G}_n \rtimes S_n$ using γ_1 and γ_2.**

Lemma 4.3. *There exists a unique Lie algebra morphism $\mathfrak{d} \to \bar{\mathfrak{t}}_{1,n} \rtimes \mathfrak{d}$, $x \mapsto [x]$, such that $[\delta_{2n}] = \delta_{2n} + \sum_{i<j}(\mathrm{ad}\,\bar{x}_i)^{2n}(\bar{t}_{ij})$, $[X] = X$, $[\Delta_0] = \Delta_0$, $[d] = d$.*

It induces a group morphism $\mathbf{G}_1 \to \mathbf{G}_n$, also denoted $g \mapsto [g]$.

Lemma 4.4. *For each map $\phi : \{1, ..., m\} \to \{1, ..., n\}$, there exists a Lie algebra morphism $\bar{\mathfrak{t}}_{1,n} \to \bar{\mathfrak{t}}_{1,m}$, $x \mapsto x^\phi$, defined by $(\bar{x}_i)^\phi := \sum_{i'\in\phi^{-1}(i)} \bar{x}_{i'}$, $(\bar{y}_i)^\phi := \sum_{i'\in\phi^{-1}(i)} \bar{y}_{i'}$, $(\bar{t}_{ij})^\phi := \sum_{i'\in\phi^{-1}(i),j'\in\phi^{-1}(j)} \bar{t}_{i'j'}$.*

It induces a group morphism $\exp(\hat{\bar{\mathfrak{t}}}_{1,n}) \to \exp(\hat{\bar{\mathfrak{t}}}_{1,m})$, also denoted $g \mapsto g^\phi$.

The proofs are immediate. We now recall the definition and properties of the KZ associator ([Dr3]).

If $\mathbf{k}$ is a field with $\mathrm{char}(\mathbf{k}) = 0$, we let $\mathfrak{t}_n^{\mathbf{k}}$ be the $\mathbf{k}$-Lie algebra generated by t_{ij}, where $i \neq j \in \{1, ..., n\}$, with relations

$$t_{ji} = t_{ij}, \quad [t_{ij} + t_{ik}, t_{jk}] = 0, \quad [t_{ij}, t_{kl}] = 0$$

for i, j, k, l distinct (in this section, we set $\mathfrak{t}_n := \mathfrak{t}_n^{\mathbb{C}}$). For each partially defined map $\{1, ..., m\} \supset D_\phi \xrightarrow{\phi} \{1, ..., n\}$, we have a Lie algebra morphism $\mathfrak{t}_n \to \mathfrak{t}_m$, $x \mapsto x^\phi$, defined by[3] $(t_{ij})^\phi := \sum_{i'\in\phi^{-1}(i),j'\in\phi^{-1}(j)} t_{i'j'}$. We also have morphisms $\mathfrak{t}_n \to \bar{\mathfrak{t}}_{1,n}$, $t_{ij} \mapsto \bar{t}_{ij}$, compatible with the maps $x \mapsto x^\phi$ on both sides.

The KZ associator $\Phi = \Phi(t_{12}, t_{23}) \in \exp(\hat{\mathfrak{t}}_3)$ is defined by $G_0(z) = G_1(z)\Phi$, where $G_i :]0, 1[\to \exp(\hat{\mathfrak{t}}_3)$ are the solutions of $G'(z)G(z)^{-1} = t_{12}/z + t_{23}/(z-1)$ with $G_0(z) \sim z^{t_{12}}$ as $z \to 0$ and $G_1(z) \sim (1-z)^{t_{23}}$ as $z \to 1$. The KZ associator satisfies the duality, hexagon and pentagon equation (37), (38) below (where $\lambda = 2\pi\,\mathrm{i}$).

Lemma 4.5. *$\gamma_2(A_2)$ and $\gamma_2(B_2)$ belong to $\exp(\hat{\bar{\mathfrak{t}}}_{1,2}) \subset \mathbf{G}_2$.*

Proof. If $F(\mathbf{z}|\tau) : H_2 \to \mathbf{G}_2$ is a solution of the KZB equation for $n = 2$, then $A_2^F = F^H(\mathbf{z} + \delta_2|\tau)F^H(\mathbf{z}|\tau)^{-1}$ is expressed as the iterated integral, from $\mathbf{z}_0 \in D_n$ to $\mathbf{z}_0 + \delta_2$, of $\bar{K}_2(\mathbf{z}|\tau) \in \hat{\bar{\mathfrak{t}}}_{1,2}$, hence $A_2^F \in \exp(\hat{\bar{\mathfrak{t}}}_{1,2})$. Since $\gamma_2(A_2)$ is a conjugate of A_2^F, it belongs to $\exp(\hat{\bar{\mathfrak{t}}}_{1,2})$ as $\exp(\hat{\bar{\mathfrak{t}}}_{1,2}) \subset \mathbf{G}_2 \rtimes S_2$ is normal. One proves similarly that $\gamma_2(B_2) \in \exp(\hat{\bar{\mathfrak{t}}}_{1,2})$. □

Set

$$\Phi_i := \Phi^{1...i-1,i,i+1...n} ... \Phi^{1...n-2,n-1,n} \in \exp(\hat{\mathfrak{t}}_n).$$

We denote by $x \mapsto \{x\}$ the morphism $\exp(\hat{\mathfrak{t}}_n) \to \exp(\hat{\bar{\mathfrak{t}}}_{1,n})$ induced by $t_{ij} \mapsto \bar{t}_{ij}$.

Proposition 4.6. *If $n \geq 2$, then*

$$\gamma_n(\Theta) = [\gamma_1(\Theta)]e^{\mathrm{i}\,\frac{\pi}{2}\sum_{i<j}\bar{t}_{ij}}, \quad \gamma_n(\Psi) = [\gamma_1(\Psi)]e^{\mathrm{i}\,\frac{\pi}{6}\sum_{i<j}\bar{t}_{ij}},$$

and if $n \geq 3$, then

$$\gamma_n(A_i) = \{\Phi_i\}^{-1}\gamma_2(A_2)^{1...i-1,i...n}\{\Phi_i\}, \quad \gamma_n(B_i) = \{\Phi_i\}^{-1}\gamma_2(B_2)^{1...i-1,i...n}\{\Phi_i\}, \ (i = 1, ..., n),$$

$$\gamma_n(\sigma_i) = \{\Phi^{1...i-1,i,i+1}\}^{-1}e^{\mathrm{i}\,\pi\bar{t}_{i,i+1}}\{\Phi^{1...i-1,i,i+1}\}, \ (i = 1, ..., n-1).$$

Proof. In the region $z_{21} \ll z_{31} \ll ... \ll z_{n1} \ll 1$, $(\mathbf{z}, \tau) \in D_n$, we have

$$F^{(n)}(\mathbf{z}|\tau) \simeq z_{21}^{\bar{t}_{12}} ... z_{n1}^{\bar{t}_{1n}+...+\bar{t}_{n-1,n}} \exp(-\frac{a_0}{2\pi\,\mathrm{i}}(\int_{\mathrm{i}}^{\tau} E_2 + C)(\sum_{i<j}\bar{t}_{ij}))[F(\tau)],$$

where $F(\tau) = F^{(1)}(z|\tau)$ for any z. Here C is the constant such that $\int_{\mathrm{i}}^{\tau} E_2 + C = \tau + o(1)$ as $\tau \to \mathrm{i}\,\infty$.

[3]We will also use the notation $x^{I_1,...,I_n}$ for x^ϕ, where $I_i = \phi^{-1}(i)$.

We have $F(\tau+1) = F(\tau)\gamma_1(\Psi)$, $F(-1/\tau) = F(\tau)\gamma_1(\Theta)$. Since $\sum_{i<j}\bar t_{ij}$ commutes with the image of $x \mapsto [x]$, we get $F^{(n)}(\mathbf{z}|\tau+1) = F^{(n)}(\mathbf{z}|\tau)\exp(-\frac{a_0}{2\pi\,\mathrm{i}}(\sum_{i<j}\bar t_{ij}))[\gamma_1(\Psi)]$, so

$$\gamma_n(\Psi) = \exp(\mathrm{i}\,\frac{\pi}{6}\sum_{i<j}\bar t_{ij})[\gamma_1(\Psi)].$$

In the same region,

$$c_{T^{-1}}(\mathbf{z}|\tau)^{-1}F^{(n)V}(-\frac{\mathbf{z}}{\tau}|-\frac{1}{\tau}) \simeq (-\tau)^d e^{\frac{2\pi\,\mathrm{i}}{\tau}(\sum_i z_i\bar x_i + X)}(-z_{21}/\tau)^{\bar t_{12}}...(-z_{n1}/\tau)^{\bar t_{1n}+...+\bar t_{n-1,n}}$$
$$\exp(-\frac{a_0}{2\pi\,\mathrm{i}}(\int_{\mathrm{i}}^{-1/\tau}E_2 + C)(\sum_{i<j}\bar t_{ij}))[F(-1/\tau)].$$

Now $E_2(-1/\tau) = \tau^2 E_2(\tau) + (6\,\mathrm{i}/\pi)\tau$, so $\int_{\mathrm{i}}^{-1/\tau}E_2 - \int_{\mathrm{i}}^{\tau}E_2 = (6\,\mathrm{i}/\pi)[\log(-1/\tau) - \log\mathrm{i}]$ (where $\log(re^{i\theta}) = \log r + \mathrm{i}\,\theta$ for $\theta \in]-\pi,\pi[$).

It follows that

$$c_{T^{-1}}(\mathbf{z}|\tau)^{-1}F^{(n)V}(-\frac{\mathbf{z}}{\tau}|-\frac{1}{\tau}) \simeq e^{2\pi\,\mathrm{i}(\sum_i z_i\bar x_i)}z_{21}^{\bar t_{12}}...z_{n1}^{\bar t_{1n}+...+\bar t_{n-1,n}}\exp(-\frac{a_0}{2\pi\,\mathrm{i}}(\int_{\mathrm{i}}^{\tau}E_2 + C)(\sum_{i<j}\bar t_{ij}))$$

$$(\exp -\frac{a_0}{2\pi\,\mathrm{i}}\frac{-6\,\mathrm{i}}{\pi}(\log\mathrm{i})(\sum_{i<j}\bar t_{ij}))[(-\tau)^d e^{(2\pi\,\mathrm{i}/\tau)X}F(-1/\tau)]$$

$$\simeq z_{21}^{\bar t_{12}}...z_{n1}^{\bar t_{1n}+...+\bar t_{n-1,n}}\exp(-\frac{a_0}{2\pi\,\mathrm{i}}(\int_{\mathrm{i}}^{\tau}E_2 + C)(\sum_{i<j}\bar t_{ij}))[F(\tau)\gamma_1(\Theta)]\exp(\frac{\mathrm{i}\,\pi}{2}\sum_{i<j}\bar t_{ij})$$

$$\simeq F^{(n)H}(\mathbf{z}|\tau)[\gamma_1(\Theta)]\exp(\frac{\mathrm{i}\,\pi}{2}\sum_{i<j}\bar t_{ij})$$

(the second $\simeq$ follows from $\sum_i z_i\bar x_i = \sum_{i>1} z_{i1}\bar x_i$ and $z_{i1} \to 0$), so

$$\gamma_n(\Theta) = [\gamma_1(\Theta)]\exp(\mathrm{i}\,\frac{\pi}{2}\sum_{i<j}\bar t_{ij}).$$

Let $G_i(\mathbf{z}|\tau)$ be the solution of the elliptic KZB system, such that

$$G_i(\mathbf{z}|\tau)$$
$$= z_{21}^{\bar t_{12}}...z_{i-1,1}^{\bar t_{12}+...+\bar t_{1,i-1}}z_{n,i}^{\bar t_{i,n}+...+\bar t_{n-1,n}}...z_{n,n-1}^{\bar t_{n-1,n}}\exp\Big(-\frac{\tau}{2\pi\,\mathrm{i}}\Big(\Delta_0 + \sum_{n\geq 0}a_{2n}\big(\delta_{2n} + \sum_{i<j}(\mathrm{ad}\,\bar x_i)^{2n}(\bar t_{ij})\big)\Big)\Big)$$

when $z_{21} \ll ... \ll z_{i-1,1} \ll 1$, $z_{n,n-1} \ll ... \ll z_{n,i} \ll 1$, $\tau \to \mathrm{i}\,\infty$ and $(\mathbf{z},\tau) \in D_n$. Then $G_i(\mathbf{z} + \sum_{j=i}^n \delta_i|\tau) = G_i(\mathbf{z}|\tau)\gamma_2(A_2)^{1...i-1,i...n}$, because in the domain considered $\bar K_i(\mathbf{z}|\tau)$ is close to $\bar K_2(z_1,z_n|\tau)^{1...i-1,i...n}$ (where $\bar K_2(...)$ corresponds to the 2-point system); on the other hand, $F(\mathbf{z}|\tau) = G_i(\mathbf{z}|\tau)\{\Phi_i\}$, which implies the formula for $\gamma_n(A_i)$. The formula for $\gamma_n(B_i)$ is proved in the same way. Finally, the behavior of $F^{(n)}(\mathbf{z}|\tau)$ for $z_{21} \ll ... \ll z_{n1} \ll 1$ is similar to that of a solution of the KZ equations, which implies the formula for $\gamma_n(\sigma_i)$. □

Remark 4.7. One checks that the composition $\mathrm{SL}_2(\mathbb{Z}) \simeq \Gamma_{1,1} \to \mathbf{G}_1 \to \mathrm{SL}_2(\mathbb{C})$ is a conjugation of the canonical inclusion. It follows that the composition $\widetilde{\mathrm{SL}_2(\mathbb{Z})} \subset \Gamma_{1,n} \to \mathbf{G}_1 \to \mathrm{SL}_2(\mathbb{C})$ is a conjugation of the canonical projection for any $n \geq 1$. □

Let us set $\tilde A := \gamma_2(A_2)$, $\tilde B := \gamma_2(B_2)$. The image of $A_2A_3^{-1} = \sigma_1^{-1}A_2^{-1}\sigma_1^{-1}$ by γ_3 yields

$$\tilde A^{12,3} = e^{\mathrm{i}\,\pi\bar t_{12}}\{\Phi\}^{3,1,2}\tilde A^{2,13}\{\Phi\}^{2,1,3}e^{\mathrm{i}\,\pi\bar t_{12}}\cdot\{\Phi\}^{3,2,1}\tilde A^{1,23}\{\Phi\}^{1,2,3} \quad (22)$$

and the image of $B_2B_3^{-1} = \sigma_1 B_2^{-1}\sigma_1$ yields

$$\tilde{B}^{12,3} = e^{-\mathrm{i}\,\pi\bar{t}_{12}}\{\Phi\}^{3,1,2}\tilde{B}^{2,13}\{\Phi\}^{2,1,3}e^{-\mathrm{i}\,\pi\bar{t}_{12}}\cdot\{\Phi\}^{3,2,1}\tilde{B}^{1,23}\{\Phi\}^{1,2,3}. \tag{23}$$

Since $(\gamma_3(A_2),\gamma_3(A_3)) = (\gamma_3(B_2),\gamma_3(B_3)) = 1$, we get

$$(\{\Phi\}^{3,2,1}\tilde{A}^{1,23}\{\Phi\},\tilde{A}^{12,3}) = (\{\Phi\}^{3,2,1}\tilde{B}^{1,23}\{\Phi\},\tilde{B}^{12,3}) = 1 \tag{24}$$

(this equation can also be directly derived from (22) and (23) by noting that the l.h.s. is invariant $x\mapsto x^{2,1,3}$ and commutes with $e^{\pm\mathrm{i}\,\pi\bar{t}_{12}}$). We have for $n=2$, $C_{12} = (B_2,A_2)$, so $(\tilde{A},\tilde{B}) = \gamma_2(C_{12})^{-1}$. Also $\gamma_1(\Theta)^4 = 1$, so $\gamma_2(C_{12}) = \gamma_2(\Theta)^4 = (e^{\mathrm{i}\,\pi\bar{t}_{12}/2}[\gamma_1(\Theta)])^4 = e^{2\pi\,\mathrm{i}\,\bar{t}_{12}}[\gamma_1(\Theta)^4] = e^{2\pi\,\mathrm{i}\,\bar{t}_{12}}$, so

$$(\tilde{A},\tilde{B}) = e^{-2\pi\,\mathrm{i}\,\bar{t}_{12}}. \tag{25}$$

For $n=3$, we have $\gamma_3(\Theta)^4 = e^{2\pi\,\mathrm{i}(\bar{t}_{12}+\bar{t}_{13}+\bar{t}_{23})} = \gamma_3(C_{12}C_{23})$; since $\gamma_3(C_{12}) = (\gamma_3(B_2),\gamma_3(A_2)) = \{\Phi\}^{-1}(\tilde{B},\tilde{A})^{1,23}\{\Phi\} = \{\Phi\}^{-1}e^{2\pi\,\mathrm{i}(\bar{t}_{12}+\bar{t}_{13})}\{\Phi\}$, we get $\gamma_3(C_{23}) = \{\Phi\}^{-1}e^{2\pi\,\mathrm{i}\,\bar{t}_{23}}\{\Phi\}$. The image by γ_3 of $(B_3,A_3A_2^{-1}) = (B_3B_2^{-1},A_3) = C_{23}$ then gives

$$(\tilde{B}^{12,3},\tilde{A}^{12,3}\{\Phi\}^{-1}(\tilde{A}^{1,23})^{-1}\{\Phi\}) = (\tilde{B}^{12,3}\{\Phi\}^{-1}(\tilde{B}^{1,23})^{-1}\{\Phi\},\tilde{A}^{12,3}) = \{\Phi\}^{-1}e^{2\pi\,\mathrm{i}\,\bar{t}_{23}}\{\Phi\} \tag{26}$$

(applying $x\mapsto x^{\emptyset,1,2}$, this identity implies (25)).

Let us set $\tilde{\Theta} := \gamma_1(\Theta)$, $\tilde{\Psi} := \gamma_1(\Theta)$. Since γ_1,γ_2 are group morphisms, we have

$$\tilde{\Theta}^4 = (\tilde{\Theta}\tilde{\Psi})^3 = (\tilde{\Theta}^2,\tilde{\Psi}) = 1, \tag{27}$$

$$[\tilde{\Theta}]e^{\mathrm{i}\,\frac{\pi}{2}\bar{t}_{12}}\tilde{A}([\tilde{\Theta}]e^{\mathrm{i}\,\frac{\pi}{2}\bar{t}_{12}})^{-1} = \tilde{B}^{-1},\quad [\tilde{\Theta}]e^{\mathrm{i}\,\frac{\pi}{2}\bar{t}_{12}}\tilde{B}([\tilde{\Theta}]e^{\mathrm{i}\,\frac{\pi}{2}\bar{t}_{12}})^{-1} = \tilde{B}\tilde{A}\tilde{B}^{-1}, \tag{28}$$

$$[\tilde{\Psi}]e^{\mathrm{i}\,\frac{\pi}{6}\bar{t}_{12}}\tilde{A}([\tilde{\Psi}]e^{\mathrm{i}\,\frac{\pi}{6}\bar{t}_{12}})^{-1} = \tilde{A},\quad [\tilde{\Psi}]e^{\mathrm{i}\,\frac{\pi}{6}\bar{t}_{12}}\tilde{B}([\tilde{\Psi}]e^{\mathrm{i}\,\frac{\pi}{6}\bar{t}_{12}})^{-1} = \tilde{B}\tilde{A}. \tag{29}$$

(27) (resp., (28), (29)) are identities in $\mathbf{G}_1$ (resp., $\mathbf{G}_2$); in (28), (29), $x\mapsto[x]$ is induced by the map $\mathfrak{d}\to\mathfrak{d}\ltimes\bar{\mathfrak{t}}_{1,2}$ defined above.

4.5. **Expression of $\tilde{\Psi}$ and of $\tilde{A}$ and $\tilde{B}$ in terms of Φ.** In this section, we compute $\tilde{A}$ and $\tilde{B}$ in terms of the KZ associator Φ. We also compute $\tilde{\Psi}$.

Recall the definition of $\tilde{\Psi}$. The elliptic KZB system for $n=1$ is

$$2\pi\,\mathrm{i}\,\partial_\tau F(\tau) + \big(\Delta_0 + \sum_{k\geq 1}a_{2k}E_{2k+2}(\tau)\delta_{2k}\big)F(\tau) = 0.$$

The solution $F(\tau) := F^{(1)}(z|\tau)$ (for any z) is determined by $F(\tau)\simeq\exp(-\frac{\tau}{2\pi\,\mathrm{i}}(\Delta_0 + \sum_{k\geq 1}a_{2k}\delta_{2k}))$. Then $\tilde{\Psi}$ is determined by $F(\tau+1) = F(\tau)\tilde{\Psi}$. We have therefore:

Lemma 4.8. $\tilde{\Psi} = \exp(-\frac{1}{2\pi\,\mathrm{i}}(\Delta_0 + \sum_{k\geq 1}a_{2k}\delta_{2k}))$.

Recall the definition of $\tilde{A}$ and $\tilde{B}$. The elliptic KZB system for $n=2$ is

$$\partial_z F(z|\tau) = -\Big(\frac{\theta(z+\operatorname{ad}x|\tau)\operatorname{ad}x}{\theta(z|\tau)\theta(\operatorname{ad}x|\tau)}\Big)(y)\cdot F(z|\tau), \tag{30}$$

$$2\pi\,\mathrm{i}\,\partial_\tau F(z|\tau) + \big(\Delta_0 + \sum_{k\geq 1}a_{2k}E_{2k+2}(\tau)\delta_{2k} - g(z,\operatorname{ad}x|\tau)(t)\big)F(z|\tau) = 0, \tag{31}$$

where $z = z_{21}$, $x = \bar{x}_2 = -\bar{x}_1$, $y = \bar{y}_2 = -\bar{y}_1$, $t = \bar{t}_{12} = -[x,y]$.

The solution $F(z|\tau) := F^{(2)}(z_1,z_2|\tau)$ is determined by its behavior $F(z|\tau)\simeq z^t\exp(-\frac{\tau}{2\pi\,\mathrm{i}}\big(\Delta_0 + \sum_{k\geq 0}a_{2k}(\delta_{2k} + (\operatorname{ad}x)^{2k})(t)\big))$ when $z\to 0^+$, $\tau\to\mathrm{i}\infty$. We then have $F^H(z+1|\tau) = F^H(z|\tau)\tilde{A}$, $e^{2\pi\,\mathrm{i}\,x}F^V(z+\tau|\tau) = F^V(z|\tau)\tilde{B}$.

Proposition 4.9. *We have*[4]
$\tilde{A} = (2\pi/\mathrm{i})^t \Phi(\tilde{y}, t) e^{2\pi \mathrm{i}\, \tilde{y}} \Phi(\tilde{y}, t)^{-1} (\mathrm{i}/2\pi)^t = (2\pi)^t\, \mathrm{i}^{-3t}\, \Phi(-\tilde{y}-t, t) e^{2\pi \mathrm{i}(\tilde{y}+t)} \Phi(-\tilde{y}-t, t)^{-1} (2\pi \mathrm{i})^{-t}$,
where $\tilde{y} = -\frac{\mathrm{ad}\, x}{e^{2\pi \mathrm{i}\, \mathrm{ad}\, x} - 1}(y)$.

Proof. $\tilde{A} = F^H(z|\tau)^{-1} F^H(z+1|\tau)$, which we will compute in the limit $\tau \to \mathrm{i}\,\infty$. For this, we will compute $F(z|\tau)$ in the limit $\tau \to \mathrm{i}\,\infty$. In this limit, $\theta(z|\tau) = (1/\pi)\sin(\pi z)[1 + O(e^{2\pi \mathrm{i}\, \tau})]$ so the system becomes

$$\partial_z F(z|\tau) = \big(\pi \operatorname{cotg}(\pi z) t - \pi \operatorname{cotg}(\pi\, \mathrm{ad}\, x)\, \mathrm{ad}\, x(y) + O(e^{2\pi \mathrm{i}\, \tau})\big) F(z|\tau) \tag{32}$$

$$2\pi \mathrm{i}\, \partial_\tau F(z|\tau) + \big(\Delta_0 + \sum_{k\geq 1} a_{2k}\delta_{2k} + (\frac{\pi^2}{\sin^2(\pi\, \mathrm{ad}\, x)} - \frac{1}{(\mathrm{ad}\, x)^2})(t) + O(e^{2\pi \mathrm{i}\, \tau})\big) F(z|\tau) = 0$$

where the last equation is

$$2\pi \mathrm{i}\, \partial_\tau F(z|\tau) + \big(\Delta_0 + a_0 t + \sum_{k\geq 1} a_{2k}(\delta_{2k} + (\mathrm{ad}\, x)^{2k}(t)) + O(e^{2\pi \mathrm{i}\, \tau})\big) F(z|\tau) = 0.$$

We set

$$\Delta := \Delta_0 + \sum_{k\geq 1} a_{2k}\delta_{2k}, \quad \text{so} \quad \Delta_0 + a_0 t + \sum_{k\geq 1} a_{2k}(\delta_{2k} + (\mathrm{ad}\, x)^{2k}(t)) = [\Delta] + a_0 t.$$

The compatibility of this system implies that $[\Delta] + a_0 t$ commutes with t and $(\pi\, \mathrm{ad}\, x)\operatorname{cotg}(\pi\, \mathrm{ad}\, x)(y) = \mathrm{i}\,\pi(-t - 2\tilde{y})$, hence with t and $\tilde{y}$; actually t commutes with each $[\delta_{2k}] = \delta_{2k} + (\mathrm{ad}\, x)^{2k}(t)$.

Equation (30) can be written $\partial_z F(z|\tau) = (t/z + O(1))F(z|\tau)$. We then let $F_0(z|\tau)$ be the solution of (30) in $V := \{(z,\tau) | \tau \in \mathfrak{H}, z = a + b\tau, a \in]0,1[, b \in \mathbb{R}\}$ such that $F_0(z|\tau) \simeq z^t$ when $z \to 0^+$, for any τ. This means that the left (equivalently, right) ratio of these quantities has the form $1 + \sum_{k>0}(\text{degree } k) O(z(\log z)^{f(k)})$ where $f(k) \geq 0$.

We now relate $F(z|\tau)$ and $F_0(z|\tau)$. Let $F(\tau) = F^{(1)}(z|\tau)$ for any z be the solution of the KZB system for $n = 1$, such that $F(\tau) \simeq \exp(-\frac{\tau}{2\pi \mathrm{i}}\Delta)$ as $\tau \to \mathrm{i}\,\infty$ (meaning that the left, or equivalently right, ratio of these quantities has the form $1 + \sum_{k>0}(\text{degree } k) O(\tau^{f(k)} e^{2\pi \mathrm{i}\, \tau})$, where $f(k) \geq 0$).

Lemma 4.10. *We have* $F(z|\tau) = F_0(z|\tau)\exp(-\frac{a_0}{2\pi \mathrm{i}}(\int_{\mathrm{i}}^{\tau} E_2 + C)t)[F(\tau)]$, *where* C *is such that* $\int_{\mathrm{i}}^{\tau} E_2 + C = \tau + O(e^{2\pi \mathrm{i}\, \tau})$.

Proof of Lemma. $F(z|\tau) = F_0(z|\tau) X(\tau)$, where $X : \mathfrak{H} \to \mathbf{G}_2$ is a map. We have $g(z, \mathrm{ad}\, x|\tau)(t) = a_0 E_2(\tau) t + \sum_{k>0} a_{2k} E_{2k+2}(\tau)(\mathrm{ad}\, x)^{2k}(t) + O(z)$ when $z \to 0^+$ and for any τ, so (31) is written as

$$2\pi \mathrm{i}\, \partial_\tau F(z|\tau) + \big(\Delta_0 + a_0 E_2(\tau) t + \sum_{k>0} a_{2k} E_{2k+2}(\tau)[\delta_{2k}] + O(z)\big) F(z|\tau) = 0$$

where $O(z)$ has degree > 0. Since Δ_0, t and the $[\delta_{2k}]$ all commute with t, the ratio $F_0(z|\tau)^{-1} F(z|\tau)$ satisfies

$$2\pi \mathrm{i}\, \partial_\tau (F_0^{-1} F(z|\tau)) + \big(\Delta_0 + a_0 E_2(\tau) t + \sum_{k>0} a_{2k} E_{2k+2}(\tau)[\delta_{2k}] + \sum_{k>0} (\text{degree } k) O(z(\log z)^{h(k)})\big)(F_0^{-1} F(z|\tau)) = 0$$

where $h(k) \geq 0$. Since $F_0(z|\tau)^{-1} F(z|\tau) = X(\tau)$ is in fact independent on z, we have

$$2\pi \mathrm{i}\, \partial_\tau (X(\tau)) + \big(\Delta_0 + a_0 E_2(\tau) t + \sum_{k>0} a_{2k} E_{2k+2}(\tau)[\delta_{2k}]\big)(X(\tau)) = 0,$$

which implies that $X(\tau) = \exp(-\frac{a_0}{2\pi \mathrm{i}}(\int_{\mathrm{i}}^{\tau} E_2 + C)t)[F(\tau)] X_0$, where X_0 is a suitable element in $\mathbf{G}_2$. The asymptotic behavior of $F(z|\tau)$ when $\tau \to \mathrm{i}\,\infty$ and $z \to 0^+$ then implies $X_0 = 1$. □

[4]By convention, if $z \in \mathbb{C} \setminus \mathbb{R}_-$ and $x \in \mathfrak{n}$, where $\mathfrak{n}$ is a pronilpotent Lie algebra, then z^x is $\exp(x \log z) \in \exp(\mathfrak{n})$, where $\log z$ is chosen with imaginary part in $]-\pi, \pi[$.

End of proof of Proposition. We then have $F(z|\tau) = F_0(z|\tau)X(\tau)$, where $X(\tau) \simeq \exp(-\frac{\tau}{2\pi\,\mathrm{i}}([\Delta]+a_0t))$ as $\tau \to \mathrm{i}\,\infty$, where this means that the left ratio (equivalently, the right ratio) of these quantities has the form $1 + \sum_{k>0}(\text{degree } k)O(\tau^{x(k)}e^{2\pi\,\mathrm{i}\,\tau})$, where $x(k) \geq 0$.

If we set $u := e^{2\pi\,\mathrm{i}\,z}$, then (30) is rewritten as

$$\partial_u \bar{F}(u|\tau) = (\tilde{y}/u + t/(u-1) + O(e^{2\pi\,\mathrm{i}\,\tau}))\bar{F}(u|\tau), \tag{33}$$

where $\bar{F}(u|\tau) = F(z|\tau)$.

Let $D' := \{u||u| \leq 1\} - [0,1]$ be the complement of the unit interval in the unit disc. Then we have a bijection $\{(z,\tau)|\tau \in \mathrm{i}\,\mathbb{R}_+^\times, z = a+\tau b, a \in [0,1], b \geq 0\} \to D' \times \mathrm{i}\,\mathbb{R}_+^\times$, given by $(z,\tau) \mapsto (u,\tau) := (e^{2\pi\,\mathrm{i}\,z},\tau)$.

Let $\bar{F}_a, \bar{F}_f$ be the solutions of (33) in $D' \times \mathrm{i}\,\mathbb{R}_+$, such that $\bar{F}_a(u|\tau) \simeq ((u-1)/(2\pi\,\mathrm{i}))^t$ when $u = 1+\mathrm{i}\,0^+$, and for any τ, and $\bar{F}_f(u|\tau) \simeq e^{\mathrm{i}\,\pi t}((1-u)/(2\pi\,\mathrm{i}))^t$ when $u = 1-\mathrm{i}\,0^+$, for any τ.

Then one checks that $F_0(z|\tau) = \bar{F}_a(e^{2\pi\,\mathrm{i}\,z}|\tau)$, $F_0(z-1|\tau) = \bar{F}_f(e^{2\pi\,\mathrm{i}\,z}|\tau)$ when $(z,\tau) \in \{(z,\tau)|\tau \in \mathrm{i}\,\mathbb{R}_+^\times, z = a+\tau b | a \in [0,1], b \geq 0\}$.

We then define $\bar{F}_b, ..., \bar{F}_e$ as the solutions of (33) in $D' \times \mathrm{i}\,\mathbb{R}_+^\times$, such that: $\bar{F}_b(u|\tau) \simeq (1-u)^t$ as $u = 1-0^+$, $\Im(u) > 0$ for any τ, $\bar{F}_c(u|\tau) \simeq u^{\tilde{y}}$ as $u \to 0^+$, $\Im(u) > 0$ for any τ, $\bar{F}_d(u|\tau) \simeq u^{\tilde{y}}$ as $u \to 0^+$, $\Im(u) < 0$ for any τ, $\bar{F}_e(u|\tau) \simeq (1-u)^t$ as $u = 1-0^+$, $\Im(u) < 0$ for any τ.

Then $\bar{F}_b = \bar{F}_a(-2\pi\,\mathrm{i})^t$, $\bar{F}_c(-|\tau) = \bar{F}_b(-|\tau)[\Phi(\tilde{y},t) + O(e^{2\pi\,\mathrm{i}\,\tau})]$, $\bar{F}_d(-|\tau) = \bar{F}_c(-|\tau)e^{-2\pi\,\mathrm{i}\,\tilde{y}}$, $\bar{F}_e(-|\tau) = \bar{F}_d(-|\tau)[\Phi(\tilde{y},t)^{-1} + O(e^{2\pi\,\mathrm{i}\,\tau})]$, $\bar{F}_f = \bar{F}_e(\mathrm{i}\,/2\pi)^t$.

So $\bar{F}_f(-|\tau) = \bar{F}_a(-|\tau)\big((-2\pi\,\mathrm{i})^t\Phi(\tilde{y},t)e^{-2\pi\,\mathrm{i}\,\tilde{y}}\Phi(\tilde{y},t)^{-1}(\mathrm{i}\,/2\pi)^t + O(e^{2\pi\,\mathrm{i}\,\tau})\big)$. It follows that $F_0(z+1|\tau) = F_0(z|\tau)A(\tau)$, where

$$A(\tau) = (-2\pi\,\mathrm{i})^t\Phi(\tilde{y},t)e^{2\pi\,\mathrm{i}\,\tilde{y}}\Phi(\tilde{y},t)^{-1}(\mathrm{i}\,/2\pi)^t + O(e^{2\pi\,\mathrm{i}\,\tau}).$$

Now

$$\tilde{A} = F(z|\tau)^{-1}F(z+1|\tau) = X(\tau)^{-1}A(\tau)X(\tau) = \big(1+\sum_{k>0}(\text{degree } k)O(\tau^{x(k)}e^{2\pi\,\mathrm{i}\,\tau})\big)^{-1}$$
$$\exp(\frac{\tau}{2\pi\,\mathrm{i}}([\Delta]+a_0t))\big((-2\pi\,\mathrm{i})^t\Phi(\tilde{y},t)e^{2\pi\,\mathrm{i}\,\tilde{y}}\Phi(\tilde{y},t)^{-1}(\mathrm{i}\,/2\pi)^t + O(e^{2\pi\,\mathrm{i}\,\tau})\big)$$
$$\exp(-\frac{\tau}{2\pi\,\mathrm{i}}([\Delta]+a_0t))\big(1+\sum_{k>0}(\text{degree } k)O(\tau^{x(k)}e^{2\pi\,\mathrm{i}\,\tau})\big).$$

As we have seen, $[\Delta]+a_0t$ commutes with $\tilde{y}$ and t; on the other hand,

$$\exp(\frac{\tau}{2\pi\,\mathrm{i}}([\Delta]+a_0t))O(e^{2\pi\,\mathrm{i}\,\tau})\exp(-\frac{\tau}{2\pi\,\mathrm{i}}([\Delta]+a_0t))$$
$$= \exp(\tau\,\mathrm{ad}(\frac{[\Delta]+a_0t}{2\pi\,\mathrm{i}}))(O(e^{2\pi\,\mathrm{i}\,\tau})) = \sum_{k\geq 0}(\text{degree } k)O(\tau^{n_1(k))}e^{2\pi\,\mathrm{i}\,\tau})$$

where $n_1(k) \geq 0$, as $[\Delta]+a_0t$ is a sum of terms of positive degree and of Δ_0, which is locally ad-nilpotent.

