FINITE SUBGROUPS OF THE PLANE CREMONA GROUP

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ABSTRACT. This paper completes the classic and modern results on classification of conjugacy classes of finite subgroups of the group of birational automorphisms of the complex projective plane.

CONTENTS

1. INTRODUCTION

The Cremona group $\mathrm{Cr}_k(n)$ over a field k is the group of birational automorphisms of the projective space $\mathbb{P}_k^n$, or, equivalently, the group of k-automorphisms of the field $k(x_1, x_2, \ldots, x_n)$ of rational functions in n independent variables. The group $\mathrm{Cr}_k(1)$ is the group of automorphisms of the projective line, and hence it is isomorphic to the projective linear group $\mathrm{PGL}_k(2)$. Already in the case $n = 2$ the group $\mathrm{Cr}_k(2)$ is not well understood in spite of extensive classical literature (e.g. [18], [34]) on the subject as well as some modern research and expositions of classical results (e.g. [1]). Very little is known about the Crenona groups in higher-dimensional spaces.

In this paper we restrict ourselves with the case of the plane Cremona group over the field of complex numbers, denoted by $\mathrm{Cr}(2)$. We return to the classical problem of classification of finite subgroups of $\mathrm{Cr}(2)$. The classification of finite subgroups of $\mathrm{PGL}_{\mathbb{C}}(2)$ is well-known and goes back to F. Klein. It consists of cyclic dihedral, tetrahedral, octahedral and icosahedral groups. Groups of the same type and order constitute a unique conjugacy class in $\mathrm{PGL}_{\mathbb{C}}(2)$. Our goal is to find a similar classification in the two-dimensional case.

The history of this problem begins with the work of E. Bertini [8] who classified conjugacy classes of subgroups of order 2 in $\mathrm{Cr}(2)$. Already in this case the answer is

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drastically different. The set of conjugacy classes is parametrized by a disconnected algebraic variety whose connected components are isomorphic to the moduli spaces of hyperelliptic curves of genus g (de Jonquiéres involutions) as well as the moduli space of canonical curves of genus 3 (Geiser involutions) and canonical curves of genus 4 with vanishing theta characteristic (Bertini involutions). Bertini's proof was considered to be incomplete even according to the standards of rigor of the 19th century algebraic geometry. A complete and short proof was published only a few years ago by L. Bayle and A. Beauville [4].

In 1894 G. Castelnuovo [14], as an application of his theory of adjoint linear systems, proved that any element of finite order in Cr(2) leaves invariant either a net of lines, or a pencil of lines, or a linear system of cubic curves with $n \leq 8$ base points. A similar result was claimed earlier by S. Kantor in his memoir which was awarded a prize by the Accademia delle Scienze di Napoli in 1883. However his arguments, as was pointed out by Castelnuovo, required justifications. Kantor went much further and announced a similar theorem for arbitrary finite subgroups of Cr(2). He proceeded to classify possible groups in each case (projective groups, groups of de Jonquiérs type, and groups of type M_n). A much clearer exposition of his results can be found in a paper of A. Wiman [49]. Unfortunately, Kantor's classification, even with some correction made by Wiman, is incomplete for two main reasons. First, only maximal groups were considered and even some of them were missed. The most notorious example is a cyclic group of order 8 of automorphisms of a cubic surface, also missed by B. Segre [47] (see [32]). Second, although Kantor was aware of the problem of conjugacy of subgroups, he did not attempt to fully investigate this problem.

The goal of our work is to complete Kantor's classification. We use a modern approach to the problem initiated in the works of Yu. Manin and the second author (see a survey of their results in [38]). It makes a clear understanding of the conjugacy problem via the concept of a rational G-surface. This is meant to be a pair (S, G) which consists of a nonsingular projective rational surface S over a field k and a finite group G acting on it by biregular automorphisms (the *geometric case*) or, acting on $S \otimes_k K \overset{\text{bir}}{\cong} \mathbb{P}^2_K$ by the Galois action (the *arithmetic case*). A birational G-equivariant k-map $S - \to \mathbb{P}^2_k$ realizes G as a finite subgroup of $\mathrm{Cr}_k(2)$ (geometric case) or a finite subgroup of $\mathrm{Cr}_K(2)$ (arithmetic case). In the geometric case, two birational isomorphic G-surfaces define conjugate subgroups of $\mathrm{Cr}_k(2)$, and conversely a conjugacy class of a finite subgroup G of $\mathrm{Cr}_k(2)$ can be realized as a birational isomorphism class of G-surfaces. In this way classification of conjugacy classes of subgroups of $\mathrm{Cr}_k(2)$ becomes equivalent to the birational classification of G-surfaces. A G-equivariant analog of a minimal surface allows one to concentrate on the study of minimal G-surfaces, i.e. surfaces which cannot be G-equivariantly birationally and regularly mapped to another G-surface. Minimal G-surfaces turn out to be G-isomorphic either to $\mathbb{P}^2_k$, or a conic bundle, or Del Pezzo surface of degree $d = 9 - n \leq 6$. This leads to groups of projective transformations, or groups of de Jonquiéres type, or groups of type M_n, respectively. To complete the classification one requires

- to classify all finite groups G which may occur in a minimal G-pair (S, G);
- to determine when two minimal G-surfaces are birationally isomorphic.

To solve the first part of the problem one has to compute the full automorphism group of a conic bundle surface or a Del Pezzo surface (in the latter case this was essentially accomplished by Kantor and Wiman), then make a list of all finite subgroups which act minimally on the surface (this did not come up in the works of Kantor and Wiman). The second part is less straightforward. For this we use the ideas from Mori's theory to decompose a birational map of rational G-surfaces into elementary links. This theory was successfully applied in the arithmetic case (see [38]) and we borrow these results with obvious modifications adjusted to the geometric case. Here we use the analogy between k-rational points in the arithmetic case (fixed points of the Galois action) and fixed points of the G-action. As an important implication of the classification of elementary G-links is the rigidity property of groups of type M_n with $n \geq 6$: any minimal Del Pezzo surface (S, G) of degree $d \leq 3$ is not isomorphic to a minimal G-surface of different type. This allows us to avoid much of the painful analysis of possible conjugacy for a lot of groups.

The large amount of group-theoretical computations needed for the classification of finite subgroups of groups of automorphisms of conic bundles and Del Pezzo surfaces makes us to be aware of some possible gaps in our final classification. This seems to be a destiny of enormous classification problems. We hope that our hard work will be useful for the future faultless classification of conjugacy classes of Cr(2).

It is appropriate to mention some recent work on classification of conjugacy classes of subgroups of $Cr(2)$. We have already mentioned the work of L. Bayle and A. Beauville on groups of order 2. The papers [6],[21], [50] study groups of prime orders, Beauville's paper [7] classifies p-elementary groups, and a thesis of J. Blanc [5] contains a classification of all finite abelian groups. The second author studies two non-conjugate classes of subgroups isomorphic to $S_3 \times \mathbb{Z}/2\mathbb{Z}$. In the work of S. Bannai and H. Tokunaga examples are given of non-conjugate subgroups isomorphic to S_4 and A_5.

This paper is partly based on the lectures by the first author in workshops on Cremona transformations in Torino in September 2005 and Lisbon in May 2006. He takes the opportunity to thank the organizers for the invitation and for providing a good audience. We both like to thank A. Beauville and, especially, J. Blanc for the correspondence which helped to correct some of our preliminary results.

2. First examples

2.1. **Homaloidal linear systems.** We will be working over the field of complex numbers. Recall that a dominant rational map $\chi : \mathbb{P}^2 - \rightarrow \mathbb{P}^2$ is given by a 2-dimensional linear system $\mathcal{H}$ equal to the proper inverse transform of the linear system of lines $\mathcal{H}' = |\ell|$ in the target plane. A choice of a basis in $\mathcal{H}$ gives an explicit formula for the map in terms of homogeneous coordinates

$$(x_0', x_1', x_2') = (P_0(x_0, x_1, x_2), P_1(x_0, x_1, x_2), P_2(x_0, x_1, x_2)),$$

where P_0, P_1, P_2 are linear independent homogeneous polynomials of the same degree d, called the (algebraic) *degree* of the map. This is the smallest number d such that $\mathcal{H}$ is contained in the complete linear system $|d\ell|$ of curves of degree d in the plane. By definition of the proper inverse transform the linear system $\mathcal{H}$ does not have fixed component, or, equivalently, the polynomials P_i's do not have a common factor of positive degree. The birational map χ is not a projective transformation if and only if the degree is larger than 1, or, equivalently, when χ has *base points*, the common zeros of the members of the linear system. A linear system defining a birational map is called a *homaloidal* linear system. Being proper transform of a general line under a birational map, its general member is an irreducible rational curve. Also two general curves from the linear system intersect outside the base points at one point. These two conditions characterize homaloidal linear systems (more about this later).

2.2. **Quadratic transformations.** A quadratic Cremona transformation is a birational map $\chi : \mathbb{P}^2 - \to \mathbb{P}^2$ of degree 2. The simplest example is the *standard quadratic transformation* defined by the formula

$$\tau_1 : (x_0, x_1, x_2) = (x_1x_2, x_0x_2, x_0x_1). \tag{2.1}$$

In affine coordinates this is given by $\tau_1 : (x, y) \mapsto (\frac{1}{x}, \frac{1}{y})$. It follows from the definition that $\tau_1^{-1} = \tau_1$, i.e., τ_1 is a birational involution of $\mathbb{P}^2$. The base points of T are the points $p_1 = (1, 0, 0), p_2 = (0, 1, 0), p_3 = (0, 0, 1)$. The transformation maps an open subset of a coordinate line $x_i = 0$ to the point p_i. The homaloidal linear system defining τ_1 is the linear system of conics through the points p_1, p_2, p_3.

The Moebius transformation $x \mapsto x^{-1}$ of $\mathbb{P}^1$ is conjugate to the transformation $x \mapsto -x$ (by means of the map $x \mapsto \frac{x-1}{x+1}$). This shows that the standard Cremona transformation τ_1 is conjugate in Cr(2) to a projective transformation given by

$$(x_0, x_1, x_2) \mapsto (x_0, -x_1, -x_2).$$

When we change the homaloidal linear system defining τ_1 to the homaloidal linear system of conics passing through the points p_1 and p_2 and tangent to the line $x_0 = 0$ we obtain the transformation

$$\tau_2 : (x_0, x_1, x_2) \mapsto (x_1^2, x_0x_1, x_0x_2). \tag{2.2}$$

In affine coordinates it is given by $(x, y) \mapsto (\frac{1}{x}, \frac{y}{x^2})$. The transformation τ_2 is also an involution, and it is conjugate to a projective involution. To see this we define a rational map $\chi : \mathbb{P}^2 - \to \mathbb{P}^3$ by the formula $(x_0, x_1, x_2) \mapsto (x_1^2, x_0x_1, x_0x_2, x_1x_2)$. The Cremona transformation τ_2 acts on $\mathbb{P}^3$ via this transformation by $(u_0, u_1, u_2, u_3) \mapsto (u_1, u_0, u_3, u_2)$. Composing with the projection of the image from the fixed point $(1, 1, 1, 1)$ we get a birational map $(x_0, x_1, x_2) \mapsto (y_0, y_1, y_2) = (x_1(x_0 - x_1), x_0x_2 - x_1^2, x_1(x_2 - x_1))$. It defines the conjugation of τ_2 with the projective transformation $(y_0, y_1, y_2) \mapsto (-y_0, y_2 - y_0, y_1 - y_0)$.