Then

$$\tilde{A} = \big(1+\sum_{k>0}(\text{degree } k)O(\tau^{x(k)}e^{2\pi\,\mathrm{i}\,\tau})\big)^{-1}\big((-2\pi\,\mathrm{i})^t\Phi(\tilde{y},t)e^{2\pi\,\mathrm{i}\,\tilde{y}}\Phi(\tilde{y},t)^{-1}(\mathrm{i}\,/2\pi)^t$$
$$+\sum_{k\geq 0}(\text{degree } k)O(\tau^{n_1(k)}e^{2\pi\,\mathrm{i}\,\tau})\big)\big(1+\sum_{k>0}(\text{degree } k)O(\tau^{x(k)}e^{2\pi\,\mathrm{i}\,\tau})\big).$$

It follows that

$$\tilde{A} = (-2\pi\,\mathrm{i})^t\Phi(\tilde{y},t)e^{2\pi\,\mathrm{i}\,\tilde{y}}\Phi(\tilde{y},t)^{-1}(\mathrm{i}\,/2\pi)^t + \sum_{k\geq 0}(\text{degree } k)O(\tau^{n_2(k)}e^{2\pi\,\mathrm{i}\,\tau}),$$

where $n_2(k) \geq 0$, which implies the first formula for $\tilde{A}$. The second formula either follows from the first one by using the hexagon identity, or can be obtained repeating the above argument using a path $1 \to +\infty \to 1$, winding around 1 and ∞. □

We now prove:

Theorem 4.11.
$$\tilde{B} = (2\pi\,\mathrm{i})^t \Phi(-\tilde{y}-t, t) e^{2\pi\,\mathrm{i}\,x} \Phi(\tilde{y}, t)^{-1} (2\pi/\,\mathrm{i})^{-t}.$$

Proof. We first define $F_0(z|\tau)$ as the solution in $V := \{a + b\tau | a \in]0,1[, b \in \mathbb{R}\}$ of (30) such that $F_0(z|\tau) \sim z^t$ as $z \to 0^+$. Then there exists $B(\tau)$ such that $e^{2\pi\,\mathrm{i}\,x} F_0(z+\tau|\tau) = F_0(z|\tau)B(\tau)$. We compute the asymptotics of $B(\tau)$ as $\tau \to \mathrm{i}\,\infty$.

We define four asymptotic zones (z is assumed to remain on the segment $[0,\tau]$, and τ in the line $\mathrm{i}\,\mathbb{R}_+$): (1) $z \ll 1 \ll \tau$, (2) $1 \ll z \ll \tau$, (3) $1 \ll \tau - z \ll \tau$, (4) $\tau - z \ll 1 \ll \tau$.

In the transition (1)-(2), the system takes the form (32), or if we set $u := e^{2\pi\,\mathrm{i}\,z}$, (33).

In the transition (3)-(4), $G(z'|\tau) := e^{2\pi\,\mathrm{i}\,x} F(\tau+z'|\tau)$ satisfies (30), so $\bar{G}(u'|\tau) = e^{2\pi\,\mathrm{i}\,x} F(\tau+z'|\tau)$ satisfies (33), where $u' = e^{2\pi\,\mathrm{i}\,z'}$.

We now compute the form of the system in the transition (2)-(3). We first prove:

Lemma 4.12. *Set $u := e^{2\pi\,\mathrm{i}\,z}$, $v := e^{2\pi\,\mathrm{i}(\tau-z)}$. When $0 < \Im(z) < \Im(\tau)$, we have $|u| < 1$, $|v| < 1$. When $k \geq 0$, $(\theta^{(k)}/\theta)(z|\tau) = (-\,\mathrm{i}\,\pi)^k + \sum_{s,t\geq 0, s+t>0} a_{st}^{(k)} u^s v^t$, where the sum in the r.h.s. is convergent in the domain $|u| < 1$, $|v| < 1$.*

Proof. This is clear if $k = 0$. Set $q = uv = e^{2\pi\,\mathrm{i}\,\tau}$. We have $\theta(z|\tau) = u^{1/2}\prod_{s>0}(1 - q^s u)\prod_{s\geq 0}(1 - q^s u^{-1}) \cdot (2\pi\,\mathrm{i})^{-1}\prod_{s>0}(1-q^s)^{-2}$, so

$$(\theta'/\theta)(z|\tau) = \mathrm{i}\,\pi - 2\pi\,\mathrm{i}\sum_{s>0} q^s u/(1-q^s u) + 2\pi\,\mathrm{i}\sum_{s\geq 0} q^s u^{-1}/(1-q^s u^{-1})$$
$$= -\,\mathrm{i}\,\pi - 2\pi\,\mathrm{i}\sum_{s\geq 0}\frac{u^{s+1}v^s}{1-u^{s+1}v^s} + 2\pi\,\mathrm{i}\sum_{s\geq 0}\frac{u^s v^{s+1}}{1-u^s v^{s+1}} = -\,\mathrm{i}\,\pi + \sum_{s+t>0} a_{st} u^s v^t,$$

where $a_{st} = 2\pi\,\mathrm{i}$ if $(s,t) = k(r, r+1)$, $k > 0$, $r \geq 0$, and $a_{st} = -2\pi\,\mathrm{i}$ if $(s,t) = k(r+1, r)$, $k > 0$, $r \geq 0$. One checks that this series is convergent in the domain $|u| < 1$, $|v| < 1$. This proves the lemma for $k = 1$.

We then prove the remaining cases by induction, using

$$\frac{\theta^{(k+1)}}{\theta}(z|\tau) = \frac{\theta^{(k)}}{\theta}(z|\tau)\frac{\theta'}{\theta}(z|\tau) + \frac{\partial}{\partial z}\frac{\theta^{(k)}}{\theta}(z|\tau).$$

□

Using the expansion

$$\frac{\theta(z+x|\tau)x}{\theta(z|\tau)\theta(x|\tau)} = \frac{x}{\theta(x|\tau)}\sum_{k\geq 0}(\theta^{(k)}/\theta)(z|\tau)\frac{x^k}{k!}$$
$$= \frac{\pi x}{\sin(\pi x)}(1 + \sum_{n>0} q^n P_n(x))\Big(\sum_{k\geq 0}((-\,\mathrm{i}\,\pi)^k + \sum_{s+t>0} a_{st}^{(k)} u^s v^t)\frac{x^k}{k!}\Big)$$
$$= \frac{\pi x}{\sin(\pi x)} e^{-\,\mathrm{i}\,\pi x} + \sum_{s+t>0} a_{st}(x) u^s v^t = \frac{2\,\mathrm{i}\,\pi x}{e^{2\,\mathrm{i}\,\pi x} - 1} + \sum_{s+t>0} a_{st}(x) u^s v^t,$$

the form of the system in the transition (2)-(3) is

$$\partial_z F(z|\tau) = \big(-\frac{2\,\mathrm{i}\,\pi\,\mathrm{ad}\,x}{e^{2\,\mathrm{i}\,\pi\,\mathrm{ad}\,x} - 1}(y) + \sum_{s,t|s+t>0} a_{st} u^s v^t\big) F(z|\tau)$$
$$= \big(2\,\mathrm{i}\,\pi\tilde{y} + \sum_{s,t|s+t>0} a_{st} u^s v^t\big) F(z|\tau), \tag{34}$$

where each homogeneous part of $\sum_{s,t} a_{st}u^s v^t$ converges for $|u| < 1$, $|v| < 1$.

Lemma 4.13. *There exists a solution $F_c(z|\tau)$ of (34) defined for $0 < \Im(z) < \Im(\tau)$, such that*

$$F_c(z|\tau) = u^{\tilde y}(1 + \sum_{k>0} \sum_{s\leq s(k)} \log(u)^s f_{ks}(u,v))$$

($\log u = i\pi z$, $u^{\tilde y} = e^{2\pi \mathrm{i} z\tilde y}$), where $f_{ks}(u,v)$ is an analytic function taking its values in the homogeneous part of the algebra of degree k, convergent for $|u| < 1$ and $|v| < 1$, and vanishing at $(0,0)$. This function is uniquely defined up to right multiplication by an analytic function of the form $1 + \sum_{k>0} a_k(q)$ (recall that $q = uv$), where $a_k(q)$ is an analytic function on $\{q||q| < 1\}$, vanishing at $q = 0$, with values in the degree k part of the algebra.

Proof of Lemma. We set $G(z|\tau) := u^{-\tilde y}F(z|\tau)$, so $G(z|\tau)$ should satisfy

$$\partial_z G(z|\tau) = \exp(-\operatorname{ad}(\tilde y)\log u)\{\sum_{s+t>0} a_{st}u^s v^t\}G(z|\tau),$$

which has the general form

$$\partial_z G(z|\tau) = \Big(\sum_{k>0}\sum_{s\leq a(k)} \log(u)^s a_{ks}(u,v)\Big)G(z|\tau),$$

where $a_{ks}(u,v)$ is analytic in $|u| < 1$, $|v| < 1$ and vanishes at $(0,0)$. We show that this system admits a solution of the form $1 + \sum_{k>0}\sum_{s\leq s(k)} \log(u)^s f_{ks}(u,v)$, with $f_{ks}(u,v)$ analytic in $|u| < 1$, $|v| < 1$, in the degree k part of the algebra, vanishing at $(0,0)$ for $s \neq 0$. For this, we solve inductively (in k) the system of equations

$$\partial_z\big(\sum_s (\log u)^s f_{ks}(u,v)\big) = \sum_{s',s'',k',k''|k'+k''=k} (\log u)^{s'+s''} a_{k's'}(u,v) f_{k''s''}(u,v). \tag{35}$$

Let $\mathcal{O}$ be the ring of analytic functions on $\{(u,v)||u| < 1, |v| < 1\}$ (with values in a finite dimensional vector space) and $\mathfrak{m} \subset \mathcal{O}$ be the subset of functions vanishing at $(0,0)$. We have an injection $\mathcal{O}[X] \to \{$analytic functions in (u,v), $|u| < 1$, $|v| < 1$, $u \notin \mathbb{R}_-\}$, given by $f(u,v)X^k \mapsto (\log u)^k f(u,v)$. The endomorphism $\frac{\partial}{\partial z} = 2\pi \mathrm{i}(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v})$ then corresponds to the endomorphism of $\mathcal{O}[X]$ given by $2\pi \mathrm{i}(\frac{\partial}{\partial X} + u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v})$. It is surjective, and restricts to a surjective endomorphism of $\mathfrak{m}[X]$. The latter surjectivity implies that equation (35) can be solved.

Let us show that the solution $G(z|\tau)$ is unique up to right multiplication by functions of q like in the lemma. The ratio of two solutions is of the form $1+\sum_{k>0}\sum_{s\leq s(k)} \log(u)^s f_{ks}(u,v)$ and is killed by ∂_z. Now the kernel of the endomorphism of $\mathfrak{m}[X]$ given by $2\pi \mathrm{i}(\frac{\partial}{\partial X} + u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v})$ is $m^*(\mathfrak{m}_1)$, where $m^*(\mathfrak{m}_1) \subset \mathfrak{m}$ is the set of all functions of the form $a(uv)$, where a is an analytic function on $\{q||q| < 1\}$ vanishing at 0. This implies that the ratio of two solutions is as above. □

End of proof of Theorem. Similarly, there exists a solution $F_d(z|\tau)$ of (34) defined in the same domain, such that

$$F_d(z|\tau) = v^{-\tilde y}(1 + \sum_{k>0}\sum_{s\leq t(k)} \log(v)^t g_{ks}(u,v)),$$

where $b_{ks}(u,v)$ is as above (and $\log v = \mathrm{i}\,\pi(\tau - z)$, $v^{-\tilde y} = \exp(2\pi \mathrm{i}(z-\tau)\tilde y))$. $F_d(z|\tau)$ is defined up to right multiplication by a function of q as above.

We now study the ratio $F_c(z|\tau)^{-1}F_d(z|\tau)$. This is a function of τ only, and it has the form

$$q^{-\tilde y}\Big(1 + \sum_{k>0}\sum_{s\leq s(k),t\leq t(k)} (\log u)^s(\log v)^t a_{kst}(u,v)\Big)$$

where $a_{kst}(u,v) \in \mathfrak{m}$ (as $v^{-\tilde y}(1+\sum_{k>0}\sum_{s\le s(k)}(\log u)^s c_{ks}(u,v))v^{\tilde y}$ has the form $1+\sum_{k>0}\sum_{s,t\le t(k)}$ $(\log u)^s(\log v)^t d_{ks}(u,v)$, where $d_{ks}(u,v)\in\mathfrak{m}$ if $c_{ks}(u,v)\in\mathfrak{m}$). Set $\log q := \log u + \log v = 2\pi\,\mathrm{i}\,\tau$, then this ratio can be rewritten $q^{-\tilde y}\{1+\sum_{k>0}\sum_{s\le s(k),t\le t(k)}(\log u)^s(\log q)^t b_{kst}(u,v)\}$ where $b_{kst}(u,v)\in\mathfrak{m}$, and since the product of this ratio with $q^{\tilde y}$ is killed by ∂_z (which identifies with the endomorphism $2\pi\,\mathrm{i}(\frac{\partial}{\partial X}+u\frac{\partial}{\partial u}-v\frac{\partial}{\partial v})$ of $\mathcal{O}[X]$), the ratio is in fact of the form

$$F_c^{-1}F_d(z|\tau) = q^{\tilde y}(1+\sum_{k>0}\sum_{s\le s(k)}(\log q)^s a_{ks}(q)),$$

where a_{ks} is analytic in $\{q||q|<1\}$, vanishing at $q=0$.

It follows that

$$F_c^{-1}F_d(z|\tau) = e^{-2\pi\,\mathrm{i}\,\tau\tilde y}(1+\sum_{k>0}(\text{degree } k)O(\tau^k e^{-2\pi\,\mathrm{i}\,\tau})). \tag{36}$$

In addition to F_c and F_d, which have prescribed behaviors in zones (2) and (3), we define solutions of (30) in V by prescribing behaviors in the remaining asymptotic zones: $F_a(z|\tau)\simeq z^t$ when $z\to 0^+$ for any τ; $F_b(z|\tau)\simeq(2\pi z/\,\mathrm{i})^t$ when $z\to\mathrm{i}\,0^+$ for any τ (in particular in zone (1)); $e^{2\pi\,\mathrm{i}\,x}F_e(z|\tau)\simeq(2\pi(\tau-z)/\,\mathrm{i})^t$ when $z=\tau-\mathrm{i}\,0^+$ for any τ; $e^{2\pi\,\mathrm{i}\,x}F_f(z|\tau)\simeq(z-\tau)^t$ when $z=\tau+0^+$ for any τ (in particular in zone (4)).

Then $F_0(z|\tau)=F_a(z|\tau)$, and $e^{-2\pi\,\mathrm{i}\,x}F_0(z-\tau|\tau)=F_f(z|\tau)$. We have $F_b=F_a(2\pi/\,\mathrm{i})^t$, $F_f=F_e(2\pi\,\mathrm{i})^{-t}$.

Let us now compute the ratio between F_b and F_c. Recall that $u=e^{2\pi\,\mathrm{i}\,z}$, $v=e^{2\pi\,\mathrm{i}(\tau-z)}$. Set $\bar F(u,v):=F(z|\tau)$. Using the expansion of $\theta(z|\tau)$, one shows that (30) has the form

$$\partial_u\bar F(u,v) = (\frac{A(u,v)}{u}+\frac{B(u,v)}{u-1})\bar F(u,v),$$

where $A(u,v)$ is holomorphic in the region $|v|<1/2$, $|u|<2$, and $A(u,0)=\tilde y$, $B(u,0)=t$. We have $\bar F_b(u,v)=(1-u)^t(1+\sum_k\sum_{s\le s(k)}\log(1-u)^k b_{ks}(u,v))$ and $\bar F_b(u,v)=u^{\tilde t}(1+\sum_k\sum_{s\le s(k)}\log(u)^k a_{ks}(u,v))$, with a_{ks},b_{ks} analytic, and $a_{ks}(0,v)=b_{ks}(1,v)=0$. The ratio $\bar F_b^{-1}\bar F_c$ is an analytic function of q only, which coincides with $\Phi(\tilde y,t)$ for $q=0$, so it has the form $\Phi(\tilde y,t)+\sum_{k>0}a_k(q)$, where $a_k(q)$ has degree k, is analytic in the neighborhood of $q=0$ and vanishes at $q=0$. Therefore

$$F_c(z|\tau)=F_b(z|\tau)\big(\Phi(\tilde y,t)+O(e^{2\pi\,\mathrm{i}\,\tau})\big).$$

In the same way, one proves that

$$F_e(z|\tau)=F_d\big(e^{-2\pi\,\mathrm{i}\,x}\Phi(-\tilde y-t,t)^{-1}+O(e^{2\pi\,\mathrm{i}\,\tau})\big).$$

Indeed, let us set $\bar G_d(u',v'):=e^{2\pi\,\mathrm{i}\,x}F_d(\tau+z'|\tau)$, $\bar G_e(u',v'):=e^{2\pi\,\mathrm{i}\,x}F_e(\tau+z'|\tau)$, where $u'=e^{2\pi\,\mathrm{i}(\tau+z')}$, $v'=e^{-2\pi\,\mathrm{i}\,z'}$, then $\bar G_d(u',v')\simeq(v')^{-\tilde y-t}e^{2\pi\,\mathrm{i}\,x}$ as $(u',v')\to(0^+,0^+)$ and $\bar G_e(u',v')\simeq(1-v')^t$ as $v'\to 1^-$ for any u', and both $\bar G_d$ and $\bar G_e$ are solutions of $\partial_{v'}\bar G(u',v')=[-(\tilde y+t)/v'+t/(v'-1)+O(u')]\bar G(v')$. Therefore $\bar G_d=\bar G_e[\Phi(-\tilde y-t,t)e^{2\pi\,\mathrm{i}\,x}+O(u')]$.

Combining these results, we get:

Lemma 4.14.

$$B(\tau)\simeq(2\pi\,\mathrm{i})^t\Phi(-\tilde y-t,t)e^{2\pi\,\mathrm{i}\,x}e^{2\,\mathrm{i}\,\pi\tau\tilde y}\Phi(\tilde y,t)^{-1}(2\pi/\,\mathrm{i})^{-t},$$

in the sense that the left (equivalently, right) ratio of these quantities has the form $1+\sum_{k>0}(\text{degree } k)O(\tau^{n(k)}e^{2\pi\,\mathrm{i}\,\tau})$ *for* $n(k)\ge 0$.

Recall that we have proved:

$$F(z|\tau)=F_0(z|\tau)\exp(-\frac{a_0}{2\pi\,\mathrm{i}}(\int_{\mathrm{i}}^{\tau}E_2+C)t)[F(\tau)],$$

where C is such that $\int_{\mathrm{i}}^{\tau}E_2+C=\tau+O(e^{2\pi\,\mathrm{i}\,\tau})$.

Set $X(\tau):=\exp(-\frac{a_0}{2\pi\,\mathrm{i}}(\int_{\mathrm{i}}^{\tau}E_2+C)t)[F(\tau)]$.

When $\tau \to \mathrm{i}\,\infty$, $X(\tau) = \exp(-\frac{\tau}{2\pi\,\mathrm{i}}([\Delta]+a_0 t))(1+\sum_{k>0}(\text{degree } k)O(\tau^{f(k)}e^{2\pi\,\mathrm{i}\,\tau}))$.
Then

$$\begin{aligned}
&\tilde{B} = F(z|\tau)^{-1}e^{2\pi\,\mathrm{i}\,x}F(z+\tau|\tau) = X(\tau)^{-1}B(\tau)X(\tau)\\
&= \mathrm{Ad}\Big((1+\sum_{k>0}(\text{degree } k)O(\tau^{f(k)}e^{2\pi\,\mathrm{i}\,\tau}))^{-1}\exp(\frac{\tau}{2\pi\,\mathrm{i}}([\Delta]+a_0 t))\Big)\\
&\Big(\big((2\pi\,\mathrm{i})^t\Phi(-\tilde{y}-t,t)e^{2\pi\,\mathrm{i}\,x}e^{2\pi\,\mathrm{i}\,\tau\tilde{y}}\Phi(\tilde{y},t)^{-1}(2\pi/\,\mathrm{i})^{-t}\big)\big(1+\sum_{k>0}(\text{degree } k)O(\tau^{n(k)}e^{2\pi\,\mathrm{i}\,\tau})\big)\Big),
\end{aligned}$$

where $\mathrm{Ad}(u)(x) = uxu^{-1}$.

$[\Delta]+a_0 t$ commutes with $\tilde{y}$ and t; assume for a moment that $\mathrm{Ad}(\exp(\frac{\tau}{2\pi\,\mathrm{i}}([\Delta]+a_0 t)))(e^{2\pi\,\mathrm{i}\,x}e^{2\pi\,\mathrm{i}\,\tau\tilde{y}}) = e^{2\pi\,\mathrm{i}\,x}$ (Lemma 4.15 below), then

$$\begin{aligned}
&\mathrm{Ad}(\exp(\frac{\tau}{2\pi\,\mathrm{i}}([\Delta]+a_0 t)))\Big((2\pi\,\mathrm{i})^t\Phi(-\tilde{y}-t,t)e^{2\pi\,\mathrm{i}\,x}e^{2\pi\,\mathrm{i}\,\tau\tilde{y}}\Phi(\tilde{y},t)^{-1}(2\pi/\,\mathrm{i})^{-t}\Big)\\
&= (2\pi\,\mathrm{i})^t\Phi(-\tilde{y}-t,t)e^{2\pi\,\mathrm{i}\,x}\Phi(\tilde{y},t)^{-1}(2\pi/\,\mathrm{i})^{-t}.
\end{aligned}$$

On the other hand, $\mathrm{Ad}(\exp(\frac{\tau}{2\pi\,\mathrm{i}}([\Delta]+a_0 t)))(1+\sum_{k>0}(\text{degree } k)O(\tau^{n(k)}e^{2\pi\,\mathrm{i}\,\tau}))$ has the form $1+\sum_{k>0}(\text{degree } k)O(\tau^{n'(k)}e^{2\pi\,\mathrm{i}\,\tau})$, where $n'(k)\geq 0$. It follows that

$$\begin{aligned}
\tilde{B} = &\,\mathrm{Ad}\big(1+\sum_{k>0}(\text{degree } k)O(\tau^{f(k)}e^{2\pi\,\mathrm{i}\,\tau})\big)\\
&\Big(\big((2\pi\,\mathrm{i})^t\Phi(-\tilde{y}-t,t)e^{2\pi\,\mathrm{i}\,x}\Phi(\tilde{y},t)^{-1}(2\pi/\,\mathrm{i})^{-t}\big)\big(1+\sum_{k>0}(\text{degree } k)O(\tau^{n'(k)}e^{2\pi\,\mathrm{i}\,\tau})\big)\Big);
\end{aligned}$$

now

$$\begin{aligned}
&\mathrm{Ad}\Big((2\pi\,\mathrm{i})^t\Phi(-\tilde{y}-t,t)e^{2\pi\,\mathrm{i}\,x}\Phi(\tilde{y},t)^{-1}(2\pi/\,\mathrm{i})^{-t}\Big)^{-1}(1+\sum_{k>0}(\text{degree } k)O(\tau^{f(k)}e^{2\pi\,\mathrm{i}\,\tau}))\\
&= 1+\sum_{k>0}(\text{degree } k)O(\tau^{f(k)}e^{2\pi\,\mathrm{i}\,\tau}),
\end{aligned}$$

so

$$\begin{aligned}
\tilde{B} =&\Big((2\pi\,\mathrm{i})^t\Phi(-\tilde{y}-t,t)e^{2\pi\,\mathrm{i}\,x}\Phi(\tilde{y},t)^{-1}(2\pi/\,\mathrm{i})^{-t}\Big)(1+\sum_{k>0}(\text{degree } k)O(\tau^{f(k)}e^{2\pi\,\mathrm{i}\,\tau}))\\
&(1+\sum_{k>0}(\text{degree } k)O(\tau^{n'(k)}e^{2\pi\,\mathrm{i}\,\tau}))\\
&= \Big((2\pi\,\mathrm{i})^t\Phi(-\tilde{y}-t,t)e^{2\pi\,\mathrm{i}\,x}\Phi(\tilde{y},t)^{-1}(2\pi/\,\mathrm{i})^{-t}\Big)(1+\sum_{k>0}(\text{degree } k)O(\tau^{n''(k)}e^{2\pi\,\mathrm{i}\,\tau}))
\end{aligned}$$

for $n''(k)\geq 0$. Since $\tilde{B}$ is constant w.r.t. τ, this implies

$$\tilde{B} = (2\pi\,\mathrm{i})^t\Phi(-\tilde{y}-t,t)e^{2\pi\,\mathrm{i}\,x}\Phi(\tilde{y},t)^{-1}(2\pi/\,\mathrm{i})^{-t},$$

as claimed.

We now prove the conjugation used above.

Lemma 4.15. *For any $\tau\in\mathbb{C}$, we have*

$$e^{\frac{\tau}{2\pi\,\mathrm{i}}([\Delta]+a_0 t)}e^{2\pi\,\mathrm{i}\,x}e^{-\frac{\tau}{2\pi\,\mathrm{i}}([\Delta]+a_0 t)}e^{2\,\mathrm{i}\,\pi\tau\tilde{y}} = e^{2\pi\,\mathrm{i}\,x}.$$

Proof. We have $[\Delta]+a_0 t = \Delta_0+\sum_{k\geq 0}a_{2k}(\delta_{2k}+(\mathrm{ad}\,x)^{2k}(t))$ (where $\delta_0 = 0$), so $[[\Delta]+a_0 t, x] = y-\sum_{k\geq 0}a_{2k}(\mathrm{ad}\,x)^{2k+1}(t)$. Recall that

$$\sum_{k\geq 0}a_{2k}u^{2k} = \frac{\pi^2}{\sin^2(\pi u)}-\frac{1}{u^2},$$

then $[[\Delta]+a_0t,x]=y-(\mathrm{ad}\,x)(\frac{\pi^2}{\sin^2(\pi\,\mathrm{ad}\,x)}-\frac{1}{(\mathrm{ad}\,x)^2})(t)$. So

$$e^{-2\pi\,\mathrm{i}\,x}(\frac{1}{2\pi\,\mathrm{i}}([\Delta]+a_0t))e^{2\pi\,\mathrm{i}\,x}=\frac{1}{2\pi\,\mathrm{i}}([\Delta]+a_0t)+\frac{e^{-2\pi\,\mathrm{i}\,\mathrm{ad}\,x}-1}{\mathrm{ad}\,x}([x,\frac{1}{2\pi\,\mathrm{i}}([\Delta]+a_0t)])$$
$$=\frac{1}{2\pi\,\mathrm{i}}([\Delta]+a_0t)-\frac{1}{2\pi\,\mathrm{i}}\frac{e^{-2\pi\,\mathrm{i}\,\mathrm{ad}\,x}-1}{\mathrm{ad}\,x}\big(y-(\mathrm{ad}\,x)(\frac{\pi^2}{\sin^2(\pi\,\mathrm{ad}\,x)}-\frac{1}{(\mathrm{ad}\,x)^2})(t)\big).$$

We have

$$-\frac{1}{2\pi\,\mathrm{i}}\frac{e^{-2\pi\,\mathrm{i}\,\mathrm{ad}\,x}-1}{\mathrm{ad}\,x}\big(y-(\mathrm{ad}\,x)(\frac{\pi^2}{\sin^2(\pi\,\mathrm{ad}\,x)}-\frac{1}{(\mathrm{ad}\,x)^2})(t)\big)=-2\pi\,\mathrm{i}\,\tilde{y},$$

therefore we get

$$e^{-2\pi\,\mathrm{i}\,x}(\frac{1}{2\pi\,\mathrm{i}}([\Delta]+a_0t))e^{2\pi\,\mathrm{i}\,x}=\frac{1}{2\pi\,\mathrm{i}}([\Delta]+a_0t)-2\pi\,\mathrm{i}\,\tilde{y}.$$

Multiplying by τ, taking the exponential, and using the fact that $[\Delta]+a_0t$ commutes with $\tilde{y}$, we get $e^{-2\pi\,\mathrm{i}\,x}e^{\frac{\tau}{2\pi\,\mathrm{i}}([\Delta]+a_0t)}e^{2\pi\,\mathrm{i}\,x}=e^{\frac{\tau}{2\pi\,\mathrm{i}}([\Delta]+a_0t)}e^{-2\pi\,\mathrm{i}\,\tau\tilde{y}}$, which proves the lemma. □

This ends the proof of Theorem 4.11. □

5. Construction of morphisms $\Gamma_{1,[n]}\to\mathbf{G}_n\rtimes S_n$

In this section, we fix a field $\mathbf{k}$ of characteristic zero. We denote the algebras $\bar{\mathfrak{t}}^{\mathbf{k}}_{1,n}$, $\mathfrak{t}^{\mathbf{k}}_n$ simply by $\bar{\mathfrak{t}}_{1,n}$, $\mathfrak{t}_n$. The above group $\mathbf{G}_n$ is the set of $\mathbb{C}$-points of a group scheme defined over $\mathbb{Q}$, and we now again denote by $\mathbf{G}_n$ the set of its $\mathbf{k}$-points.

5.1. **Construction of morphisms $\Gamma_{1,[n]}\to\mathbf{G}_n\rtimes S_n$ from a 5-uple $(\Phi_\lambda,\tilde{A},\tilde{B},\tilde{\Theta},\tilde{\Psi})$.** Let Φ_λ be a λ-associator defined over $\mathbf{k}$. This means that $\Phi_\lambda\in\exp(\hat{\mathfrak{t}}_3)$ (the Lie algebras are now over $\mathbf{k}$),

$$\Phi_\lambda^{3,2,1}=\Phi_\lambda^{-1},\quad \Phi_\lambda^{2,3,4}\Phi_\lambda^{1,23,4}\Phi_\lambda^{1,2,3}=\Phi_\lambda^{1,2,34}\Phi_\lambda^{12,3,4},\tag{37}$$

$$e^{\lambda t_{31}/2}\Phi_\lambda^{2,3,1}e^{\lambda t_{23}/2}\Phi_\lambda e^{\lambda t_{12}/2}\Phi_\lambda^{3,1,2}=e^{\lambda(t_{12}+t_{23}+t_{13})/2}.\tag{38}$$

E.g., the KZ associator is a $2\pi\,\mathrm{i}$-associator over $\mathbb{C}$.

Proposition 5.1. *If $\tilde{\Theta},\tilde{\Psi}\in\mathbf{G}_1$ and $\tilde{A},\tilde{B}\in\exp(\hat{\bar{\mathfrak{t}}}_{1,2})$ satisfy: the "$\Gamma_{1,1}$ identities" (27), the "$\Gamma_{1,2}$ identities" (28), (29), and the "$\Gamma_{1,[3]}$ identities" (23), (22), (26) (with $2\pi\,\mathrm{i}$ replaced by λ), as well as $\tilde{A}^{\emptyset,1}=\tilde{A}^{1,\emptyset}=\tilde{B}^{\emptyset,1}=\tilde{B}^{1,\emptyset}=1$, then one defines a morphism $\Gamma_{1,[n]}\to\mathbf{G}_n\rtimes S_n$ by*

$$\Theta\mapsto[\tilde{\Theta}]e^{\mathrm{i}\,\frac{\pi}{2}\sum_{i<j}\bar{t}_{ij}},\quad \Psi\mapsto[\tilde{\Psi}]e^{\mathrm{i}\,\frac{\pi}{6}\sum_{i<j}\bar{t}_{ij}},\quad \sigma_i\mapsto\{\Phi_\lambda^{1\ldots i-1,i,i+1}\}^{-1}e^{\lambda\bar{t}_{i,i+1}/2}(i,i+1)\{\Phi_\lambda^{1\ldots i-1,i,i+1}\},$$

$$C_{jk}\mapsto\{\Phi_{\lambda,j}^{-1}\Phi_\lambda^{j,j+1,\ldots n}\ldots\Phi_\lambda^{j\ldots,k-1,\ldots n}(e^{\lambda t_{12}})^{j\ldots k-1,k\ldots n}(\Phi_\lambda^{j,j+1,\ldots n}\ldots\Phi_\lambda^{j\ldots,k-1,\ldots n})^{-1}\Phi_{\lambda,j}\},$$

$$A_i\mapsto\{\Phi_{\lambda,i}\}^{-1}\tilde{A}^{1\ldots i-1,i\ldots n}\{\Phi_{\lambda,i}\},\quad B_i\mapsto\{\Phi_{\lambda,i}\}^{-1}\tilde{B}^{1\ldots i-1,i\ldots n}\{\Phi_{\lambda,i}\},$$

where $\Phi_{\lambda,i}=\Phi_\lambda^{1\ldots i-1,i,i+1\ldots n}\ldots\Phi_\lambda^{1\ldots n-2,n-1,n}$.

According to Section 4.4, the representations γ_n are obtained by the procedure described in this proposition from the KZ associator, $\tilde{\Theta},\tilde{\Psi}$ arising from γ_1, and $\tilde{A},\tilde{B}$ arising from γ_2.

Note also that the analogue of (22) is equivalent to the pair of equations

$$e^{\lambda\bar{t}_{12}/2}\tilde{A}^{2,1}e^{\lambda\bar{t}_{12}/2}\tilde{A}=1,\quad (e^{\lambda\bar{t}_{12}/2}\tilde{A})^{3,12}\Phi_\lambda^{3,1,2}(e^{\lambda\bar{t}_{12}/2}\tilde{A})^{2,31}\Phi_\lambda^{2,3,1}(e^{\lambda\bar{t}_{12}/2}\tilde{A})^{1,23}\Phi_\lambda^{1,2,3}=1,$$

and similarly (23) is equivalent to the same equations, with $\tilde{A},\lambda$ replaced by $\tilde{B},-\lambda$.

Remark 5.2. One can prove that it Φ_λ satisfies only the pentagon equation and $\tilde\Theta, \tilde\Psi, \tilde A, \tilde B$ satisfy the the "$\Gamma_{1,1}$ identities" (27), the "$\Gamma_{1,2}$ identities" (28), (29), and the "$\Gamma_{1,3}$ identities" (24), (26), then the above formulas (removing σ_i) define a morphism $\Gamma_{1,n} \to \mathbf{G}_n$. In the same way, if Φ_λ satisfies all the associator conditions and $\tilde A, \tilde B$ satisfy the $\Gamma_{1,[3]}$ identities (22), (23), (26), then the above formulas (removing Θ, Ψ) define a morphism $\overline{\mathrm{B}}_{1,n} \to \exp(\hat{\bar{\mathfrak{t}}}_{1,n}) \rtimes S_n$.

Proof. Let us prove that the identity $(A_i, A_j) = 1$ $(i < j)$ is preserved. Applying $x \mapsto x^{1...i-1,i...j-1,j...n}$ to the first identity of (24), we get

$$(\tilde A^{1...i-1,i...n}, \Phi_\lambda^{1...,i...j-1,...n} \tilde A^{1...j-1,j...n} (\Phi_\lambda^{-1})^{1...,i...j-1,...n}) = 1.$$

The pentagon identity implies

$$\Phi_\lambda^{1...,i,...n} ... \Phi_\lambda^{1...,j-1,...n} = (\Phi_\lambda^{i,i+1,...n} ... \Phi_\lambda^{i...,j-1,..,n}) \Phi_\lambda^{1...,i...j-1,...n} (\Phi_\lambda^{1...,i,...j-1} ... \Phi_\lambda^{1...,j-2,j-1}), \tag{39}$$

so the above identity is rewritten

$$\big(\Phi_\lambda^{i,i+1,...n} ... \Phi_\lambda^{i...,j-1,..,n} \tilde A^{1...i-1,i...n} (\Phi_\lambda^{i,i+1,...n} ... \Phi_\lambda^{i...,j-1,..,n})^{-1}, \Phi_\lambda^{1...,i,...n} ... \Phi_\lambda^{1...,j-1,...n}$$
$$(\Phi_\lambda^{1...,i,...j-1} ... \Phi_\lambda^{1...,j-2,...j-1})^{-1} \tilde A^{1...j-1,j...n} \Phi_\lambda^{1...,i,...j-1} ... \Phi_\lambda^{1...,j-2,...j-1} (\Phi_\lambda^{1...,i,...n} ... \Phi_\lambda^{1...,j-1,...n})^{-1}\big) = 1.$$

Now $\Phi_\lambda^{i,i+1,...n}, ..., \Phi_\lambda^{i...,j-1,..,n}$ commute with $\tilde A^{1...i-1,i...n}$, and $\Phi_\lambda^{1...,i,...j-1}, ..., \Phi_\lambda^{1...,j-2,...j-1}$ commute with $\Phi_\lambda^{1...,i,...j-1} ... \Phi_\lambda^{1...,j-2,...j-1}$, which implies

$$(\tilde A^{1...i-1,i...n}, \Phi_\lambda^{1...,i,...n} ... \Phi_\lambda^{1...,j-1,...n} \tilde A^{1...j-1,j...n} (\Phi_\lambda^{1...,i,...n} ... \Phi_\lambda^{1...,j-1,...n})^{-1}) = 1,$$

so that $(A_i, A_j) = 1$ is preserved. In the same way, one shows that $(B_i, B_j) = 1$ is preserved.

Let us show that $(B_k, A_k A_j^{-1}) = C_{jk}$ is preserved (if $j \le k$).

$$(\Phi_{\lambda,k}^{-1} \tilde B^{1...k-1,k...n} \Phi_{\lambda,k}, \Phi_{\lambda,k}^{-1} \tilde A^{1...k-1,k...n} \Phi_{\lambda,k} \Phi_{\lambda,j}^{-1} (\tilde A^{1...j-1,j...n})^{-1} \Phi_{\lambda,j})$$
$$= \Phi_{\lambda,j}^{-1} \big((\Phi_\lambda^{1...,j,...n} ... \Phi_\lambda^{1...,k-1,...n}) \tilde B^{1...k-1,k...n} (\Phi_\lambda^{1...,j,...n} ... \Phi_\lambda^{1...,k-1,...n})^{-1},$$
$$(\Phi_\lambda^{1...,j,...n} ... \Phi_\lambda^{1...,k-1,...n}) \tilde A^{1...k-1,k...n} (\Phi_\lambda^{1...,j,...n} ... \Phi_\lambda^{1...,k-1,...n})^{-1} (\tilde A^{1...j-1,j...n})^{-1}\big) \Phi_{\lambda,j}$$
$$= \Phi_{\lambda,j}^{-1} \big(\Phi_\lambda^{j,j+1,...n} ... \Phi_\lambda^{j...,k-1,...n} \Phi_\lambda^{1...,j...k-1,...n} \tilde B^{1...k-1,k...n} (\Phi_\lambda^{j,j+1,...n} ... \Phi_\lambda^{j...,k-1,...n} \Phi_\lambda^{1...,j...k-1,...n})^{-1},$$
$$\Phi_\lambda^{j,j+1,...n} ... \Phi_\lambda^{j...,k-1,...n} \Phi_\lambda^{1...,j...k-1,...n} \tilde A^{1...k-1,k...n} (\Phi_\lambda^{j,j+1,...n} ... \Phi_\lambda^{j...,k-1,...n} \Phi_\lambda^{1...,j...k-1,...n})^{-1}$$
$$(\tilde A^{1...j-1,j...n})^{-1}\big) \Phi_{\lambda,j} = \Phi_{\lambda,j}^{-1} \Phi_\lambda^{j,j+1,...n} ... \Phi_\lambda^{j...,k-1,...n} \big(\Phi_\lambda^{1...,j...k-1,...n} \tilde B^{1...k-1,k...n} (\Phi_\lambda^{1...,j...k-1,...n})^{-1},$$
$$\Phi_\lambda^{1...,j...k-1,...n} \tilde A^{1...k-1,k...n} (\Phi_\lambda^{1...,j...k-1,...n})^{-1} (\tilde A^{1...j-1,j...n})^{-1}\big) (\Phi_\lambda^{j,j+1,...n} ... \Phi_\lambda^{j...,k-1,...n})^{-1} \Phi_{\lambda,j}$$
$$= \Phi_{\lambda,j}^{-1} \Phi_\lambda^{j,j+1,...n} ... \Phi_\lambda^{j...,k-1,...n} \{\Phi(\tilde B^{12,3}, \tilde A^{12,3} \Phi_\lambda^{-1} (\tilde A^{1,23})^{-1} \Phi_\lambda) \Phi_\lambda^{-1}\}^{1...,j...k-1,...n}$$
$$(\Phi_\lambda^{j,j+1,...n} ... \Phi_\lambda^{j...,k-1,...n})^{-1} \Phi_{\lambda,j}$$
$$= \Phi_{\lambda,j}^{-1} \Phi_\lambda^{j,j+1,...n} ... \Phi_\lambda^{j...,k-1,...n} (e^{2\pi \mathrm{i} \bar t_{12}})^{j...k-1,k...n} (\Phi_\lambda^{j,j+1,...n} ... \Phi_\lambda^{j...,k-1,...n})^{-1} \Phi_{\lambda,j},$$

where the second identity uses (39) and the invariance of Φ_λ, the third identity uses the fact that $\Phi_\lambda^{j,j+1,...n}, ..., \Phi_\lambda^{j...,k-1,...n}$ commute with $\tilde A^{1..j-1,j..n}$ (again by the invariance of Φ_λ), and the last identity uses (26). So $(B_k, A_k A_j^{-1}) = C_{jk}$ is preserved. One shows similarly that

$$\big(\Phi_{\lambda,k}^{-1} \tilde B^{1...k-1,k...n} \Phi_{\lambda,k} \Phi_{\lambda,j}^{-1} (\tilde B^{1...j-1,j...n})^{-1} \Phi_{\lambda,j}, \Phi_{\lambda,k}^{-1} \tilde A^{1...k-1,k...n} \Phi_{\lambda,k}\big)$$
$$= \Phi_j^{-1} \Phi^{j,j+1,...n} ... \Phi^{j...,k-1,...n} (e^{2\pi \mathrm{i} \bar t_{12}})^{j...k-1,k...n} (\Phi_\lambda^{j,j+1,...n} ... \Phi_\lambda^{j...,k-1,...n})^{-1} \Phi_{\lambda,j},$$

so that $(B_k B_j^{-1}, A_k) = C_{jk}$ is preserved.

Let us show that $(A_i, C_{jk}) = 1$ $(i \leq j \leq k)$ is preserved. We have

$$\big(\Phi_{\lambda,i}^{-1}\tilde{A}^{1\ldots i-1,i\ldots n}\Phi_{\lambda,i}, \Phi_{\lambda,j}^{-1}\Phi_\lambda^{j,j+1,\ldots n}\ldots\Phi_\lambda^{j\ldots,k-1,\ldots n}(e^{2\pi\,\mathrm{i}\,\bar{t}_{12}})^{j\ldots k-1,k\ldots n}(\Phi_\lambda^{j,j+1,\ldots n}\ldots\Phi_\lambda^{j\ldots,k-1,\ldots n})^{-1}\Phi_{\lambda,j}\big)$$
$$= \Phi_{\lambda,i}^{-1}\big(\tilde{A}^{1\ldots i-1,i\ldots n}, \Phi_\lambda^{1\ldots,i,\ldots n}\ldots\Phi_\lambda^{1\ldots,j-1,\ldots n}\Phi_\lambda^{j,j+1,\ldots n}\ldots\Phi_\lambda^{j\ldots,k-1,\ldots n}(e^{2\pi\,\mathrm{i}\,\bar{t}_{12}})^{j\ldots k-1,k\ldots n}$$
$$(\Phi_\lambda^{1\ldots,i,\ldots n}\ldots\Phi_\lambda^{1\ldots,j-1,\ldots n}\Phi_\lambda^{j,j+1,\ldots n}\ldots\Phi_\lambda^{j\ldots,k-1,\ldots n})^{-1}\big)\Phi_{\lambda,i}$$
$$= \Phi_{\lambda,i}^{-1}\big(\tilde{A}^{1\ldots i-1,i\ldots n}, \Phi_\lambda^{i,i+1,\ldots n}\ldots\Phi_\lambda^{i\ldots,j-1,\ldots n}\Phi_\lambda^{1\ldots,i\ldots j-1,\ldots n}\Phi_\lambda^{1\ldots,i,\ldots j-1}\ldots\Phi_\lambda^{1\ldots,j-2,j-1}$$
$$\Phi_\lambda^{j,j+1,\ldots n}\ldots\Phi_\lambda^{j\ldots,k-1,\ldots n}(e^{2\pi\,\mathrm{i}\,\bar{t}_{12}})^{j\ldots k-1,k\ldots n}$$
$$(\Phi_\lambda^{i,i+1,\ldots n}\ldots\Phi_\lambda^{i\ldots,j-1,\ldots n}\Phi_\lambda^{1\ldots,i\ldots j-1,\ldots n}\Phi_\lambda^{1\ldots,i,\ldots j-1}\ldots\Phi_\lambda^{1\ldots,j-2,j-1}\Phi_\lambda^{j,j+1,\ldots n}\ldots\Phi_\lambda^{j\ldots,k-1,\ldots n}\big)^{-1}\big)\Phi_{\lambda,i}$$
$$= \Phi_{\lambda,i}^{-1}\big(\tilde{A}^{1\ldots i-1,i\ldots n}, \Phi_\lambda^{i,i+1,\ldots n}\ldots\Phi_\lambda^{i\ldots,j-1,\ldots n}\Phi_\lambda^{1\ldots,i\ldots j-1,\ldots n}\Phi_\lambda^{j,j+1,\ldots n}\ldots\Phi_\lambda^{j\ldots,k-1,\ldots n}$$
$$(e^{2\pi\,\mathrm{i}\,\bar{t}_{12}})^{j\ldots k-1,k\ldots n}(\Phi_\lambda^{i,i+1,\ldots n}\ldots\Phi_\lambda^{i\ldots,j-1,\ldots n}\Phi_\lambda^{1\ldots,i\ldots j-1,\ldots n}\Phi_\lambda^{j,j+1,\ldots n}\ldots\Phi_\lambda^{j\ldots,k-1,\ldots n})^{-1}\big)\Phi_{\lambda,i}$$
$$= \Phi_{\lambda,i}^{-1}\Phi_\lambda^{i,i+1,\ldots n}\ldots\Phi_\lambda^{i\ldots,j-1,\ldots n}\big(\tilde{A}^{1\ldots i-1,i\ldots n}, \Phi_\lambda^{1\ldots,i\ldots j-1,\ldots n}\Phi_\lambda^{j,j+1,\ldots n}\ldots\Phi_\lambda^{j\ldots,k-1,\ldots n}$$
$$(e^{2\pi\,\mathrm{i}\,\bar{t}_{12}})^{j\ldots k-1,k\ldots n}(\Phi_\lambda^{1\ldots,i\ldots j-1,\ldots n}\Phi_\lambda^{j,j+1,\ldots n}\ldots\Phi_\lambda^{j\ldots,k-1,\ldots n})^{-1}\big)(\Phi_\lambda^{i,i+1,\ldots n}\ldots\Phi_\lambda^{i\ldots,j-1,\ldots n})^{-1}\Phi_{\lambda,i}$$
$$= \Phi_{\lambda,i}^{-1}\Phi_\lambda^{i,i+1,\ldots n}\ldots\Phi_\lambda^{i\ldots,j-1,\ldots n}(\tilde{A}^{1\ldots i-1,i\ldots n}, \Phi_\lambda^{j,j+1,\ldots n}\ldots\Phi_\lambda^{j\ldots,k-1,\ldots n}\Phi_\lambda^{1\ldots,i\ldots j-1,\ldots n}$$
$$(e^{2\pi\,\mathrm{i}\,\bar{t}_{12}})^{j\ldots k-1,k\ldots n}(\Phi_\lambda^{j,j+1,\ldots n}\ldots\Phi_\lambda^{j\ldots,k-1,\ldots n}\Phi_\lambda^{1\ldots,i\ldots j-1,\ldots n})^{-1})(\Phi_\lambda^{i,i+1,\ldots n}\ldots\Phi_\lambda^{i\ldots,j-1,\ldots n})^{-1}\Phi_{\lambda,i}$$
$$= 1,$$

where the second equality follows from the generalized pentagon identity (39), the third equality follows from the fact that $\Phi_\lambda^{1\ldots,i,\ldots j-1}$, ..., $\Phi_\lambda^{1\ldots,j-2,j-1}$ commute with $(e^{2\pi\,\mathrm{i}\,\bar{t}_{12}})^{j\ldots k-1,k\ldots n}$, $\Phi_\lambda^{j,j+1,\ldots n}$, ..., $\Phi_\lambda^{j\ldots,k-1,\ldots n}$, the fourth equality follows from the fact that $\Phi_\lambda^{i,i+1,\ldots n}$, ..., $\Phi_\lambda^{i\ldots,j-1,\ldots n}$ commute with $\tilde{A}^{1\ldots i-1,i\ldots n}$ (as Φ_λ is invariant), the last equality follows from the fact that $\Phi_\lambda^{1\ldots,i\ldots j-1,j\ldots n}$ commutes with $\Phi_\lambda^{j,j+1,\ldots n}$, ..., $\Phi_\lambda^{j\ldots,k-1,\ldots n}$ (again as Φ_λ is invariant) and with $(e^{2\pi\,\mathrm{i}\,\bar{t}_{12}})^{j\ldots k-1,k\ldots n}$ (as t_{34} commutes with the image of $\mathfrak{t}_3 \to \mathfrak{t}_4$, $x \mapsto x^{1,2,34}$). Therefore $(A_i, C_{jk}) = 1$ is preserved. One shows similarly that $(B_i, C_{jk}) = 1$ $(i \leq j \leq k)$, $X_{i+1} = \sigma_i X_i \sigma_i$ and $Y_{i+1} = \sigma_i^{-1} Y_i \sigma_i^{-1}$ are preserved.

The fact that the relations $\Theta A_i \Theta^{-1} = B_i^{-1}$, $\Theta B_i \Theta^{-1} = B_i A_i B_i^{-1}$, $\Psi A_i \Psi^{-1} = A_i$, $\Psi B_i \Psi^{-1} = B_i A_i$, are preserved follows from the identities (28), (29) and that if we denote by $x \mapsto [x]_n$ the morphism $\mathfrak{d} \to \mathfrak{d} \ltimes \bar{\mathfrak{t}}_{1,n}$ defined above, then: (a) Φ_i commutes with $\sum_{i,j|i<j} \bar{t}_{ij}$ and with the image of $\mathfrak{d} \to \mathfrak{d} \ltimes \bar{\mathfrak{t}}_{1,n}$, $x \mapsto [x]_n$; (b) for $x \in \mathfrak{d}$, $y \in \bar{\mathfrak{t}}_{1,2}$, we have $[[x]_n, y^{1\ldots i-1,i\ldots n}] = [[x]_2, y]^{1\ldots i-1,i\ldots n}$. Let us prove (a): the first part follows from the fact that Φ commutes with $t_{12}+t_{13}+t_{23}$; the second part follows from the fact that X, d, Δ_0 and $\delta_{2n}+\sum_{k<l}(\mathrm{ad}\,\bar{x}_k)^{2n}(\bar{t}_{kl})$ commute with $\bar{t}_{ij}$ for any $i<j$. Let us prove (b): the identity holds for $[x, x']$ whenever it holds for x and for x', so it suffices to check it for x a generator of $\mathfrak{d}$; x being such a generator, both sides are (as functions of y) derivations $\bar{\mathfrak{t}}_{1,2} \to \bar{\mathfrak{t}}_{1,n}$ w.r.t. the morphism $\bar{\mathfrak{t}}_{1,2} \to \bar{\mathfrak{t}}_{1,n}$, $y \mapsto y^{1\ldots i-1,i\ldots n}$, so it suffices to check the identity for y a generator of $\bar{\mathfrak{t}}_{1,2}$. The identity is obvious if $x \in \{\Delta_0, d, X\}$ and $y \in \{\bar{x}_1, \bar{y}_1, \bar{x}_2, \bar{y}_2\}$. If $x = \delta_{2s}$ and $y = \bar{x}_1$, then the identity holds because we have

$$[\delta_{2s} + (\mathrm{ad}\,\bar{x}_1)^{2s}(\bar{t}_{12}), \bar{x}_1]^{1\ldots i-1,i\ldots n} = -\big((\mathrm{ad}\,\bar{x}_1)^{2s+1}(\bar{t}_{12})\big)^{1\ldots i-1,i\ldots n}$$
$$= -(\mathrm{ad}(\sum_{u'=1}^{i-1}\bar{x}_{u'}))^{2s+1}(\sum_{1\leq u<i\leq v\leq n}\bar{t}_{uv}) = -\sum_{1\leq u<i\leq v\leq n}(\mathrm{ad}\,\bar{x}_u)^{2s+1}(\bar{t}_{uv}),$$

while

$$[\delta_{2s}+\sum_{1\le u<v\le n}(\mathrm{ad}\,\bar x_u)^{2s}(\bar t_{uv}),\sum_{u'=1}^{i-1}\bar x_{u'}]=[\sum_{1\le u<i\le v\le n}(\mathrm{ad}\,\bar x_u)^{2s}(\bar t_{uv}),\sum_{u'=1}^{i-1}\bar x_{u'}]$$
$$=-\sum_{1\le u<i\le v\le n}(\mathrm{ad}\,\bar x_u)^{2s+1}(\bar t_{uv})$$

where the first equality follows from the fact that $(\mathrm{ad}\,\bar x_u)^{2s}(\bar t_{uv})$ commutes with $\sum_{u'=1}^{i-1}\bar x_{u'}$ whenever $u<v<i$ or $i\le u<v$. If $x=\delta_{2s}$ and $y=\bar x_2$, then the identity follows because $[\delta_{2s}+(\mathrm{ad}\,\bar x_1)^{2s}(\bar t_{12}),\bar x_1+\bar x_2]=0$ and $[\delta_{2s}+\sum_{1\le u<v\le n}(\mathrm{ad}\,\bar x_u)^{2s}(\bar t_{uv}),\sum_{u'=1}^{n}\bar x_{u'}]=0$.

If $x=\delta_{2s}$ and $y=\bar y_1$, then

$$[\delta_{2s}+(\mathrm{ad}\,\bar x_1)^{2s}(\bar t_{12}),\bar y_1]^{1\dots i-1,i\dots n}$$
$$=\{\frac12\sum_{p+q=2s-1}[(\mathrm{ad}\,\bar x_1)^p(\bar t_{12}),(-\mathrm{ad}\,\bar x_1)^q(\bar t_{12})]+[(\mathrm{ad}\,\bar x_1)^{2s}(\bar t_{12}),\bar y_1]\}^{1\dots i-1,i\dots n}$$
$$=\frac12\sum_{p+q=2s-1}[\sum_{1\le u<i\le v\le n}(\mathrm{ad}\,\bar x_u)^p(\bar t_{uv}),\sum_{1\le u'<i\le v'\le n}(\mathrm{ad}\,\bar x_{u'})^q(\bar t_{u'v'})]$$
$$+[\sum_{1\le u<i\le v\le n}(\mathrm{ad}\,\bar x_u)^{2s}(\bar t_{uv}),\bar y_1+\dots+\bar y_{i-1}];$$

on the other hand,

$$[\delta_{2s}+\sum_{1\le u<v\le n}(\mathrm{ad}\,\bar x_u)^{2s}(\bar t_{uv}),\bar y_1+\dots+\bar y_{i-1}]$$
$$=\sum_{1\le u<v\le n}[(\mathrm{ad}\,\bar x_u)^{2s}(\bar t_{uv}),\bar y_1+\dots+\bar y_{i-1}]+\sum_{u=1}^{i-1}\sum_{v|v\neq u}\sum_{p+q=2s-1}\frac12[(\mathrm{ad}\,\bar x_u)^p(\bar t_{uv}),(-\mathrm{ad}\,\bar x_u)^q(\bar t_{uv})]$$
$$=\sum_{1\le u<v\le n}[(\mathrm{ad}\,\bar x_u)^{2s}(\bar t_{uv}),\bar y_1+\dots+\bar y_{i-1}]+\sum_{1\le u<i\le v\le n}\sum_{p+q=2s-1}\frac12[(\mathrm{ad}\,\bar x_u)^p(\bar t_{uv}),(-\mathrm{ad}\,\bar x_u)^q(\bar t_{uv})],$$

where the second equality follows from the fact that $[(\mathrm{ad}\,\bar x_u)^p(\bar t_{uv}),(-\bar x_u)^q(\bar t_{uv})]+[(\mathrm{ad}\,\bar x_v)^p(\bar t_{uv}),(-\mathrm{ad}\,\bar x_v)^q(\bar t_{uv})]=0$ as $p+q$ is odd.

Then

$$[\delta_{2s}+(\mathrm{ad}\,\bar x_1)^{2s}(\bar t_{12}),\bar y_1]^{1\dots i-1,i\dots n}-[\delta_{2s}+\sum_{1\le u<v\le n}(\mathrm{ad}\,\bar x_u)^{2s}(\bar t_{uv}),\bar y_1+\dots+\bar y_{i-1}]$$
$$=-\sum_{1\le u<v<i}[(\mathrm{ad}\,\bar x_u)^{2s}(\bar t_{uv}),\bar y_1+\dots+\bar y_{i-1}]-\sum_{i\le u<v\le n}[(\mathrm{ad}\,\bar x_u)^{2s}(\bar t_{uv}),\bar y_1+\dots+\bar y_{i-1}]$$
$$\frac12\sum_{p+q=2s-1}\sum_{\substack{1\le u<i\le v\le n\\ 1\le u'<i\le v'\le n,(u,v)\neq(u',v')}}[(\mathrm{ad}\,\bar x_u)^p(\bar t_{uv}),(-\mathrm{ad}\,\bar x_{u'})^q(\bar t_{u'v'})]$$
$$=\sum_{1\le u<v<i}[(\mathrm{ad}\,\bar x_u)^{2s}(\bar t_{uv}),\bar y_i+\dots+\bar y_n]-\sum_{i\le u<v\le n}[(\mathrm{ad}\,\bar x_u)^{2s}(\bar t_{uv}),\bar y_1+\dots+\bar y_{i-1}]$$
$$+\frac12\sum_{p+q=2s-1}\sum_{\substack{1\le u<i\le v\le n\\ 1\le u<i\le v'\le n,v\neq v'}}[(\mathrm{ad}\,\bar x_u)^p(\bar t_{uv}),(-\mathrm{ad}\,\bar x_u)^q(\bar t_{uv'})]$$
$$+\frac12\sum_{p+q=2s-1}\sum_{\substack{1\le u<i\le v\le n\\ 1\le u'<i\le v\le n,u\neq u'}}[(\mathrm{ad}\,\bar x_u)^p(\bar t_{uv}),(-\mathrm{ad}\,\bar x_{u'})^q(\bar t_{u'v})]$$

where the second equality follows from the centrality of $\bar y_1+...+\bar y_n$, the last equality follows for the fact that $(\mathrm{ad}\,\bar x_u)^p(\bar t_{uv})$ and $(-\mathrm{ad}\,\bar x_{u'})^q(\bar t_{u'v'})$ commute for u,v,u',v' all distinct. Since $p+q$ is odd, it follows that

$$[\delta_{2s}+(\mathrm{ad}\,\bar x_1)^{2s}(\bar t_{12}),\bar y_1]^{1...i-1,i...n}-[\delta_{2s}+\sum_{1\le u<v\le n}(\mathrm{ad}\,\bar x_u)^{2s}(\bar t_{uv}),\bar y_1+...+\bar y_{i-1}]$$
$$=\sum_{1\le u<v<i}[(\mathrm{ad}\,\bar x_u)^{2s}(\bar t_{uv}),\bar y_i+...+\bar y_n]-\sum_{i\le u<v\le n}[(\mathrm{ad}\,\bar x_u)^{2s}(\bar t_{uv}),\bar y_1+...+\bar y_{i-1}]$$
$$+\sum_{p+q=2s-1}\ \sum_{1\le u<i\le v<v'\le n}[(\mathrm{ad}\,\bar x_u)^p(\bar t_{uv}),(-\mathrm{ad}\,\bar x_u)^q(\bar t_{uv'})]$$
$$+\sum_{p+q=2s-1}\ \sum_{1\le u<u'<i\le v\le n}[(\mathrm{ad}\,\bar x_u)^p(\bar t_{uv}),(-\mathrm{ad}\,\bar x_{u'})^q(\bar t_{u'v})].$$

Now if $1\le u<v<i$, we have

$$[(\mathrm{ad}\,\bar x_u)^{2s}(\bar t_{uv}),\bar y_i+...+\bar y_n]=\sum_{p+q=2s-1}(\mathrm{ad}\,\bar x_u)^p\,\mathrm{ad}(\bar t_{ui}+...+\bar t_{un})(\mathrm{ad}\,\bar x_u)^q(\bar t_{uv})$$
$$=\sum_{w=i}^n\sum_{p+q=2s-1}(\mathrm{ad}\,\bar x_u)^p[\bar t_{uw},(-\mathrm{ad}\,\bar x_v)^q(\bar t_{uv})]=\sum_{w=i}^n\sum_{p+q=2s-1}(\mathrm{ad}\,\bar x_u)^p(-\mathrm{ad}\,\bar x_v)^q([\bar t_{uw},\bar t_{uv}])$$
$$=-\sum_{w=i}^n\sum_{p+q=2s-1}(\mathrm{ad}\,\bar x_u)^p(-\mathrm{ad}\,\bar x_v)^q([\bar t_{uw},\bar t_{vw}])=-\sum_{w=i}^n\sum_{p+q=2s-1}[(\mathrm{ad}\,\bar x_u)^p(\bar t_{uw}),(-\mathrm{ad}\,\bar x_v)^q(\bar t_{vw})];$$

one shows in the same way that if $i\le u<v\le n$, then $[(\mathrm{ad}\,\bar x_u)^{2s}(\bar t_{uv}),\bar y_1+...+\bar y_{i-1}]=\sum_{w=1}^{i-1}\sum_{p+q=2s-1}[(\mathrm{ad}\,\bar x_u)^p(\bar t_{uw}),(-\mathrm{ad}\,\bar x_v)^q(\bar t_{vw})]$; all this implies that

$$[\delta_{2s}+(\mathrm{ad}\,\bar x_1)^{2s}(\bar t_{12}),\bar y_1]^{1...i-1,i...n}-[\delta_{2s}+\sum_{1\le u<v\le n}(\mathrm{ad}\,\bar x_u)^{2s}(\bar t_{uv}),(\bar y_1)^{1...i-1}].$$

Since $[\delta_{2s}+(\mathrm{ad}\,\bar x_1)^{2s}(\bar t_{12}),\bar y_1+\bar y_2]=0$ and $[\delta_{2s}+\sum_{1\le u<v\le n}(\mathrm{ad}\,\bar x_u)^{2s}(\bar t_{uv}),\bar y_1+...+\bar y_n]=0$, this equality implies

$$[\delta_{2s}+(\mathrm{ad}\,\bar x_1)^{2s}(\bar t_{12}),\bar y_2]^{1...i-1,i...n}-[\delta_{2s}+\sum_{1\le u<v\le n}(\mathrm{ad}\,\bar x_u)^{2s}(\bar t_{uv}),(\bar y_2)^{1...i-1}],$$

which ends the proof of (b) above, and therefore of the fact that the identities $\Theta A_i\Theta^{-1}=B_i^{-1}$, ..., $\Psi B_i\Psi^{-1}=B_iA_i$ are preserved.

The relation $(\Theta,\Psi^2)=1$ is preserved because

$$([\tilde\Theta]e^{\mathrm{i}\frac{\pi}{2}\sum_{i<j}\bar t_{ij}},([\tilde\Psi]e^{\mathrm{i}\frac{\pi}{6}\sum_{i<j}\bar t_{ij}})^2)=([\tilde\Theta]e^{\mathrm{i}\frac{\pi}{2}\sum_{i<j}\bar t_{ij}},[\tilde\Psi]^2e^{\mathrm{i}\frac{\pi}{3}\sum_{i<j}\bar t_{ij}})=([\tilde\Theta],[\tilde\Psi]^2)=[(\tilde\Theta,\tilde\Psi^2)]=1,$$

where the two first identities follow from the fact that $\sum_{i<j}\bar t_{ij}$ commutes with the image of $\mathfrak{d}\to\mathfrak{d}\ltimes\bar{\mathfrak{t}}_{1,n}$, $x\mapsto[x]$, the third identity follows from the fact that $\mathbf{G}_1\to\mathbf{G}_n$, $g\mapsto[g]$ is a group morphism, and the last identity follows from (27).

The image of $C_{i,i+1}$ is $\Phi_{\lambda,i}^{-1}(e^{2\pi\mathrm{i}\,\bar t_{12}})^{i,i+1...n}\Phi_{\lambda,i}$, to the product of the images of $C_{12},...,C_{n-1,n}$ is

$$\Phi_{\lambda,1}^{-1}(e^{2\pi\mathrm{i}\,\bar t_{12}})^{1,2...n}(\Phi_{\lambda,1}\Phi_{\lambda,2}^{-1})(e^{2\pi\mathrm{i}\,\bar t_{12}})^{2,3...n}(\Phi_{\lambda,2}\Phi_{\lambda,3}^{-1})(e^{2\pi\mathrm{i}\,\bar t_{12}})^{3,4...n}...(\Phi_{\lambda,n-1}\Phi_{\lambda,n}^{-1})e^{2\pi\mathrm{i}\,\bar t_{n-1,n}}\Phi_{\lambda,n}$$
$$=\Phi_{\lambda,1}^{-1}(e^{2\pi\mathrm{i}\,\bar t_{12}})^{1,2...n}(e^{2\pi\mathrm{i}\,\bar t_{12}})^{2,3...n}\Phi_\lambda^{1,2,3...n}(e^{2\pi\mathrm{i}\,\bar t_{12}})^{3,4...n}...\Phi_\lambda^{1...,i-1,...n}(e^{2\pi\mathrm{i}\,\bar t_{12}})^{i,i+1...n}$$
$$...\Phi_\lambda^{1...,n-2,n-1\ n}e^{2\pi\mathrm{i}\,\bar t_{n-1,n}}$$
$$=\Phi_{\lambda,1}^{-1}(e^{2\pi\mathrm{i}\,\bar t_{12}})^{1,2...n}(e^{2\pi\mathrm{i}\,\bar t_{12}})^{2,3...n}(e^{2\pi\mathrm{i}\,\bar t_{12}})^{3,4...n}...(e^{2\pi\mathrm{i}\,\bar t_{12}})^{i,i+1...n}...e^{2\pi\mathrm{i}\,\bar t_{n-1,n}}$$
$$\Phi_\lambda^{1,2,3...n}...\Phi_\lambda^{1...,i-1,...n}...\Phi_\lambda^{1...,n-2,n-1\ n}=\Phi_{\lambda,1}^{-1}e^{2\pi\mathrm{i}\sum_{i<j}\bar t_{ij}}\Phi_{\lambda,1}=e^{2\pi\mathrm{i}\sum_{i<j}\bar t_{ij}},$$

where the second equality follows from the fact that $\Phi^{1...,i,...n}$ commutes with $(e^{2\pi\mathrm{i}\,\bar t_{12}})^{j,j+1...n}$ whenever $j>i$, and the last equality follows from the fact that $\sum_{i<j}t_{ij}$ is central is $\mathfrak{t}_n$.

So the product of the images of $C_{12}...C_{n-1,n}$ is $e^{2\pi\mathrm{i}\sum_{i<j}\bar t_{ij}}$.

The relation $(\Theta\Psi)^3 = C_{12}...C_{n-1,n}$ is then preserved because $([\tilde\Theta]e^{\mathrm{i}\frac{\pi}{2}\sum_{i<j}\bar t_{ij}}[\tilde\Psi]e^{\mathrm{i}\frac{\pi}{6}\sum_{i<j}\bar t_{ij}})^3 = ([\tilde\Theta][\tilde\Psi])^3 e^{2\pi\mathrm{i}\sum_{i<j}\bar t_{ij}} = [(\tilde\Theta\tilde\Psi)^3]e^{2\pi\mathrm{i}\sum_{i<j}\bar t_{ij}} = e^{2\pi\mathrm{i}\sum_{i<j}\bar t_{ij}}$, where the first equality follows from the fact that $\sum_{i<j}\bar t_{ij}$ commutes with the image of $\mathbf{G}_1 \to \mathbf{G}_n$, $g\mapsto[g]$, the second equality follows from the fact that $g\mapsto[g]$ is a group morphism and the last equality follows from (27). In the same way, one proves that $\Theta^4 = C_{12}...C_{n-1,n}$, $\sigma_i^2 = C_{i,i+1}C_{i+1,i+2}C_{i,i+1}^{-1}$ and $(\Theta,\sigma_i) = (\Psi,\sigma_i) = 1$ are preserved. □

5.2. **Construction of morphisms $\overline{\mathrm{B}}_{1,n} \to \exp(\widehat{\bar{\mathfrak{t}}^{\mathbf{k}}_{1,n}}) \rtimes S_n$ using an associator Φ_λ.** Let us keep the notation of the previous section. Set $a_{2n}(\lambda) := -(2n+1)B_{2n+2}\lambda^{2n+2}/(2n+2)!$, $\tilde y_\lambda := -\frac{\mathrm{ad}\,x}{e^{\lambda\,\mathrm{ad}\,x}-1}(y)$,

$$\tilde A_\lambda := \Phi_\lambda(\tilde y_\lambda,t)e^{\lambda\tilde y_\lambda}\Phi_\lambda(\tilde y_\lambda,t)^{-1} = e^{-\lambda t/2}\Phi_\lambda(-\tilde y_\lambda - t,t)e^{\lambda(\tilde y_\lambda+t)}\Phi_\lambda(-\tilde y_\lambda-t,t)^{-1}e^{-\lambda t/2},$$

$$\tilde B_\lambda := e^{\lambda t/2}\Phi_\lambda(-\tilde y_\lambda - t,t)e^{\lambda x}\Phi_\lambda(\tilde y_\lambda,t)^{-1}$$

(the identity in the definition of A_λ follows from the hexagon relation).

Proposition 5.3. *We have*

$$\tilde A_\lambda^{12,3} = e^{\lambda\bar t_{12}/2}\{\Phi_\lambda\}^{3,1,2}\tilde A_\lambda^{2,13}\{\Phi_\lambda\}^{2,1,3}e^{\lambda\bar t_{12}/2}\cdot\{\Phi_\lambda\}^{3,2,1}\tilde A_\lambda^{1,23}\{\Phi_\lambda\}^{1,2,3},$$

$$\tilde B_\lambda^{12,3} = e^{-\lambda\bar t_{12}/2}\{\Phi_\lambda\}^{3,1,2}\tilde B_\lambda^{2,13}\{\Phi_\lambda\}^{2,1,3}e^{-\lambda\bar t_{12}/2}\cdot\{\Phi_\lambda\}^{3,2,1}\tilde B_\lambda^{1,23}\{\Phi_\lambda\}^{1,2,3},$$

$$(\tilde B_\lambda^{12,3}, e^{\lambda\bar t_{12}/2}\{\Phi_\lambda\}^{3,1,2}\tilde A_\lambda^{2,13}\{\Phi_\lambda\}^{2,1,3}e^{\lambda\bar t_{12}/2}) = (e^{-\lambda\bar t_{12}/2}\{\Phi_\lambda\}^{3,1,2}\tilde B_\lambda^{2,13}\{\Phi_\lambda\}^{2,1,3}e^{-\lambda\bar t_{12}/2}, \tilde A_\lambda^{12,3})$$
$$= \{\Phi_\lambda\}^{3,2,1}e^{\lambda\bar t_{23}}\{\Phi_\lambda\}^{1,2,3},$$

so the formulas of Proposition 5.1 (restricted to the generators $A_i, B_i, \sigma_i, C_{jk}$) induce a morphism $\overline{\mathrm{B}}_{1,n} \to \exp(\widehat{\bar{\mathfrak{t}}^{\mathbf{k}}_{1,n}}) \rtimes S_n$ (here $\widehat{\bar{\mathfrak{t}}^{\mathbf{k}}_{1,n}}$ is the degree completion of $\bar{\mathfrak{t}}^{\mathbf{k}}_{1,n}$).

Proof. In this proof, we shift the indices of the generators of $\mathfrak{t}_{n+1}$ by 1, so these generators are now t_{ij}, $i\neq j\in\{0,...,n\}$ (recall that $\mathfrak{t}_{n+1} = \mathfrak{t}^{\mathbf{k}}_{n+1}$, $\bar t_{1,n} = \bar{\mathfrak{t}}^{\mathbf{k}}_{1,n}$).