Finally, we could further "degenerate" τ_1 by considering the linear system of conics passing through the point p_3 and intersecting each other at this point with multiplicity 3. This linear system defines a birational involution

$$\tau_3 : (x_0, x_1, x_2) \mapsto (x_0^2, x_0x_1, x_1^2 - x_0x_2). \tag{2.3}$$

Again it can be shown that τ_3 is conjugate to a projective involution.

Recall that a birational transformation is not determined by the choice of a homaloidal linear system, one has to choose additionally a basis of the linear system. In the above examples, the basis is chosen to make the transformation an involution.

2.3. **De Jonquiéres involutions.** Here we exhibit a series of birational involutions which are not conjugate to each other and not conjugate to a projective involution. In affine coordinates they are given by the formula

$$\mathrm{dj}_P : (x, y) \mapsto (x, \frac{P(x)}{y}), \tag{2.4}$$

where $P(x)$ a polynomial of degree $2g+1$ or $2g+2$ without multiple roots. The conjugation by the transformationn $(x,y)\mapsto (\frac{ax+b}{cx+d},y)$ shows that the conjugacy class of dj_P depends only on the orbit of the set of roots of P with respect to the group PGL(2), or, in other words on the birational class of the hyperelliptic curve

$$y^2 - P(x) = 0. \tag{2.5}$$

The transformation dj_P has the following beautiful geometric interpretation. Consider the projective model H_{g+2} of the hyperelliptic curve (2.5) given by the homogeneous equation of degree $g+2$

$$T_2^2F_g(T_0,T_1) + 2T_2F_{g+1}(T_0,T_1) + F_{g+2}(T_0,T_1) = 0, \tag{2.6}$$

where

$$D = F_{g+1}^2 - F_gF_{g+2} = T_0^{2g+2}P(T_1/T_0)$$

is the homogenization of the polynomial $P(x)$. The curve H_{g+2} has an ordinary singular point of multiplicity g at $q=(0,0,1)$ and the projection from this point to $\mathbb{P}^1$ exhibits the curve as a double cover of $\mathbb{P}^1$ branched over the $2g+2$ zeroes of D.

Consider the affine set $T_2 = 1$ with affine coordinates (x,y). A general line $y = kx$ intersects the curve H_{g+2} at the point $q = (0,0)$ with multiplicity g and two other points $(\alpha, k\alpha)$ and $(\alpha', k\alpha')$, where α,α' are the roots of the quadratic equation

$$t^2F_{g+2}(1,k) + 2tF_{g+1}(1,k) + F_g(1,k) = 0.$$

Take a general point $p = (x,kx)$ on the line and define the point $p' = (x',kx')$ such that the pairs $(\alpha,k\alpha),(\alpha',k\alpha')$ and $(x,kx),(x',kx')$ are harmonic conjugate. This means that x,x' are the roots of the equation $at^2+2bt+c=0$, where $aF_g(1,k) + cF_{g+2}(1,k) - 2bF_{g+1}(1,k) = 0$. Since $x+x' = -2b/c, xx' = c/a$ we get $F_g(1,k) + xx'F_{g+2}(1,k) + (x+x)F_{g+1}(1,k) = 0$. We express x' as $(ax+b)/(cx+d)$ and solve for (a,b,c,d) to obtain

$$x' = \frac{-F_{g+1}(1,k)x - F_g(1,k)}{xF_{g+2}(1,k) + F_{g+1}(1,k)}.$$

Recall now that $k = y/x$ and changing the affine coordinates $(x,y) = (x_0/x_2, x_1/x_2)$ to $(X,Y) = (x_1/x_0, x_2/x_0) = (y/x, 1/x)$, we get

$$IH_{g+2} : (X',Y') = \big(X, \frac{-YP_{g+1}(X) - P_{g+2}(X)}{P_g(X)Y + P_{g+1}(X)}\big), \tag{2.7}$$

where $P_i(X) = F_i(1,X)$. Let $T : (x,y) \mapsto (x, yP_g + P_{g+1})$. Then, taking $P(x) = P_{g+1}^2 - P_gP_{g+2}$ we check that $T^{-1}\circ \mathrm{dj}_P \circ T = IH_{g+2}$. This shows that our geometric de Jonquiéres involution IH_{g+2} given by (2.7) is conjugate to the de Jonquiéres involution dj_P defined by (2.4).

Let us rewrite the formula (2.7) in homogeneous coordinates:

$$\begin{aligned} x_0' &= x_0(x_2F_g(x_0,x_1) + F_{g+1}(x_0,x_1)) \\ x_1' &= x_1(x_2F_g(x_0,x_1) + F_{g+1}(x_0,x_1)) \\ x_2' &= -x_2F_{g+1}(x_0,x_1) - F_{g+2}(x_0,x_1), \end{aligned} \tag{2.8}$$

Now it is clear that the homoloidal linear system defining IH_{g+2} consists of curves of degree $g+2$ which pass through the singular point q of the hyperelliptic curve (2.6) with multiplicity g. Other base points satisfy

$$x_2F_g(x_0,x_1) + F_{g+1}(x_0,x_1) = -x_2F_{g+1}(x_0,x_1) - F_{g+2}(x_0,x_1) = 0.$$

Eliminating x_2, we get the equation $F_{g+1}^2 - F_g F_{g+2} = 0$ which defines the set of the $2g+2$ ramification points $p_1, \ldots, p_{2g+2}$ of the projection $H_{g+2} \setminus \{q\} \to \mathbb{P}^1$.

The lines $\langle q, p_i \rangle$ and the first polar Γ of H_{g+2} with respect to the point q, i.e. the curve given by the equation

$$T_2 F_g(T_0, T_1) + F_{g+1}(T_0, T_1) = 0,$$

are blown down to points under IH_{g+2}. It follows immediately from (2.8) that the set of fixed points of the involution outside the base locus is the hyperelliptic curve (2.6). Also we see that the pencil of lines through q is invariant with respect to IH_{g+2}.

Let $\sigma : S \to \mathbb{P}^2$ be the blow-up of the point q and the points $p_1, \ldots, p_{2g+2}$. The pre-image of a line $\ell_i = \langle q, p_i \rangle$ is the union of two components isomorphic to $\mathbb{P}^1$ which intersect transversally at one point. One component is the exceptional curve $R_i = \sigma^{-1}(p_i)$, another one is the proper inverse transform R_i' of the line ℓ_i. The proper transform of H_{g+2} intersects $\sigma^{-1}(\ell_i)$ at its singular point. Thus the proper transform of the hyperelliptic curve H_{g+2} intersects the exceptional curve $E = \sigma^{-1}(q)$ at the same points where the proper transform of lines ℓ_i intersect E. The proper inverse transform $\bar{\Gamma}$ of Γ intersects R_i at one nonsingular point, and intersects E at g points, the same points where the proper inverse transform $\bar{H}_{g+2}$ of H_{g+2} intersects E. The involution IH_{g+2} lifts to a biregular automorphism τ of S. It switches the components R_i and R_i' of $\sigma^{-1}(\ell_i)$, switches E with $\bar{\Gamma}$ and fixes the curve $\bar{H}_{g+2}$ pointwiseley. The pencil of lines through q defines a morphism $\phi : S \to \mathbb{P}^1$ whose fibres over the points corresponding to the lines ℓ_i are isomorphic to a bouquet of two $\mathbb{P}^1$'s. All other fibres are isomorphic to $\mathbb{P}^1$. This is an example of a *conic bundle* or a Mori fibration (or in the archaic terminology of [37], a mimimal rational surface with a pencil of rational curves).

To show that the birational involutions $IH_{g+2}, g > 0$, are not conjugate to each other or to a projective involution we use the following.

Lemma 2.1. *Let G be a finite subgroup of* $\mathrm{Cr}(2)$ *and let $C_1, \ldots, C_k$ be nonrational irreducible curves on $\mathbb{P}^2$ such that each of them contains an open subset C_i^0 whose points are fixed under all $g \in G$. Then the birational isomorphism classes of the curves C_i is an invariant of the conjugacy class of G in* $\mathrm{Cr}(2)$.

Proof. Suppose $G = T \circ G' \circ T^{-1} \cong$ is conjugate to another subgroup G' of $\mathrm{Cr}(2)$. Then, replacing C_i^0 by a smaller open subset we may assume that $T^{-1}(C_i^0)$ is defined and consists of fixed points of G'. Since C_i is not rational, $T^{-1}(C_i^0)$ is not a point, and hence its Zariski closure is a rational irreducible curve C_i' birationally isomorphic to C_i which contains an open subset of fixed points of G'. □

Note that any connected component of the fixed locus of a finite group of projective transformations is a line or a point. This shows that IH_{g+2} is not conjugate to a subgroup of projective transformations for any $g > 0$. Since IH_{g+2} is conjugate to an involution (2.4), where $P(x)$ is determined by the birational isomorphis class of H_{g+2}, we see from the previous lemma that IH_{g+2} is conjugate to $IH'_{g'+2}$ if and only if $g = g'$ and the curves H_{g+2} and H'_{g+2} are birationally isomorphic. Finally, let us look at the involution IH_2. It is a quadratic transformation which is conjugate to the transformation $(x, y) \mapsto (x, x/y)$. It is easy to check that the birational transformation $(x_0, x_1, x_2) \mapsto (x_0 x_1 - x_2^2, (x_0 - x_2)x_2, (x_1 - x_2)x_2)$ conjugates IH_2 with a projective transformation.