We have a morphism $\alpha_n : \mathfrak{t}_{n+1} \to \bar{\mathfrak{t}}_{1,n}$, defined by $t_{ij}\mapsto\bar t_{ij}$ if $1\le i<j\le n$ and $t_{0i}\mapsto\tilde y_i := -\frac{\mathrm{ad}\,\bar x_i}{e^{\lambda\,\mathrm{ad}\,\bar x_i}-1}(\bar y_i)$ if $1\le i\le n$ (it takes the central element $\sum_{0\le i<j\le n}t_{ij}$ to 0).

Let $\phi : \{1,...,m\}\to\{1,...,n\}$ be a map and $\phi' : \{0,...,m\}\to\{0,...,n\}$ be given by $\phi'(1) = 1$, $\phi'(i) = \phi(i)$ for $i = 1,...,m$. The diagram

$$\begin{array}{ccc} \mathfrak{t}_{n+1} & \overset{x\mapsto x^{\phi'}}{\to} & \mathfrak{t}_{m+1} \\ \alpha_n\downarrow & & \downarrow\alpha_m \\ \bar{\mathfrak{t}}_{1,n} & \overset{x\mapsto x^{\phi}}{\to} & \bar{\mathfrak{t}}_{1,m} \end{array}$$

is not commutative, we have instead the identity

$$\alpha_m(x^{\phi'}) = \alpha_n(x)^\phi - \sum_{i=1}^n \xi_i(x)(\sum_{i',j'\in\phi^{-1}(i)|i'<j'}\bar t_{i'j'}),$$

where $\xi_i : \bar{\mathfrak{t}}_{1,n}\to\mathbf{k}$ is the linear form defined by $\xi_i(t_{0i}) = 1$, ξ_i(any other homogeneous Lie polynomial in the t_{kl}) $= 0$.

Since the various $\sum_{i',j'\in\phi^{-1}(i)|i'<j'}\bar t_{i'j'}$ commute with each other and with the image of $x\mapsto x^\phi$, this implies

$$\alpha_m(g^{\phi'}) = \alpha_n(g)^\phi\prod_{i=1}^n e^{-\xi_i(\log g)(\sum_{i',j'\in\phi^{-1}(i),i'<j'}\bar t_{i'j'})}$$

for $g\in\exp(\hat{\mathfrak{t}}_{n+1})$.

Set $\bar{A}_\lambda := \Phi_\lambda^{0,1,2} e^{\lambda t_{01}} (\Phi_\lambda^{0,1,2})^{-1} \in \exp(\hat{\mathfrak{t}}_3)$. One proves that

$$\bar{A}_\lambda^{0,12,3} e^{\lambda t_{12}} = e^{\lambda t_{12}/2} \Phi_\lambda^{3,1,2} \bar{A}_\lambda^{0,2,13} \Phi_\lambda^{2,1,3} e^{\lambda t_{12}/2} \cdot \Phi_\lambda^{3,2,1} \bar{A}_\lambda^{0,1,23} \Phi_\lambda^{1,2,3}$$

(relation in $\exp(\hat{\mathfrak{t}}_4)$). We then have $\alpha_2(\bar{A}_\lambda) = \tilde{A}_\lambda$, $\alpha_3(\Phi_\lambda^{1,2,3}) = \Phi_\lambda^{1,2,3}$, and the relation between the α_i and coproducts implies $\alpha_3(\bar{A}_\lambda^{0,1,23}) = \tilde{A}_\lambda^{1,23}$ and $\alpha_3(\bar{A}_\lambda^{0,12,3} e^{\lambda t_{12}}) = \tilde{A}_\lambda^{12,3}$. Taking the image by α_3, we get the first identity.

As we have already mentioned, this identity implies $(\Phi_\lambda^{-1} \tilde{A}_\lambda^{1,23} \Phi_\lambda, \tilde{A}_\lambda^{12,3}) = 1$.

Let $\exp(\hat{\mathfrak{t}}_{n+1}) * \mathbb{Z}^n / I_n$ be the quotient of the free product of $\exp(\hat{\mathfrak{t}}_{n+1})$ with $\mathbb{Z}^n = \oplus_{i=1}^n \mathbb{Z} X_i$ by the normal subgroup generated by the rations of the exponentials of the sides of each of the equations

$$X_i t_{0i} X_i^{-1} = \sum_{0 \le \alpha \le n, \alpha \ne i} t_{\alpha i}, \quad X_i (t_{0j} + t_{ij}) X_i^{-1} = t_{0j},\ X_i t_{jk} X_i^{-1} = t_{jk},\ X_j X_k t_{jk} (X_j X_k)^{-1} = t_{jk}$$

where i, j, k are distinct in $\{1, ..., n\}$. Then the morphism $\alpha_n : \mathfrak{t}_{n+1} \to \bar{\mathfrak{t}}_{1,n}$ extends to $\tilde{\alpha}_n : \exp(\hat{\mathfrak{t}}_{n+1}) * \mathbb{Z}^n / I_n \to \exp(\hat{\bar{\mathfrak{t}}}_{1,n})$ by $X_i \mapsto e^{\lambda x_i}$.

If $\phi : \{1, ..., m\} \to \{1, ..., n\}$ is a map, then the Lie algebra morphism $\mathfrak{t}_{n+1} \to \mathfrak{t}_{m+1}$, $x \mapsto x^{\phi'}$ extends to a group morphism $\exp(\hat{\mathfrak{t}}_n) * \mathbb{Z}^n / I_n \to \exp(\hat{\mathfrak{t}}_m) * \mathbb{Z}^m / I_m$ by $X_i \mapsto \prod_{i' \in \phi^{-1}(i)} X_{i'}$.

Let

$$\bar{B}_\lambda := e^{\lambda t_{12}/2} \Phi_\lambda^{0,2,1} X_1 \Phi_\lambda^{2,1,0} \in \exp(\hat{\mathfrak{t}}_3) * \mathbb{Z}^2 / I_2,$$

then $\alpha_2(\bar{B}_\lambda) = \tilde{B}_\lambda$.

We will prove that

$$\bar{B}_\lambda^{0,12,3} = e^{-\lambda t_{12}/2} \Phi_\lambda^{3,1,2} \bar{B}_\lambda^{0,2,13} \Phi_\lambda^{2,1,3} e^{-\lambda t_{12}/2} \cdot \Phi^{3,2,1} \bar{B}_\lambda^{0,1,23} \Phi_\lambda^{1,2,3}. \tag{40}$$

The l.h.s. is

$$\bar{B}_\lambda^{0,12,3} = e^{\lambda t_{3,12}/2} \Phi_\lambda^{0,3,12} X_1 X_2 \Phi_\lambda^{3,21,0}$$

and the r.h.s. is

$$e^{-\lambda t_{12}/2} \Phi_\lambda^{3,1,2} e^{\lambda t_{31,2}/2} \Phi_\lambda^{0,13,2} X_2 \Phi_\lambda^{13,2,0} \Phi_\lambda^{2,1,3} e^{-\lambda t_{12}/2} \Phi_\lambda^{3,2,1} e^{\lambda t_{23,1}/2} \Phi_\lambda^{0,23,1} X_1 \Phi_\lambda^{32,1,0} \Phi_\lambda^{1,2,3}.$$

The equality between these terms is rewritten as

$$X_1 X_2 = \Phi_\lambda^{03,1,2} \Phi_\lambda^{1,3,0} e^{-\lambda t_{13}/2} X_2 \Phi_\lambda^{13,2,0} e^{\lambda t_{13}/2} \Phi_\lambda^{2,3,1} \Phi_\lambda^{0,23,1} X_1 \Phi_\lambda^{01,2,3} \Phi_\lambda^{2,1,0},$$

or, using the fact that X_i commutes with t_{jk} $(i, j, k$ distinct), as

$$X_1 X_2 = \Phi_\lambda^{03,1,2} \Phi_\lambda^{1,3,0} X_2 \Phi_\lambda^{02,3,1} \Phi_\lambda^{3,2,0} X_1 \Phi_\lambda^{01,2,3} \Phi_\lambda^{2,1,0}.$$

Now $X_2 \Phi_\lambda^{02,3,1} = \Phi_\lambda^{0,3,1} X_2$, $X_1 \Phi_\lambda^{01,2,3} = \Phi_\lambda^{0,2,3} X_1$ and $X_1 X_2 \Phi_\lambda^{2,1,0} = \Phi_\lambda^{2,1,03} X_1 X_2$, so the r.h.s. is rewritten as $\Phi_\lambda^{03,1,2} \Phi_\lambda^{1,3,0} \Phi_\lambda^{0,3,1} X_2 \Phi_\lambda^{3,2,0} \Phi_\lambda^{0,2,3} X_1 \Phi_\lambda^{2,1,0} = X_1 X_2$. This ends the proof of (40). Taking the image by α_4, we then get the second identity of the Proposition.

Let us prove the next identity. We have

$$\begin{aligned}
&(\bar{B}_\lambda^{0,12,3}, e^{\lambda \bar{t}_{12}/2} \Phi_\lambda^{3,1,2} \bar{A}_\lambda^{0,2,13} \Phi_\lambda^{2,1,3} e^{\lambda \bar{t}_{12}/2}) \\
&= e^{\lambda t_{12,3}/2} \Phi_\lambda^{0,3,12} X_1 X_2 \Phi_\lambda^{3,12,0} e^{\lambda \bar{t}_{12}/2} \Phi_\lambda^{3,1,2} \Phi_\lambda^{0,2,13} e^{\lambda t_{0,2}} \Phi_\lambda^{13,2,0} \Phi_\lambda^{2,1,3} e^{\lambda \bar{t}_{12}/2} \Phi_\lambda^{0,12,3} (X_1 X_2)^{-1} \\
&\Phi_\lambda^{12,3,0} e^{-\lambda t_{12,3}/2} e^{-\lambda \bar{t}_{12}/2} \Phi_\lambda^{3,1,2} \Phi_\lambda^{0,2,13} e^{-\lambda t_{0,2}} \Phi_\lambda^{13,2,0} \Phi_\lambda^{2,1,3} e^{-\lambda \bar{t}_{12}/2}.
\end{aligned}$$

Now

$$\begin{aligned}
&X_1 X_2 \Phi_\lambda^{3,12,0} e^{\lambda \bar{t}_{12}/2} \Phi_\lambda^{3,1,2} \Phi_\lambda^{0,2,13} e^{\lambda t_{0,2}} \Phi_\lambda^{13,2,0} \Phi_\lambda^{2,1,3} e^{\lambda \bar{t}_{12}/2} \Phi_\lambda^{0,12,3} (X_1 X_2)^{-1} \\
&= e^{\lambda \bar{t}_{12}/2} X_1 X_2 \Phi_\lambda^{3,12,0} \Phi_\lambda^{3,1,2} \Phi_\lambda^{0,2,13} e^{\lambda t_{0,2}} \Phi_\lambda^{13,2,0} \Phi_\lambda^{2,1,3} \Phi_\lambda^{0,12,3} (X_1 X_2)^{-1} e^{\lambda \bar{t}_{12}/2} \\
&= e^{\lambda \bar{t}_{12}/2} X_1 X_2 \Phi_\lambda^{0,2,1} \Phi_\lambda^{3,1,02} e^{\lambda t_{0,2}} \Phi_\lambda^{02,1,3} \Phi_\lambda^{0,2,1} (X_1 X_2)^{-1} e^{\lambda \bar{t}_{12}/2} \\
&= e^{\lambda \bar{t}_{12}/2} X_1 X_2 \Phi_\lambda^{0,2,1} e^{\lambda t_{0,2}} \Phi_\lambda^{0,2,1} (X_1 X_2)^{-1} e^{\lambda \bar{t}_{12}/2} \\
&= e^{\lambda \bar{t}_{12}/2} \Phi_\lambda^{03,2,1} X_1 X_2 e^{\lambda t_{0,2}} (X_1 X_2)^{-1} \Phi_\lambda^{03,2,1} e^{\lambda \bar{t}_{12}/2} \\
&= e^{\lambda \bar{t}_{12}/2} \Phi_\lambda^{03,2,1} e^{\lambda t_{03,2}} \Phi_\lambda^{03,2,1} e^{\lambda \bar{t}_{12}/2}.
\end{aligned}$$

Plugging this in the above expression for $(\bar B_\lambda^{0,12,3}, e^{\lambda\bar t_{12}/2}\Phi_\lambda^{3,1,2}\bar A_\lambda^{0,2,13}\Phi_\lambda^{2,1,3}e^{\lambda t_{12}/2})$, one then finds $(\bar B_\lambda^{0,12,3}, e^{\lambda\bar t_{12}/2}\Phi_\lambda^{3,1,2}\bar A_\lambda^{0,2,13}\Phi_\lambda^{2,1,3}e^{\lambda\bar t_{12}/2}) == \Phi_\lambda^{3,2,1}e^{\lambda t_{23}}\Phi_\lambda^{1,2,3}$. Taking the image by α_4, we then obtain $(\tilde B_\lambda^{12,3}, e^{\lambda\bar t_{12}/2}\Phi_\lambda^{3,1,2}\tilde A_\lambda^{2,13}\Phi_\lambda^{2,1,3}e^{\lambda\bar t_{12}/2}) = \Phi_\lambda^{3,2,1}e^{\lambda\bar t_{23}}\Phi_\lambda^{1,2,3}$.

Let us prove that last identity. For this, we will show

$$(e^{-\lambda t_{12}/2}\Phi_\lambda^{3,1,2}\bar B_\lambda^{0,2,13}\Phi_\lambda^{2,1,3}e^{-\lambda t_{12}/2}, \bar A_\lambda^{0,12,3}e^{\lambda t_{12}}) = \Phi_\lambda^{3,2,1}e^{\lambda t_{23}}\Phi_\lambda^{1,2,3}$$

and take the image by α_4.

We have

$$(e^{-\lambda t_{12}/2}\Phi_\lambda^{3,1,2}\bar B_\lambda^{0,2,13}\Phi_\lambda^{2,1,3}e^{-\lambda t_{12}/2}, \bar A_\lambda^{0,12,3}e^{\lambda t_{12}})$$
$$= e^{-\lambda t_{12}/2}\Phi_\lambda^{3,1,2}e^{\lambda t_{2,13}/2}\Phi_\lambda^{0,13,2}X_2\Phi_\lambda^{13,2,0}\Phi_\lambda^{2,1,3}e^{-\lambda t_{12}/2}\Phi_\lambda^{0,12,3}e^{\lambda t_{0,12}}\Phi_\lambda^{3,12,0}e^{\lambda t_{12}}e^{\lambda t_{12}/2}\Phi_\lambda^{3,1,2}\Phi_\lambda^{0,2,13}X_2^{-1}$$
$$\Phi_\lambda^{2,13,0}e^{-\lambda t_{2,13}/2}\Phi_\lambda^{2,1,3}e^{\lambda t_{12}/2}\Phi_\lambda^{0,12,3}e^{-\lambda t_{0,12}}\Phi_\lambda^{3,12,0}e^{-\lambda t_{12}}$$
$$= e^{-\lambda t_{12}/2}\Phi_\lambda^{3,1,2}e^{\lambda t_{2,13}/2}\Phi_\lambda^{0,13,2}X_2\Phi_\lambda^{13,2,0}\Phi_\lambda^{2,1,3}\Phi_\lambda^{0,12,3}e^{\lambda t_{0,12}+\lambda t_{12}}\Phi_\lambda^{3,12,0}\Phi_\lambda^{3,1,2}\Phi_\lambda^{0,2,13}X_2^{-1}$$
$$\Phi_\lambda^{2,13,0}e^{-\lambda t_{2,13}/2}\Phi_\lambda^{2,1,3}e^{-\lambda t_{12}/2}\Phi_\lambda^{0,12,3}e^{-\lambda t_{0,12}}\Phi_\lambda^{3,12,0}.$$

Now

$$X_2\Phi_\lambda^{13,2,0}\Phi_\lambda^{2,1,3}\Phi_\lambda^{0,12,3}e^{\lambda t_{0,12}+\lambda t_{12}}\Phi_\lambda^{3,12,0}\Phi_\lambda^{3,1,2}\Phi_\lambda^{0,2,13}X_2^{-1}$$
$$= X_2\Phi_\lambda^{02,1,3}\Phi_\lambda^{1,2,0}e^{\lambda t_{0,12}+\lambda t_{12}}\Phi_\lambda^{0,2,1}\Phi_\lambda^{3,1,02}X_2^{-1} = \Phi_\lambda^{0,1,3}X_2\Phi_\lambda^{1,2,0}e^{\lambda t_{0,12}+\lambda t_{12}}\Phi_\lambda^{0,2,1}X_2^{-1}\Phi_\lambda^{3,1,0}$$
$$= \Phi_\lambda^{0,1,3}X_2e^{\lambda(t_{01}+t_{02}+t_{12})}X_2^{-1}\Phi_\lambda^{3,1,0} = \Phi_\lambda^{0,1,3}e^{\lambda(t_{01}+t_{02}+t_{12}+t_{23})}\Phi_\lambda^{3,1,0}.$$

So

$$(e^{-\lambda t_{12}/2}\Phi_\lambda^{3,1,2}\bar B_\lambda^{0,2,13}\Phi_\lambda^{2,1,3}e^{-\lambda t_{12}/2}, \bar A_\lambda^{0,12,3}e^{\lambda t_{12}})$$
$$= e^{-\lambda t_{12}/2}\Phi_\lambda^{3,1,2}e^{\lambda t_{2,13}/2}\Phi_\lambda^{0,13,2}\Phi_\lambda^{0,1,3}e^{\lambda(t_{01}+t_{02}+t_{12}+t_{23})}$$
$$\Phi_\lambda^{3,1,0}\Phi_\lambda^{2,13,0}e^{-\lambda t_{2,13}/2}\Phi_\lambda^{2,1,3}e^{-\lambda t_{12}/2}\Phi_\lambda^{0,12,3}e^{-\lambda t_{0,12}}\Phi_\lambda^{3,12,0};$$

after some computation, we find that this equals $\Phi_\lambda^{3,2,1}e^{\lambda t_{23}}\Phi_\lambda^{1,2,3}$. □

In particular, $(\Phi_\lambda, \tilde A_\lambda, \tilde B_\lambda)$ give rise to a morphism $\overline{\mathrm B}_{1,n} \to \exp(\widehat{\bar{\mathfrak t}^{\mathbf k}_{1,n}}) \rtimes S_n$; one proves as in Section 2 that it induces an isomorphism of filtered Lie algebras $\mathrm{Lie}(\overline{\mathrm{PB}}_{1,n})_{\mathbf k} \simeq \widehat{\bar{\mathfrak t}^{\mathbf k}_{1,n}}$. Taking Φ_λ to be a rational associator ([Dr3]), we then obtain:

Corollary 5.4. *We have a filtered isomorphism* $\mathrm{Lie}(\overline{\mathrm{PB}}_{1,n})_{\mathbb Q} \simeq \widehat{\bar{\mathfrak t}^{\mathbb Q}_{1,n}}$, *which can be extended to an isomorphism* $\overline{\mathrm B}_{1,n}(\mathbb Q) \simeq \exp(\widehat{\bar{\mathfrak t}^{\mathbb Q}_{1,n}}) \rtimes S_n$.

5.3. **Construction of morphisms $\Gamma_{1,[n]} \to \mathbf G_{1,n} \rtimes S_n$ using a pair $(\Phi_\lambda, \tilde\Theta_\lambda)$.** Keep the notation of the previous section and set

$$\tilde\Psi_\lambda := \exp(-\frac{1}{\lambda}(\Delta_0 + \sum_{k\geq 1} a_{2k}(\lambda)\delta_{2k})).$$

Proposition 5.5. *We have*

$$[\tilde\Psi_\lambda]e^{\lambda\bar t_{12}/12}\tilde A_\lambda([\tilde\Psi_\lambda]e^{\lambda\bar t_{12}/12})^{-1} = \tilde A_\lambda, \quad [\tilde\Psi_\lambda]e^{\lambda\bar t_{12}/12}\tilde B_\lambda([\tilde\Psi_\lambda]e^{\lambda\bar t_{12}/12})^{-1} = \tilde B_\lambda\tilde A_\lambda.$$

Proof. The first identity follows from the fact that $\Delta_0 + \sum_{k\geq 1} a_{2k}(\lambda)[\delta_{2k}] - \lambda^2 t/12$ commutes with t and $\tilde y_\lambda$; the second identity follows from these facts and the analogue of Lemma 4.15, where $2\pi\mathrm i$ is replaced by λ. □

Assume that $\tilde\Theta_\lambda \in \mathbf G_1$ satisfies

$$\tilde\Theta_\lambda^4 = (\tilde\Theta_\lambda\tilde\Psi_\lambda)^3 = (\tilde\Theta_\lambda^2, \tilde\Psi_\lambda) = 1,$$
$$[\tilde\Theta_\lambda]e^{\lambda\bar t_{12}/4}\tilde A_\lambda([\tilde\Theta_\lambda]e^{\lambda\bar t_{12}/4})^{-1} = \tilde B_\lambda^{-1}, \quad [\tilde\Theta_\lambda]e^{\lambda\bar t_{12}/4}\tilde B_\lambda([\tilde\Theta_\lambda]e^{\lambda\bar t_{12}/4})^{-1} = \tilde B_\lambda\tilde A_\lambda\tilde B_\lambda^{-1}$$

(one can show that the two last equations are equivalent), then $\Theta \mapsto [\tilde\Theta_\lambda]e^{\lambda(\sum_{i<j}\bar t_{ij})/4}$, $\Psi \mapsto [\tilde\Psi_\lambda]e^{\lambda(\sum_{i<j}\bar t_{ij})/12}$ extends the morphism defined in Proposition 5.3 to a morphism $\Gamma_{1,[n]} \to \mathbf{G}_n \rtimes S_n$.

We do not know whether for each Φ_λ defined over $\mathbf{k}$, there exists a $\tilde\Theta_\lambda$ defined over $\mathbf{k}$, satisfying the above conditions.

5.4. **Elliptic structures over QTQBA's.** Let $(H, \Delta_H, R_H, \Phi_H)$ be a quasitriangular quasibialgebra (QTQBA). Recall that this means that ([Dr2]): (H, m_H) is an algebra, $\Delta_H : H \to H^{\otimes 2}$ is an algebra morphism, $R_H \in H^{\otimes 2}$ and $\Phi_H \in H^{\otimes 3}$ are invertible, and

$$\Delta_H(x)^{2,1} = R_H\Delta_H(x)R_H^{-1}, \quad (\mathrm{id}\otimes\Delta_H)\circ\Delta_H(x) = \Phi_H(\Delta_H\otimes\mathrm{id})\circ\Delta_H(x)\Phi_H^{-1},$$

$$R_H^{12,3} = \Phi_H^{3,1,2}R_H^{1,3}(\Phi_H^{1,3,2})^{-1}R_H^{2,3}\Phi_H^{1,2,3}, \quad R_H^{1,23} = (\Phi_H^{2,3,1})^{-1}R_H^{1,3}\Phi_H^{2,1,3}R_H^{1,2}(\Phi_H^{1,2,3})^{-1},$$

$$\Phi_H^{1,2,34}\Phi_H^{12,3,4} = \Phi_H^{2,3,4}\Phi_H^{1,23,4}\Phi_H^{1,2,3}.$$

One also assumes the existence of a unit 1_H and a counit ε_H.

If $\mathbf{A}$ is an algebra and $J_1, J_2 \subset \mathbf{A}$ are left ideals, define the Hecke bimodule $\mathcal{H}(\mathbf{A}|J_1, J_2)$ or $\mathcal{H}(J_1, J_2)$ as $\mathrm{Hom}_{\mathbf{A}}(\mathbf{A}/J_1, \mathbf{A}/J_2) = (\mathbf{A}/J_2)^{J_1}$ where J_1 acts on the quotient from the left; we have thus $\mathcal{H}(J_1, J_2) = \{x \in \mathbf{A}|J_1x \subset J_2\}/J_2$. The product of $\mathbf{A}$ induces a product $\mathcal{H}(J_1, J_2) \otimes \mathcal{H}(J_2, J_3) \to \mathcal{H}(J_1, J_3)$. When $J_1 = J_2 = J$, $\mathcal{H}(J) := \mathcal{H}(J, J)$ is the usual Hecke algebra, and $\mathcal{H}(J_1, J_2)$ is a $(\mathcal{H}(J_1), \mathcal{H}(J_2))$-bimodule. Recall that we have a functor $\mathbf{A}$-mod $\to \mathcal{H}(J)$-mod, $V \mapsto V^J := \{v \in V|Jv = 0\}$.

If H is an algebra with unit equipped with a morphism $\Delta_H : H \to H^{\otimes 2}$ and $a : H \to D$ is a morphism of algebras with unit, we define for each $n \geq 1$ and each pair of words w, w' in the free magma generated by $1, ..., n$ containing $1, ..., n$ exactly once (recall that a magma is a set with a non-necessarily associative binary operation) the Hecke bimodule

$$\mathcal{H}^{w,w'}(D, H) := \mathcal{H}(D \otimes H^{\otimes n}|J_w, J_{w'}),$$

(or simply $\mathcal{H}^{w,w'}$) where $J_w \subset D \otimes H^{\otimes n}$ is the left ideal generated by the image of $(a \otimes \Delta_H^w) \circ \Delta_H : H_+ \to D \otimes H^{\otimes n}$. Here $H_+ = \mathrm{Ker}(H \stackrel{\varepsilon_H}{\to} \mathbf{k})$ and for example $\Delta_H^{(21)3} = (213) \circ (\Delta_H \otimes \mathrm{id}_H) \circ \Delta_H$, etc. We have products $\mathcal{H}^{w,w'} \otimes \mathcal{H}^{w',w''} \to \mathcal{H}^{w,w''}$. We denote the Hecke algebra $\mathcal{H}^{w,w}$ by $\mathcal{H}^w(D, H)$ or $\mathcal{H}^w$; we denote by 1_w its unit. We denote by $(\mathcal{H}^{w,w'})^\times$ the set of invertible elements of $\mathcal{H}^{w,w'}$, i.e., the set of elements X such that for some $X' \in \mathcal{H}^{w',w}$, $X'X = 1_{w'}$, $XX' = 1_w$. The symmetric group S_n acts on the system of bimodules $\mathcal{H}^{w,w'}$ by permuting the factors, so we get maps $\mathrm{Ad}(\sigma) : \mathcal{H}^{w,w'} \to \mathcal{H}^{\sigma(w),\sigma(w')}$ (where $\sigma(w)$ is the word w, where i is replaced by $\sigma(i)$). If $w_0 = ((12)...)n$, we define an algebra structure on $\oplus_{\sigma\in S_n}\mathcal{H}^{w_0,\sigma(w_0)}\sigma$ by $(\sum_{\sigma\in S_n} h_\sigma\sigma)(\sum_{\tau\in S_n} h'_\tau\tau) := \sum_{\sigma,\tau\in S_n} h_\sigma \mathrm{Ad}(\sigma)(h'_\tau)\sigma\tau$. Then $\sqcup_{\sigma\in S_n}(\mathcal{H}^{w_0,\sigma(w_0)})^\times\sigma \subset \oplus_{\sigma\in S_n}\mathcal{H}^{w_0,\sigma(w_0)}\sigma$ is a group with unit 1_{w_0}. We have an exact sequence $1 \to (\mathcal{H}^{w_0})^\times \to \sqcup_{\sigma\in S_n}(\mathcal{H}^{w_0,\sigma(w_0)})^\times\sigma \to S_n$, but the last map is not necessarily surjective (and if it is, does not necessarily split).

If H is a quasibialgebra, then Φ_H gives rise to an element of $\mathcal{H}^{1(23),(12)3}(D, H)$, which we also denote Φ_H; similarly Φ_H^{-1} gives rise to the inverse (w.r.t. composition of Hecke bimodules) element $\Phi_H^{-1} \in \mathcal{H}^{(12)3,1(23)}(D, H)$. We have algebra morphisms $\mathcal{H}^{12}(D, H) \to \mathcal{H}^{(12)3}(D, H)$ induced by $X \mapsto X^{0,12,3} := (\mathrm{id}_H \otimes(\Delta_H \otimes \mathrm{id}_H) \circ \Delta_H)(X)$ (0 is the index of D) and similarly morphisms $\mathcal{H}^{12}(D, H) \to \mathcal{H}^{2(13)}(D, H)$, $X \mapsto X^{0,2,13}$, $\mathcal{H}^{12}(D, H) \to \mathcal{H}^1(D, H)$, $X \mapsto X^{0,1,\emptyset}$ and $X^{0,\emptyset,1}$, etc. If moreover H is quasitriangular, then $R_H \in \mathcal{H}^{21,12}(D, H)$, $R_H^{-1} \in \mathcal{H}^{12,21}(D, H)$, so in that case $\sqcup_{\sigma\in S_n}\mathcal{H}^{w_0,\sigma(w_0)}\sigma \to S_n$ is surjective, and we have a morphism $\mathrm{B}_n \to \sqcup_{\sigma\in S_n}\mathcal{H}^{w_0,\sigma(w_0)}\sigma$ such that the composition $\mathrm{B}_n \to \sqcup_{\sigma\in S_n}\mathcal{H}^{w_0,\sigma(w_0)}\sigma \to S_n$ is the canonical projection.

Definition 5.6. *If H is a QTQBA, an elliptic structure on H is a triple (D, A, B), where D is an algebra with unit, equipped with an algebra morphism $a : H \to D$, and $A, B \in$*

$\mathcal{H}^{12}(D,H)$ *are invertible such that* $A^{0,1,\emptyset}=A^{0,\emptyset,1}=B^{0,1,\emptyset}=B^{0,\emptyset,1}=1_D\otimes 1_H$,

$$A^{0,12,3}=R_H^{2,1}(\Phi_H^{2,1,3})^{-1}A^{0,2,13}\Phi_H^{2,1,3}R_H^{1,2}(\Phi_H^{1,2,3})^{-1}A^{0,1,23}\Phi_H^{1,2,3}, \tag{41}$$

$$B^{0,12,3}=(R_H^{1,2})^{-1}(\Phi_H^{2,1,3})^{-1}B^{0,2,13}\Phi_H^{2,1,3}(R_H^{2,1})^{-1}(\Phi_H^{1,2,3})^{-1}B^{0,1,23}\Phi_H^{1,2,3} \tag{42}$$

and

$$\begin{aligned}&(B^{0,12,3},R_H^{2,1}(\Phi_H^{2,1,3})^{-1}A^{0,2,13}\Phi_H^{2,1,3}R_H^{1,2})\\&=((R_H^{1,2})^{-1}(\Phi_H^{2,1,3})^{-1}B^{0,2,13}\Phi_H^{2,1,3}(R_H^{2,1})^{-1},A^{0,12,3})=(\Phi_H^{1,2,3})^{-1}R_H^{3,2}R_H^{2,3}\Phi_H^{1,2,3}\end{aligned}$$

(identities in $\mathcal{H}^{(12)3}(D,H)$*).*

The pair of identities (41), (42) is equivalent to

$$R_H^{2,1}A^{0,2,1}R_H^{1,2}A^{0,1,2}=1,\quad R_H^{3,12}A^{0,3,12}\Phi_H^{3,1,2}R_H^{2,31}A^{0,2,31}\Phi_H^{2,3,1}R_H^{1,23}A^{0,1,23}\Phi_H^{1,2,3}=1,$$

and

$$(R_H^{1,2})^{-1}B^{0,2,1}(R_H^{2,1})^{-1}B^{0,1,2}=1,\quad (R_H^{-1})^{12,3}B^{0,3,12}\Phi_H^{3,1,2}(R_H^{-1})^{31,2}B^{0,2,31}\Phi_H^{2,3,1}(R_H^{-1})^{23,1}B^{0,1,23}\Phi_H^{1,2,3}=1,$$

so the invertibility conditions on A,B follow from (41), (42).

If $F\in H^{\otimes 2}$ is invertible with $(\varepsilon_H\otimes \mathrm{id}_H)(F)=(\mathrm{id}_H\otimes\varepsilon_H)(F)=1_H$, then the twist of H by F is the quasi-Hopf algebra ${}^F H$ with product m_H, coproduct $\tilde\Delta_H(x)=F\Delta_H(x)F^{-1}$, R-matrix $\tilde R_H=F^{2,1}R_HF^{-1}$ and associator $\tilde\Phi_H=F^{2,3}F^{1,23}\Phi_H(F^{1,2}F^{12,3})^{-1}$. If $a:H\to D$ is an algebra morphism, it can be viewed as a morphism ${}^F H\to D$, and we have an algebra isomorphism $\mathcal{H}^{(12)3}(D,H)\to\mathcal{H}^{(12)3}(D,{}^F H)$, induced by $X\mapsto F^{1,2}F^{0,12}X(F^{1,2}F^{0,12})^{-1}$ (more generally, we have an isomorphism of the systems of bimodules $\mathcal{H}^{w,w'}(D,H)\to\mathcal{H}^{w,w'}(D,{}^F H)$ induced by $X\mapsto F_wXF_{w'}^{-1}$ for suitable F_w).

If (D,A,B) is an elliptic structure on H, then an elliptic structure ${}^F H$ is $(D,\tilde A,\tilde B)$, where $\tilde A=F^{1,2}F^{0,12}A(F^{1,2}F^{0,12})^{-1}$ and $\tilde B=F^{1,2}F^{0,12}B(F^{1,2}F^{0,12})^{-1}$.

An elliptic structure (D,A,B) over H gives rise to a unique group morphism

$$\overline{\mathrm{B}}_{1,n}\to\sqcup_{\sigma\in S_n}\mathcal{H}^{w_0,\sigma(w_0)}(D,H)^\times\sigma,$$

such that

$$\sigma_i\mapsto\left(\Phi_H^{(((12)3)\ldots i-1),i,i+1}\right)^{-1}R_H^{i,i+1}(i,i+1)\Phi_H^{(((12)3)\ldots i-1),i,i+1},$$

$$A_i\mapsto\Phi_{H,i}^{-1}A^{0,(((12)3)\ldots i-1),(i\ldots(n-1,n))}\Phi_{H,i},\quad B_i\mapsto\Phi_{H,i}^{-1}B^{0,(((12)3)\ldots i-1),(i\ldots(n-1,n))}\Phi_{H,i},$$

where

$$\Phi_{H,i}=\Phi_H^{((12)\ldots i-1),i,(i+1(\ldots(n-1,n)))}\ldots\Phi_H^{((12)\ldots n-2),n-1,n};$$

here we have for example $x^{((12)3)}=(\Delta_H\otimes\mathrm{id}_H)\circ\Delta_H(x)$ for $x\in H$.

If $\mathfrak{g}$ is a Lie algebra and $t_{\mathfrak{g}}\in S^2(\mathfrak{g})^{\mathfrak{g}}$ is nondegenerate, then $H=U(\mathfrak{g})[[\hbar]]$ is a QTQBA, with m_H,Δ_H are the undeformed product and coproduct, $R_H=e^{\hbar t_{\mathfrak{g}}/2}$ and $\Phi_H=\Phi(\hbar t_{\mathfrak{g}}^{1,2},\hbar t_{\mathfrak{g}}^{2,3})$, where Φ is an 1-associator. The results of next Section then imply that (D,A,B) is an elliptic structure over H, where $D=D(\mathfrak{g})[[\hbar]]$ ($D(\mathfrak{g})$ is the algebra of algebraic differential operators on $\mathfrak{g}$) and A,B are given by the formulas for $\tilde A_\lambda,\tilde B_\lambda$ with t replaced by $\hbar t_{\mathfrak{g}}^{1,2}$, x replaced by $\hbar\sum_\alpha \mathrm{x}_\alpha\otimes(e_\alpha^1+e_\alpha^2)$, y replaced by $\hbar\sum_\alpha\partial_\alpha\otimes(e_\alpha^1+e_\alpha^2)$.