A Jonquiéres involution (2.4) is a special case of a Cremona transformation of the form

$$(x,y) \mapsto (\frac{ax+b}{cx+d}, \frac{r_1(x)y+r_2(x)}{r_3(x)y+r_4(x)}),$$

where $a,b,c,d \in \mathbb{C}, ad-bc \neq 0$ and $r_i(x) \in \mathbb{C}(x)$ with $r_1(x)r_4(x)-r_2(x)r_3(x) \neq 0$. These transformations form a subgroup of $\mathrm{Cr}(2)$ called a *de Jonquiéres subgroup* and denoted by $\mathrm{dJ}(2)$. Of course, its definition requires a choice of a transcendental base of the field $\mathbb{C}(\mathbb{P}^2)$. If we identify $\mathrm{Cr}(2)$ with the group $\mathrm{Aut}_{\mathbb{C}}(\mathbb{C}(x,y))$, and consider the field $\mathbb{C}(x,y)$ as a field $K(y)$, where $K = \mathbb{C}(x)$, then

$$\mathrm{dJ}(2) \cong \mathrm{PGL}_{\mathbb{C}(x)}(2) \rtimes \mathrm{PGL}_{\mathbb{C}}(2)$$

where $\mathrm{PGL}_{\mathbb{C}(x)}(2)$ is the normal subgroup and $\mathrm{PGL}_{\mathbb{C}}(2)$ acts on $\mathrm{PGL}_{\mathbb{C}(x)}(2)$ via Moebius transformations of the variable x.

It is clear that all elements from $\mathrm{dJ}(2)$ leave the pencil of lines parallel to the y-axis invariant. One can show that a subgroup of $\mathrm{Cr}(2)$ which leaves a pencil of rational curves invariant is conjugate to $\mathrm{dJ}(2)$.

2.4. **Geiser and Bertini involutions.** The classical definition of a Geiser involution is as follows [28]. Fix 7 points $p_1,\ldots,p_7$ in $\mathbb{P}^2$ in general position (we will make this more precise later). The linear system L of cubic curves through the seven points is two-dimensional. A general point p defines a pencil in this linear system of curves passing through p. A pencil of cubic curves has 9 base points, so define $\gamma(p)$ as the ninth common point of curves from the pencil. One can also see this transformations as follows. The linear system L defines a rational map of degree 2

$$f : \mathbb{P}^2 - \to |L|^* \cong \mathbb{P}^2.$$

The points p and $\gamma(p)$ lie in the same fibre. Thus γ is a birational deck transformation of this cover. Blowing up the seven points, we obtain a Del Pezzo surface S of degree 2 (more about this later), and a regular map of degree 2 from S to $\mathbb{P}^2$. The Geiser involution γ becomes an automorphism of the surface S.

It is easy to see that the fixed points of a Geiser involution lie on the ramification curve of f. This curve is a curve of degree 6 with double points at the points $p_1,\ldots,p_7$ birationally isomorphic to a canonical curve of genus 3. Applying Lemma 2.1, we obtain that a Geiser involution is not conjugate to any de Jonquiéres involution IH_{g+2}. Also, as we will see later, the conjugacy classes of Geiser involution are in a bijective correspondence with the moduli space of canonical curves of genus 3 (isomorphic to nonsingular plane quartics).

The algebraic degree of a Geiser involution is equal to 8.

Let us now fix 8 points in $\mathbb{P}^2$ in general position and consider the pencil of cubic curves through these points. It has the ninth base point p_9. For any general point p there will be a unique cubic curve $C(P)$ from the pencil which passes through p. Take p_9 for the zero in the group law of the cubic $C(P)$ and define $\beta(P)$ as the negative $-p$ with respect to the group law. This defines a birational involution on $\mathbb{P}^2$, a *Bertini involution* [8]. We will show later that the fixed points of a Bertini involution lie on a canonical curve of genus 4 with vanishing theta characteritic (isomorphic to a nonsingular intersection of a cubic surface and a quadratic cone in $\mathbb{P}^3$). So, a Bertini involution is not conjugate to a Geiser involution or a de Jonquiéres involution. It can be realized as an automorphism of the blow-up of the eight points (a Del Pezzo surface of degree 1), and the quotient of this involution is the quadratic cone.

The algebraic degree of a Bertini involution is equal to 17.

3. Rational G-surfaces

3.1. **Resolution of indeterminacy points.** Let $\chi : S - \to S'$ be a birational map of nonsingular projective surfaces. It is well-known (see [31]) that there exist birational morphisms $\sigma : X \to S$ and $\phi : X \to S'$ of nonsingulart surfaces such that the following diagram is commutative

(3.1)
$$\begin{array}{ccc} & X & \\ {}^{\sigma}\swarrow & & \searrow^{\phi} \\ S & \dashrightarrow & S' \end{array}$$

It is called a *resolution of indeterminacy points* of χ. Recall also that one can decompose any birational morphism into a composition of blow-ups with center at a point. Let

$$\sigma : X = X_N \xrightarrow{\sigma_N} X_{N-1} \xrightarrow{\sigma_{N-1}} \ldots \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X_0 = S \tag{3.2}$$

be such a composition. Here $\sigma_i : X_i \to X_{i-1}$ is the blow-up of a point $x_i \in X_{i-1}$. Let

$$E_i = \sigma_i^{-1}(x_i), \quad \mathcal{E}_i = (\sigma_{i+1} \circ \ldots \sigma_N)^{-1}(E_i). \tag{3.3}$$

Let H' be a very ample divisor class on S' and $\mathcal{H}'$ be the corresponding complete linear system $|H'|$. Let $\mathcal{H}_N = \phi^*(\mathcal{H}')$. Define $m(x_N)$ as the smallest positive number such that $\mathcal{H}_N + m(x_N)E_N = \sigma_N^*(\mathcal{H}_{N-1})$ for some linear system $\mathcal{H}_{N-1}$ on X_{N-1}. Then proceed inductively to define linear systems $\mathcal{H}_k$ on each X_k such that $\mathcal{H}_{k+1} + m(x_{k+1})E_{k+1} = \sigma_{k+1}^*(\mathcal{H}_k)$, and finally a linear system $\mathcal{H} = \mathcal{H}_0$ on S such that $\mathcal{H}_1 + m(x_1)E_1 = \sigma_1^*(\mathcal{H})$. It follows from the definition that

$$\phi^*(\mathcal{H}') = \sigma^*(\mathcal{H}) - \sum_{i=1}^{N} m(x_i)\mathcal{E}_i. \tag{3.4}$$

The proper inverse transform of $\mathcal{H}'$ on S under χ (whose general member, by definition, is the closure of the pre-image of a general member of $\mathcal{H}'$ on the open subset of S where χ is defined) is contained in the linear system $\mathcal{H}$. It consists of curves which pass through the points x_i with multiplicity $\geq m_i$. We denote it by

$$\chi^{-1}(\mathcal{H}') = |H - m(x_1)x_1 - \ldots - m(x_N)x_N|,$$

where $\mathcal{H} \subset |H|$. Here for a curve on S to pass through a point $x_i \in X_{i-1}$ with multiplicity $\geq m(x_i)$ means that the proper inverse transform of the curve on X_{i-1} has x_i as a point of multiplicity $\geq m(x_i)$. The divisors $\mathcal{E}_i$ are called the *exceptional configurations* of the resolution $\sigma : X \to S$ of the birational map χ. Note that $\mathcal{E}_i$ is an irreducible curve if and only if $\sigma_{i+1} \circ \ldots \circ \sigma_N : X \to X_i$ is an isomorphism over $E_i = \sigma^{-1}(x_i)$.

The set of points $x_i \in X_i, i = 1, \ldots, N$, is called the set of *indeterminacy points*, or *base points*, or *fundamental points* of χ. Note that, strictly speaking, only one of them, x_1, lies in $\mathbb{P}^2$. However, if $\sigma_1 \circ \ldots \circ \sigma_i : X_i \to S$ is an isomorphism in a neighborhhod of x_{i+1} we can identify this point with a point on S. Let $\{x_i, i \in I\}$

be the set of such points. Points $x_j, j \notin I$, are *infinitely near points*. A precise meaning of this classical notion is as follows.

Let S be a nonsingular projective surface and $\mathcal{B}(S)$ be the category of birational morphisms $\pi : S' \to S$ of nonsingular projective surfaces. Recall that a morphism from $(S' \xrightarrow{\pi'} S)$ to $(S'' \xrightarrow{\pi''} S)$ in this category is a regular map $\phi : S' \to S''$ such that $\pi'' \circ \phi = \pi'$.

Definition 3.1. The *bubble space* S^{bb} of a nonsingular surface S is the factor set

$$S^{\mathrm{bb}} = \Big(\bigcup_{(S' \xrightarrow{\pi'} S) \in \mathcal{B}(S)} S' \Big)/R,$$

where R is the following equivalence relation: $x' \in S'$ is equivalent to $x'' \in S''$ if the rational map $\pi''^{-1} \circ \pi' : S' - \to S''$ maps isomorphically an open neighborhood of x' to an open neighborhood of x''.

It is clear that for any $\pi : S' \to S$ from $\mathcal{B}(S)$ we have an injective map $i_{S'} : S' \to S^{\mathrm{bb}}$. We will identify points of S' with their images. If $\phi : S'' \to S'$ is a morphism in $\mathcal{B}(S)$ which is isomorphic in $\mathcal{B}(S')$ to the blow-up of a point $x' \in S'$, any point $x'' \in \phi^{-1}(x')$ is called *infinitely near point* to x' of the first order. This is denoted by $x'' \succ x'$. By induction, one defines an infinitely near point of order k, denoted by $x'' \succ_k x'$. This defines a partial order on S^{bb}.

We say that a point $x \in S^{\mathrm{bb}}$ is of height k, if $x \succ_k x_0$ for some $x_0 \in S$. This defines the *height function* on the bubble space

$$\mathrm{ht}_S : S^{\mathrm{bb}} \to \mathbb{N}.$$

Clearly, $S = \mathrm{ht}^{-1}(0)$.

It follows from the known behavior of the canonical class under a blow-up that

$$K_X = \sigma^*(K_S) + \sum_{i=1}^{N} \mathcal{E}_i. \tag{3.5}$$

The intersection theory on a nonsingular surface gives

$$\begin{aligned} \mathcal{H}'^2 &= (\phi^*(\mathcal{H}'))^2 = (\sigma^*(\mathcal{H}) - \sum_{i=1}^{N} m(x_i)\mathcal{E}_i)^2 = \mathcal{H}^2 - \sum_{i=1}^{N} m(x_i)^2, \\ K_{S'} \cdot \mathcal{H}' &= K_S \cdot \mathcal{H} - \sum_{i=1}^{N} m(x_i). \end{aligned} \tag{3.6}$$

Example 3.2. Let $\chi : \mathbb{P}^2 - \to \mathbb{P}^2$ be a Cremona transformation, $\mathcal{H}' = |\ell|$ be the linear system of lines in $\mathbb{P}^2$, and $\mathcal{H} \subset |n\ell|$. The formulas (3.6) give

$$\begin{aligned} n^2 - \sum_{i=1}^{N} m(x_i)^2 &= 1 \\ 3n - \sum_{i=1}^{N} m(x_i) &= 3 \end{aligned} \tag{3.7}$$

The linear system $\mathcal{H}$ is written in this situation as $\mathcal{H} = |n\ell - \sum_{i=1}^{N} m_i x_i|$. For example, a quadratic transformation with 3 base points p_1, p_2, p_3 is given by the linear system $|2\ell - p_1 - p_2 - p_3|$. Note in the case of the transformation τ_1 the curves $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ are irreducible and the maps $\sigma_1 : X_1 \to \mathbb{P}^2$ is an isomorphism in a neighborhood of p_2, p_3, so we can identify p_1, p_2, p_3 with points on $\mathbb{P}^2$. In the case of the transformation τ_2, we have $\sigma_1(p_2) = p_1$ and p_3 can be identified with a point on $\mathbb{P}^2$. So in this case $p_2 \succ p_1$. In the case of τ_3 we have $p_3 \succ p_2 \succ p_1$.