Remark 5.7. If H is a Hopf algebra, we have an isomorphism

$$\mathcal{H}^{w_0}(D,H)\simeq(D\otimes H^{\otimes n-1})^H,$$

where the right side is the commutant of the diagonal map $H\to D\otimes H^{\otimes n-1}$, $h\mapsto(a\otimes\mathrm{id}_H^{\otimes n-1})\circ\Delta_H^{(n)}(h)$. This map takes the class of $d\otimes h_1\otimes\ldots\otimes h_n$ to $da(S_H(h_n^{(n)}))\otimes h_1S_H(h_n^{(n-1)})\otimes\ldots\otimes h_{n-1}S_H(h_n^{(1)})$ (S_H is the antipode of H). So A,B identify with elements $\mathcal{A},\mathcal{B}\in(D\otimes H)^H$; the conditions are then

$$\mathcal{A}^{0,12}=R_H^{2,1}\mathcal{A}^{0,2}R_H^{1,2}\mathcal{A}^{0,1},\quad\mathcal{B}^{0,12}=(R_H^{1,2})^{-1}\mathcal{B}^{0,2}(R_H^{2,1})^{-1}\mathcal{B}^{0,1},$$

$$(\mathcal{B}^{0,12}, R_H^{2,1}\mathcal{A}^{0,2}R_H^{1,2}) = ((R_H^{1,2})^{-1}\mathcal{B}^{0,2}(R_H^{2,1})^{-1}, \mathcal{A}^{0,12}) = (R_H^{3,2}R_H^{1,2}R_H^{0,2}R_H^{2,0}R_H^{2,1}R_H^{2,3})^{\tilde{0},\tilde{1},2\cdot\tilde{3}}$$

(conditions in $(D \otimes H^{\otimes 2})^H$), where the superscript $\mathrm{B}'_n \rtimes \mathbb{Z}^{n-1} \to \mathrm{B}_{n-1} \rtimes \mathbb{Z}^{n-1}$ is the map $x_0 \otimes ... \otimes x_3 \mapsto S_H(x_0) \otimes S_H(x_1) \otimes x_2 S_H(x_3)$.

Moreover, the morphism $\mathrm{PB}_n \to (\mathcal{H}^{w_0})^\times \simeq (D \otimes H^{\otimes n-1})^H$ factors through $\mathrm{PB}_n \to \mathrm{PB}_{n-1} \times \mathbb{Z}^{n-1} \to (D \otimes H^{\otimes n-1})^H$, where: (a) the first morphism is induced by $\mathbb{Z}^{n-1} \rtimes \mathrm{B}'_n \to \mathbb{Z}^{n-1} \rtimes \mathrm{B}_{n-1}$ (where $\mathrm{B}'_n = \mathrm{B}_n \times_{S_n} S_{n-1}$ is the group of braids leaving the last strand fixed), constructed as follows: we have a composition $\mathrm{B}'_{n+1} \to \pi_1((\mathbb{P}^1)^{n+1} - \text{diagonals}/S_n) \to \pi_1(\mathbb{C}^n - \text{diagonals}/S_n) = \mathrm{B}_n$, where the first map is induced by $\mathbb{C} \subset \mathbb{P}^1$, and the middle map comes from the fibration $\mathbb{C}^n - \text{diagonals} \to (\mathbb{P}^1)^{n+1} - \text{diagonals} \to \mathbb{P}^1$, $(z_1, ..., z_n) \to (z_1, ..., z_n, \infty)$ and $(z_1, ..., z_{n+1}) \to z_{n+1}$ [the second projection has a section so the map between π_1's is an isomorphism]; viewing $\mathbb{Z}^{n-1} \rtimes \mathrm{B}'_n$, $\mathbb{Z}^{n-1} \rtimes \mathrm{B}_{n-1}$ as fundamental groups of configuration spaces of points equipped with a nonzero tangent vector, we then get the morphism $\mathbb{Z}^{n-1} \rtimes \mathrm{B}'_n \to \mathbb{Z}^{n-1} \rtimes \mathrm{B}_{n-1}$ (which does not restrict to a morphism $\mathrm{B}'_n \to \mathrm{B}_{n-1}$); (b) the second map is induced by the standard map $\mathrm{PB}_{n-1} \times \mathbb{Z}^{n-1} \to (H^{\otimes n-1})^\times$ induced by $R_H = \sum_\alpha r'_\alpha \otimes r''_\alpha$ and the map taking the ith generator of $\mathbb{Z}^{n-1}$ to $1 \otimes ... \otimes u S_H(u) \otimes ... \otimes 1$, where $u = \sum_i S_H(r''_\alpha) r'_\alpha$ (see [Dr1]). The morphism $\mathrm{B}_n \to \mathrm{Aut}((\mathcal{H}^{w_0})^\times) = \mathrm{Aut}((D \otimes H^{\otimes n-1})^H)$ extends the inner action of PB_n by

$$\sigma_{n-1} \cdot X := \{R_H^{n-1,n...2n-1} X^{0,1,...,n-2,n...2n-1} R_H^{n...2n-1,n-1}\}^{0\cdot\widetilde{2n-1},...,n-1\cdot\tilde{n}}$$

(where the superscript means that $x_0 \otimes ... \otimes x_{2n-1}$ maps to $x_0 S_H(x_{2n-1}) \otimes ... \otimes x_{n-1} S_H(x_n)$).

We have then $\sqcup_{\sigma \in S_n} (\mathcal{H}^{w_0,\sigma(w_0)})^\times \sigma \simeq ((D \otimes H^{\otimes n-1})^\times)^H \rtimes_{\mathrm{PB}_n} \mathrm{B}_n$ (the index means that $\mathrm{PB}_n \subset \mathrm{B}_n$ is identified with its image in $((D \otimes H^{\otimes n-1})^\times)^H$).

Then if $(\mathcal{A}, \mathcal{B})$ is an elliptic structure over $a : H \to D$, the morphism $\mathrm{B}_n \to ((D \otimes H^{\otimes n-1})^\times)^H \rtimes_{\mathrm{PB}_n} \mathrm{B}_n$ extends to a morphism

$$\overline{\mathrm{B}}_{1,n} \to ((D \otimes H^{\otimes n-1})^\times)^H \rtimes_{\mathrm{PB}_n} \mathrm{B}_n$$

via $A_i \mapsto \mathcal{A}^{0,1...i-1}$, $B_i \mapsto \mathcal{B}^{0,1...i-1}$.

This interpretation of $\mathcal{H}^{w_0}$ and of the relations between $\mathcal{A}, \mathcal{B}$ can be extended to the case when H is a quasi-Hopf algebra.

Remark 5.8. Let $\mathcal{C}$ be a rigid braided monoidal category. We define an elliptic structure on $\mathcal{C}$ as a quadruple $(\mathcal{E}, A, B, F)$, where $\mathcal{E}$ is a category, $F : \mathcal{E} \to \mathcal{C}$ is a functor, and A, B are functorial automorphisms of $F(?) \otimes ?$, which reduce to the identity if the second factor is the neutral object $\mathbf{1}$, and such that the following equalities of automorphisms of $F(M) \otimes (X \otimes Y)$ hold (we write them omitting associativity maps, as they can be put in automatically):

$$A_{M,X\otimes Y} = \beta_{Y,X} A_{M,Y} \beta_{X,Y} A_{M,X},$$
$$B_{M,X\otimes Y} = \beta_{X,Y}^{-1} B_{M,Y} \beta_{Y,X}^{-1} B_{M,X},$$

$$(B_{M,X\otimes Y}, \beta_{Y,X} A_{M,Y} \beta_{X,Y}) = (\beta_{Y,X}^{-1} B_{M,Y} \beta_{X,Y}^{-1}, A_{M,X\otimes Y})$$
$$= \beta_{(M\otimes X\otimes Y)^*,Y} \beta_{Y,(M\otimes X\otimes Y)^*} \circ \mathrm{can}_{M\otimes X\otimes Y},$$

where $\mathrm{can}_X \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X \otimes X^*)$ is the canonical map and the r.h.s. of the last identity is viewed as an element of $\mathrm{End}_{\mathcal{C}}(M \otimes X \otimes Y)$ using its identification with $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, (M \otimes X \otimes Y) \otimes (M \otimes X \otimes Y)^*)$. An elliptic structure on a quasitriangular quasi-Hopf algebra H gives rise to an elliptic structure on H-mod. An elliptic structure over a rigid braided monoidal category $\mathcal{C}$ gives rise to representations of $\overline{\mathrm{B}}_{1,n}$ by $\mathcal{C}$-automorphisms of $F(M) \otimes X^{\otimes n-1}$.

6. The KZB connection as a realization of the universal KZB connection

6.1. **Realizations of $\bar{\mathfrak{t}}_{1,n}$.** Let $\mathfrak{g}$ be a Lie algebra and $t_{\mathfrak{g}} \in S^2(\mathfrak{g})^{\mathfrak{g}}$ be nondegenerate. We denote by $(a, b) \mapsto \langle a, b \rangle$ the corresponding invariant pairing.

Let $D(\mathfrak{g})$ be the algebra of algebraic differential operators on $\mathfrak{g}$. It has generators x_a, ∂_a, $a \in \mathfrak{g}$, and relations: $a \mapsto \mathrm{x}_a$, $a \mapsto \partial_a$ are linear, $[\mathrm{x}_a, \mathrm{x}_b] = [\partial_a, \partial_b] = 0$, $[\partial_a, \mathrm{x}_b] = \langle a, b \rangle$.

There is a unique Lie algebra morphism $\mathfrak{g} \to D(\mathfrak{g})$, $a \mapsto X_a$, where $X_a := \sum_\alpha \mathrm{x}_{[a,e_\alpha]} \partial_{e_\alpha}$, and $t_{\mathfrak{g}} = \sum_\alpha e_\alpha \otimes e_\alpha$ (it is the infinitesimal of the adjoint action). We also have a Lie algebra morphism $\mathfrak{g} \to A_n := D(\mathfrak{g}) \otimes U(\mathfrak{g})^{\otimes n}$, $a \mapsto Y_a := X_a \otimes 1 + 1 \otimes (\sum_{i=1}^n a^{(i)})$. We denote by $\mathfrak{g}^{\mathrm{diag}}$ the image of this morphism. We denote by $\mathcal{H}_n(\mathfrak{g})$ the Hecke algebra of $(A_n, \mathfrak{g}^{\mathrm{diag}})$. It is defined as the quotient $\{x \in A_n | \forall a \in \mathfrak{g}, Y_a x \in A_n \mathfrak{g}^{\mathrm{diag}}\}/A_n \mathfrak{g}^{\mathrm{diag}}$. We have a natural action of S_n on A_n, which induces an action of S_n on $\mathcal{H}_n(\mathfrak{g})$.

If $(V_i)_{i=1,\ldots,n}$ are $\mathfrak{g}$-modules, then $(S(\mathfrak{g}) \otimes (\otimes_{i=1}^n V_i))^{\mathfrak{g}}$ is a module over $\mathcal{H}_n(\mathfrak{g})$. If moreover $V_1 = \ldots = V_n$, this is a module over $\mathcal{H}_n(\mathfrak{g}) \rtimes S_n$.

Proposition 6.1. *There is a unique Lie algebra morphism* $\rho_{\mathfrak{g}} : \bar{\mathfrak{t}}_{1,n} \to \mathcal{H}_n(\mathfrak{g})$, $\bar{x}_i \mapsto \sum_\alpha \mathrm{x}_\alpha \otimes e_\alpha^{(i)}$, $\bar{y}_i \mapsto -\sum_\alpha \partial_\alpha \otimes e_\alpha^{(i)}$, $\bar{t}_{ij} \mapsto 1 \otimes t_{\mathfrak{g}}^{(ij)}$ *(we set* $\mathrm{x}_\alpha := \mathrm{x}_{e_\alpha}$, $\partial_\alpha := \partial_{e_\alpha}$*).*

Proof. The images of all the generators of $\bar{\mathfrak{t}}_{1,n}$ are contained in the commutant of $\mathfrak{g}^{\mathrm{diag}}$ in A_n, therefore also in its normalizer. According to Lemma 2.1, we will use the following presentation of $\bar{\mathfrak{t}}_{1,n}$. Generators are $\bar{x}_i, \bar{y}_i, \bar{t}_{ij}$, relations are $[\bar{x}_i, \bar{x}_j] = [\bar{y}_i, \bar{y}_j] = 0$, $[\bar{x}_i, \bar{y}_j] = \bar{t}_{ij}$ $(i \neq j)$, $\bar{t}_{ij} = \bar{t}_{ji}$, $\sum_i \bar{x}_i = \sum_i \bar{y}_i = 0$, $[\bar{x}_i, \bar{t}_{jk}] = [\bar{y}_i, \bar{t}_{jk}] = 0$ $(i, j, k$ distinct).

The relations $[\bar{x}_i, \bar{x}_j] = [\bar{y}_i, \bar{y}_j] = 0$, $[\bar{x}_i, \bar{y}_j] = \bar{t}_{ij}$ $(i \neq j)$, $\bar{t}_{ij} = \bar{t}_{ji}$ and $[\bar{x}_i, \bar{t}_{jk}] = [\bar{y}_i, \bar{t}_{jk}] = 0$ are obviously preserved. Let us check that $\sum_i \bar{x}_i = \sum_i \bar{y}_i = 0$ are preserved.

We have

$$\sum_i \rho_{\mathfrak{g}}(\bar{x}_i) = \sum_\alpha \mathrm{x}_\alpha \otimes (\sum_i e_\alpha^{(i)}) = \sum_\alpha (\mathrm{x}_\alpha \otimes 1)(Y_\alpha - X_\alpha \otimes 1)$$
$$\equiv -\sum_\alpha \mathrm{x}_\alpha X_\alpha \otimes 1 = \sum_{\alpha,\beta} \mathrm{x}_{e_\alpha} \mathrm{x}_{[e_\alpha,e_\beta]} \partial_{e_\beta} \otimes 1 = 0$$

since x_α commutes with $\mathrm{x}_{[e_\alpha,e_\beta]}$ and $\sum_\beta e_\beta \otimes e_\beta = t_{\mathfrak{g}}$ is invariant. We also have

$$\sum_i \rho_{\mathfrak{g}}(\bar{y}_i) = -\sum_\alpha \partial_\alpha \otimes (\sum_i e_\alpha^{(i)}) = -\sum_\alpha (\partial_\alpha \otimes 1)(Y_\alpha - X_\alpha \otimes 1) \equiv \sum_\alpha \partial_\alpha X_\alpha \otimes 1$$
$$= -\sum_{\alpha,\beta} \partial_{e_\alpha} \mathrm{x}_{[e_\alpha,e_\beta]} \partial_{e_\beta} = -\sum_{\alpha,\beta} \langle e_\alpha, [e_\alpha, e_\beta] \rangle \partial_{e_\beta} - \sum_{\alpha,\beta} \mathrm{x}_{[e_\alpha,e_\beta]} \partial_{e_\alpha} \partial_{e_\beta};$$

since $t_{\mathfrak{g}}$ is invariant and $\langle -, - \rangle$ is symmetric, we have $\sum_\alpha \langle e_\alpha, [e_\alpha, e_\beta] \rangle = 0$ for any β, and since $[\partial_{e_\alpha}, \partial_{e_\beta}] = 0$, we have $\sum_{\alpha,\beta} \mathrm{x}_{[e_\alpha,e_\beta]} \partial_{e_\alpha} \partial_{e_\beta}$, so $\sum_i \rho_{\mathfrak{g}}(\bar{y}_i) = 0$. □

6.2. **Realizations of $\bar{\mathfrak{t}}_{1,n} \rtimes \mathfrak{d}$.** Let $(\mathfrak{g}, t_{\mathfrak{g}})$ be as in Subsection 6.1. We keep the same notations.

Proposition 6.2. *The Lie algebra morphism* $\rho_{\mathfrak{g}} : \bar{\mathfrak{t}}_{1,n} \to \mathcal{H}_n(\mathfrak{g})$ *of Proposition 6.1 extends to a Lie algebra morphism* $\bar{\mathfrak{t}}_{1,n} \rtimes \mathfrak{d} \to \mathcal{H}_n(\mathfrak{g})$*, defined by* $\Delta_0 \mapsto -\frac{1}{2}(\sum_\alpha \partial_\alpha^2) \otimes 1$, $X \mapsto \frac{1}{2}(\sum_\alpha \mathrm{x}_\alpha^2) \otimes 1$, $d \mapsto \frac{1}{2}(\sum_\alpha \mathrm{x}_\alpha \partial_\alpha + \partial_\alpha \mathrm{x}_\alpha) \otimes 1$*, and*

$$\delta_{2m} \to \frac{1}{2} \sum_{\alpha_1,\ldots,\alpha_{2m},\alpha} \mathrm{x}_{\alpha_1} \cdots \mathrm{x}_{\alpha_{2m}} \otimes (\sum_{i=1}^n (\mathrm{ad}(e_{\alpha_1}) \cdots \mathrm{ad}(e_{\alpha_{2m}})(e_\alpha) \cdot e_\alpha)^{(i)})$$

for $m \geq 1$*. This morphism further extends to a morphism* $U(\bar{\mathfrak{t}}_{1,n} \rtimes \mathfrak{d}) \rtimes S_n \to \mathcal{H}_n(\mathfrak{g}) \rtimes S_n$ *by* $\sigma \mapsto \sigma$*.*

Proof. We have

$$[\rho_{\mathfrak{g}}(\delta_{2m}), \rho_{\mathfrak{g}}(\bar{x}_i)] = \frac{1}{2} \sum_{\alpha_1,\ldots,\alpha_{2m},\alpha,\beta} \mathrm{x}_{\alpha_1} \cdots \mathrm{x}_{\alpha_{2m}} \mathrm{x}_\beta \otimes [e_\beta, \mathrm{ad}(e_{\alpha_1}) \cdots \mathrm{ad}(e_{\alpha_{2m}})(e_\alpha) e_\alpha]^{(i)}$$
$$= \frac{1}{2} \sum_{\alpha_1,\ldots,\alpha_{2m},\alpha,\beta} \mathrm{x}_{\alpha_1} \cdots \mathrm{x}_{\alpha_{2m}} \mathrm{x}_\beta \otimes \sum_{\ell=1}^{2m} \big(\mathrm{ad}(e_{\alpha_1}) \cdots \mathrm{ad}([e_\beta, e_{\alpha_\ell}]) \cdots \mathrm{ad}(e_{\alpha_{2m}})(e_\alpha) e_\alpha\big)^{(i)} = 0$$

the second equality follows from the invariance of $t_{\mathfrak g}$, and the last equality follows from the fact that the first factor is symmetric in (β,α_ℓ) while the second is antisymmetric in (β,α_l).

$\rho_{\mathfrak g}$ preserves the relation $[\delta_{2m},\bar t_{ij}]=[\bar t_{ij},\mathrm{ad}(\bar x_i)^{2m}(\bar t_{ij})]$, because $\rho_{\mathfrak g}(\delta_{2m}+\sum_{i<j}\mathrm{ad}(\bar x_i)^{2m}(\bar t_{ij}))$ belongs to $D(\mathfrak g)\otimes\mathrm{Im}(\Delta^{(n)}:U(\mathfrak g)\to U(\mathfrak g)^{\otimes n})$, where $\Delta^{(n)}$ is the n-fold coproduct and $U(\mathfrak g)$ is equipped with its standard bialgebra structure.

Now

$$[\rho_{\mathfrak g}(\delta_{2m}),\rho_{\mathfrak g}(\bar y_i)]=\frac12\sum_{\alpha_1,\dots,\alpha_{2m},\alpha,\beta}\Big(\sum_j[\partial_\beta,\mathrm x_{\alpha_1}\cdots\mathrm x_{\alpha_{2m}}]\otimes e_\beta^{(i)}\mathrm{ad}(e_{\alpha_1})\cdots\mathrm{ad}(e_{\alpha_{2m}})(e_\alpha)^{(j)}e_\alpha^{(j)}$$

$$+\,\mathrm x_{\alpha_1}\cdots\mathrm x_{\alpha_{2m}}\,\partial_\beta\otimes[e_\beta,\mathrm{ad}(e_{\alpha_1})\cdots\mathrm{ad}(e_{\alpha_{2m}})(e_\alpha)\cdot e_\alpha]^{(i)}\Big)$$

$$=\frac12\sum_{l=1}^{2m}\sum_{\alpha_1,\dots,\alpha_{2m},\alpha}\Big(\sum_j\mathrm x_{\alpha_1}\cdots\check{\mathrm x}_{\alpha_l}\cdots\mathrm x_{\alpha_{2m}}\otimes e_{\alpha_l}^{(i)}\mathrm{ad}(e_{\alpha_1})\cdots\mathrm{ad}(e_{\alpha_{2m}})(e_\alpha)^{(j)}e_\alpha^{(j)}$$

$$+\,\mathrm x_{\alpha_1}\cdots\mathrm x_{\alpha_{2m}}\,\partial_\beta\otimes\mathrm{ad}(e_{\alpha_1})\cdots\mathrm{ad}([e_\beta,e_{\alpha_l}])\cdots\mathrm{ad}(e_{\alpha_{2m}})(e_\alpha)^{(i)}e_\alpha^{(i)}\Big)$$

$$\equiv\frac12\sum_{l=1}^{2m}\sum_{\alpha_1,\dots,\alpha_{2m},\alpha}\sum_j\Big(\mathrm x_{\alpha_1}\cdots\check{\mathrm x}_{\alpha_l}\cdots\mathrm x_{\alpha_{2m}}\otimes e_{\alpha_l}^{(i)}\mathrm{ad}(e_{\alpha_1})\cdots\mathrm{ad}(e_{\alpha_{2m}})(e_\alpha)^{(j)}e_\alpha^{(j)}$$

$$-\,\mathrm x_{\alpha_1}\cdots\check{\mathrm x}_{\alpha_l}\cdots\mathrm x_{\alpha_{2m}}\otimes\mathrm{ad}(e_{\alpha_1})\cdots\mathrm{ad}(e_{\alpha_{2m}})(e_\alpha)^{(i)}e_\alpha^{(i)}e_{\alpha_l}^{(j)}\Big).$$

The term corresponding to $j=i$ is

$$\frac12\sum_{l=1}^{2m}\sum_{\alpha_1,\dots,\alpha_{2m},\alpha}\mathrm x_{\alpha_1}\cdots\check{\mathrm x}_{\alpha_l}\cdots\mathrm x_{\alpha_{2m}}\otimes[e_{\alpha_l},\mathrm{ad}(e_{\alpha_1})\cdots\mathrm{ad}(e_{\alpha_{2m}})(e_\alpha)\cdot e_\alpha]^{(i)}$$

It corresponds to the linear map $S^{2m-1}(\mathfrak g)\to U(\mathfrak g)$, such that for $x\in\mathfrak g$,

$$x^{2m-1}\mapsto\frac12\sum_{p+q=2m-1}\sum_{\alpha,\beta}[e_\beta,\mathrm{ad}(x)^p\mathrm{ad}(e_\beta)\mathrm{ad}(x)^q(e_\alpha)\cdot e_\alpha]$$

$$=\frac12\sum_{\alpha,\beta}\sum_{p+q+r=2m-2}\mathrm{ad}(x)^p\mathrm{ad}([e_\beta,x])\mathrm{ad}(x)^q\mathrm{ad}(e_\beta)\mathrm{ad}(x)^r(e_\alpha)\cdot e_\alpha$$

$$+\,\mathrm{ad}(x)^p\mathrm{ad}(e_\beta)\mathrm{ad}(x)^q\mathrm{ad}([e_\beta,x])\mathrm{ad}(x)^r(e_\alpha)\cdot e_\alpha$$

since $\mu(t_{\mathfrak g})=0$ ($\mu:\mathfrak g^{\otimes2}\to\mathfrak g$ is the Lie bracket) and $t_{\mathfrak g}$ is $\mathfrak g$-invariant. Now this is zero since $t_{\mathfrak g}=\sum_\beta e_\beta\otimes e_\beta$ is invariant.

The term corresponding to $j\neq i$ corresponds to the map $S^{2m-1}(\mathfrak g)\to U(\mathfrak g)^{\otimes n}$, such that for $x\in\mathfrak g$

$$x^{2m-1}\mapsto-\frac12\sum_{l=1}^{2m}\sum_{\alpha,\beta}\big((\mathrm{ad}x)^{l-1}(\mathrm{ad}e_\beta)(\mathrm{ad}x)^{2m-l}(e_\alpha)\cdot e_\alpha\big)^{(i)}e_\beta^{(j)}-(i\leftrightarrow j)$$

$$=\frac12\sum_{l=1}^{2m}(-1)^{l+1}\sum_{\alpha,\beta}\big((\mathrm{ad}x)^{l-1}([e_\beta,e_\alpha])\cdot(\mathrm{ad}x)^{2m-l}(e_\alpha)\big)^{(i)}e_\beta^{(j)}-(i\leftrightarrow j)$$

$$=\frac12\sum_{l=1}^{2m}(-1)^{l-1}\sum_{\alpha,\beta}\big((\mathrm{ad}x)^{l-1}(e_\beta)\cdot(\mathrm{ad}x)^{2m-l}(e_\alpha)\big)^{(i)}[e_\alpha,e_\beta]^{(j)}-(i\leftrightarrow j)$$

$$=\frac12\sum_{l=1}^{2m}(-1)^l\Big[\sum_\alpha\big((\mathrm{ad}x)^{l-1}(e_\alpha)\big)^{(i)}e_\alpha^{(j)},\sum_\beta\big((\mathrm{ad}x)^{2m-l}(e_\beta)\big)^{(i)}e_\beta^{(j)}\Big],$$

which coincides with the image of $\frac12\sum_{p+q=2m-1}(-1)^q[(\mathrm{ad}\bar x_i)^p(\bar t_{ij}),(\mathrm{ad}\bar x_i)^q(\bar t_{ij})]$.

It is then clear that $\rho_{\mathfrak g}$ preserves the commutation relations of Δ_0, X and d with δ_{2m}. □

6.3. **Reductions.** Assume that $\mathfrak{g}$ is finite dimensional and we have a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$, i.e., $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra and $\mathfrak{n} \subset \mathfrak{g}$ is a vector subspace such that $[\mathfrak{h}, \mathfrak{n}] \subset \mathfrak{n}$; assume also that $t_{\mathfrak{g}} = t_{\mathfrak{h}} + t_{\mathfrak{n}}$, where $t_{\mathfrak{h}} \in S^2(\mathfrak{h})^{\mathfrak{h}}$ and $t_{\mathfrak{n}} \in S^2(\mathfrak{n})^{\mathfrak{h}}$.

We assume that for a generic $h \in \mathfrak{h}$, $\mathrm{ad}(h)_{|\mathfrak{n}} \in \mathrm{End}(\mathfrak{n})$ is invertible. This condition is equivalent to the nonvanishing of $P(\lambda) := \det(\mathrm{ad}(\lambda^\vee)_{|\mathfrak{n}}) \in S^{\dim \mathfrak{n}}(\mathfrak{h})$, where $\lambda \mapsto \lambda^\vee$ is the map $\mathfrak{h}^* \to \mathfrak{h}$, with $\lambda^\vee := (\lambda \otimes \mathrm{id})(t_{\mathfrak{h}})$. If G is a Lie group with Lie algebra $\mathfrak{g}$, an equivalent condition is that a generic element of $\mathfrak{g}^*$ is conjugate to some element in $\mathfrak{h}^*$ (see [EE]).

Let us set, for $\lambda \in \mathfrak{h}^*$,

$$r(\lambda) := (\mathrm{id} \otimes (\mathrm{ad}\, \lambda^\vee)^{-1}_{|\mathfrak{n}})(t_{\mathfrak{n}}),$$

Then $r : \mathfrak{h}^*_{\mathrm{reg}} \to \wedge^2(\mathfrak{n})$ is an $\mathfrak{h}$-equivariant map (here $\mathfrak{h}^*_{\mathrm{reg}} = \{\lambda \in \mathfrak{h}^* | P(\lambda) \neq 0\}$), satisfying the classical dynamical Yang-Baxter (CDYB) equation

$$\mathrm{CYB}(r) - \mathrm{Alt}(\mathrm{d}\, r) = 0$$

(see [EE]). Here for $r = \sum_\alpha a_\alpha \otimes b_\alpha \otimes \ell_\alpha \in (\mathfrak{n}^{\otimes 2} \otimes S(\mathfrak{h})[1/P])^{\mathfrak{h}}$, we set $\mathrm{CYB}(r) = \sum_{\alpha,\alpha'}([a_\alpha, a_{\alpha'}] \otimes b_\alpha \otimes b_{\alpha'} + a_\alpha \otimes [b_\alpha, a_{\alpha'}] \otimes b_{\alpha'} + a_\alpha \otimes a_{\alpha'} \otimes [b_\alpha, b_{\alpha'}]) \otimes \ell_\alpha \ell_{\alpha'}$, $\mathrm{d}\, r := \sum_\alpha a_\alpha \otimes b_\alpha \otimes \mathrm{d}\, \ell_\alpha$, where d extends $S(\mathfrak{h}) \to \mathfrak{h} \otimes S(\mathfrak{h})$, $x^k \mapsto kx \otimes x^{k-1}$ and $\mathrm{Alt}(X \otimes \ell) = (X + X^{2,3,1} + X^{3,1,2}) \otimes \ell$.

We also set

$$\psi(\lambda) := (\mathrm{id} \otimes (\mathrm{ad}\, \lambda^\vee)^{-2}_{|\mathfrak{n}})(t_{\mathfrak{n}}).$$

We write $\psi(\lambda) = \sum_\alpha A_\alpha \otimes B_\alpha \otimes L_\alpha$.

Let $D(\mathfrak{h})[1/P]$ be the localization at P of the algebra $D(\mathfrak{h})$ of differential operators on $\mathfrak{h}$; the latter algebra is generated by $\bar{\mathrm{x}}_h$, $\bar\partial_h$, $h \in \mathfrak{h}$, with relations $h \mapsto \bar{\mathrm{x}}_h$, $h \mapsto \bar\partial_h$ linear, $[\bar{\mathrm{x}}_h, \bar{\mathrm{x}}_{h'}] = [\bar\partial_h, \bar\partial_{h'}] = 0$, and $[\bar\partial_h, \bar{\mathrm{x}}_{h'}] = \langle h, h' \rangle$.

Set $B_n := D(\mathfrak{h})[1/P] \otimes U(\mathfrak{g})^{\otimes n}$. For $h \in \mathfrak{h}$, we define $\bar X_h := \sum_\nu \bar{\mathrm{x}}_{[h,h_\nu]} \bar\partial_{h_\nu} \in D(\mathfrak{h})$, where $t_{\mathfrak{h}} = \sum_\nu h_\nu \otimes h_\nu$. We then set $\bar Y_h := \bar X_h + \sum_{i=1}^n h^{(i)}$. The map $\mathfrak{h} \to B_n$ is a Lie algebra morphism; we denote by $\mathfrak{h}^{\mathrm{diag}}$ its image.

We denote by $\mathcal{H}_n(\mathfrak{g}, \mathfrak{h})$ the Hecke algebra of B_n relative to $\mathfrak{h}^{\mathrm{diag}}$. Explicitly, $\mathcal{H}_n(\mathfrak{g}, \mathfrak{h}) = \{x \in B_n | \forall h \in \mathfrak{h}, \bar Y_h x \in B_n \mathfrak{h}^{\mathrm{diag}}\} / B_n \mathfrak{h}^{\mathrm{diag}}$.

Proposition 6.3. *There is a unique Lie algebra morphism*

$$\rho_{\mathfrak{g},\mathfrak{h}} : \bar{\mathfrak{t}}_{1,n} \to \mathcal{H}_n(\mathfrak{g}, \mathfrak{h}),$$

such that $\bar x_i \mapsto \sum_\nu \bar{\mathrm{x}}_\nu \otimes h_\nu^{(i)}$, $\bar y_i \mapsto -\sum_\nu \bar\partial_\nu \otimes h_\nu^{(i)} + \sum_j \sum_\alpha \ell_\alpha \otimes a_\alpha^{(i)} b_\alpha^{(j)}$, $\bar t_{ij} \mapsto t_{\mathfrak{g}}^{(ij)}$. *Here* $r(\lambda) = \sum_\alpha \ell_\alpha(\lambda)(a_\alpha \otimes b_\alpha)$.

If $V_1, ..., V_n$ are $\mathfrak{g}$-modules, then $S(\mathfrak{h})[1/P] \otimes (\otimes_i V_i)$ is a module over $D(\mathfrak{h})[1/P] \otimes U(\mathfrak{g})^{\otimes n}$, and $(S(\mathfrak{h})[1/P] \otimes (\otimes_i V_i))^{\mathfrak{h}}$ is a module over $H_n(\mathfrak{g}, \mathfrak{h})$.

Moreover, we have a restriction morphism $(S(\mathfrak{g}) \otimes (\otimes_i V_i))^{\mathfrak{g}} \to (S(\mathfrak{h})[1/P] \otimes (\otimes V_i))^{\mathfrak{h}}$. Note that $(S(\mathfrak{g}) \otimes (\otimes_i V_i))^{\mathfrak{g}}$ is a $\bar{\mathfrak{t}}_{1,n}$-module using the morphism $\bar{\mathfrak{t}}_{1,n} \to \mathcal{H}_n(\mathfrak{g})$, while $(S(\mathfrak{h})[1/P] \otimes (\otimes V_i))^{\mathfrak{h}}$ is a $\bar{\mathfrak{t}}_{1,n}$-module using the morphism $\bar{\mathfrak{t}}_{1,n} \to \mathcal{H}_n(\mathfrak{g}, \mathfrak{h})$. Then one checks that the restriction morphism $(S(\mathfrak{g}) \otimes (\otimes_i V_i))^{\mathfrak{g}} \to (S(\mathfrak{h})[1/P] \otimes (\otimes V_i))^{\mathfrak{h}}$ is a $\bar{\mathfrak{t}}_{1,n}$-modules morphism.

Proof. The images of the above elements are all $\mathfrak{h}$-invariant. To lighten the notation, we will imply summation over repeated indices and denote elements of B_n as follows: $\bar\partial_\nu \otimes 1$ by $\bar\partial_\nu$, $\bar{\mathrm{x}}_\nu \otimes 1$ by $\langle \lambda, h_\nu \rangle$, $1 \otimes x^{(i)}$ by x^i. Then $\rho_{\mathfrak{g},\mathfrak{h}}(\bar x_i) = (\lambda^\vee)^i$, $\rho_{\mathfrak{g},\mathfrak{h}}(\bar y_i) = -h_\nu^i \bar\partial_\nu + \sum_{j=1}^n r(\lambda)^{ij}$ (here for $x \otimes y \in \mathfrak{g}^{\otimes 2}$, $(x \otimes y)^{ii} := x^i y^i$).

We will use the same presentation of $\bar{\mathfrak{t}}_{1,n}$ as in Proposition 6.1. The relations $[\bar x_i, \bar x_j] = 0$ and $\bar t_{ij} = \bar t_{ji}$ are obviously preserved.

Let us check that $[\bar x_i, \bar y_j] = \bar t_{ij}$ is preserved. We have for $i \neq j$, $[\rho_{\mathfrak{g},\mathfrak{h}}(\bar x_i), \rho_{\mathfrak{g},\mathfrak{h}}(\bar y_j)] = [\bar x_\nu h_\nu^i, -h_\nu^j \bar\partial_\nu + \sum_k r(\lambda)^{jk}] = t_{\mathfrak{h}}^{ij} + [\lambda^i, r(\lambda)^{ji}] = t_{\mathfrak{h}}^{ij} + t_{\mathfrak{n}}^{ij} = t_{\mathfrak{g}}^{ij} = \rho_{\mathfrak{g},\mathfrak{h}}(\bar t_{ij})$.

Let us check that $\sum_i \bar x_i = \sum_i \bar y_i = 0$ are preserved. We have $\sum_i \rho_{\mathfrak{g},\mathfrak{h}}(\bar x_i) = 0$ by the same argument as above and $\sum_i \rho_{\mathfrak{g},\mathfrak{h}}(\bar y_i) = \sum_i (\lambda^\vee)^i$ (by the antisymmetry of $r(\lambda)$), which vanishes by the same argument as above.