For a Geiser involution (resp. Bertini involution) we have $\mathcal{H} = |8\ell - 3p_1 - \ldots - 3p_7|$ (resp. $\mathcal{H} = |17\ell - 6p_1 - \ldots - 6p_8|$).

3.2. ***G*-surfaces.** Assume now that a finite group G acts on surfaces S. Strictly speaking this means that there is given a homomorphism of groups $\phi : G \to \mathrm{Aut}(S)$. However, for simplicity of notation, we will always assume that the homomorphisms is injective and we will often identify G with their image. A pair (S, ρ) (or (S, G)) will be called a *G-surface*. The definition of a morphism (or a rational map) of G-surfaces is straightforward.

Let $\chi : S \to S'$ be a birational map of G-surfaces. Then one can G-equivariantly resolve χ, in the sense that one can find the diagram (3.1) where all maps are maps of G-surfaces. The group G acts on the surface X permuting the exceptional configurations $\mathcal{E}_i$ in such a way that $\mathcal{E}_i \subset \mathcal{E}_j$ implies $g(\mathcal{E}_i) \subset g(\mathcal{E}_j)$. This defines an action of G on the set of indeterminacy points of χ ($g(x_i) = x_j$ if $g(\mathcal{E}_i) = g(\mathcal{E}_j)$). The action preserves the order, i.e. $x_i \succ x_j$ implies $g(x_i) \succ g(x_j)$, so the function $\mathrm{ht} : \{x_1, \ldots, x_N\} \to \mathbb{N}$ is constant on each orbit Gx_i.

Let $\mathcal{H}' = |H'|$ be an ample linear system on S' and $\phi^*(\mathcal{H}') = \sigma^*(\mathcal{H}) - \sum_{i=1}^N m(x_i)\mathcal{E}_i$ be its inverse transform on X as above. Everything here is G-invariant, so $\mathcal{H}$ is a G-invariant linear system on S and the multiplicities $m(x_i)$ are constant on the G-orbits. So we can rewrite the system in the form

$$\phi^*(\mathcal{H}') = \sigma^*(\mathcal{H}) - \sum_{\kappa \in \mathcal{I}} m(\kappa)\mathcal{E}_\kappa,$$

where $\mathcal{I} = \{\kappa_1, \ldots, \kappa_{N'}\}$ is the set of G-orbits of indeterminacy points, $m(\kappa) = m(x_i)$ for any $x_i \in \kappa$ and $\mathcal{E}_\kappa, = \sum_{x_i \in \kappa} \mathcal{E}_i$. Similarly one can rewrite the proper inverse transform of $\mathcal{H}'$ on S

$$|H - \sum_{\kappa \in \mathcal{I}} m(\kappa)\kappa|. \tag{3.8}$$

We can now rewrite the intersection formula (3.6) in the form

$$\begin{aligned} H'^2 &= H^2 - \sum_{\kappa \in \mathcal{I}} m(\kappa)^2 d(\kappa) \\ K_{S'} \circ H' &= K_S \cdot H - \sum_{\kappa \in \mathcal{I}} m(\kappa) d(\kappa), \end{aligned} \tag{3.9}$$

where $d(\kappa) = \#\{i : x_i \in \kappa\}$.

Remark 3.3. In the arithmetical analog of the previous theory all the notations become much more natural. Our maps are maps over a perfect ground field k. A blow-up is the blow-up of a closed point in scheme-theoretical sense, not necessary k-rational. The exceptional configuration is defined obver k but when we replace k to its algebraic closure $\bar{k}$, it decomposes into the union of conjugate exceptional configurations over $\bar{k}$. So, in the above notation κ means a closed point on S or on one of x_i's. The analog of $d(\kappa)$ is of course the degree of a point, i.e. the extension degree $[k(x) : k]$, where $k(x)$ is the residue field of x.

3.3. **The bubble space.** Here we recall Manin's formalism of the theory of linear systems with basis condition in its G-equivariant form (see [44]).

First we define the G-*equivariant bubble space* of G-surface S as a G-equivariant version $(S,G)^{\mathrm{bb}}$ of Definition 3.1. One replaces the category $\mathcal{B}(S)$ of birational morphisms $S' \to S$ with the category $\mathcal{B}(S,G)$ of birational morphisms of G-surfaces. In this way the group G acts on the bubble space $(S,G)^{\mathrm{bb}}$ and the height function becomes constant on G-orbits. Let

$$Z^*(S,G) = \varinjlim \mathrm{Pic}(S'), \tag{3.10}$$

where the inductive limit is taken with respect to the functor Pic from the category $\mathcal{B}(S,G)$ with values in the category of abelian groups defined by $S' \to \mathrm{Pic}(S')$. The group $Z*(S,G)$ is equiiped with a natural structure of G-module. Also the following natural structures are defined on $Z^*(X)$

(a) There is a symmetric G-invariant pairing

$$Z^*(S,G) \times Z^*(S,G) \to \mathbb{Z}$$

induced by the intersection pairing on each $\mathrm{Pic}(S')$.

(b) $Z^*(S,G)$ contains a distinguished cone of effective divisors classes

$$Z^*_+(S,G) = \varinjlim \mathrm{Pic}_+(S'),$$

where $\mathrm{Pic}_+(S')$ is the cone of effective divisor classes on each S' from $\mathcal{B}(S,G)$

(c) There is a distinguished G-equivariant homomorphism

$$K : Z^*(S,G) \to \mathbb{Z}, \quad K(z) = K_{S'} \cdot z, \text{for any } S' \to S \text{ from } \mathcal{B}(S,G).$$

Let $f : S' \to S$ be a morphism from $\mathcal{B}(S,G)$ and $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be its exceptional configurations. We have a natural splitting

$$\mathrm{Pic}(S') = f^*(\mathrm{Pic}(S)) \oplus \mathbb{Z}[\mathcal{E}_1] \oplus \ldots \oplus \mathbb{Z}[\mathcal{E}_n].$$

Now let $Z_0(S,G) = \mathbb{Z}^{(S,G)^{\mathrm{bb}}}$ be the free abelian group generated by the set $(S,G)^{\mathrm{bb}}$. Identifying exceptional configurations with points in the bubble space, and passing to the limit we obtain a natural splitting

$$Z^*(S,G) = Z_0(S,G) \oplus \mathrm{Pic}(S). \tag{3.11}$$

Passing to invariants we get the splitting

$$Z^*(S,G)^G = Z_0(S,G)^G \oplus \mathrm{Pic}(S)^G \tag{3.12}$$

Let us write an element of $Z^*(S,G)^G$ in the form

$$z = D + \sum_{\kappa \in O} m(\kappa)\kappa,$$

where O is the set of G-orbits in $Z_0(S,G)^G$ and D is G-invariant divisor class on S. Then

(a) $z \cdot z' = D \cdot D' - \sum m(\kappa)m'(\kappa)d(\kappa)$;
(b) $z \in Z^*_+(S,G)$ if and only if $D \in \mathrm{Pic}_+(S)^G, m(\kappa) \geq 0$ and $m(\kappa') \leq m(\kappa)$ if $\kappa' \succ \kappa$;
(c) $K(z) = D \cdot K_S - \sum_{\kappa \in O} m(\kappa)d(\kappa)$.

Let $\phi : S' \to S$ be an object of $\mathcal{B}(S,G)$. Then we have a natural map $\phi_{\mathrm{bb}} : (S',G)^{\mathrm{bb}} \to (S,G)^{\mathrm{bb}}$ which induces an isomorphism $\phi_{\mathrm{bb}}^* : Z(S,G) \to Z(S',G)$. We also have a natural isomorphism $\phi_*^{\mathrm{bb}} : Z(S',G) \to Z(S,G)$. Both of these maps do not preserve the splitting (3.11). Resolving indeterminacy points of any birational map $\chi : (S,G) - \to (S',G')$ we can define

- *proper direct transform map* $\chi_* : Z^*(S,G) \xrightarrow{\sim} Z^*(S',G)$;
- *proper inverse transform map* $\chi^* : Z^*(S',G) \xrightarrow{\sim} Z^*(S,G)$.

The group $Z^*(S,G)$ equipped with all above structures is one of the main G-birational invariants of S. It can be viewed as the Picard group of the bubble space $(S,G)^{\mathrm{bb}}$.

The previous machinery gives a convenient way to consider the linear systems defining rational maps of surfaces. Thus we can rewrite (3.4) in the form $|z|$, where $z = H - \sum m_i x_i$ is consider as an element of $Z_+^*(S,G)$. The condition that $|z|$ is homaloidal is equivalent to the conditions

$$
\begin{aligned}
z^2 &= H^2 - \sum m_i^2 = H'^2 \\
K(z) &= H \cdot K_S - \sum m_i = H' \cdot K_{S'}.
\end{aligned}
\tag{3.13}
$$

When $S = S' = \mathbb{P}^2$ we get the equalities (3.7).

3.4. **Minimal rational G-surfaces.** Let (S,G) be a rational G-surface. Choose a birational map $\phi : S - \to \mathbb{P}^2$. For any $g \in G$ the map $\phi \circ g \circ \phi^{-1} : \mathbb{P}^2 - \to \mathbb{P}^2$ is a birational map of $\mathbb{P}^2$. This defines an injective homomorphism

$$
\iota_\phi : G \to \mathrm{Cr}(2). \tag{3.14}
$$

Suppose (S',G) is another rational G-surface and $\phi' : S' - \to \mathbb{P}^2$ be a birational map.

Lemma 3.4. *The subgroups $\iota_\phi(G)$ and $\iota_{\phi'}(G)$ of* Cr(2) *are conjugate if and only if there exists a birational map of G-surfaces $\chi : S' - \to S$.*

Proof. Suppose $\iota'_\phi(G) = T \circ \iota_\phi \circ T^{-1}$ for some $T \in \mathrm{Cr}(2)$. Set $\chi = \phi'^{-1} \circ T \circ \phi : S - \to S'$. Then χ is a birational map of G-surfaces if and only if for any $g \in G$ one has $\chi \circ g = g \circ \chi$. Plugging in $\chi = \phi'^{-1} \circ T \circ \phi$, we see that this is equivalent to $T \circ \iota_\phi(g) = \iota_{\phi'}(g) \circ T$.

□

The lemma shows that a birational isomorphism class of G-surfaces defines a conjugacy class of subgroups of Cr(2) isomorphic to G. The next lemma shows that any conjugacy class is obtained in this way.

Lemma 3.5. *Let G be a finite subgroup of* Cr(2)*. Then there exists a rational G-surface (S,ρ) and a birational map $\phi : S \to \mathbb{P}^2$ such that, for all $g \in G$,*

$$
g = \phi \circ \rho(g) \circ \phi^{-1}.
$$

Proof. We give two proofs. The fist one is after A. Verra. Let $U(g)$ be an open invariant subset of $\mathbb{P}^2$ on which $g \in G$ acts biregularly, and $U = \cap_{g \in G} U(g)$. Order G in some way and consider a copy of $\mathbb{P}_g^2$ indexed by $g \in G$. For any $u \in U$ let $g(u) \in \mathbb{P}_g^2$. We have the morphism

$$
\phi : U \to \prod_{g \in G} \mathbb{P}_g^2, \ u \mapsto (g(u))_{g \in G}.
$$

Define an action of G on the product by $g'((x_g)_{g\in G}) = (x_{gg'})_{g\in G}$. Then ϕ is obviously G-equivariant. Now define V as the Zariski closure of $\phi(U)$ in the product. It is obviously a G-invariant surface which contains an open G-invariant subset G-isomorphic to U. It remains to replace V by its G-equivariant resolution of singularities (which always exists).

The second proof is standard. Let U be as above and $U' = U/G$ be the orbit space. It is a normal algebraic surface. Choose any normal projective completion X' of U'. Let S' be the normalization of X' in the field of rational functions of U. This is a normal projective surface on which G acts by biregular transformations. It remains to define S to be a G-invariant resolution of singularities (see also [21]). □

Summing up, we obtain the following result.

Theorem 3.6. *There is a natural bijective correspondence between birational isomorphism classes of rational G-surfaces and conjugate classes of subgroups of* $\mathrm{Cr}(2)$ *isomorphic to G.*

So our goal is to classify G-surfaces (S,ρ) up to birational isomorphism of G-surfaces.

Definition 3.7. A *minimal* G-surface is a G-surface (S,ρ) such that any birational morphism of G-surfaces $(S,\rho)\to(S',\rho')$ is an isomorphism. A group G of automorphisms of a rational surface S is called a *minimal group of automorphisms* if the pair (S,ρ) is minimal.

Obviously, it is enough to classify minimal rational G-surfaces up to birational isomorphism of G-surfaces.

Before we state the next fundamental result, let us recall some terminology.

A *conic bundle structure* on a rational G-surface (S,G) is a G-equivariant morphism $\phi: S\to\mathbb{P}^1$ such that the fibres are isomorphic to a reduced conic on $\mathbb{P}^2$. A *Del Pezzo surface* is a surface with ample anti-canonical divisor $-K_S$.

Theorem 3.8. *Let S be a minimal rational G-surface. Then either S admits a structure of a conic bundle with* $\mathrm{Pic}(S)^G\cong\mathbb{Z}^2$*, or S is isomorphic to a Del Pezzo surface with* $\mathrm{Pic}(S)\cong\mathbb{Z}$.

A historical analog of this theorem makes use of the method of the termination of adjoints, first introduced for linear system of plane curves in the work of G. Castelnuovo. It consists in replacing a linear system $|D|$ with the linear system $|D+K_S|$ until it becomes empty. The application of this method to finding a G-invariant linear system of curves on the plane was initiated in the works of S. Kantor [41], who essentially stated the theorem above but without the concept of minimality. In arithmetical situation this method was first applied by Enriques [27]. A first modern proof of the theorem was given by Manin [44] and by the second author [37] (an earlier proof of Manin used the assumption that G is an abelian group). Nowdays the theorem follows easily from a G-equivariant version of Mori's theory (see [42], Example 2.18) and the proof can be found in literarure ([6], [22]). For this reason we omit the proof.

Recall the classification of Del Pezzo surfaces (see [20]). The number $d = K_S^2$ is called the degree. It takes the value between 1 and 9. For $d\ge 3$, the anti-canonical linear system $|-K_S|$ maps S in $\mathbb{P}^d$ onto a nonsingular surface of degree d. If $d = 9$, $S\cong\mathbb{P}^2$. If $d = 8$, then $S\cong\mathbb{P}^1\times\mathbb{P}^1$, or the minimal ruled surface $\mathbf{F}_1$. For $d\le 7$, a

Del Pezzo surface S is isomorphic to the blow-up of $n = 9-d$ points in $\mathbb{P}^2$ satisfying the following conditions

- no three are on a line;
- no six are on a conic;
- not contained on a plane cubic with one of them being its singular point $(n = 8)$.

For $d = 2$, the linear system $|-K_S|$ defines a finite morphism of degree 2 from S to $\mathbb{P}^2$ with branch curve a nonsingular quartic. Finally, for $d = 1$, the linear system $|-2K_S|$ defines a finite morphism of degree 2 onto a quadric cone Q with branch curve cut out by a cubic.

For a minimal Del Pezzo G-surface the group $\mathrm{Pic}(S)^G$ is generated by K_S if S is not isomorphic to $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. In the latter cases it is generated by $\frac{1}{3}K_S$ or $\frac{1}{2}K_S$, respectively.

A conic bundle surface is either minimal ruled surface $\mathbf{F}_n$ or a surface obtained from one these surfaces by blowing up a finite set of points, no two lying in a fibre of the ruling. The number of blow-ups is equal to the number of singular fibres of the conic bundle fibration. We will exclude the surfaces $\mathbf{F}_0$ and $\mathbf{F}_1$, considering them as Del Pezzo surfaces.

There are minimal conic bundles with ample $-K_S$ (see Proposition 5.2).

4. Automorphisms of minimal ruled surfaces

4.1. **Some of group theory.** We employ the standard notations for groups used by group-theorists (see [16]):

n means a cyclic group C_n of order n;

$n^r = C_n^r$, the direct sum of r copies of C_n;

$A.B$ is an extension of a group B with help of a normal subgroup A;

$A : B$ is a split extension $A.B$, i.e. a semi-direct product $A \rtimes B$,

$A^{\cdot}B$ nonsplit extension $A.B$,

$A \wr S_n$ is the wreath product, i.e. $A^n : S_n$, where S_n is the symmetric product acting on A^n by permuting the factors;

$L_n(q) = \mathrm{PSL}(n, \mathbb{F}_q)$, where $q = p^r$ is a power of a prime number p;

$D_{2n} = n : 2$, the dihedral group of order $2n$,

S_n, the permutation group of degree n,

A_n, the alternating group of even permutations of degree n,

μ_n is the group of nth roots of unity with generator $\epsilon_n = e^{2\pi i/n}$.

We will often use the following simple result from group theory which is known as Goursat's Lemma.

Lemma 4.1. *Let G be a finite subgroup of the product $A \times B$ of two groups A and B. Let $p_1 : A \times B \to A, p_2 : A \times B \to B$ be the projection homomorphims. Let $G_i = p_i(G), H_i = \mathrm{Ker}(p_j|G), i \neq j = 1, 2$. Then $H_i \trianglelefteq G_i$ is a normal subgroup in G_i. The map $f : G_1/H_1 \to G_2/H_2$ defined by $\alpha(aH_1) = p_2(a)H_2$ is an isomorphism, and*

$$G = \{(a, b) \in G_1 \times G_2 : \alpha(aH_1) = bH_2\}.$$

The data $(H_1 \trianglelefteq G_1, H_2 \trianglelefteq G_2, \alpha)$ determines G uniquely. If $a \in A, b \in B$, then the conjugate subgroup $(a, b)G(a, b)^{-1}$ is determined by the data

$$(aH_1a^{-1} \trianglelefteq aG_1a^{-1}, bH_2b^{-1} \trianglelefteq bG_2b_2^{-1}, \alpha''),$$

where α' is the composition of α with the conjugation automorphisms of G_1/H_1 and G_2/H_2.

We denote the group defined by the data $(H_1 \trianglelefteq G_1, H_2 \trianglelefteq G_2, \alpha)$ by $(G_1, H_1, G_2, H_2)_\alpha$. In the case when H_i is a unique normal subgroup of G_i and α is a unique (up to conjugation by an element of $A \times B$) isomorphism $\alpha : G_1/H_1 \to G_2/H_2$, we skip the subscript.

Note some special cases:

$$(G_1, G_1, G_2, G_2) = G_1 \times G_2, \quad (G_1, 1, G_2, 1)_\phi = \{(g, \phi(g)), g \in G_1\}.$$

We will be dealing with various group extensions. The following lemma will be often in use.

Lemma 4.2. *Let $A.B$ be an extension of groups. Suppose that the orders of A and B are coprime. Then the extension splits. If moreover A or B is solvable, then all subgroups of $A.B$ defining splittings are conjugate.*

Proof. This is known in group theory as the Schur-Zassenhaus Theorem. Its proof can be found in [30], 6.2. □

4.2. **Finite groups of projective automorphisms.** We start with the case $S = \mathbb{P}^2$, where $\mathrm{Aut}(S) \cong \mathrm{PGL}_{\mathbb{C}}(3)$. To save space we will often denote a projective transformation

$$(x_0, x_1, x_2) \mapsto (L_0(x_0, x_1, x_2), L_1(x_0, x_1, x_2), L_2(x_0, x_1, x_2))$$

by $[L_0(x_0, x_1, x_2), L_1(x_0, x_1, x_2), L_2(x_0, x_1, x_2)]$.

Recall some standard terminology from the theory of linear groups. Let G be a subgroup of the general linear group $\mathrm{GL}(V)$ of a complex vector space of dimension $d = \dim V$. The group G is called *intransitive* if the representation of G in $\mathrm{GL}(V)$ is reducible. Otherwise it is called *transitive*. A transitive group G is called *imprimitive* if it contains an intransitive normal subgroup G'. In this case V decomposes into a direct sum of G'-invariant proper subspaces, and elements from G permute them. A group is *primitive* if it is neither intransitive, nor imprimitive. We reserve this terminology for subgroups of $\mathrm{PGL}(V)$ keeping in mind that each such group can be represented by a subgroup of $\mathrm{GL}(V)$.

Let G be an intransitive subgroup of $\mathrm{PGL}_{\mathbb{C}}(3)$ and G' be its pre-image in $\mathrm{GL}(3, \mathbb{C})$. Then G' is conjugate to a subgroup $\mathbb{C}^* \times \mathrm{GL}(2, \mathbb{C})$ of block matrices.