Let us check that $[\bar y_i,\bar y_j]=0$ is preserved, for $i\neq j$. We have

$$\begin{aligned}
&[\rho_{\mathfrak{g},\mathfrak{h}}(\bar y_i),\rho_{\mathfrak{g},\mathfrak{h}}(\bar y_j)]\\
&=\sum_{k|k\neq i,j}\big(-h^i_\nu(\partial_\nu r(\lambda))^{jk}+h^j_\nu(\partial_\nu r(\lambda))^{ik}+[r(\lambda)^{ij},r(\lambda)^{jk}]+[r(\lambda)^{ik},r(\lambda)^{jk}]+[r(\lambda)^{ik},r(\lambda)^{ji}]\big)\\
&+[(h^i_\nu+h^j_\nu)\bar\partial_\nu,r(\lambda)^{ij}]-[h^i_\nu\bar\partial_\nu,r(\lambda)^{jj}]+[h^j_\nu\bar\partial_\nu,r(\lambda)^{ii}]+[r(\lambda)^{ij},r(\lambda)^{ii}+r(\lambda)^{jj}]\\
&=\big(\sum_{k|k\neq i,j}h^k_\nu(\partial_\nu r(\lambda))^{ij}\big)+[(h^i_\nu+h^j_\nu)\bar\partial_\nu,r(\lambda)^{ij}]-[h^i_\nu\bar\partial_\nu,r(\lambda)^{jj}]+[h^j_\nu\bar\partial_\nu,r(\lambda)^{ii}]+[r(\lambda)^{ij},r(\lambda)^{ii}+r(\lambda)^{jj}]\\
&\equiv(\partial_\nu r(\lambda))^{ij}(-h^i_\nu-h^j_\nu-\bar X_\nu)+[(h^i_\nu+h^j_\nu)\bar\partial_\nu,r(\lambda)^{ij}]-h^i_\nu(\partial_\nu r(\lambda))^{jj}+h^j_\nu(\partial_\nu r(\lambda))^{ii}\\
&+[r(\lambda)^{ij},r(\lambda)^{ii}+r(\lambda)^{jj}]=[h^i_\nu+h^j_\nu,r(\lambda)^{ij}]\bar\partial_\nu-(\partial_\nu r^{ij}(\lambda))\bar X_\nu\\
&+[h^i_\nu+h^j_\nu,\partial_\nu r(\lambda)^{ij}]-h^i_\nu(\partial_\nu r(\lambda))^{jj}+h^j_\nu(\partial_\nu r(\lambda))^{ii}+[r(\lambda)^{ij},r(\lambda)^{ii}+r(\lambda)^{jj}].
\end{aligned}$$

The second equality follows from the CDYBE and the antisymmetry on $r(\lambda)$. Then

$$[h^i_\nu+h^j_\nu,r(\lambda)^{ij}]\bar\partial_\nu-(\partial_\nu r^{ij}(\lambda))\bar X_\nu=\big([h^i_{\nu'}+h^j_{\nu'},r(\lambda)^{ij}]-\partial_\nu r^{ij}(\lambda)\langle\lambda,[h_\nu,h_{\nu'}]\rangle\big)\bar\partial_{\nu'}=0$$

using the $\mathfrak{h}$-invariance of $r(\lambda)$. Applying $x^iy^jz^k\mapsto x^i(yz)^i$ to the CDYB identity

$$[r(\lambda)^{ij},r(\lambda)^{ik}]+[r(\lambda)^{ij},r(\lambda)^{jk}]+[r(\lambda)^{ik},r(\lambda)^{jk}]-h^i_\nu\partial_\nu r(\lambda)^{jk}+h^j_\nu\partial_\nu r(\lambda)^{ik}-h^j_\nu\partial_\nu r(\lambda)^{ij}=0,$$

we get

$$(1/2)\sum_{\alpha,\beta}\ell_\alpha\ell'_\beta(\lambda)[a_\alpha,a_\beta]^i[b_\alpha,b_\beta]^j+[r(\lambda)^{ij},r(\lambda)^{ii}]-h^i_\nu(\partial_\nu r(\lambda))^{jj}+[h^j_\nu,\partial_\nu r(\lambda)^{ij}]=0.$$

Since $r(\lambda)$ is antisymmetric, the sum $(1/2)\sum_{\alpha,\beta}...$ is symmetric in (i,j); antisymmetrizing in (i,j), we get

$$[h^i_\nu+h^j_\nu,\partial_\nu r(\lambda)^{ij}]-h^i_\nu(\partial_\nu r(\lambda))^{jj}+h^j_\nu(\partial_\nu r(\lambda))^{ii}+[r(\lambda)^{ij},r(\lambda)^{ii}+r(\lambda)^{jj}]=0.$$

All this implies that $[\rho_{\mathfrak{g},\mathfrak{h}}(\bar y_i),\rho_{\mathfrak{g},\mathfrak{h}}(\bar y_j)]=0$.

Let us check that $[\bar x_i,\bar t_{jk}]=0$ is preserved (i,j,k distinct). We have $[\rho_{\mathfrak{g},\mathfrak{h}}(\bar x_i),\rho_{\mathfrak{g},\mathfrak{h}}(\bar t_{jk})]=[(\lambda^\vee)^i,t^{jk}_{\mathfrak{g}}]=0$.

Let us prove that $[\bar y_i,\bar t_{jk}]=0$ is preserved (i,j,k distinct). We have $[\rho_{\mathfrak{g},\mathfrak{h}}(\bar y_i),\rho_{\mathfrak{g},\mathfrak{h}}(\bar t_{jk})]=[-h^i_\nu\bar\partial_\nu+\sum_l r(\lambda)^{il},t^{jk}_{\mathfrak{g}}]=[r(\lambda)^{ij}+r(\lambda)^{ik},t^{jk}_{\mathfrak{g}}]=0$ because $t_{\mathfrak{g}}$ is $\mathfrak{g}$-invariant. □

Proposition 6.4. *If $V_1,...,V_n$ are $\mathfrak{g}$-modules, then $(S(\mathfrak{h})[1/P]\otimes(\otimes_iV_i))^{\mathfrak{h}}$ is a $\bar{\mathfrak{t}}_{1,n}\rtimes\mathfrak{d}$-module. The $\bar{\mathfrak{t}}_{1,n}$-module structure is induced by the morphism $\bar{\mathfrak{t}}_{1,n}\to\mathcal{H}_n(\mathfrak{g},\mathfrak{h})$ of Proposition 6.3, so*

$$\rho_{(V_i)}(\bar x_i)(f(\lambda)\otimes(\otimes_iv_i))=(\lambda^\vee)^i(f(\lambda)\otimes(\otimes_iv_i)),$$
$$\rho_{(V_i)}(\bar y_i)(f(\lambda)\otimes(\otimes_iv_i))=(-h^i_\nu\partial_\nu+\sum_j r(\lambda)^{ij})(f(\lambda)\otimes(\otimes_iv_i)),$$
$$\rho_{(V_i)}(\bar t_{ij})(f(\lambda)\otimes(\otimes_iv_i))=t^{ij}_{\mathfrak{g}}(f(\lambda)\otimes(\otimes_iv_i)),$$

and the $\mathfrak{d}$-module structure is given by

$$\rho_{(V_i)}(\delta_{2m})(f(\lambda)\otimes(\otimes_iv_i))=\frac12(\sum_i\{(\mathrm{ad}\,\lambda^\vee)^{2m}(e_\alpha)\cdot e_\alpha\}^i)(f(\lambda)\otimes(\otimes_iv_i)),$$

$$\begin{aligned}
&\rho_{(V_i)}(\Delta_0)(f(\lambda)\otimes(\otimes_iv_i))\\
&=\Big(-\frac12\partial_\nu^2+\frac12\langle\mu(r(\lambda)),h_\nu\rangle\partial_\nu+\{\frac12\psi(\lambda)^{11}-\frac12(\mathrm{ad}\,\lambda^\vee)^{-1}_{|\mathfrak{n}}(\mu(r(\lambda))_{\mathfrak{n}})\}^{12\dots n}\Big)(f(\lambda)\otimes(\otimes_iv_i)),
\end{aligned}$$

$$\rho_{(V_i)}(d)(f(\lambda)\otimes(\otimes_iv_i))=\frac12(\langle\lambda,h_\nu\rangle\partial_\nu+\partial_\nu\langle\lambda,h_\nu\rangle+\langle\mu(r(\lambda)),\lambda^\vee\rangle)(f(\lambda)\otimes(\otimes_iv_i)),$$
$$\rho_{(V_i)}(X)(f(\lambda)\otimes(\otimes_iv_i))=(1/2)\langle\lambda^\vee,\lambda^\vee\rangle(f(\lambda)\otimes(\otimes_iv_i)).$$

Here $x_{\mathfrak{n}}$ is the projection of $x\in\mathfrak{g}$ on $\mathfrak{n}$ along $\mathfrak{h}$.

To summarize, we have a diagram

$$\begin{array}{ccccc} \bar{\mathfrak{t}}_{1,n} & \to & \mathcal{H}_n(\mathfrak{g},\mathfrak{h}) & \to & \mathrm{End}((S(\mathfrak{h})[1/P]\otimes(\otimes_i V_i))^{\mathfrak{h}}) \\ & \subset\searrow & {\scriptstyle (1)}\uparrow & \nearrow & \\ & & \bar{\mathfrak{t}}_{1,n}\rtimes\mathfrak{d} & & \end{array}$$

As before, the restriction morphism $(S(\mathfrak{g})\otimes(\otimes_i V_i))^{\mathfrak{g}}\to(S(\mathfrak{h})[1/P]\otimes(\otimes_i V_i))^{\mathfrak{h}}$ extends to a $\bar{\mathfrak{t}}_{1,n}\rtimes\mathfrak{d}$-modules morphism.

The action of $\bar{\mathfrak{t}}_{1,n}\rtimes\mathfrak{d}$ factors through a morphism $\tilde\rho_{\mathfrak{g},\mathfrak{h}}:\bar{\mathfrak{t}}_{1,n}\rtimes\mathfrak{d}\to\mathcal{H}_n(\mathfrak{g},\mathfrak{h})$ extending $\rho_{\mathfrak{g},\mathfrak{h}}:\bar{\mathfrak{t}}_{1,n}\to\mathcal{H}_n(\mathfrak{g},\mathfrak{h})$ (denoted by (1) in the diagram).

Proof. Let $\lambda\in\mathfrak{h}^*_{\mathrm{reg}}$. Then if V is a $\mathfrak{g}$-module, we have $(\hat{\mathcal{O}}_{\mathfrak{g}^*,\lambda}\otimes V)^{\mathfrak{g}}=(\hat{\mathcal{O}}_{\mathfrak{h}^*,\lambda}\otimes V)^{\mathfrak{h}}$ (where $\hat{\mathcal{O}}_{X,x}$ is the completed local ring of a variety X at the point x). We then have a morphism $\bar{\mathfrak{t}}_{1,n}\rtimes\mathfrak{d}\to\mathcal{H}_n(\mathfrak{g})\to\mathrm{End}((\hat{\mathcal{O}}_{\mathfrak{g}^*,\lambda}\otimes(\otimes_i V_i))^{\mathfrak{g}})$ for any $\lambda\in\mathfrak{g}^*$, so when $\lambda\in\mathfrak{h}^*_{\mathrm{reg}}$ we get a morphism $\bar{\mathfrak{t}}_{1,n}\rtimes\mathfrak{d}\to\mathrm{End}((\hat{\mathcal{O}}_{\mathfrak{h}^*,\lambda}\otimes(\otimes_i V_i))^{\mathfrak{h}})$.

Let show that the images of the generators of $\bar{\mathfrak{t}}_{1,n}\rtimes\mathfrak{d}$ under this morphism are given by the above formulas.

Since the actions of $\bar{x}_i$, $\bar{t}_{ij}$ and X on $(\hat{\mathcal{O}}_{\mathfrak{g}^*,\lambda}\otimes(\otimes_i V_i))^{\mathfrak{g}}$ are given by multiplication by elements of $(\hat{\mathcal{O}}_{\mathfrak{g}^*,\lambda}\otimes U(\mathfrak{g})^{\otimes n})^{\mathfrak{g}}$, their actions on $(\hat{\mathcal{O}}_{\mathfrak{h}^*,\lambda}\otimes(\otimes_i V_i))^{\mathfrak{h}}$ are given by multiplication by restrictions of these elements to $\mathfrak{h}^*$.

Let us compute the action of $\bar{y}_i$. Let $\tilde f(\lambda)\in(\hat{\mathcal{O}}_{\mathfrak{h}^*,\lambda}\otimes(\otimes_i V_i))^{\mathfrak{h}}$ and $\tilde F(\lambda)\in(\hat{\mathcal{O}}_{\mathfrak{g}^*,\lambda}\otimes(\otimes_i V_i))^{\mathfrak{g}}$ be its equivariant extension to a formal map $\mathfrak{g}^*\to\otimes_i V_i$. Then for $x\in\mathfrak{n}$, we have $(\partial_{x^\wedge}+\sum_i(\mathrm{ad}\,\lambda^\vee)^{-1}(x)^i)(\tilde F(\lambda))_{|\mathfrak{h}^*}=0$ (the map $x\mapsto x^\wedge$ is the inverse of $\mathfrak{g}^*\to\mathfrak{g}$, $\lambda\mapsto\lambda^\vee$). Then $\rho_{(V_i)}(\bar{y}_i)(\tilde f(\lambda))=\Big(-h^i_\nu\partial_\nu+\sum_j e^i_\beta\big((\mathrm{ad}\,\lambda^\vee)^{-1}(e_\beta)\big)^j\Big)\tilde f(\lambda)=(-h^i_\nu\partial_\nu+\sum_j r(\lambda)^{ij})(\tilde f(\lambda))$.

Let us now compute the action of Δ_0. Let $\lambda_0\in\mathfrak{h}^*$ be such that $\lambda_0^\vee\in U$ and $\lambda\in\mathfrak{g}^*$ be close to λ_0. We set $\delta\lambda:=\lambda-\lambda_0$. We then have $\lambda=e^{\mathrm{ad}\,x}(\lambda_0+h^\wedge)$, where $x\in\mathfrak{n}$ and $h\in\mathfrak{h}$ are close to 0. We have the expansions

$$h=(\delta\lambda)^\vee_{\mathfrak{h}}+\frac{1}{2}[(\mathrm{ad}\,\lambda_0^\vee)^{-1}_{|\mathfrak{n}}((\delta\lambda)^\vee_{\mathfrak{n}}),(\delta\lambda)^\vee_{\mathfrak{n}}]_{\mathfrak{h}},$$

$$x=-(\mathrm{ad}\,\lambda_0^\vee)^{-1}_{|\mathfrak{n}}\Big((\delta\lambda)^\vee_{\mathfrak{n}}+[(\mathrm{ad}\,\lambda_0^\vee)^{-1}_{|\mathfrak{n}}((\delta\lambda)^\vee_{\mathfrak{n}}),(\delta\lambda)^\vee_{\mathfrak{h}}]+\frac{1}{2}[(\mathrm{ad}\,\lambda_0^\vee)^{-1}_{|\mathfrak{n}}((\delta\lambda)^\vee_{\mathfrak{n}}),(\delta\lambda)^\vee_{\mathfrak{n}}]_{\mathfrak{n}}\Big)$$

up to terms of order >2; here the indices $u_{\mathfrak{n}}$ and $u_{\mathfrak{h}}$ mean the projections of $u\in\mathfrak{g}$ to $\mathfrak{n}$ and $\mathfrak{h}$. If now $\tilde f(\lambda):\mathfrak{h}^*\supset V(\lambda_0,\mathfrak{h}^*)\to\otimes_i V_i$ is an $\mathfrak{h}$-equivariant function defined at the vicinity of λ_0 and $\tilde F(\lambda):\mathfrak{g}^*\supset V(\lambda_0,\mathfrak{g}^*)\to\otimes_i V_i$ it its $\mathfrak{g}$-equivariant extension to a neighborhood of λ_0 in $\mathfrak{g}^*$, then $\tilde F(\lambda)=(e^x)^{1\dots n}\tilde f(\lambda_0+h)$, which implies the expansion

$$\begin{aligned}\tilde F(\lambda)&=\tilde f(\lambda_0)+\Big((\delta\lambda)_\nu+\frac{1}{2}\langle[(\mathrm{ad}\,\lambda_0^\vee)^{-1}_{|\mathfrak{n}}(e_\beta),e_{\beta'}],h_\nu\rangle(\delta\lambda)_\beta(\delta\lambda)_{\beta'}\Big)\partial_\nu\tilde f(\lambda_0)+\frac{1}{2}(\delta\lambda)_\nu(\delta\lambda)_{\nu'}\partial^2_{\nu\nu'}\tilde f(\lambda_0)\\&+\Big(-(\mathrm{ad}\,\lambda_0^\vee)^{-1}_{|\mathfrak{n}}(e_\beta)(\delta\lambda)_\beta-(\mathrm{ad}\,\lambda_0^\vee)^{-1}([(\mathrm{ad}\,\lambda_0^\vee)^{-1}_{|\mathfrak{n}}(e_\beta),h_\nu])(\delta\lambda)_\nu(\delta\lambda)_\beta\\&-\frac{1}{2}(\mathrm{ad}\,\lambda_0^\vee)^{-1}_{|\mathfrak{n}}([(\mathrm{ad}\,\lambda_0^\vee)^{-1}_{|\mathfrak{n}}(e_\beta),e_{\beta'}]_{\mathfrak{n}})(\delta\lambda)_\beta(\delta\lambda)_{\beta'}+\frac{1}{2}(\mathrm{ad}\,\lambda_0^\vee)^{-1}_{|\mathfrak{n}}(e_\beta)(\mathrm{ad}\,\lambda_0^\vee)^{-1}_{|\mathfrak{n}}(e_{\beta'})(\delta\lambda)_\beta(\delta\lambda)_{\beta'}\Big)^{1\dots n}\tilde f(\lambda_0)\\&-(\mathrm{ad}\,\lambda_0^\vee)^{-1}_{|\mathfrak{n}}(e_\beta)^{1\dots n}(\delta\lambda)_\beta(\delta\lambda)_\nu\partial_\nu\tilde f(\lambda_0)\end{aligned}$$

up to terms of order >2.

Then

$$\begin{aligned}(\partial^2_\alpha F)(\lambda_0)&=(\partial^2_\nu\tilde f)(\lambda_0)+\langle[(\mathrm{ad}\,\lambda_0^\vee)^{-1}_{|\mathfrak{n}}(e_\beta),e_\beta],h_\nu\rangle\partial_\nu\tilde f(\lambda_0)\\&+\Big(-(\mathrm{ad}\,\lambda_0^\vee)^{-1}_{|\mathfrak{n}}([(\mathrm{ad}\,\lambda_0^\vee)^{-1}_{|\mathfrak{n}}(e_\beta),e_\beta]_{\mathfrak{n}})+((\mathrm{ad}\,\lambda_0^\vee)^{-1}_{|\mathfrak{n}}(e_\beta))^2\Big)^{1\dots n}\tilde f(\lambda_0),\end{aligned}$$

which implies the formula for the action of Δ_0.

Then $(S(\mathfrak{h})[1/P]\otimes(\otimes_i V_i))^{\mathfrak{h}} \subset \prod_{\lambda\in\mathfrak{h}^*_{\rm reg}}(\hat{\mathcal{O}}_{\mathfrak{h}^*,\lambda}\otimes(\otimes_i V_i))^{\mathfrak{h}}$ is preserved by the action of the generators of $\bar{\mathfrak{t}}_{1,n}\rtimes\mathfrak{d}$-module, hence it is a sub-$(\bar{\mathfrak{t}}_{1,n}\rtimes\mathfrak{d})$-module, with action given by the above formulas. □

6.4. **Realization of the universal KZB system.** The realization of the flat connection $\mathrm{d}-\sum_i \bar{K}_i(\mathbf{z}|\tau)\,\mathrm{d}\,z_i-\bar{\Delta}(\mathbf{z}|\tau)\,\mathrm{d}\,\tau$ on $(\mathfrak{H}\times\mathbb{C}^n)-\mathrm{Diag}_n$ is a flat connection on the trivial bundle with fiber $(\mathcal{O}_{\mathfrak{h}^*_{\rm reg}}\otimes(\otimes_i V_i))^{\mathfrak{h}}$.

We now compute this realization, under the assumption that $\mathfrak{h}\subset\mathfrak{g}$ is a maximal abelian subalgebra. In this case, two simplifications occur:

(a) $(\mathrm{ad}\,\lambda^\vee)(h_\nu)=0$ since $\mathfrak{h}$ is abelian,

(b) $[(\mathrm{ad}\,\lambda^\vee)^{-1}_{|\mathfrak{n}}(e_\beta),e_\beta]_{\mathfrak{n}}=0$ since $[(\mathrm{ad}\,\lambda^\vee)^{-1}_{|\mathfrak{n}}(e_\beta),e_\beta]$ commutes with any element in $\mathfrak{h}$, so that it belongs to $\mathfrak{h}$.

The image of $\bar{K}_i(\mathbf{z}|\tau)$ is then the operator

$$K_i^{(V_i)}(\mathbf{z}|\tau)=h^i_\nu\partial_\nu-\sum_j r(\lambda)^{ij}+\sum_{j|j\neq i}k(z_{ij},(\mathrm{ad}\,\lambda^\vee)^i|\tau)(t^{ij}_{\mathfrak{n}}+t^{ij}_{\mathfrak{h}})$$

$$=h^i_\nu\partial_\nu-r(\lambda)^{ii}+\sum_{j|j\neq i}\frac{\theta(z_{ij}+(\mathrm{ad}\,\lambda^\vee)^i|\tau)}{\theta(z_{ij}|\tau)\theta((\mathrm{ad}\,\lambda^\vee)^i|\tau)}(t^{ij}_{\mathfrak{n}})+\sum_{j|j\neq i}\frac{\theta'}{\theta}(z_{ij}|\tau)t^{ij}_{\mathfrak{h}}$$

The image of $2\pi\,\mathrm{i}\,\bar{\Delta}(\mathbf{z}|\tau)$ is the operator

$$2\pi\,\mathrm{i}\,\Delta^{(V_i)}(\mathbf{z}|\tau)=\frac{1}{2}\partial_\nu^2+\frac{1}{2}\langle[(\mathrm{ad}\,\lambda^\vee)^{-1}(e_\beta),e_\beta],h_\nu\rangle\partial_\nu-g(0,0|\tau)\sum_i\frac{1}{2}t^{ii}_{\mathfrak{g}}$$

$$+\sum_{i,j}\frac{1}{2}\big([g(z_{ij},\mathrm{ad}\,\lambda^\vee|\tau)-(\mathrm{ad}\,\lambda^\vee)^{-2}](e_\beta)\big)^i e^j_\beta+\sum_{i,j}\frac{1}{2}g(z_{ij},0|\tau)h^i_\nu h^j_\nu$$

and the connection is now

$$\nabla^{(V_i)}=\mathrm{d}-\sum_i K_i^{(V_i)}(\mathbf{z}|\tau)-\Delta^{(V_i)}(\mathbf{z}|\tau).$$

Recall that $P(\lambda)=\det((\mathrm{ad}\,\lambda^\vee)_{|\mathfrak{n}})$. We compute the conjugation $P^{1/2}\nabla^{(V_i)}P^{-1/2}$, where $P^{\pm1/2}$ is the operator of multiplication by (inverse branches of) $P^{\pm1/2}$ on $\mathcal{O}_{\mathfrak{h}^*_{\rm reg}}\otimes(\otimes_i V_i)^{\mathfrak{h}}$.

Lemma 6.5. $\partial_\nu\log P(\lambda)=-\langle h_\nu,\mu(r(\lambda))\rangle$, $P^{1/2}[h^i_\nu\partial_\nu-r(\lambda)^{ii}]P^{-1/2}=h^i_\nu\partial_\nu$, $P^{1/2}[\partial_\nu^2+\langle[(\mathrm{ad}\,\lambda^\vee)^{-1}_{|\mathfrak{n}}(e_\beta),e_\beta],h_\nu\rangle\partial_\nu]P^{-1/2}=\partial_\nu^2+\partial_\nu\big(\langle h_\nu,\frac{1}{2}\mu(r(\lambda))\rangle\big)-\langle h_\nu,\frac{1}{2}\mu(r(\lambda))\rangle^2$.

Proof. $\partial_\nu\log P(\lambda)=(d/dt)_{|t=0}\det[(\mathrm{ad}(\lambda^\vee+th_\nu)_{|\mathfrak{n}})(\mathrm{ad}\,\lambda^\vee)^{-1}_{|\mathfrak{n}}]=\mathrm{tr}[(\mathrm{ad}\,h_\nu)_{|\mathfrak{n}}\circ(\mathrm{ad}\,\lambda^\vee)^{-1}_{|\mathfrak{n}}]=\langle e_\beta,(\mathrm{ad}\,h_\nu)\circ(\mathrm{ad}\,\lambda^\vee)^{-1}_{|\mathfrak{n}}(e_\beta)\rangle=\langle[(\mathrm{ad}\,\lambda^\vee)^{-1}_{|\mathfrak{n}}(e_\beta),e_\beta],h_\nu\rangle=-\langle h_\nu,\mu(r(\lambda))\rangle$. The next equality follows from $\mu(r(\lambda))^i=2r(\lambda)^{ii}$. The last equality is a direct consequence. □

We then get:

Proposition 6.6. $P^{1/2}\nabla^{(V_i)}P^{-1/2}=\mathrm{d}-\sum_i\tilde{K}_i(\mathbf{z}|\tau)\,\mathrm{d}\,z_i-\tilde{\Delta}(\mathbf{z}|\tau)\,\mathrm{d}\,\tau$, *where*

$$\tilde{K}_i(\mathbf{z}|\tau)=h^i_\nu\partial_\nu+\sum_{j|j\neq i}\frac{\theta(z_{ij}+(\mathrm{ad}\,\lambda^\vee)^i|\tau)}{\theta(z_{ij}|\tau)\theta((\mathrm{ad}\,\lambda^\vee)^i|\tau)}(t^{ij}_{\mathfrak{n}})+\sum_{j|j\neq i}\frac{\theta'}{\theta}(z_{ij}|\tau)t^{ij}_{\mathfrak{h}}$$

$$2\pi\,\mathrm{i}\,\tilde{\Delta}(\mathbf{z}|\tau)=\frac{1}{2}\partial_\nu^2+\partial_\nu\big(\langle h_\nu,\frac{1}{2}\mu(r(\lambda))\rangle\big)-\langle h_\nu,\frac{1}{2}\mu(r(\lambda))\rangle^2-g(0,0|\tau)\sum_i\frac{1}{2}t^{ii}_{\mathfrak{g}}$$

$$+\sum_{i,j}\frac{1}{2}\Big(\big(g(z_{ij},\mathrm{ad}\,\lambda^\vee|\tau)-(\mathrm{ad}\,\lambda^\vee)^{-2}\big)(e_\beta)\Big)^i e^j_\beta+\sum_{i,j}\frac{1}{2}g(z_{ij},0|\tau)h^i_\nu h^j_\nu,$$

where

$$g(z,0|\tau)=\frac{1}{2}\frac{\theta''}{\theta}(z|\tau)-2\pi\,\mathrm{i}\,\frac{\partial_\tau\eta}{\eta}(\tau)$$

and

$$g(z,\alpha|\tau) - \alpha^{-2} = \frac{1}{2}\frac{\theta(z+\alpha|\tau)}{\theta(x|\tau)\theta(\alpha|\tau)}(\frac{\theta'}{\theta}(z+\alpha|\tau) - \frac{\theta'}{\theta}(\alpha|\tau))$$

The term in $\sum_i (1/2)t_{\mathfrak{g}}^{ii}$ is central and can be absorbed by a suitable further conjugation. Rescaling $t_{\mathfrak{g}}$ into $\kappa^{-1}t_{\mathfrak{g}}$, where $\kappa \in \mathbb{C}^\times$, $\tilde{K}_i(\mathbf{z}|\tau)$ and $\tilde{\Delta}(\mathbf{z}|\tau)$ get multiplied by κ. Moreover, we have:

Lemma 6.7. *When $\mathfrak{g}$ is simple and $\mathfrak{h} \subset \mathfrak{g}$ is the Cartan subalgebra, $\partial_\nu\{\langle h_\nu, \frac{1}{2}\mu(r(\lambda))\rangle\} = \langle h_\nu, \frac{1}{2}\mu(r(\lambda))\rangle^2$.*

Proof. Let $D(\lambda) := \prod_{\alpha\in\Delta^+}(\alpha,\lambda)$, where Δ^+ is the set of positive roots of $\mathfrak{g}$. Then $D(\lambda)$ is W-antiinvariant, where W is the Weyl group. Therefore $\partial_\nu^2 D(\lambda)$ is also W-antiinvariant, so it is divisible (as a polynomial on $\mathfrak{h}^*$) by all the (α,λ), where $\alpha \in \Delta^+$, so it is divisible by $D(\lambda)$; since $\partial_\nu^2 D(\lambda)$ has degree strictly lower than $D(\lambda)$, we get $\partial_\nu^2 D(\lambda) = 0$.

Now if $(e_\alpha, f_\alpha, h_\alpha)$ is a basis of the $\mathfrak{sl}_2$-triple associated with α, we have $r(\lambda) = \sum_{\alpha\in\Delta^+} -(e_\alpha \otimes f_\alpha - f_\alpha \otimes e_\alpha)/(\alpha,\lambda)$, so $\frac{1}{2}\mu(r(\lambda)) = -\sum_{\alpha\in\Delta^+} h_\alpha/(\alpha,\lambda)$. Therefore $\frac{1}{2}\mu(r(\lambda)) = -\partial_\nu \log D(\lambda) h_\nu$. Then $\partial_\nu^2 D(\lambda) = 0$ implies that $\partial_\nu^2 \log D + (\partial_\nu \log D)^2 = 0$, which implies the lemma. □

The resulting flat connection then coincides with that of [Be1, FW].

7. The universal KZB connection and representations of Cherednik algebras

7.1. The rational Cherednik algebra of type A_{n-1}. Let k be a complex number, and $n \geq 1$ an integer. The rational Cherednik algebra $H_n(k)$ of type A_{n-1} is the quotient of the algebra $\mathbb{C}[S_n] \ltimes \mathbb{C}[\mathrm{x}_1, ..., \mathrm{x}_n, \mathrm{y}_1, ..., \mathrm{y}_n]$ by the relations

$$\sum_i \mathrm{x}_i = 0,\ \sum_i \mathrm{y}_i = 0,\ [\mathrm{x}_i, \mathrm{x}_j] = 0 = [\mathrm{y}_i, \mathrm{y}_j],$$

$$[\mathrm{x}_i, \mathrm{y}_j] = \frac{1}{n} - k s_{ij},\ i \neq j,$$

where $s_{ij} \in S_n$ is the permutation of i and j (see e.g. [EG]). [5]

Let $e := \frac{1}{n!}\sum_{\sigma\in S_n}\sigma \in \mathbb{C}[S_n]$ be the Young symmetrizer. The spherical subalgebra $B_n(k)$ (often called the spherical Cherednik algebra) is defined to be the algebra $eH_n(k)e$.

We define an important element

$$\mathbf{h} := \frac{1}{2}\sum_i (\mathrm{x}_i\mathrm{y}_i + \mathrm{y}_i\mathrm{x}_i).$$

We recall that category $\mathcal{O}$ is the category of $H_n(k)$-modules which are locally nilpotent under the action of the operators y_i and decompose into a direct sum of finite dimensional generalized eigenspaces of $\mathbf{h}$. Similarly, one defines category $\mathcal{O}$ over $B_n(k)$ to be the category of $B_n(k)$-modules which are locally nilpotent under the action of $\mathbb{C}[\mathrm{y}_1, ..., \mathrm{y}_n]^{S_n}$ and decompose into a direct sum of finite dimensional generalized eigenspaces of $\mathbf{h}$.

7.2. The homorphism from $\bar{\mathfrak{t}}_{1,n}$ to the rational Cherednik algebra.

Proposition 7.1. *For each $k, a, b \in \mathbb{C}$, we have a homomorphism of Lie algebras $\xi_{a,b}$: $\bar{\mathfrak{t}}_{1,n} \to H_n(k)$, defined by the formula*

$$\bar{x}_i \mapsto a\mathrm{x}_i, \quad \bar{y}_i \mapsto b\mathrm{y}_i, \quad \bar{t}_{ij} \mapsto ab\left(\frac{1}{n} - k s_{ij}\right).$$

Proof. Straightforward. □

[5] The generators $\mathrm{x}_\alpha, \partial_\alpha$ of Section 6.1 will be henceforth renamed q_α, p_α.

Remark 7.2. Obviously, a, b can be rescaled independently, by rescaling the generators $\bar{x}_i$ and $\bar{y}_i$ of the source algebra $\bar{\mathfrak{t}}_{1,n}$. On the other hand, if we are only allowed to apply automorphisms of the target algebra $H_n(k)$, then a, b can only be rescaled in such a way that the product ab is preserved. □

This shows that any representation V of the rational Cherednik algebra $H_n(k)$ yields a family of realizations for $\bar{\mathfrak{t}}_{1,n}$ parametrized by $a, b \in \mathbb{C}$, and gives rise to a family of flat connections $\nabla_{a,b}$ over the configuration space $\bar{C}(E_\tau, n)$.

7.3. **Monodromy representations of double affine Hecke algebras.** Let $\mathcal{H}_n(q,t)$ be Cherednik's double affine Hecke algebra of type A_{n-1}. By definition, $\mathcal{H}_n(q,t)$ is the quotient of the group algebra of the orbifold fundamental group $\overline{\mathrm{B}}_{1,n}$ of $\bar{C}(E_\tau, n)/S_n$ by the additional relations

$$(T - q^{-1}t)(T + q^{-1}t^{-1}) = 0,$$

where T is any element of $\overline{\mathrm{B}}_{1,n}$ homotopic (as a free loop) to a small loop around the divisor of diagonals in the counterclockwise direction.

Let V be a representation of $H_n(k)$, and let $\nabla_{a,b}(V)$ be the universal connection $\nabla_{a,b}$ evaluated in V. In some cases, for example if a, b are formal, or if V is finite dimensional, we can consider the monodromy of this connection, which obviously gives a representation of $\mathcal{H}_n(q,t)$ on V, with

$$q = e^{-2\pi \mathrm{i} ab/n}, \ t = e^{-2\pi \mathrm{i} kab}.$$

In particular, taking $a = b$, $V = H_n(k)$, this monodromy representation defines an homomorphism $\theta_a : \mathcal{H}_n(q,t) \to H_n(k)[[a]]$, where

$$q = e^{-2\pi \mathrm{i} a^2/n}, \ t = e^{-2\pi \mathrm{i} k a^2}.$$

It is easy to check that this homomorphism becomes an isomorphism upon inverting a. The existence of such an isomorphism was pointed out by Cherednik (see [Ch2], end of Section 6, and the end of [Ch1]), but his proof is different.

Example 7.3. *Let $k = r/n$, where r is an integer relatively prime to n. In this case, it is known (see e.g.* [BEG1]*) that the algebra $H_n(k)$ admits an irreducible finite dimensional representation $Y(r,n)$ of dimension r^{n-1}. By virtue of the above construction, the space $Y(r,n)$ carries an action of $\mathcal{H}_n(q,t)$ with any nonzero q, t such that $q^r = t$. This finite dimensional representation of $\mathcal{H}_n(q,t)$ is irreducible for generic q, and is called a perfect representation; it was first constructed in* [E]*, p. 500, and later in* [Ch2]*, Theorem 6.5, in a greater generality.*

7.4. **The modular extension of $\xi_{a,b}$.** Assume that $a, b \neq 0$.

Proposition 7.4. *The homomorphism $\xi_{a,b}$ can be extended to the algebra $U(\bar{\mathfrak{t}}_{1,n} \rtimes \mathfrak{d}) \rtimes S_n$ by the formulas*

$$\xi_{a,b}(s_{ij}) = s_{ij},$$

$$\xi_{a,b}(d) = \mathbf{h} = \frac{1}{2}\sum_i (\mathrm{x}_i \mathrm{y}_i + \mathrm{y}_i \mathrm{x}_i), \quad \xi_{a,b}(X) = -\frac{1}{2} a b^{-1} \sum_i \mathrm{x}_i^2,$$

$$\xi_{a,b}(\Delta_0) = \frac{1}{2} b a^{-1} \sum_i \mathrm{y}_i^2, \quad \xi_{a,b}(\delta_{2m}) = -\frac{1}{2} a^{2m-1} b^{-1} \sum_{i<j} (\mathrm{x}_i - \mathrm{x}_j)^{2m}.$$

Proof. Direct computation. □

Thus, the flat connections $\nabla_{a,b}$ extend to flat connections on $\mathcal{M}_{1,[n]}$.

This shows that the monodromy representation of the connection $\nabla_{a,b}(V)$, when it can be defined, is a representation of the double affine Hecke algebra $\mathcal{H}_n(q,t)$ with a compatible action of the extended modular group $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$. In particular, this is the case if $V = Y(r,n)$.

Such representations of $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ were considered by Cherednik, [Ch2]. The element T of $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ acts in this representation by "the Gaussian", and the element S by the "Fourier-Cherednik transform". They are generalizations of the $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$-action on Verlinde algebras.