To classify such subgroups we have to classify subgroups of $\mathrm{GL}(2, \mathbb{C})$. We will use the well-known classification of finite subgroups of $\mathrm{PGL}_{\mathbb{C}}(2)$. They are isomorphic to one of the following *polyhedral groups*

- a cyclic group C_n;
- a dihedral group D_{2n} of order $2n \geq 2$;
- the tetrahedral group $T \cong A_4$ of order 12;
- the octahedron group $O \cong S_4$ of order 24;
- the icosahedron group $I \cong A_5$ of order 60.

Two isomorphic subgroups are conjugate subgroups of $\mathrm{PGL}_{\mathbb{C}}(2)$.

The pre-image of such group in $\mathrm{SL}(2, \mathbb{C})$ under the natural map $\mathrm{SL}(2, \mathbb{C}) \to \mathrm{SL}(2, \mathbb{C})/(\pm 1) \cong \mathrm{PGL}_{\mathbb{C}}(2)$ is a double extension $\bar{G} = 2.G$ (except when G is a cyclic group of odd order). The group $\bar{G} = 2.G$ is called a *binary polyhedral group.* A cyclic group of odd order is isomorphic to a subgroup $\mathrm{SL}(2, \mathbb{C})$ intersecting trivially the center. We will also denote it by $\bar{G}$.

Consider a natural surjective homomorphism of groups

$$\beta : \mathbb{C}^* \times \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(2,\mathbb{C}), \quad (c, A) \mapsto cA.$$

Its kernel is the subgroup $\{(1 \times I_2), (-1 \times -I_2)\} \cong C_2$.

Let G be a finite subgroup of $\mathrm{GL}(2,\mathbb{C})$ with center Z and μ_m be the image of the determinant character $\det : G \to \mathbb{C}^*$. The pre-image $\beta^{-1}(G)$ of G under the map β is a subgroup of $\mu_{2m} \times \mathrm{SL}(2,\mathbb{C})$. Its image under the second projection is the group $\bar{G'}$, where G' is the image of G in $\mathrm{PGL}_{\mathbb{C}}(2)$. Let H_1 be the kernel of the second projection $\beta^{-1}(G) \to \bar{G'}$. The homomorphism β maps H_1 isomorphically onto $Z \subset G$. Let H_2 be the kernel of the first projection. Under the map β it is mapped isomorphically onto $G_0 = \mathrm{Ker}(\det : G \to \mathbb{C}^*)$. Applying Lemma 4.1, we obtain

$$\beta^{-1}(G) \cong (\mu_{2m}, Z, \bar{G'}, G_0)_\alpha.$$

Since $\mathrm{Ker}(\beta)$ is a group of order 2, $\mathrm{Ker}(\beta : \beta^{-1}(G) \to G)$ is either trivial or is the group of order 2 generated by $(-1 \times -I_2)$. The first case happens if and only if $-I_2 \notin \bar{G'}$, i.e., G' is a cyclic group of odd order.

Lemma 4.3. *Let G be a finite non-abelian subgroup of $\mathrm{GL}(2,\mathbb{C})$. Then $G = \beta(\tilde{G})$, where $\tilde{G} \subset \mathbb{C}^* \times \mathrm{SL}(2,\mathbb{C})$ is conjugate to one of the following groups*

- $\tilde{G} = (\mu_{2m}, \mu_{2m}, \bar{I}, \bar{I}),\ G \cong \mu_m \times \bar{I}$;
- $\tilde{G} = (\mu_{2m}, \mu_{2m}, \bar{O}, \bar{O}),\ G \cong \mu_m \times \bar{O}$;
- $\tilde{G} = (\mu_{2m}, \mu_{2m}, \bar{T}, \bar{T}),\ G \cong \mu_m \times \bar{T}$;
- $\tilde{G} = (\mu_{2m}, \mu_{2m}, \bar{D}_{2n}, \bar{D}_{2n}),\ G \cong \mu_m \times \bar{D}_{2n}$;
- $\tilde{G} = (\mu_{4m}, \mu_{2m}, \bar{O}, \bar{T}),\ G \cong (\mu_m \times \bar{T}) \cdot 2$;
- $\tilde{G} = (\mu_{6m}, \mu_{2m}, \bar{T}, \bar{D}_4),\ G \cong (\mu_m \times \bar{D}_4) \cdot 3$;
- $\tilde{G} = (\mu_{4m}, \mu_{2m}, \bar{D}_{2n}, \bar{C}_n), G \cong \mu_m \times D_{2n},$ *n is even;*
- $\tilde{G} = (\mu_{4m}, \mu_m, \bar{D}_{2n}, \bar{C}_n),\ G \cong \mu_m \times D_{2n},$ *n is odd;*
- $\tilde{G} = (\mu_{2km}, \mu_{2m}, \bar{D}_{2kn}, \bar{D}_{2n})_\alpha,\ G \cong (\mu_{2m} \times \bar{D}_{2n}) \cdot k,\ \alpha \in \mathrm{Isom}(\bar{D}_{2kn}/\bar{D}_{2n} \to C_k)$,

An abelian subgroup $G \subset \mathrm{GL}(2,\mathbb{C})$ is conjugate to a subgroup of diagonal matrice of the form $(\epsilon_m^a, \epsilon_n^b)$, where ϵ_m, ϵ_n are primitive roots of unity and $a, b \in \mathbb{Z}$. Let $d = (m,n), m = du, n = dv, d = kq$ for some fixed positive integer k. Let $H_1 = \langle \epsilon_m^k \rangle \subset \langle \epsilon_m \rangle, H_2 = \langle \epsilon_n^k \rangle \subset \langle \epsilon_n \rangle$ be cyclic subgroups of index k. Applying Lemma 4.1 we obtain

$$G \cong (\langle \epsilon_m \rangle, \langle \epsilon_m^k \rangle, \langle \epsilon_n \rangle, \langle \epsilon_n^k \rangle)_\alpha,$$

where α is an automorphism of the cyclic group $\langle \epsilon_k \rangle$ defined by a choice of a new generator $\epsilon_m^s, (s,k) = 1$. In this case

$$G = (\langle \epsilon_m^k \rangle \times \langle \epsilon_n^k \rangle).\langle \epsilon_k \rangle \tag{4.1}$$

is of order $mn/k = uvkq^2$. In other words, G consists of diagonal matrices of the form $(\epsilon_m^a, \epsilon_n^b)$, where $a \equiv sb \mod k$.

Corollary 4.4. *Let G be an intransitive finite subgroup of $\mathrm{GL}(3,\mathbb{C})$. Then its image in $\mathrm{PGL}_{\mathbb{C}}(3)$ consists of transformations $[ax_0 + bx_1, cx_0 + dx_1, x_2]$, where the matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ form a nonabelian finite subgroup H of $\mathrm{GL}(2,\mathbb{C})$ from Lemma 4.3 or an abelian group of the form (4.1).*

Now suppose G is transitive but imprimitive subgroup of $\mathrm{PGL}_{\mathbb{C}}(3)$. Let G' be its largest intransitive normal subgroup. Then G/G' permutes transitively the invariant subspaces of G', hence we may assume that all of them are one-dimensional. Replacing G by a conjugate group we may assume that G' is a subgroup of diagonal matrices. We can represent its elements by diagonal matrices $g = (\epsilon_m^a, \epsilon_n^b, 1)$, where $a \equiv sb \mod k$ as in (4.1). The group G contains a cyclic permutation τ of coordinates. Since G' is a normal subgroup of G, we get $\tau^{-1} g \tau = (\epsilon_n^{-b}, \epsilon_n^{-b}\epsilon_m^a, 1) \in G'$. This implies that $n|bm, m|an$, hence $u|b, v|a$. Since $(\epsilon_m, \epsilon_n^s, 1)$ or $(\epsilon_m^{s'}, \epsilon_n, 1), ss' \equiv 1 \mod k$, belongs to G we must have $u = v = 1$, i.e. $m = n = d$. Therefore G' consists of diagonal matrices $g = (\epsilon_d^a, \epsilon_d^{sa}, 1)$. Since $\tau^{-1} g \tau = (\epsilon_d^{-sa}, \epsilon_d^{a-sa}, 1) \in G'$, we get $a - sa \equiv -s^2 a \mod k$ for all $a \in \mathbb{Z}/m\mathbb{Z}$. Hence the integers s satisfy the congruence $s^2 - s + 1 \equiv 0 \mod k$. If, moreover, $G/G' \cong S_3$, then we have an additional condition $s^2 \equiv 1 \mod k$, and hence either $k = 1$ and $G' = \mu_n \times \mu_n$ or $k = 3$, $s = 2$ and $G' = n \times n/k$.

This gives the following.

Theorem 4.5. *Let G be a transitive imprimitive finite subgroup of $\mathrm{PGL}_{\mathbb{C}}(3)$. Then G is conjugate to one of the following groups*

- $G \cong n^2 : 3$ *generated by transformations*
$$[\epsilon_n x_0, x_1, x_2],\ [x_0, \epsilon_n x_1, x_2],\ [x_2, x_0, x_1];$$
- $G \cong n^2 : S_3$ *generated by transformations*
$$[\epsilon_n x_0, x_1, x_2],\ [x_0, \epsilon_n x_1, x_2],\ [x_0, x_2, x_1],\ [x_2, x_0, x_1];$$
- $G = G_{n,k,s} \cong (n \times n/k) : 3$, *where* $k > 1, k|n$ *and* $s^2 - s + 1 = 0 \mod k$. *It is generated by transformations*
$$[\epsilon_{n/k} x_0, x_1, x_2],\ [\epsilon_n^s x_0, \epsilon_n x_1, x_2],\ [x_2, x_0, x_1].$$
- $G \simeq (n \times n/3) : S_3$ *generated by transformations*
$$[\epsilon_{n/3} x_0, x_1, x_2],\ [\epsilon_n^2 x_0, \epsilon_n x_1, x_2],\ [x_0, x_2, x_1],\ [x_0, x_1, x_2].$$[1]

The next theorem is a well-known result of Blichfeldt [9].