8. Explicit realizations of certain highest weight representations of the rational Cherednik algebra of type A_{n-1}

8.1. **The representation V_N.** Let N be a divisor of n, and $\mathfrak{g} = \mathfrak{sl}_N(\mathbb{C})$, $G = \mathrm{SL}_N(\mathbb{C})$. Let $V_N = (\mathbb{C}[\mathfrak{g}] \otimes (\mathbb{C}^N)^{\otimes n})^{\mathfrak{g}}$ (the divisor condition is needed for this space to be nonzero). It turns out that V_N has a natural structure of a representation of $H_n(k)$ for $k = N/n$.

Proposition 8.1. *We have a homomorphism $\zeta_N : H_n(N/n) \to \mathrm{End}(V_N)$, defined by the formulas*

$$\zeta_N(s_{ij}) = s_{ij}, \quad \zeta_N(\mathrm{x}_i) = X_i, \quad \zeta_N(\mathrm{y}_i) = Y_i, \quad (i = 1, ..., n)$$

where for $f \in V_N, A \in \mathfrak{g}$ we have

$$(X_i f)(A) = A_i f(A),$$

$$(Y_i f)(A) = \frac{N}{n} \sum_p (b_p)_i \frac{\partial f}{\partial b_p}(A),$$

where $\{b_p\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to the trace form.

Proof. Straightforward verification. □

The relationship of the representation V_N to other results in this paper is described by the following proposition.

Proposition 8.2. *The connection $\nabla_{a,1}(V_N)$ corresponding to the representation V_N is the usual KZB connection for the n-point correlation functions on the elliptic curve for the Lie algebra $\mathfrak{sl}_N$ and n copies of the vector representation $\mathbb{C}^N$, at level $K = -\frac{n}{aN} - N$.*

Proof. We have a sequence of maps

$$U(\bar{\mathfrak{t}}_{1,n} \rtimes \mathfrak{d}) \rtimes S_n \to H_n(N/n) \to \mathcal{H}_n(\mathfrak{g}) \rtimes S_n \to \mathrm{End}(V_N),$$

where the first map is $\xi_{a,b}$, the second map sends s_{ij} to s_{ij}, x_i to the class of $\sum_\alpha q_\alpha \otimes e^i_\alpha$, and y_i to the class of $\sum_\alpha p_\alpha \otimes e^i_\alpha$ (recall that the x_a, ∂_a of Section 6.1 have been renamed q_a, p_a), and the last map is explained in Section 6.1. The composition of the two first maps is then that of Proposition 6.2, and the composition of the two last maps is the map ζ_N of Proposition 8.1. This implies the statement. □

Remark 8.3. Suppose that K is a nonnegative integer, i.e. $a = -\frac{n}{N(K+N)}$, where $K \in \mathbb{Z}_+$. Then the connection $\nabla_{a,1}$ on the infinite dimensional vector bundle with fiber V_N preserves a finite dimensional subbundle of conformal blocks for the WZW model at level K. Th subbundle gives rise to a finite dimensional monodromy representation V_N^K of the Cherednik algebra $\mathcal{H}_n(q, t)$ with

$$q = e^{\frac{2\pi \mathrm{i}}{N(K+N)}}, t = q^N,$$

(so both parameters are roots of unity). The dimension of V_N^K is given by the Verlinde formula, and it carries a compatible action of $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ to the action of the Cherednik algebra. Representations of this type were studied by Cherednik in [Ch2].

8.2. **The spherical part of V_N.** Note that

$$((\sum_{i=1}^{n} X_i^p)f)(A) = \frac{n}{N}(\operatorname{tr} A^p)f(A), \tag{43}$$

$$((\sum_{i=1}^{n} Y_i^p)f)(A) = \left(\frac{N}{n}\right)^{p-1}(\operatorname{tr}\partial_A^p)f(A) \tag{44}$$

Consider the space $U_N = eV_N = (\mathbb{C}[\mathfrak{g}]\otimes S^n\mathbb{C}^N)^{\mathfrak{g}}$ as a module over the spherical subalgebra $B_n(k)$. It is known (see e.g. [BEG2]) that the spherical subalgebra is generated by the elements $(\sum \mathrm{x}_i^p)e$ and $(\sum \mathrm{y}_i^p)e$. Thus formulas (43,44) determine the action of $B_n(k)$ on U_N.

We note that by restriction to the set $\mathfrak{h}$ of diagonal matrices $\operatorname{diag}(\lambda_1,...,\lambda_N)$, and dividing by $\Delta^{n/N}$, where $\Delta = \prod_{i<j}(\lambda_i-\lambda_j)$, one identifies U_N with $\mathbb{C}[\mathfrak{h}]^{S_N}$. Moreover, it follows from [EG] that formulas (43,44) can be viewed as defining an action of another spherical Cherednik algebra, namely $B_N(1/k)$, on $\mathbb{C}[\mathfrak{h}]^{S_N}$. Moreover, this representation is the symmetric part W of the standard polynomial representation of $H_N(1/k)$, which is faithful and irreducible since $1/k = n/N$ is an integer ([GGOR]). In other words, we have the following proposition.

Proposition 8.4. *There exists a surjective homomorphism $\phi : B_n(N/n) \to B_N(n/N)$, such that $\phi^*W = U_N$. In particular, U_N is an irreducible representation of $B_n(N/n)$.*

Proposition 8.4 can be generalized as follows. Let $0 \le p \le n/N$ be an integer. Consider the partition $\mu(p) = (n - p(N-1), p, ..., p)$ of n. The representation of $\mathfrak{g}$ attached to $\mu(p)$ is $S^{n-pN}\mathbb{C}^N$.

Let $e(p)$ be a primitive idempotent of the representation of S_n attached to $\mu(p)$. Let $U_N^p = e(p)V_N = (\mathbb{C}[\mathfrak{g}] \otimes S^{n-pN}\mathbb{C}^N)^{\mathfrak{g}}$. Then the algebra $e(p)H_n(N/n)e(p)$ acts on U_N^p, and the above situation of U_N is the special case $p = 0$.

Proposition 8.5. *There exists a surjective homomorphism $\phi_p : e(p)H_n(N/n)e(p) \to B_N(n/N-p)$, such that $\phi_p^*W = U_N^p$. In particular, U_N^p is an irreducible representation of $B_n(N/n-p)$.*

Proof. Similar to the proof of Proposition 8.4. □

Example 8.6. *$p = 1$, $n = N$. In this case $e(p) = e_- = \frac{1}{n!}\sum_{\sigma\in S_n}\varepsilon(\sigma)\sigma$, the antisymmetrizer, and the map ϕ_p is the shift isomorphism $e_-H_N(1)e_- \to eH_N(0)e$.*

8.3. **Coincidence of the two $\mathfrak{sl}_2$ actions.** As before, let $\{b_p\}$ be an orthonormal basis of $\mathfrak{g}$ (under some invariant inner product). Consider the $\mathfrak{sl}_2$-triple

$$H = \sum b_p \frac{\partial}{\partial b_p} + \frac{\dim\mathfrak{g}}{2} \tag{45}$$

(the shifted Euler field),

$$F = \frac{1}{2}\sum_p b_p^2, \quad E = \frac{1}{2}\Delta_{\mathfrak{g}}, \tag{46}$$

where $\Delta_{\mathfrak{g}}$ is the Laplace operator on $\mathfrak{g}$. Recall also (see e.g. [BEG2]) that the rational Cherednik algebra contains the $\mathfrak{sl}_2$-triple $\mathbf{h} = \frac{1}{2}\sum_i(\mathrm{x}_i\mathrm{y}_i + \mathrm{y}_i\mathrm{x}_i)$, $\mathbf{e} = \frac{1}{2}\sum_i \mathrm{y}_i^2$, $\mathbf{f} = \frac{1}{2}\sum_i \mathrm{x}_i^2$.

The following proposition shows that the actions of these two $\mathfrak{sl}_2$ algebras on V_N essentially coincide.

Proposition 8.7. *On V_N, one has*

$$\mathbf{h} = H, \quad \mathbf{e} = \frac{N}{n}E, \quad \mathbf{f} = \frac{n}{N}F.$$

Proof. The last two equations follow from formulas (43,44), and the first one follows from the last two by taking commutators. □

8.4. **The irreducibility of V_N.** Let $\Delta(n,N)$ be the representation of the symmetric group S_n corresponding to the rectangular Young diagram with N rows (and correspondingly n/N columns), i.e. to the partition $(\frac{n}{N},...,\frac{n}{N})$; e.g., $\Delta(n,1)$ is the trivial representation.

For a representation π of S_n, let $L(\pi)$ denote the irreducible lowest weight representation of $H_n(k)$ with lowest weight π.

Theorem 8.8. *The representation V_N is isomorphic to $L(\Delta(n,N))$.*

Proof. The representation V_N is graded by the degree of polynomials, and in degree zero we have $V_N[0] = ((\mathbb{C}^N)^{\otimes n})^{\mathfrak{g}} = \Delta(n,N)$ by the Weyl duality.

Let us show that the module V_N is semisimple. It is sufficient to show that V_N is a unitary representation, i.e. admits a positive definite contravariant Hermitian form. Such a form can be defined by the formula

$$(f,g) = \langle f(\partial_A), g(A)\rangle|_{A=0},$$

where $\langle -,-\rangle$ is the Hermitian form on $(\mathbb{C}^N)^{\otimes n}$ obtained by tensoring the standard forms on the factors. This form is obviously positive definite, and satisfies the contravariance properties:

$$(Y_i f, g) = \frac{N}{n}(f, X_i g),\ (f, Y_i g) = \frac{N}{n}(X_i f, g).$$

The existence of the form $(-,-)$ implies the semisimplicity of V_N. In particular, we have a natural inclusion $L(\Delta(n,N)) \subset V_N$.

Next, formula (43) implies that V_N is a torsion-free module over $R := \mathbb{C}[\mathrm{x}_1,...,\mathrm{x}_N]^{S_N} = \mathbb{C}[\sum_{i=1}^N \mathrm{x}_i^p, 2 \le p \le N]$. Since V_N is semisimple, this implies that $V_N/L(\Delta(n,N))$ is torsion-free as well.

On the other hand, we will now show that the quotient $V_N/L(\Delta(n,N))$ is a torsion module over R. This will imply that the quotient is zero, as desired.

Let $v_1,...,v_N$ be the standard basis of $\mathbb{C}^N$, and for each sequence $J=(j_1,...,j_n)$, $j_i \in \{1,...,N\}$, let $v_J := v_{j_1}\otimes ... \otimes v_{j_n}$. Let us say that a sequence J is balanced if it contains each of its members exactly n/N times. Let B be the set of balanced sequences. The set B has commuting left and right actions S_N and S_n, $\sigma * (j_1,...,j_n) * \tau = (\sigma(j_{\tau(1)}),...,\sigma(j_{\tau(n)}))$. Let $J_0 = (1...1, 2...2, ..., N...N)$, then any $J \in B$ has the form $J = J_0 * \tau$ for some $\tau \in S_n$.

Let $f \in V_N$. Then f is a function $\mathfrak{h} \to ((\mathbb{C}^N)^{\otimes n})^{\mathfrak{h}}$, equivariant under the action of S_N (here $\mathfrak{h} \subset \mathfrak{g}$ is the Cartan subalgebra, so $\mathfrak{h} = \{(\lambda_1,...,\lambda_N)|\sum_i \lambda_i = 0\}$), so

$$f(\lambda) = \sum_{J\in B} f_J(\lambda) v_J, \tag{47}$$

where $\lambda = (\lambda_1,...,\lambda_N)$, and f_J are scalar functions (the summation is over B since $f(\lambda)$ must have zero weight). By the S_N-invariance, we have $f_{\sigma * J}(\sigma(\lambda)) = f_J(\lambda)$. We then decompose $f(\lambda) = \sum_{o\in S_N\setminus B} f_o(\lambda)$, where $f_o(\lambda) = \sum_{J\in o} f_J(\lambda) v_J$.

For each $o \in S_N \setminus B$, we construct a nonzero $\phi_o \in \mathbb{C}[\mathrm{x}_1,....,\mathrm{x}_n]$ such that $\phi_o \cdot f_o(\lambda) \in L(\Delta(n,N))$. Then $\phi := \prod_{o\in S_N\setminus B}\prod_{\sigma\in S_N}\sigma(\phi_o) \in R$ is nonzero and such that $\phi \cdot f(\lambda) \in L(\Delta(n,N))$.

We first construct ϕ_o when $o = o_0$, the class of J_0. By S_N-invariance, $f_{o_0}(\lambda)$ has the form

$$f_{o_0}(\lambda) = \sum_{\sigma\in S_N} g(\lambda_{\sigma(1)},...,\lambda_{\sigma(N)}) v_{\sigma(1)}^{\otimes n/N} \otimes ... \otimes v_{\sigma(N)}^{\otimes n/N}, \quad \text{where} \quad g(\lambda,...,\lambda_N) \in \mathbb{C}[\lambda_1,...,\lambda_N].$$

For $\phi_{o_0} \in \mathbb{C}[\mathrm{x}_1,...,\mathrm{x}_N]$, we have

$$\phi_{o_0} \cdot f_{o_0}(\lambda) = \sum_{\sigma\in S_N} (\phi_{o_0} g)(\lambda_{\sigma(1)},...,\lambda_{\sigma(N)}) v_{\sigma(1)}^{\otimes n/N} \otimes ... \otimes v_{\sigma(N)}^{\otimes n/N}. \tag{48}$$

On the other hand, let $v \in \Delta(n,N)$; expand $v = \sum_{J\in B} c_J v_J$. One checks that v can be chosen such that $c_{J_0} \neq 0$ (one starts with a nonzero vector v' and $J' \in B$ such that the coordinate of v' along J' is nonzero, and then acts on v' by an element of S_n bringing J' to J_0). Then since v is $\mathfrak{g}$-invariant (and therefore S_N-invariant), we have

$$c_{\sigma(1)...\sigma(1)...\sigma(N)...\sigma(N)} = c_{J_0} \tag{49}$$

for any $\sigma \in S_N$.

If $Q \in \mathbb{C}[\mathrm{x}_1, ..., \mathrm{x}_n]$, then

$$(Q \cdot v)(\lambda) = \sum_{(j_1,...,j_n)\in B} c_{j_1...j_n} Q(\lambda_{j_1}, ..., \lambda_{j_n}) v_{j_1} \otimes ... \otimes v_{j_n} \in L(\Delta(n, N)). \tag{50}$$

Set $Q_0(\lambda_1, ..., \lambda_n) := \prod_{1\le a<b\le n, j_a^0 \ne j_b^0} (\lambda_a - \lambda_b)$, where $J_0 = (1...1, ..., N...N) = (j_1^0, ..., j_n^0)$, $q_0(\lambda_1, ..., \lambda_N) := Q_0(\lambda_1...\lambda_1, ..., \lambda_N...\lambda_N)$, so $q_0(\lambda_1, ..., \lambda_N) = \big(\prod_{1\le i<j\le N} (\lambda_i - \lambda_j)\big)^{(n/N)^2}$.

Set $\phi_{o_0}(\lambda_1, ..., \lambda_N) := q_0(\lambda_1, ..., \lambda_N)$ and

$$Q(\lambda_1, ..., \lambda_n) := Q_0(\lambda_1, ..., \lambda_n) q(\lambda_1, \lambda_{(n/N)+1}, ..., \lambda_{(N-1)\frac{n}{N}+1}).$$

Then (48) and (50) coincide, as: (a) for $J \notin o_0$, $Q_0(\lambda_{j_1}, ..., \lambda_{j_n}) = 0$ so the coefficient of v_J in both expressions is zero, (b) the coefficients of v_{J_0} in both expressions coincide, (c) for $J \in o_0$, the coefficients of v_J coincide because of (b) and of (49). The functions ϕ_o are constructed in the same way for a general $o \in S_N \setminus B$. This ends the proof of the theorem. □

Remark 8.9. Theorem 8.8 is a special case of a much more general (but much less elementary) Theorem 9.8, which is proved below.

8.5. **The character formula for V_N.** For each partition μ of n, let $V(\mu)$ be the representation of $\mathfrak{g}$, and $\pi(\mu)$ the representation of S_n corresponding to μ.

Let $P_\mu(q)$ be the q-analogue of the weight multiplicity of the zero weight in $V(\mu)$. Namely, we have a filtration $F^\bullet$ on $V(\mu)[0]$ such that F^i is the space of vectors in $V(\mu)[0]$ killed by the $i+1$-th power of the principal nilpotent element $\sum e_i$ of $\mathfrak{g}$. Then $P_\mu(q) = \sum_{j\ge 0} \dim(F^j/F^{j-1}) q^j$. The coefficients of $P_\mu(q)$ are called the generalized exponents of $V(\mu)$ (see [K, He, Lu1] for more details).

We have $V_N = \oplus_\mu \pi(\mu) \otimes (\mathbb{C}[\mathfrak{g}] \otimes V(\mu))^{\mathfrak{g}}$. This together with Theorem 8.8 implies the following.

Corollary 8.10. *The character of $L(\Delta(n, N))$ is given by the formula*

$$\mathrm{Tr}|_{L(\Delta(n,N))}(w \cdot q^{\mathbf{h}}) = q^{(N^2-1)/2} \frac{\sum_\mu \chi_{\pi(\mu)}(w) P_\mu(q)}{(1-q^2)...(1-q^N)},$$

where $w \in S_n$, and $\chi_{\pi(\mu)}$ is the character of $\pi(\mu)$. Here the summation is over partitions μ of n with at most N parts.

Proof. The formula follows, using Proposition 8.7, from Kostant's result ([K]) that $(\mathbb{C}[\mathfrak{g}] \otimes V(\mu))^{\mathfrak{g}}$ is a free module over $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$, and the fact that the Hilbert polynomial of the space of generators for this module is the q-weight multiplicity of the zero weight, $P_\mu(q)$ ([K, Lu1, He]). □

Remark 8.11. It would be interesting to compare this formula with the character formula of [Ro] for the same module.

9. Equivariant D-modules and representations of the rational Cherednik algebra

9.1. **The category of equivariant D-modules on the nilpotent cone.** The theory of equivariant D-modules on the nilpotent cone arose from Harish-Chandra's work on invariant distributions on nilpotent orbits of real groups, and was developed further in many papers, see e.g. [HK, LS, L, Mi] and references therein. Let us recall some of the basics of this theory.

Let G be a simply connected simple algebraic group over $\mathbb{C}$, and $\mathfrak{g}$ its Lie algebra. Let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone of $\mathfrak{g}$. We denote by $\mathcal{D}(\mathfrak{g})$ the category of finitely generated D-modules on $\mathfrak{g}$, by $\mathcal{D}_G(\mathfrak{g})$ the subcategory of G-equivariant D-modules, and by $\mathcal{D}_G(\mathcal{N})$ the category of G-equivariant D-modules which are set-theoretically supported on $\mathcal{N}$ (here we do not make a distinction between a D-module on an affine space and the space of its global sections). Since G acts on $\mathcal{N}$ with finitely many orbits, it is well known that any object in $\mathcal{D}_G(\mathcal{N})$ is regular and holonomic.

Moreover, the category $\mathcal{D}_G(\mathcal{N})$ has finitely many simple objects, and every object of this category has finite length (so this category is equivalent to the category of modules over a finite dimensional algebra).

9.2. **Simple objects in $\mathcal{D}_G(\mathcal{N})$.** Recall (see e.g. [Mi] and references) that irreducible objects in the category $\mathcal{D}_G(\mathcal{N})$ are parametrized by pairs (O,χ), where O is a nilpotent orbit of G in $\mathfrak{g}$, and χ is an irreducible representation of the fundamental group $\pi_1(O)$, which is clearly isomorphic to the component group $A(O)$ of the centralizer G_x of a point $x \in O$. Namely, χ defines a local system L_χ on O, and the simple object $M(O,\chi) \in \mathcal{D}_G(\mathcal{N})$ is the direct image of the Goresky-Macpherson extension of L_χ to the closure $\bar{O}$ of O, under the inclusion of $\bar{O}$ into $\mathfrak{g}$.

9.3. **Semisimplicity of $\mathcal{D}_G(\mathcal{N})$.** The proof of the following theorem was explained to us by G. Lusztig.

Theorem 9.1. *The category $\mathcal{D}_G(\mathcal{N})$ is semisimple.*

Proof. We may replace the category $\mathcal{D}_G(\mathcal{N})$ by the category of G-equivariant perverse sheaves (of complex vector spaces) on $\mathfrak{g}$ supported on $\mathcal{N}$, $\mathrm{Perv}_G(\mathcal{N})$, as these two categories are known to be equivalent. We must show that $\mathrm{Ext}^1(P,Q) = 0$ for every two simple objects $P, Q \in \mathrm{Perv}_G(\mathcal{N})$.

Let P', Q' be the Fourier transforms of P, Q. Then P', Q' are character sheaves on $\mathfrak{g}$, and it suffices to show that $\mathrm{Ext}^1(P', Q') = 0$.

Recall that to each character sheaf S one can naturally attach a conjugacy class of pairs (L, θ), where L is a Levi subgroup of G, and θ is a cuspidal local system on a nilpotent orbit for L. It is shown by arguments parallel to those in [Lu3] (which treats the more difficult case of character sheaves on the group) that if (L_i, θ_i) corresponds to S_i, $i = 1, 2$, and (L_1, θ_1) is not conjugate to (L_2, θ_2) then $\mathrm{Ext}^*(S_1, S_2) = 0$. Thus it is sufficient to assume that the pair (L, θ) attached to P' and Q' is the same.

Using standard properties of constructible sheaves (in particular, Poincaré duality), we have

$$\mathrm{Ext}^1(P', Q') = H^1(\mathfrak{g}, \underline{\mathrm{Hom}}(P', Q')) =$$

$$H_c^{2\dim\mathfrak{g}-1}(\mathfrak{g}, \underline{\mathrm{Hom}}(P', Q')^*)^* = H_c^{2\dim\mathfrak{g}-1}(\mathfrak{g}, (Q')^* \otimes P')^*,$$

where $*$ for sheaves denotes the Verdier duality functor.

Recall that to each character sheaf one can attach an irreducible representation of a certain Weyl group, via the generalized Springer correspondence. Let R be the direct sum of all character sheaves corresponding to a given pair (L, θ) with multiplicities given by the dimensions of the corresponding representations. Then it is sufficient to show that $H_c^{2\dim\mathfrak{g}-1}(\mathfrak{g}, (R')^* \otimes R') = 0$.

This fact is essentially proved in [Lu2]. Namely, it follows from the computations of [Lu2] that $H^i_c(\mathfrak{g}, (R')^* \otimes R')$ is the cohomology with compact support of a certain generalized Steinberg variety with twisted coefficients, and it is shown that this cohomology is concentrated in even degrees.[6] The theorem is proved. ☐

9.4. **Monodromicity.** We will need the following lemma.

Lemma 9.2. *Let $Q \in \mathcal{D}_G(\mathcal{N})$. Then for any finite dimensional representation U of $\mathfrak{g}$, the action of the shifted Euler operator H defined by (45) on $(Q \otimes U)^{\mathfrak{g}}$ is locally finite (so Q is a monodromic D-module), and has finite dimensional generalized eigenspaces. Moreover, the eigenvalues of H on $(Q \otimes U)^{\mathfrak{g}}$ are bounded from above. In particular, $(Q \otimes U)^{\mathfrak{g}}$ belongs to category $\mathcal{O}$ for the $\mathfrak{sl}_2$-algebra spanned by H and the elements E, F given by (46).*

Proof. Since Q has finite length, it is sufficient to assume that Q is irreducible. We may further assume that Q is generated by an irreducible G-submodule V, annihilated by multiplication by any invariant polynomial on $\mathfrak{g}$ of positive degree. Indeed, let V_0 be an irreductible G-submodule of Q, let $J_{V_0} := \{f \in \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} | fV_0 = 0\}$ and for any $v \in V_0$, let $J_v := \{f \in \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} | fv = 0\}$. Then if $v \in V_0$ is nonzero, $J_v = J_{V_0}$ as $Gv = V_0$. Moreover, the support condition implies that $J_v \subset \mathfrak{m}^k$ for some $k \geq 0$, where $\mathfrak{m} = \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}_+$. So $J_{V_0} \subset \mathfrak{m}^k$ and is an ideal of $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$. Let $f \in \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ be such that $f \notin J_{V_0}$ and $f\mathfrak{m} \subset J_{V_0}$; we set $V := fV_0$.

Then Q is a quotient of the D-module $\tilde{Q} \otimes V$ by a G-stable submodule, where

$$\tilde{Q} := D(\mathfrak{g})/(D(\mathfrak{g})\mathrm{ad}(\mathrm{Ann}(V)) + D(\mathfrak{g})I),$$

$\mathrm{Ann}(V)$ is the annihilator of V in $U(\mathfrak{g})$, and I is the ideal in $\mathbb{C}[\mathfrak{g}]$ generated by invariant polynomials on $\mathfrak{g}$ of positive degree. Thus, it suffices to show that the lemma holds for the module $\tilde{Q}$ (which is only weakly G-equivariant, i.e. the group action and the Lie algebra action coming from differential operators do not agree, in general).

The algebra $D(\mathfrak{g})$ has a grading in which $\deg(\mathfrak{g}^*) = -1$, $\deg(\mathfrak{g}) = 1$. This grading descends to a grading on $\tilde{Q}$. We will show that for each U, this grading on $(\tilde{Q} \otimes U)^{\mathfrak{g}}$ has finite dimensional pieces, and is bounded from above. This implies the lemma, since the Euler operator preserves the grading.

Consider the associated graded module $\tilde{Q}_0$ of $\tilde{Q}$ under the Bernstein filtration. This is a bigraded module over $\mathbb{C}[\mathfrak{g} \oplus \mathfrak{g}]$ (where we identify $\mathfrak{g}$ and $\mathfrak{g}^*$ using the trace form). We have to show that the homogeneous subspaces of $(\tilde{Q}_0 \otimes U)^{\mathfrak{g}}$ under the grading defined by $\deg(\mathfrak{g} \oplus 0) = -1$, $\deg(0 \oplus \mathfrak{g}) = 1$ are finite dimensional.

The associated graded of the ideal $\mathrm{Ann}(V) \subset U(\mathfrak{g})$ is such that $\mathbb{C}[\mathfrak{g}]^k_+ \subset \mathrm{grAnn}(V) \subset \mathbb{C}[\mathfrak{g}]_+$ for some $k \geq 1$, therefore

$$\tilde{Q}_0 = \mathbb{C}[\mathfrak{g} \oplus \mathfrak{g}]/J,$$

where J is a (not necessarily radical) ideal whose zero set is the variety $\mathcal{Z}$ of pairs $(u, v) \in \mathcal{N} \times \mathfrak{g}$ such that $[u, v] = 0$. Let

$$Q'_0 = \mathbb{C}[\mathfrak{g} \oplus \mathfrak{g}]/\sqrt{J}.$$

Because of the Hilbert basis theorem, it suffices to prove that the homogeneous subspaces of $(Q'_0 \otimes U)^{\mathfrak{g}}$ are finite dimensional, and the degree is bounded above. But Q'_0 is the algebra of regular functions on $\mathcal{Z}$. By the result of [J], one has $\mathbb{C}[\mathcal{Z}]^{\mathfrak{g}} = \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$, the algebra of invariant polynomials of Y. But it follows from the Hilbert's theorem on invariants that every isotypic component of $\mathbb{C}[\mathcal{Z}]$ is a finitely generated module over $\mathbb{C}[\mathcal{Z}]^{\mathfrak{g}}$. This implies the result. ☐

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[6] More precisely, in the arguments of [Lu2] the vanishing of odd cohomology is proved for G-equivariant cohomology with compact supports, and in the non-equivariant case one should use parallel arguments, rather than exactly the same arguments.

9.5. **Characters.** Lemma 9.2 allows one to define the character of an object $M \in \mathcal{D}_G(\mathcal{N})$. Namely, let $\mu = (\mu_1, ..., \mu_N)$ be a dominant integral weight for $\mathfrak{g}$, and $V(\mu)$ the irreducible representation of $\mathfrak{g}$ with highest weight μ. Let $K_M(\mu) = (M \otimes V(\mu))^{\mathfrak{g}}$. Then the character of M is defined by the formula

$$\mathrm{Ch}_M(t, g) = \mathrm{Tr}_{|M}(gt^{-H}) = \sum_{\mu} \mathrm{Tr}_{|K_M(\mu)}(t^{-H})\chi_\mu(g), \; g \in G,$$

where χ_μ denotes the character of μ. It can be viewed as a linear functional from $\mathbb{C}[G]^G$ to $\mathbb{F} := \oplus_{\beta\in\mathbb{C}} t^\beta \mathbb{C}[[t]]$, via the integration pairing.

In other words, the multiplicity spaces $K_M(\mu)$ are representations from category $\mathcal{O}$ of the Lie algebra $\mathfrak{sl}_2$ spanned by E, F, H, and the character of M carries the information about the characters of these representations.

The problem of computing characters of simple objects in $\mathcal{D}_G(\mathcal{N})$ is interesting and, to our knowledge, open. Below we will show how these characters for $G = \mathrm{SL}_N(\mathbb{C})$ can be expressed via characters of irreducible representations of the rational Cherednik algebra.

Example 9.3. *Recall (see e.g.* [Mi]*) that an object $M \in \mathcal{D}_G(\mathcal{N})$ is cuspidal iff $\mathcal{F}(M) \in \mathcal{D}_G(\mathcal{N})$, where $\mathcal{F}$ is the Fourier transform (Lusztig's criterion). If follows that in the case of cuspidal objects M, the spaces $K_M(\mu)$ are also in the category $\mathcal{O}$ for the opposite Borel subalgebra of $\mathfrak{sl}_2$, hence are finite dimensional representations of $\mathfrak{sl}_2$, and, in particular, their dimensions are of interest.*

9.6. **The functors F_n, F_n^*.** The representation V_N is a special case of representations of the rational Cherednik algebra which can be constructed via a functor similar to the one defined in [GG1]. Namely, the construction of V_N can be generalized as follows.

Let n and N be positive integers (we no longer assume that N is a divisor of n), and $k = N/n$. We again consider the special case $G = \mathrm{SL}_N(\mathbb{C})$, $\mathfrak{g} = \mathfrak{sl}_N(\mathbb{C})$. Then we have a functor $F_n : \mathcal{D}(\mathfrak{g}) \to H_n(k)$-mod defined by the formula

$$F_n(M) = (M \otimes (\mathbb{C}^N)^{\otimes n})^{\mathfrak{g}},$$

where $\mathfrak{g}$ acts on M by adjoint vector fields. The action of $H_n(k)$ on $F_n(M)$ is defined by the same formulas as in Proposition 8.1, and Proposition 8.7 remains valid.

Note that $F_n(M) = F_n(M_{\mathrm{fin}})$, where M_{fin} is the set of $\mathfrak{g}$-finite vectors in M. Clearly M_{fin} is a G-equivariant D-module. Thus, it is sufficient to consider the restriction of F_n to the subcategory $\mathcal{D}_G(\mathfrak{g})$, which we will do from now on.

In general, $F_n(M)$ does not belong to category $\mathcal{O}$. However, we have the following lemma.

Lemma 9.4. *If the Fourier transform $\mathcal{F}(M)$ of M is set-theoretically supported on the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$, then $F_n(M)$ belongs to the category $\mathcal{O}$.*

Proof. Since $\mathcal{F}(M)$ is supported on $\mathcal{N}$, invariant polynomials on $\mathfrak{g}$ act locally nilpotently on $\mathcal{F}(M)$. Hence invariant differential operators on $\mathfrak{g}$ with constant coefficients act locally nilpotently on M. Thus, it follows from formula (44) that the algebra $\mathbb{C}[\mathrm{y}_1, ..., \mathrm{y}_n]^{S_n}$ acts locally nilpotently on $F_n(M)$. Also, by Lemma 9.2, the operator $\mathbf{h}$ acts with finite dimensional generalized eigenspaces on $F_n(M)$. This implies the statement. □

Thus we obtain an exact functor $F_n^* = F_n \circ \mathcal{F} : \mathcal{D}_G(\mathcal{N}) \to \mathcal{O}(H_n(k))$.

9.7. **The symmetric part of F_n.** Consider the symmetric part $eF_n(M)$ of $F_n(M)$. We have $eF_n(M) = (M \otimes S^n\mathbb{C}^N)^{\mathfrak{g}}$, and we have an action of the spherical subalgebra $B_n(k)$ on $eF_n(M)$, given by formulas (43,44).

This allows us to relate the functor F_n with the functor defined in [GG1]. Namely, recall from [GG1] that for any $c \in \mathbb{Z}$, one may define the category $\mathcal{D}_c(\mathfrak{g} \times \mathbb{P}^{N-1})$ of coherent D-modules on $\mathfrak{g} \times \mathbb{P}^{N-1}$ which are twisted by the c-th power of the tautological line bundle on the second factor (this makes sense for all complex c even though the c-th power is defined only for integer c). Then the paper [GG1][7] defines a functor

$$\mathbb{H} : \mathcal{D}_c(\mathfrak{g} \times \mathbb{P}^{N-1}) \to B_N(c/N)\text{-mod},$$

given by $\mathfrak{H}(M) = M^{\mathfrak{g}}$.

Proposition 9.5. *(i) If n is divisible by N then one has a functorial isomorphism $eF_n(M) \simeq \phi^*\mathbb{H}(M \otimes S^n\mathbb{C}^N)$, where $S^n\mathbb{C}^N$ is regarded as a twisted D-module on $\mathbb{P}^{N-1}$ (with $c = n$).*

(ii) For any n, the actions of $B_n(N/n)$ and $B_N(n/N)$ on the space $eF_n(M) = \mathbb{H}(M \otimes S^n\mathbb{C}^N)$ have the same image in the algebra of endomorphisms of this space.

Proof. This follows from the definition of $\mathbb{H}$ and formulas (43,44). □

Corollary 9.6. *The functor eF_n^* on the category $\mathcal{D}_G(\mathcal{N})$ maps irreducible objects into irreducible ones.*

Proof. This follows from Proposition 9.5, (ii) and Proposition 7.4.3 of [GG1], which states that the functor $\mathbb{H}$ maps irreducible objects to irreducible ones. □

Formulas 43,44 can also be used to study the support of $F_n^*(M)$ for $M \in \mathcal{D}_G(\mathcal{N})$, as a $\mathbb{C}[\mathrm{x}_1, ..., \mathrm{x}_n]$-module. Namely, we have the following proposition.

Proposition 9.7. *Let $q = GCD(n, N)$ be the greatest common divisor of n and N. Then the support S of $F_n^*(M)$ is contained in the union of the S_n-translates of the subspace E_q of $\mathbb{C}^n$ defined by the equations $\sum_{i=1}^n x_i = 0$ and $x_i = x_j$ if $\frac{n}{q}(l-1)+1 \le i, j \le \frac{nl}{q}$ for some $1 \le l \le q$.*

Proof. It follows from equation (44) that for any $(x_1, ..., x_n) \in S$ there exists a point $(z_1, ..., z_N) \in \mathbb{C}^N$ such that one has

$$\frac{1}{n}\sum_{i=1}^n x_i^p = \frac{1}{N}\sum_{j=1}^N z_j^p$$

for all positive integer p. In particular, writing generating functions, we find that

$$N\sum_{i=1}^n \frac{1}{1-tx_i} = n\sum_{j=1}^N \frac{1}{1-tz_j}.$$

In particular, every fraction occurs on both sides at least $LCM(n, N)$ times, and hence the numbers x_i fall into n/q-tuples of equal numbers (and the numbers z_j into N/q-tuples of equal numbers). The proposition is proved. □

9.8. **Irreducible equivariant D-modules on the nilpotent cone for $G = \mathrm{SL}_N(\mathbb{C})$.** Nilpotent orbits for $\mathrm{SL}_N(\mathbb{C})$ are labelled by Young diagrams, or partitions. Namely, if $x \in \mathfrak{sl}_N(\mathbb{C})$ is a nilpotent element, then we let μ_i be the sizes of its Jordan blocks enumerated in the decreasing order. The partition $\mu = (\mu_1, ..., \mu_m)$ and the corresponding Young diagram whose rows have lengths μ_i are attached to x. If O is the orbit of x then we will denote μ by $\mu(O)$. For instance, if $O = \{0\}$ then $\mu = (1^N)$ and if O is the open orbit then $\mu = (N)$.

It is known (and easy to show) that the group $A(O)$ is naturally isomorphic to $\mathbb{Z}/d\mathbb{Z}$, where d is the greatest common divisor of the μ_i. Namely, let $Z = \mathbb{Z}/N\mathbb{Z}$ be the center of G (we identify $\mathbb{Z}/N\mathbb{Z}$ with Z by $p \to e^{2\pi i p/N}\mathrm{Id}$). Then we have a natural surjective homomorphism $\theta : Z \to A(O)$ induced by the inclusion $Z \to G_x$, $x \in O$. This homomorphism sends d to 0, and thus $A(O)$ gets identified with $\mathbb{Z}/d\mathbb{Z}$.

Thus, any character $\chi : A(O) \to \mathbb{C}^*$ is defined by the formula $\chi(p) = e^{-2\pi i p s/d}$, where $0 \le s < d$. We will denote this character by χ_s.

[7]There seems to be a misprint in [GG1]: in the definition of $\mathbb{H}$, c should be replaced by c/N.

9.9. **The action of F_n^* on irreducible objects.** Obviously, the center Z of G acts on $F_n^*(M)$ by $z \to z^{-sN/d}$. Thus, a necessary condition for $F_n^*(M(O,\chi_s))$ to be nonzero is

$$n = N\left(p + \frac{s}{d}\right), \tag{51}$$

where p is a nonnegative integer.

Our main result in this section is the following theorem.

Theorem 9.8. *The functor F_n^* maps irreducible objects into irreducible ones or zero. Specifically, if condition (51) holds, then we have*

$$F_n^*(M(O,\chi_s)) = L(\pi(n\mu(O)/N)),$$

the irreducible representation of $H_n(k)$ whose lowest weight is the representation of S_n corresponding to the partition $n\mu(O)/N$.

Remark 9.9. Here if μ is a partition and $c \in \mathbb{Q}$ is a rational number, then we denote by $c\mu$ the partition whose parts are $c\mu_i$, provided that these numbers are all integers. In our case, this integrality condition holds since all parts of $\mu(O)$ are divisible by d. □

Corollary 9.10. *Let λ be a partition of n into at most N parts. Let $M = M(O_\mu, \chi_s)$, and assume that condition (51) is satisfied. Then*

$$(M \otimes V(\lambda))^{\mathfrak{g}} = \mathrm{Hom}_{S_n}(\pi(\lambda), L(\pi(n\mu/N)))$$

as graded vector spaces.

This corollary allows us to express the characters of the irreducible D-modules $M(O,\chi)$ in terms of characters of certain special lowest weight irreducible representations of $H_n(k)$. We note that characters of lowest weight irreducible representations of rational Cherednik algebras of type A have been computed by Rouquier, [Ro].

Remark 9.11. Note that Theorem 8.8 is the special case of Theorem 9.8 for $O = \{0\}$.

9.10. **Proof of Theorem 9.8.** Our proof of Theorem 9.8 is based on the following result of [GS].

Theorem 9.12. *Let $k > 0$. Then the functor $V \mapsto eV$ is an equivalence of categories between $H_n(k)$-modules and $B_n(k)$-modules.*

Remark 9.13. We note that Theorem 9.12 is proved in [GS] under the technical assumption $k \notin \mathbb{Z} + 1/2$. It was noticed by V. Ginzburg that this assumption is really unnecessary. Indeed, the only place where this assumption is used is in the proof of Lemma 3.5. Namely, it is used in the proof of this lemma that Hom between Verma modules over $H_n(k)$ is isomorphic to Hom between the corresponding dual Specht modules, which is known, from [GGOR], only for $k \notin \mathbb{Z} + 1/2$. However, it is sufficient for the proof of Lemma 3.5 of [GS] to know just that the first Hom injects into the second one, which is known for all positive k thanks to a lemma by Opdam and Rouquier (Lemma 2.10 of [BEG2]).

Theorem 9.12 implies the first statement of the theorem, i.e. that if (51) holds then $F_n^*(M(O_\mu,\chi_s))$ is irreducible. Indeed, it follows from Corollary 9.6 that $eF_n^*(M(O_\mu,\chi_s))$ is irreducible over $B_n(k)$. Thus, it remains to find the lowest weight of $F_n^*(M(O_\mu,\chi_s))$.

Let $\mu = (\mu_1, ..., \mu_N)$ be a partition of N ($\mu_i \geq 0$). Let O_μ be the nilpotent orbit of $\mathfrak{g}$ corresponding to the partition μ. Denote by d the greatest common divisor of μ_i, and by m a divisor of d. Define the following function f on O_μ with values in $\otimes_{i=1}^N S^{\mu_i}\mathbb{C}^N$:

$$f(X, \xi_1, ..., \xi_N) = \bigwedge_{i=1}^{N} \bigwedge_{j=0}^{\mu_i - 1} \xi_i X^j,$$

$\xi_i \in (\mathbb{C}^N)^*$ (here $X^j \in \mathrm{M}_N(\mathbb{C})$ is the jth power of X, so $\xi_i X^j \in \mathbb{C}^N$).

Lemma 9.14. *(i) For any $X \in O_\mu$, $f(X,\dots)^{1/m}$ is a polynomial in $\xi_1,...,\xi_N$. Thus, $f^{1/m}$ is a regular function on the universal cover $\tilde{O}_\mu$ of O_μ with values in $\otimes_{i=1}^N S^{\mu_i/m}\mathbb{C}^N$.*

(ii) For any $X \in O_\mu$, the function $f(X,\dots)^{1/m}$ generates a copy of the representation $V(\mu/m)$ inside $\otimes_{i=1}^N S^{\mu_i/m}\mathbb{C}^N$.

(iii) Specifically, let the standard basis $u_1,...,u_N$ of $(\mathbb{C}^N)^*$ be filled into the squares of the Young diagram of μ (filling the first column top to bottom, then the second one, etc.), and let X be the matrix J acting by the horizontal shift to the right on this basis. Then $f(J,\dots)^{1/m}$ is a highest weight vector of the representation $V(\mu/m)$.

Proof. It is sufficient to prove (iii). Let $\mu^* = (\mu_1^*,...,\mu_N^*)$ be the conjugate partition. Let p_j be the number of times the part j occurs in this partition. Clearly, p_j is divisible by m. By looking at the matrix whose determinant is f, we see that we have, up to sign:

$$f(J,\xi_1,...,\xi_N) = \prod_j \Delta_j(\xi_1,...,\xi_N)^{p_j},$$

where Δ_j is the left upper j-by-j minor of the matrix $(\xi_1,...,\xi_N)$. Thus $f^{1/m} = \prod_j \Delta_j^{p_j/m}$ is clearly a highest weight vector of weight $\sum_j p_j\varpi_j/m$, where ϖ_j are the fundamental weights. But $\sum p_j\varpi_j = \mu$, so we are done. $\square$

Corollary 9.15. *The function f gives rise to a G-equivariant regular map $f : \tilde{O}_\mu \to V(\mu/d)$, whose image is the orbit of the highest weight vector. In particular, we have a G-equivariant inclusion of commutative algebras*

$$f^* : \oplus_{\ell\ge 0} V(\ell\mu/d)^* \to \mathbb{C}[\tilde{O}_\mu].$$

Now let $0 \le s \le d-1$, and denote by $\mathbb{C}[\tilde{O}_\mu]_s$ the subspace of $\mathbb{C}[\tilde{O}_\mu]$, on which central elements $z \in G$ act by $z \to z^{-s}$. Then we have an inclusion

$$f^* : \oplus_{\ell: d^{-1}(\ell-s)\in\mathbb{Z}} V(\ell\mu/d)^* \to \mathbb{C}[\tilde{O}_\mu]_s.$$

Now recall that by construction, $\mathbb{C}[\tilde{O}_\mu]_s$ sits inside $M = M(O_\mu,\chi_s)$ as a $\mathbb{C}[O_\mu]$-submodule. In particular, the operators X_i act on the space $(\mathbb{C}[\tilde{O}_\mu]_s \otimes (\mathbb{C}^N)^{\otimes n})^{\mathfrak{g}}$.

Let $\pi(\mu)$ be the representation of S_n corresponding to μ, and regard $V(\lambda)\otimes\pi(\lambda)$, for any partition λ of n, as a subspace of $(\mathbb{C}^N)^{\otimes n}$ using the Weyl duality. Then for any $u \in \pi(n\mu/N)$, we can define the element $a(u) \in F_n^*(M)$ by $a(u) = f_n^* \otimes u$, where $f_n^* \in \mathbb{C}[\tilde{O}_\mu]_s \otimes V(n\mu/N)$ is the homogeneous part of f^* of degree n.

Lemma 9.16. *$a(u)$ is annihilated by the elements y_i of $H_n(k)$.*

Proof. We need to show that the operators X_i (or, equivalently, the elements $\mathrm{x}_i \in H_n(k)$) annihilate $a(u) \in F_n(M)$. Since $a(u)$ is G-invariant, it is sufficient to prove the statement at the point $X = J$. This boils down to showing that for any j not exceeding the number of parts of μ (i.e. $j \le \mu_1^*$), the application of J in any component annihilates the element $\Delta_j(\xi_1,...,\xi_N) \in \wedge^j\mathbb{C}^N \subset (\mathbb{C}^N)^{\otimes j}$. This is clear, since the first μ_1^* columns of J are zero. $\square$

This implies that the lowest weight of $F_n^*(M(O_\mu,\chi_s))$ is $\pi(n\mu/N)$, as desired. The theorem is proved.

Remark 9.17. Here is another, short proof of Theorem 9.8 for $n = N$. We have

$$e_- F_N^*(M(O,1)) = \mathcal{F}(M(O,1))^G.$$

According to [L, LS],

$$\mathcal{F}(M(O,1))^G = (\mathbb{C}[\mathfrak{h}] \otimes \pi(\mu(O)))^{S_N}$$

as a module over $D(\mathfrak{h})^W = e_- H_N(1)e_-$. Thus, $e_- F_N^*(M(O,1)) = e_- L(\pi(\mu(O)))$ as $e_- H_N(1)e_-$-modules. But the functor $V \to e_- V$ is an equivalence of categories $H_N(1)$-mod $\to e_- H_N(1)e_-$-mod (see [BEG2]). Thus, $F_N^*(M(O,1)) = L(\pi(\mu(O)))$ as $H_N(1)$-modules, as desired.

9.11. **The support of $L(\pi(n\mu/N))$.**

Corollary 9.18. *Let μ be a partition of N such that $n\mu_i/N$ are integers. Then the support of the representation $L(\pi(n\mu/N))$ of $H_n(N/n)$ as a module over $\mathbb{C}[\mathrm{x}_1,...,\mathrm{x}_n]$ is contained in the union of S_n-translates of E_q, $q = GCD(n,N)$.*

Proof. This follows from Theorem 9.8 and Proposition 9.7. □

We note that in the case when $\mu = (N)$, Corollary 9.18 follows from Theorem 3.2 from [CE].

9.12. **The cuspidal case.** An interesting special case of Theorem 9.8 is the cuspidal case. In this case N and n are relatively prime, $d = N$ (i.e., O is the open orbit), and s is relatively prime to N.

Here is a short proof of Theorem 9.8 in the cuspidal case.

Since the Fourier transform of $M(O,\chi_s)$ in the cuspidal case is supported on the nilpotent cone, $F_n^*(M(O,\chi_s))$ belongs not only to the category $\mathcal{O}$ generated by lowest weight modules, but also to the "dual" category $\mathcal{O}_-$ generated by highest weight modules over $H_n(k)$. Thus, by the results of [BEG1], $F_n^*(M(O,\chi_s))$ is a multiple of the unique finite dimensional irreducible $H_n(k)$-module $L(\mathbb{C}) = Y(N,n)$, of dimension N^{n-1}. But this multiple must be a single copy by Corollary 9.6, so the theorem is proved.

Theorem 9.8 implies the following formula for the characters of the cuspidal D-modules $M(O,\chi_s)$.

Let μ be a dominant integral weight for $\mathfrak{g}$, such that the center Z of G acts on $V(\mu)$ via $z \to z^s = z^n$. Let ρ be the half-sum of positive roots of $\mathfrak{g}$. Let $K_s(\mu) = (M(O,\chi_s)\otimes V(\mu))^{\mathfrak{g}}$ be the isotypic components of $M(O,\chi_s)$.

Theorem 9.19. *We have*

$$\mathrm{Tr}_{|K_s(\mu)}(q^{2H}) = \frac{q-q^{-1}}{q^N-q^{-N}}\varphi_\mu(q),$$

where

$$\varphi_\mu(q) := \prod_{1\le p<r\le N} \frac{q^{\mu_r-\mu_p+r-p}-q^{\mu_p-\mu_r+p-r}}{q^{r-p}-q^{p-r}} = \chi_{V(\mu)}(q^{2\rho}),$$

where $\chi_{V(\mu)}$ is the character of $V(\mu)$. In particular,

$$\dim K_s(\mu) = \frac{1}{N}\prod_{1\le p<r\le N}\frac{\mu_r-\mu_p+r-p}{r-p} = \frac{1}{N}\dim V(\mu).$$

Proof. We extend the representation $V(\mu)$ to $\mathrm{GL}_N(\mathbb{C})$ by setting $z \to z^n$ for all scalar matrices z, so that its $\mathrm{GL}_N(\mathbb{C})$-highest weight is

$$\tilde{\mu} := (\mu_1 + n/N, ..., \mu_N + n/N).$$

Note that we automatically have $\mu_i + n/N \in \mathbb{Z}$. Assume that n is so big that $\tilde{\mu}$ is a partition of n (i.e., $\mu_i + n/N \ge 0$).

It follows from the results of [BEG1] that the character of the irreducible representation $L(\mathbb{C})$ of the rational Cherednik algebra $H_n(k)$, $k = N/n$, is given by the formula

$$\mathrm{Tr}_{|L(\mathbb{C})}(gq^{2\mathbf{h}}) = \frac{q-q^{-1}}{q^N-q^{-N}}\frac{\det(q^{-N}-q^Ng)}{\det(q^{-1}-qg)}, \quad g\in S_n, \tag{52}$$

where the determinants are taken in $\mathbb{C}^n$.

Let us equip $\mathbb{C}^N$ with the structure of an irreducible representation of $\mathfrak{sl}_2$ with basis e, f, h. Let $g \in S_n$. Then

$$\mathrm{Tr}_{|\mathrm{Hom}_{S_n}(\pi(\tilde{\mu}),(\mathbb{C}^N)^{\otimes n})}(q^h) = \mathrm{Tr}_{|V(\mu)}(q^{2\rho}) = \varphi_\mu(q),$$

by the Weyl character formula. On the other hand, it is easy to show that

$$\mathrm{Tr}_{|(\mathbb{C}^N)^{\otimes n}}(gq^h) = \frac{\det(q^{-N} - q^N g)}{\det(q^{-1} - qg)}.$$

Thus,

$$\begin{aligned}\mathrm{Tr}_{|\mathrm{Hom}_{S_n}(\pi(\tilde{\mu}),L(\mathbb{C}))}(q^{2\mathbf{h}}) &= \frac{q - q^{-1}}{q^N - q^{-N}} \mathrm{Tr}_{|\mathrm{Hom}_{S_n}(\pi(\tilde{\mu}),(\mathbb{C}^N)^{\otimes n})}(q^h) \\ &= \frac{q - q^{-1}}{q^N - q^{-N}} \varphi_\mu(q).\end{aligned}$$

By Theorem 9.8 and Weyl duality, this implies that

$$\mathrm{Tr}_{|(M(O,\chi_s)\otimes V(\mu))^{\mathfrak{g}}}(q^{2H}) = \frac{q - q^{-1}}{q^N - q^{-N}} \varphi_\mu(q),$$

as desired. □

Example 9.20. *Let $N = 2$, $s = 1$. In this case Theorem 9.19 gives us the following decomposition of $M(O, \chi_s)$:*

$$M(O, \chi_s) = \oplus_{j\geq 1} N_j \otimes V_{2j-1},$$

where V_j is the irreducible representation of $\mathfrak{sl}_2$ of dimension $j+1$, and the spaces N_j satisfy the equation

$$\mathrm{Tr}_{|N_j}(q^{2H}) = \frac{q^{2j} - q^{-2j}}{q^2 - q^{-2}}.$$

This shows that $N_j = V_{j-1}$ as a representation of the $\mathfrak{sl}_2$-subalgebra spanned by E, F, H, which commutes with $\mathfrak{g}$.

9.13. **The case of general orbits.** Let $W = S_N$ the Weyl group of G, $\lambda \in \mathfrak{h}/W$, and $\mathcal{N}_\lambda$ be the closure in $\mathfrak{g}$ of the adjoint orbit of a regular element of $\mathfrak{g}$ whose semisimple part is λ. Denote by $\mathcal{D}_G(\mathcal{N}_\lambda)$ the category of G-equivariant D-modules on G which are concentrated on $\mathcal{N}_\lambda$. We also let $\mathcal{O}_\lambda$ be the category of finitely generated $H_n(k)$-modules in which the subalgebra $\mathbb{C}[\mathrm{y}_1, ..., \mathrm{y}_n]^{S_n}$ acts through the character λ. Then one can show, similarly to the above, that the functor F_n^* restricts to a functor $F_{n,\lambda}^* : \mathcal{D}_G(\mathcal{N}_\lambda) \to \mathcal{O}_\lambda$. The functor considered above is $F_{n,0}^*$. We plan to study the functor $F_{n,\lambda}^*$ for general λ in a future work.

9.14. **The trigonometric case.** Our results about rational Cherednik algebras can be extended to the trigonometric case. For this purpose, D-modules on the Lie algebra $\mathfrak{g}$ should be replaced with D-modules on the group G. Let us describe this generalization.

First, let us introduce some notation. As above, we let $G = \mathrm{SL}_N(\mathbb{C})$. For $b \in \mathfrak{g}$, let L_b be the right invariant vector field on G equal to b at the identity element; that is, L_b generates the group of left translations by e^{tb}. As before, we let $k = N/n$.

Now let M be a D-module on G. Similarly to the above, we define $F_n(M)$ to be the space

$$F_n(M) = (M \otimes (\mathbb{C}^N)^{\otimes n})^G,$$

where G acts on itself by conjugation.

Consider the operators X_i, Y_i, $i = 1, ..., n$, on $F_n(M)$, defined by the formulas similar to the rational case:

$$X_i = \sum_{j,l} A_{jl} \otimes (E_{lj})_i, \;\; Y_i = \frac{N}{n} \sum_p L_{b_p} \otimes (b_p)_i,$$

where A_{jl} is the jl-th matrix element of $A \in G$ regarded as the multiplication operator in M by a regular function on G.

Proposition 9.21. *The operators X_i, Y_i satisfy the following relations:*

$$\prod_i X_i = 1,\ \sum_i Y_i + k\sum_{i<j} s_{ij} = 0,$$

$$s_{ij}X_i = X_j s_{ij},\ s_{ij}Y_i = Y_j s_{ij},\ [s_{ij}, X_l] = [s_{ij}, Y_l] = 0,$$
$$[X_i, X_j] = 0,\ [Y_i, Y_j] = ks_{ij}(Y_i - Y_j),$$
$$[Y_i, X_j] = \left(ks_{ij} - \frac{1}{n}\right) X_j,$$

where i, j, l denote distinct indices.

Proof. Straightforward computation. □

Corollary 9.22. *The operators $\bar{Y}_i = Y_i + k\sum_{j<i} s_{ij}$ pairwise commute.*

The relations of Proposition 9.21 are nothing but the defining relations of the *degenerate double affine Hecke algebra* of type A_{n-1}, which we will denote $H_n^{\mathrm{tr}}(k)$ (where "tr" stands for trigonometric, to illustrate the fact that this algebra is a trigonometric deformation of the rational Cherednik algebra $H_n(k)$). Thus we have defined an exact functor $F_n : \mathcal{D}(G) \to H_n^{\mathrm{tr}}(k)$-mod. As before, it is sufficient to consider the restriction of this functor to the category of equivariant finitely generated D-modules, $\mathcal{D}_G(G)$.

This allows us to generalize much of our story for rational Cherednik algebras to the trigonometric case. In particular, let $\mathcal{U}$ be the unipotent variety on G, and $\mathcal{D}_G(\mathcal{U})$ be the category of finitely generated G-equivariant D-modules on G concentrated on $\mathcal{U}$. If we restrict the functor F_n to this category, we get a situation identical to that in the rational case. Indeed, one can show that for any M in this category, $F_n(M)$ belongs to the category $\mathcal{O}_-^{\mathrm{tr}}$ of finitely generated modules over $H_n^{\mathrm{tr}}(k)$ which are locally unipotent with respect to the action of X_i. The latter category is equivalent to the category $\mathcal{O}_-$ over the rational Cherednik algebra $H_n(k)$, because the completion of $H_n^{\mathrm{tr}}(k)$ with respect to the ideal generated by $X_i - 1$ is isomorphic to the completion of $H_n(k)$ with respect to the ideal generated by x_i. On the other hand, the exponential map identifies the categories $\mathcal{D}_G(\mathcal{U})$ and $\mathcal{D}_G(\mathcal{N})$. It is clear that after we make these two identifications, the functor F_n becomes the functor F_n in the rational case that we considered above.

On the other hand, because of the absence of Fourier transform on the group (as opposed to Lie algebra), the trigonometric story is richer than the rational one. Namely, we can consider another subcategory of $\mathcal{D}_G(G)$, the category of character sheaves. By definition, a character sheaf on G is an object M in $\mathcal{D}_G(G)$ which is locally finite with respect to the action of the algebra of biinvariant differential operators, $U(\mathfrak{g})^G$. This category is denoted by $\mathrm{Char}(G)$. It is known that one has a decomposition

$$\mathrm{Char}(G) = \oplus_{\lambda \in T^\vee/W} \mathrm{Char}_\lambda(G),$$

where $T^\vee$ is dual torus, and $\mathrm{Char}_\lambda(G)$ the category of those $M \in \mathcal{D}_G(G)$ for which the generalized eigenvalues of $U(\mathfrak{g})^G$ (which we identify with $U(\mathfrak{h})^W$ via the Harish-Chandra homomorphism) project to λ under the natural projection $\mathfrak{h}^* \to T^\vee$.

On the other hand, one can define the category $\mathrm{Rep}_{Y-\mathrm{fin}}(H_n^{\mathrm{tr}}(k))$ of modules over $H_n^{\mathrm{tr}}(k)$ on which the commuting elements $\bar{Y}_i$ act in a locally finite manner. We have a similar decomposition

$$\mathrm{Rep}_{Y-\mathrm{fin}}(H_n^{\mathrm{tr}}(k)) = \oplus_{\lambda \in T^\vee/W} \mathrm{Rep}_{Y-\mathrm{fin}}(H_n^{\mathrm{tr}}(k))_\lambda,$$

where $\mathrm{Rep}_{Y-\mathrm{fin}}(H_n^{\mathrm{tr}}(k))_\lambda$ is the subcategory of all objects where the generalized eigenvalues of $\bar{Y}_i$ project to $\lambda \in T^\vee/W$. Then one can show, similarly to the rational case, that the functor F_n gives rise to the functors

$$F_{n,\lambda} : \mathrm{Char}_\lambda(G) \to \mathrm{Rep}_{Y-fin}(H_n^{\mathrm{tr}}(k))_\lambda$$

for each $\lambda \in T^\vee/W$. The most interesting case is $\lambda = 0$ (unipotent character sheaves). We plan to study these functors in subsequent works.

9.15. **Relation with the Arakawa-Suzuki functor.** Note that the elements Y_i and s_{ij} generate the degenerate affine Hecke algebra $\mathcal{H}_n$ of Drinfeld and Lusztig (of type A_{n-1}). To define the action of this algebra on $F_n(M) = (M \otimes (\mathbb{C}^N)^{\otimes n})^{\mathfrak{g}}$ by the formula of Proposition 9.21, we only need the action of the operators L_b, $b \in \mathfrak{g}$ in M. So M can be taken to be an arbitrary $\mathfrak{g}$-bimodule which is locally finite with respect to the diagonal action of $\mathfrak{g}$ (in this case, $\sum_i Y_i + \sum_{i<j} s_{ij}$ is a central element which does not necessarily act by zero, so we get a representation of a central extension $\widetilde{\mathcal{H}}_n$ of $\mathcal{H}_n$). In particular, we have an exact functor $F_n : \mathrm{HC}(\mathfrak{g}) \to \widetilde{\mathcal{H}}_n$-mod from the category of Harish-Chandra bimodules over $\mathfrak{g}$ to the category of finite dimensional representations of the degenerate affine Hecke algebra $\widetilde{\mathcal{H}}_n$. This functor was essentially considered in [AS] (where it was applied to the Harish-Chandra modules of the form $M = \mathrm{Hom}_{\mathfrak{g}-\mathrm{finite}}(M_1, M_2)$, where M_1 and M_2 are modules from category $\mathcal{O}$ over $\mathfrak{g}$). We note that the paper [AST] describes the extension of this construction to affine Lie algebras, which yields representations of degenerate double affine Hecke algebras.

9.16. **Directions of further study.** In conclusion we would like to discuss (in a fairly speculative manner) several directions of further study and generalizations (we note that these generalizations can be combined with each other).

1. The q-case: the group G is replaced with the corresponding quantum group, D-modules with q-D-modules, and degenerate double affine Hecke algebras with the usual double affine Hecke algebras (defined by Cherednik). It is especially interesting to consider this generalization if q is a root of unity.

2. The quiver case. This generalization was suggested by Ginzburg, and will be studied in his subsequent work with the third author. In this case, one has a finite subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$, and one should consider equivariant D-modules on the representation space of the affine quiver attached to Γ (with some orientation). Then there should exist an analog of the functor F_n, which takes values in the category of representations of an appropriate symplectic reflection algebra for the wreath product $S_n \ltimes \Gamma^n$, [EG] (or, equivalently, the Gan-Ginzburg algebra, [GG2]). This generalization should be especially nice in the case when Γ is a cyclic group, when the symplectic reflection algebra is a Cherednik algebra for a complex reflection group, and one has the notion of category $\mathcal{O}$ for it.

3. The symmetric space case. This is the trigonometric version of the previous generalization for $\Gamma = \mathbb{Z}/2$. In this generalization one considers (monodromic) equivariant D-modules on the symmetric space $\mathrm{GL}_{p+q}(\mathbb{C})/(\mathrm{GL}_p \times \mathrm{GL}_q)(\mathbb{C})$ (see [Gin]), and one expects a functor from this category to the category of representations of an appropriate degenerate double affine Hecke algebra of type $C^\vee C_n$. This functor should be related, similarly to the previous subsection, to an analog of the Arakawa-Suzuki functor, which would attach to a Harish-Chandra module for the pair $(\mathrm{GL}_{p+q}(\mathbb{C}), \mathrm{GL}_p(\mathbb{C}) \times \mathrm{GL}_q(\mathbb{C}))$, a finite dimensional representation of the degenerate double affine Hecke algebra of type BC_n.

Appendix A

Let $\mathcal{O}$ be the ring $\mathbb{C}[[u_1, ..., u_n]][\ell_1, ..., \ell_n]$. Define commuting derivations D_i of $\mathcal{O}$ by $D_i(u_j) = \delta_{ij} u_i$, $D_i(\ell_j) = \delta_{ij}$ (we will later think of ℓ_i and D_i as $\log u_i$ and $u_i \frac{\partial}{\partial u_i}$).

We set $\mathcal{O}_+ := \mathfrak{m}[\ell_1, ..., \ell_n]$, where $\mathfrak{m} = \mathrm{Ker}(\mathbb{C}[[u_1, ..., u_n]] \to \mathbb{C})$ is the augmentation ideal. Let $A = \oplus_{k \geq 0} A_k$ be a graded ring with finite dimensional homogeneous components.

Proposition A.1. *Let $X_i(u_1, ..., \ell_n) \in \hat{\oplus}_{k>0}(A_k \otimes \mathcal{O}_+)$ be such that $D_i(X_j) = D_j(X_i)$. Then there exists a unique $F(u_1, ..., \ell_n) \in \hat{\oplus}_{k>0}(A_k \otimes \mathcal{O}_+)$ such that $D_i(F) = X_i$ for $i = 1, ..., n$.*

Let us say that $f \in \mathcal{O}$ has radius of convergence $R > 0$ if $f = \sum_{k_1,...,k_n \geq 0} f_{k_1,...,k_n}(u_1, ..., u_n) \ell_1^{k_1} ... \ell_n^{k_n}$, where each $f_{k_1,...,k_n}(u_1, ..., u_n)$ converges for $|u_1|, ..., |u_n| \leq R$. Then if $X_1, ..., X_n$ have radius of convergence R, so does F.

Proof. For each i, D_i restricts to an endomorphism of $\mathcal{O}_+$; one checks that $\cap_{i=1}^n \mathrm{Ker}(D_i : \mathcal{O}_+ \to \mathcal{O}_+) = 0$ which implies the uniqueness. To prove the existence, we work by induction.

One proves that $D_n : \mathcal{O}_+ \to \mathcal{O}_+$ is surjective, and its kernel is $\mathfrak{m}_{n-1}[\ell_1, ..., \ell_{n-1}]$, where $\mathfrak{m}_{n-1} = \mathrm{Ker}(\mathbb{C}[[u_1, ..., u_{n-1}]] \to \mathbb{C})$. Let G be a solution of $D_n(G) = X_n$, then the system $D_i(F') = X_i - D_i(G)$ $(i = 1, ..., n)$ is compatible, which implies $D_n(X_i') = 0$, where $X_i' := X_i - D_i(G)$, so $X_i' \in \oplus_{k>0}(A_k \otimes \mathcal{O}_+^{(n-1)})$, where $\mathcal{O}_+^{(n-1)}$ is the analogue of $\mathcal{O}_+$ at order $n-1$. Hence the system $D_i(F') = X_i - D_i(G)$ $(i = 1, ..., n-1)$ is compatible and we may apply to it the result at order $n-1$ to obtain a solution F'. Then a solution of $D_i(F) = X_i$ is $F' + G$.

Let $D : u\mathbb{C}[[u]] \to u\mathbb{C}[[u]]$ be the map $u\frac{\partial}{\partial u}$ and let $I := D^{-1}$. The map $D_1 : u\mathbb{C}[[u]][\ell] \to u\mathbb{C}[[u]][\ell]$ is bijective and its inverse is given by $D_1^{-1}(F(u)\ell^a) = \sum_{k=0}^{a}(-1)^k a(a-1)...(a-k+1)(I^{k+1}(F))(u)\ell^{a-k}$.

We have $\mathcal{O}_+ = \mathcal{O}^{(n-1)}\hat{\otimes}u_n\mathbb{C}[[u_n]][\ell_n] \oplus \mathfrak{m}^{(n-1)}\hat{\otimes}\mathbb{C}[\ell_n]$ (where $\mathcal{O}^{(n-1)}, \mathfrak{m}^{(n-1)}$ are the analogues of $\mathcal{O}, \mathfrak{m}$ at order $n-1$, $\hat{\otimes}$ is the completed tensor product). The endomorphism D_n preserves this decomposition and a section of D_n is given by $(\mathrm{id} \otimes D_1^{-1}) \oplus (\mathrm{id} \otimes J)$, where $J \in \mathrm{End}(\mathbb{C}[\ell])$ is a section of $\partial/\partial\ell$.

It follows from the fact that I preserves the radius of convergence of a series that the same holds for the section of D_n defined above. One then follows the above construction of a solution X of $D_i(X) = X_i$ and uses the fact that D_i also preserves the radius of convergence to show by induction that X has radius R if the X_i do. □

Proposition A.2. *Let $X_i(u_1, ..., \ell_n) \in \hat{\oplus}_{k>0}(A_k \otimes \mathcal{O}_+)$ be such that $D_i(X_j) - D_j(X_i) = [X_i, X_j]$. Then there exists a unique $F(u_1, ..., \ell_n) \in 1 + \hat{\oplus}_{k>0}(A_k \otimes \mathcal{O}_+)$ such that $D_i(F) = X_iF$ for $i = 1, ..., n$. If the X_i have radius R, then so does F.*

Proof. Let us prove the uniqueness. If F, F' are two solutions, then $F^{-1}F'$ is a constant (as $\cap_{i=0}^{n} \mathrm{Ker}(D_i : \mathcal{O} \to \mathcal{O}) = 0$), and it also belongs to $1 + \hat{\oplus}_{k>0}(A_k \otimes \mathcal{O}_+)$, which implies that $F = F'$. To prove the existence, one sets $F = 1 + f_1 + f_2 + ...$, $X_i = x_1^{(i)} + ...$, where $f_k, x_k^{(i)} \in A_k \otimes \mathcal{O}_+$ and solves by induction the system $D_i(f_k) = x_1^{(i)} f_{k-1} + ... + x_k^{(i)}$ using Proposition A.1. □

Proposition A.3. *Let $C_i(u_1, ..., u_n) \in \hat{\oplus}_{k>0}A_k[[u_1, ..., u_n]]$ $(i = 1, ..., n)$ be such that $u_i\partial_{u_i}(C_j) - u_j\partial_{u_j}(C_i) = [C_i, C_j]$ for any i, j. Assume that the series C_i have radius R.*

Then there exists a unique solution of the system $u_i\partial_{u_i}(X) = C_iX$, analytic in the domain $\{u||u| \leq R, u \notin \mathbb{R}_-\}^n$, such that the ratio $(u_1^{C_0^1}...u_n^{C_0^n})^{-1}X(u_1, ..., u_n)$ (we set $C_0^i := C_i(0, ..., 0)$) has the form $1 + \sum_{k>0}\sum_{a_1,...,a_n,i} r_k^{a_1,...,a_n,i}(u_1, ..., u_n)$ (the second sum is finite for any k), $r_k^{a_1,...,a_n,i}$ has degree k, $a_i \geq 0$, $i \in \{1, ..., n\}$, and $r_k^{a_1,...,a_n,i}(u_1, ..., u_n) = O(u_i(\log u_1)^{a_1}...(\log u_n)^{a_n})$. The same is then true of the ratio $X(u_1, ..., u_n)(u_1^{C_0^1}...u_n^{C_0^n})^{-1}$; we write $X(u_1, ..., u_n) \simeq u_1^{C_0^1}...u_n^{C_0^n}$.

Proof. Let us show the existence of X. The compatibility condition implies that $[C_0^i, C_0^j] = 0$. If we set $Y(u_1, ..., u_n) := (u_1^{C_0^1}...u_n^{C_0^n})^{-1}X(u_1, ..., u_n)$, then X is a solution iff Y is a solution of $u_i\partial_{u_i}(Y) = \exp(-\sum_{j=1}^{n}(\log u_j)C_j^0)(C_i - C_i^0) \cdot Y$.

Let us set $X_i(u_1, ..., \ell_n) := \exp(-\sum_{j=1}^{n} \ell_j C_j^0)(C_i(u_1, ..., u_n) - C_i(0, ..., 0))$, then $X_i(u_1, ..., \ell_n) \in \hat{\oplus}_{k>0}(A_k \otimes \mathcal{O}_+)$. We then apply Proposition A.2 and find a solution $Y \in 1 + \hat{\oplus}_{k>0}A_k \otimes \mathcal{O}_+$ of $D_i(Y) = X_iY$. Let Y_k be the component of Y of degree k. Since Y has radius R, the replacement $\ell_i = \log u_i$ in Y_k for $u_i \in \{u||u| \leq R, u \notin \mathbb{R}_-\}$ gives an analytic function on $\{u||u| \leq R, u \notin \mathbb{R}_-\}^n$. Moreover, $\mathcal{O}_+ = \sum_{i=1}^{n} u_i\mathbb{C}[[u_1, ..., u_n]][\ell_1, ..., \ell_n]$, which gives a decomposition $Y_k = \sum_{i,a_1,...,a_n} u_i\ell_1^{a_1}...\ell_n^{a_n} y_{i,a_1,...,a_n}^k(u_1, ..., u_n)$ and leads (after substitution $\ell_i = \log u_i$) to the above estimates.

The ratio $X(u_1, ..., u_n)(u_1^{C_0^1}...u_n^{C_0^n})^{-1}$ is then $1 + \exp(\sum_j C_0^j \log u_j)(Y(u_1, ..., u_n) - 1)$; the term of degree k has finitely many contributions to which we apply the above estimates.

Let us prove the uniqueness of X. Any other solution has the form $X = X(1 + c_k + ...)$ where $c_j \in A_j$, and $c_k \neq 0$. Then the degree k term is transformed by the addition of c_k, which cannot be split as a sum of terms in the various $O(u_i(\log u_1)^{a_1}...(\log u_n)^{a_n})$. □

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