Theorem 4.6. *Any primitive finite subgroup G of* $\mathrm{PGL}(3)$ *is conjugate to one of the following groups.*

1. *The icosahedron group A_5 isomorphic to $L_2(5)$. It leaves invariant a nonsingular conic.*
2. *The* Hessian group *of order 216 isomorphic to* $3^2 : L_2(3)$. *It is realized as the group of automorphisms of the Hesse pencil of cubics*
$$x^3 + y^3 + z^3 + txyz = 0.$$
The subgroup $L_2(3)$ is isomorphic to $2.A_4 \cong \bar{T}$ and permutes the four reducible members of the pencil.
3. *The* Klein group *of order 168 isomorphic to $L_2(7)$ (realized as the group of automorphisms of the Klein quartic $x^3y + y^3z + z^3x = 0$).*
4. *The* Valentiner group *of order 360 isomorphic to A_6. It can be realized as a group of automorphisms of the nonsingular plane sextic*
$$10x^3y^3 + 9zx^5 + y^5 - 45x^2y^2z^2 - 135xyz^4 + 27z^6 = 0.$$

[1]This case was omitted in the first version of the article. The authors are grateful to Chenyang Xu who has pointed out this gap.

(5) *Subgroups of the Hessian group:*
- $3^2 : 4$;
- $3^2 : Q_8$, *where* $Q_8 \cong 2.2^2$ *is the quaternion group of order 8.*

Now assume that S is a minimal ruled surface $\mathbf{F}_n, n \neq 1$. The following theorem is of course well-known.

Theorem 4.7. *Let* $S = \mathbf{F}_n, n \neq 1$. *If* $n = 0$, *then* $\mathrm{Aut}(S)$ *is isomorphic to the wreath product* $\mathrm{PGL}_{\mathbb{C}}(2) \wr S_2$. *If* $n > 1$, *then* $\mathrm{Aut}(S) \cong \mathrm{Aut}(\mathbb{P}(1,1,n))$. *If* t_0, t_1, t_2 *are homogeneous coordinates of degree* $1, 1, n$, *then* $g \in \mathrm{Aut}(S)$ *acts by the formulas*

$$(t_0, t_1, t_2) \mapsto (at_0 + bt_1, ct_0 + dt_1, et_2 + f_n(t_0, t_1)),$$

where f_n *is a homogeneous polynomial of degree* n. *The automorphisms of the form* $(t_0, t_1, t_2) \mapsto (at_0, at_1, a^n t_2)$ *act identically.*

Proof. The assertion is obvious for $n = 0$. If $n > 1$ we use that S has the unique exceptional section with self-intersection $-n$. The automorphism group is isomorphic to the automorphism group of the surface $\bar{\mathbf{F}}_n$ obtained by blowing down the section. It is well-known that $\bar{\mathbf{F}}_n \cong \mathbb{P}(1,1,n)$. □

For completeness sake note that $\mathrm{Aut}(\mathbf{F}_1)$ is isomorphic to a subgroup of $\mathrm{Aut}(\mathbb{P}^2)$ leaving one point fixed.

Corollary 4.8. *Let* G *be a finite subgroup of* $\mathrm{Aut}(\mathbf{F}_n)$. *If* $n = 0$, G *contains a subgroup* G^0 *of index* ≤ 2 *isomorphic to a finite subgroup of the product* $\mathrm{PGL}_{\mathbb{C}}(2) \times \mathrm{PGL}_{\mathbb{C}}(2)$. *If* $n > 1$, *then* G *is isomorphic to* $m.H$, *where* H *is a finite subgroup of* $\mathrm{PGL}_{\mathbb{C}}(2)$.

The classification of finite subgroups of $\mathrm{Aut}(\mathbf{F}_0)$ is based on Lemma 4.1. First we note the following special subgroups of $\mathrm{Aut}(\mathbb{P}^1) \times \mathrm{Aut}(\mathbb{P}^1)$.

(1) $G = (G_1, G_1, G_2, G_2) = G_1 \times G_2$ is the *product subgroup.*
(2) $(G, 1, G, 1)_\alpha = \{(g_1, g_2) \in G \times G : \alpha(g_1) = g_2\} \cong G$ is a α-*twisted diagonal subgroup.* If $\alpha = \mathrm{id}_G$, we get a *diagonal subgroup.*

Note that α-twisted diagonal groups are conjugate in $\mathrm{Aut}(\mathbf{F}_0)$ if $\alpha(g) = xgx^{-1}$for some x in the normalizer of G inside $\mathrm{Aut}(\mathbb{P}^1)$. In particular, we may always assume that α is an exterior automorphism of G.

The following proposition is left as an exercise in group theory.

Proposition 4.9. *Let* $c(G)$ *be the number of conjugacy classes of twisted diagonal subgroups isomorphic to* G *inside of* $\mathrm{Aut}(\mathbf{F}_0)$. *Then*

$$c(C_n) = c(D_{2n}) = \phi(n)/2,\ c(T) = c(O) = 1,\ c(I) = 2,$$

where ϕ *is the Euler function.*

We fix an embedding of one of the polyhedral groups C_n, D_{2n}, O, T, I in $\mathrm{PGL}_{\mathbb{C}}(2)$.

Theorem 4.10. *Let* G *be a finite subgroup of* $\mathrm{PGL}_{\mathbb{C}}(2) \times \mathrm{PGL}_{\mathbb{C}}(2)$. *Then* G *is conjugate to a group from one of the special cases from above, or to one of the following groups or its image under the switching of the factors.*

- $(O, T, O, T) \cong (T \times T) : 2 \cong T.O$;
- $(O, D_4, O, D_4) \cong 2^4 : S_3 \cong 2^2 : S_4$;
- $(D_{2m}, C_m, O, T) \cong (m \times T) : 2 \cong m.O$;
- $(D_{4m}, D_{2m}, O, T) \cong (D_{2m} \times T) : 2 \cong D_{2m}.O$;

- $(D_{6m}, D_{2m}, O, D_4) \cong (D_{2m} \times D_4) : S_3 \cong D_{2m}.O$;
- $(C_{2m}, C_m, O, T) \cong (m \times T) : 2$ ($\not\cong (D_{2m}, C_m, O, T)$);
- $(T, D_4, T, D_4)_\alpha \cong 2^4 : 3 \cong 2^2.T$;
- $(C_{3m}, C_m, T, D_4) \cong (m \times 2^2) : 3 \cong m.T$;
- $(D_{2m}, C_m, D_{4n}, D_{2n}) \cong (m \times D_{2n}) : 2 \cong m.D_{4n}$;
- $(D_{4m}, D_{2m}, D_{4n}, D_{2n}) \cong (D_{2m} \times D_{2n}) : 2 \cong D_{2m}.D_{4n}$;
- $(D_{2mk}, C_m, D_{2nk}, C_n)_\alpha \cong (m \times n) : D_{2k}$;
- $(C_{mk}, C_m, C_{nk}, C_n)_\alpha \cong (m, n) \times mnk/(m, n)$.

All other finite subgroups of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ are conjugate to a group $G' : 2$, where the subgroup 2 is generated by the involution which switches the factors and $G' = (G, H, G, H)_\alpha$, where $\alpha^2 = \mathrm{id}_G$. We get the following groups

- $G' \times 2$, where $G' \subset \mathrm{PGL}_{\mathbb{C}}(2)$;
- $(G' \times G') : 2$, where $G' \subset \mathrm{PGL}_{\mathbb{C}}(2)$;
- $((T \times T) : 2) : 2$;
- $(2^4 : S_3) : 2$;
- $((D_4 \times D_4) : S_3) : 2$;
- $(2^4 : 3) : 2$;
- $((D_{2m} \times D_{2m} : 2) : 2$;
- $(m^2 : D_{2k}) : 2$.

Finally let us describe finite subgroups G of $\mathrm{Aut}(\mathbf{F}_e), e \geq 2$. It follows from Theorem 4.7 that G is a finite subgroup $\tilde{G}$ of $\mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^*$. The transformation $-1 \times (-1)^e$ defines the identity transformation of $\mathbf{F}_e$. Thus, if e is odd, we get the list of groups G from Lemma 4.3. If e is even, G is a subgroup of $\mathrm{PSL}(2, \mathbb{C}) \times \mathbb{C}^*$. So, we can search our groups among the groups listed in Theorem 4.10.

This gives the following.

Theorem 4.11. *Let G be a finite subgroup of $\mathrm{Aut}(\mathbf{F}_e)$, $e \geq 2$. Then G is conjugate to one of the following groups.*

(1) *e is even.*
- $P \times \mu_m$, *where* $P \subset \mathrm{PGL}_{\mathbb{C}}(2)$;
- $(\mu_m, 1, \mu_m, 1)_\alpha$, *where* $\alpha \in \mathrm{Aut}(\mu_m)$;
- $(C_{2m}, C_m, O, T) \cong (m \times T) : 2$;
- $(C_{3m}, C_m, T, D_4) \cong (m \times 2^2) : 3$;
- $(C_{mk}, C_m, C_{nk}, C_n)_\alpha \cong (m, n) \times mnk/(m, n)$.

(2) *e is odd*
- $\bar{P} \times \mu_m$, *where $\bar{P}$ is a binary polyhedral group;*
- $(\mu_m \times \bar{T})^{\cdot}2$;
- $(\mu_m \times \bar{D}_4)^{\cdot}3$;
- $\mu_m \times D_{2n}$;
- $(\mu_m \times \bar{D}_{2n})^{\cdot}d$, $\alpha \in \mathrm{Isom}(\bar{D}_{2dn}/\bar{D}_{2n} \to C_d)$;
- $(\langle \epsilon_m^k \rangle \times \langle \epsilon_n^k \rangle)^{\cdot}\langle \epsilon_k \rangle$.

5. Automorphisms of conic bundles

5.1. **Geometry of conic bundles.** Now let us consider a conic bundle $\pi : S \to \mathbb{P}^1$ with $k > 0$ reducible fibres.

Let

$$\Sigma = \{x_1, \ldots, x_k\}$$

be the set of points on the base of the fibration such that the fibres $F_{x_i} = \pi^{-1}(x_i)$ are singular. Recall that each singular fibre is the union of two (-1)-curves $R_i + R_i'$ with $R_i \cdot R_i' = 1$. Let E be a section of the conic bundle fibration π.

The Picard group of S is freely generated by the divisor classes of E, the class F of a fibre, and the classes of k components of singular fibres, no two in the same fibre. The next lemma follows easily from the intersection theory on S.

Lemma 5.1. *Let E and E' be two sections with negative self-intersection $-n$. Let r be the number of components of reducible fibres which intersect both E and E'. Then $k - r$ is even and*

$$E \cdot E' = -n + \frac{k-r}{2}.$$

In particular,

$$2n \leq k - r.$$

Since a conic bundle S with $k > 0$ is isomorphic to a blow-up of a minimal ruled surface, it always contains a section E with negative self-intersection. Suppose (S, G) is a minimal G-surface. The group G cannot fix a section E since otherwise it leaves invariant the disjoint subset of components of singular fibres which intersect E, and hence S is not G-minimal. Thus we can always apply the previous lemma to obtain that $k \geq 2n + r$.

5.2. **Exceptional conic bundles.** We give three different constructions of the same conic bundle, which we will call an *exceptional conic bundle.*

First construction.

Choose a ruling $p : \mathbf{F}_0 \to \mathbb{P}^1$ on $\mathbf{F}_0$ and fix two points on the base, say 0 and ∞. Let F_0 and F_∞ be the corresponding fibres. Take $g + 1$ points $a_1, \ldots, a_{g+1}$ on F_0 and $g + 1$ points $a_{g+2}, \ldots, a_{2g+2}$ on F_∞ such that no two lie in the same fibre of the second ruling $q : \mathbf{F}_0 \to \mathbb{P}^1$. Let $\sigma : S \to \mathbf{F}_0$ be the blow-up of the points $a_1, \ldots, a_{2g+2}$. The composition $\pi = q \circ \sigma : S \to \mathbb{P}^1$ is a conic bundle with $2g + 2$ singular fibres $R_i + R_i'$ over the points $x_i = q(a_i), i = 1, \ldots, 2g + 2$. For $i = 1, \ldots, g + 1$, $R_i = \sigma^{-1}(a_i)$ and R_{n+i} are the proper inverse transform of the fibre $q^{-1}(a_i)$. Similarly, for $i = 1, \ldots, n$, R_i' are the proper inverse transform of the fibre $q^{-1}(a_i)$ and $R'_{g+1+i} = \sigma^{-1}(a_{g+1+i})$.

Let E_0 (resp. E_∞) be the proper inverse transforms of F_0 and F_∞ on S. These are sections of the conic bundle φ. The section E_0 intersects $R_1, \ldots, R_{2g+2}$, and the section E_∞ intersects $R_1', \ldots, R_{2g+2}'$.

Let

$$D_0 = 2E_0 + \sum_{i=1}^{2g+2} R_i, \quad D_\infty = 2E_\infty + \sum_{i=1}^{2g+2} R_i'.$$

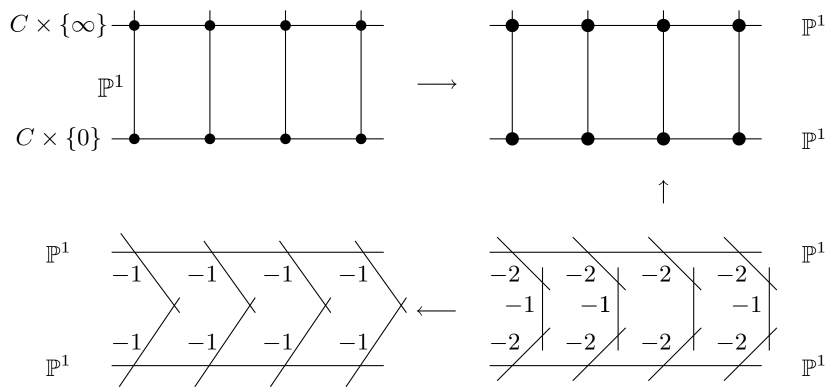
It is easy to check that $D_0 \sim D_\infty$. Consider the pencil $\mathcal{P}$ spanned by the curves D_0 and D_∞. It has $2g + 2$ simple base points $p_i = R_i \cap R_i'$. Its general member is a nonsingular curve C. In fact, a standard formula for computing the Euler characteristic of a fibred surface in terms of the Euler characteristics of fibres shows that all members except D_0 and D_∞ are nonsingular curves. Let F be a fibre of the conic bundle. Since $C \cdot F = 2$, the linear system $|F|$ cuts a g_2^1 on C, so it is a hyperelliptic curve or the genus g of C is 0 or 1. The points p_i are obviously the ramification points of the g_2^1. Computing the genus of C we find that it is equal to g, thus $p_1, \ldots, p_{2g+2}$ is the set of ramification points. Obviously all nonsingular members are isomorphic curves. Let $\sigma : S' \to S$ be the blow-up the base points

$p_1, \ldots, p_{2g+2}$ and let $\bar{D}$ denote the proper inverse transform of a curve on S. We have

$$2\bar{E}_0 + 2\bar{E}_\infty + \sum_{i=1}^{2g+2} (\bar{R}_i + \bar{R}'_i + 2\sigma^{-1}(p_i)) \sim 2\sigma^*(C).$$

This shows that there exists a double over $X' \to S'$ branched along the divisor $\sum_{i=1}^{2g+2}(\bar{R}_i + \bar{R}'_i)$. Since $\bar{R}_i^2 = \bar{R}'^2_i = -2$, their ramification divisor on X' consists of $4g+4$ (-1)-curves. Blowing them down we obtain a surface X isomorphic to the product $C \times \mathbb{P}^1$. This gives us

Second construction. Let C be a curve of genus $g \geq 0$ with an involution $h \in \mathrm{Aut}(C)$ with quotient $\mathbb{P}^1$ (i.e. a hyperelliptic curve if $g \geq 2$). Let δ be an involution of $\mathbb{P}^1$ defined by $(t_0, t_1) \mapsto (t_0, -t_1)$. Consider the involution $\tau = h \times \delta$ of the product $X = C \times \mathbb{P}^1$. Its fixed points are $4g+4$ points $c_i \times \{0\}$ and $c_i \times \{\infty\}$, where $c_1, \ldots, c_{2g+2}$ are fixed points of h. Let X' be a minimal resolution of $X/(\tau)$. It is easy to see that the images of the curves $\{c_i\} \times \mathbb{P}^1$ are (-1)-curves on X'. Blowing them down we obtain our exceptional conic bundle.



Third construction.

Let us consider a quasi-smooth hypersurface Y of degree $2g+2$ in weighted projective space $\mathbb{P} = \mathbb{P}(1, 1, g+1, g+1)$ given by an equation

$$F_{2g+2}(T_0, T_1) + T_2 T_3 = 0, \tag{5.1}$$

where $F_{2g+2}(T_0, T_1)$ is a homogeneous polynomial of degree $2g+2$ without multiple roots. The surface is a double cover of $\mathbb{P}(1, 1, g+1)$ (the cone over a Veronese curve of degree $g+1$) branched over the curve $F_{2g+2}(T_0, T_1) + T_2^2 = 0$. The pre-images of the singular point of $\mathbb{P}(1, 1, g+1)$ with coordinate $(0, 0, 1)$ is a pair of singular points of Y with coordinates $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$. The singularities are locally isomorphic to the singular points of a cone of the Veronese surface of degree $g+1$. Let S be a minimal resolution of Y. The pre-images of the singular points are disjoint smooth rational curves E and E' with self-intersection $-(g+1)$. The projection $\mathbb{P}(1, 1, g+1, g+1) \to \mathbb{P}^1, (t_0, t_1, t_2, t_3) \mapsto (t_0, t_1)$ lifts to a conic bundle on S with sections E, E'. The pencil $\lambda T_2 + \mu T_3 = 0$ cuts out a pencil of curves on Y which lifts to a pencil of bisections of the conic bundle S with $2g+2$ base points $(t_0, t_1, 0, 0)$, where $F_{2g+2}(t_0, t_1) = 0$.

It is easy to see that this is a general example of an exceptional conic bundle. In construction 2, we blow down the sections E_0, E_∞ to singular points. Then consider an automorphism g_0 of the surface which is a descent of the automorphism of the product $C \times \mathbb{P}^1$ given by $\mathrm{id}_C \times \psi$, where $\psi : (t_0, t_1) \mapsto (t_1, t_0)$. The quotient by this automorphism gives $\mathbb{P}(1, 1, g+1)$ and the ramification divisor is the image on S of the curve $C \times (1, 1)$ or $C \times (1, -1)$. On one of these curves g_0 acts identically, on the other one it acts as the involution of the g_2^1.

We denote by $\mathcal{C}_g^{ex}$ the set of isomorphism classes of exceptional conic bundles which contain a section E with $E^2 = -g - 1$.

Proposition 5.2. *Let $\phi : S \to \mathbb{P}^1$ be a minimal conic G-bundle with $k \leq 5$ singular fibres. Then S is a Del Pezzo surface, unless $k = 4$ and S is an exceptional conic bundle.*

Proof. First, if $k = 4$ and S is an exceptional conic bundle, then S is not a Del Pezzo surface since it has sections with self-intersection -2. To show that S is a Del Pezzo surface in the remaining cases we have to show that S does not have smooth rational curves with self-intersection ≤ 2. Let C be the union of smooth rational curves with self-intersection < -2. It is obviously a G-invariant curve, so we can write $C \sim -aK_S - bf$, where f is the divisor class of a fibre of ϕ. Intersecting with f we get $a > 0$. Intersecting with K_S, we get $2b > ad$, where $d = 8 - k \geq 3$. It follows from Lemma 5.1, that S contains a section E with self-intersection -2 or -1. Intersecting C with E we get $0 \leq C \cdot E = a(-K_S \cdot E) - b \leq a - b$. This contradicts the previous inequality. Now let us take C to be the union of (-2)-curves. Similarly, we get $2b = ad$ and $C^2 = -aK_S \cdot C - bC \cdot f = -bC \cdot f = -2ab$. Let r be the number of irreducible components of C. We have $2a = C \cdot f \geq r$ and $-2r \leq C^2 = -2ab \leq -br$. If $b = 2$, we have the equality everywhere, hence C consists of $r = 2a$ disjoint sections, and $8 = rd$. Since $d \geq 3$, the only solution is $d = 4, r = 2$, and this leads to the exceptional conic bundle. Assume $b = 1$. Since $C^2 = -2a$ is even, a is a positive integer, and we get $2 = ad$. Since $d \geq 3$, this is impossible. □

5.3. **Automorphisms of an exceptional conic bundle.** Let use describe the automorphism group of an exceptional conic bundle. The easiest way to do it using Construction 3. We denote by Y_g an exceptional conic bundle given by equation (5.1). Since we are interested only in minimal groups we assume that $g \geq 1$.

Since $K_{Y_g} = \mathcal{O}_{\mathbb{P}}(-2)$, any automorphism σ of Y_g is a restriction of an automorphism of $\mathbb{P}$. An automorphism σ of Y_n acts on the variables (t_0, t_1) via a linear transformation A_σ and on the variables (t_2, t_3) via a linear transformation B_σ preserving T_2T_3 up to a scalar factor.

Let $\mathcal{A}$ be the subgroup of $\mathrm{GL}(2, \mathbb{C})$ which preserves the set of zeroes $V(F_{2g+2})$ of the binary form $F_{2g+2}(T_0, T_1)$. There is a natural character $\chi_{\mathcal{A}} : \mathcal{A} \to \mathbb{C}^*$ defined by $\sigma^*(F_{2g+2}) = \chi(\sigma)F_{2g+2}$. The kernel $\mathcal{A}'$ of $\chi_{\mathcal{A}}$ contains the subgroup of diagonal matrices cI_2, where $c \in \mu_{2g+2}$.

Similarly, we define the subgroup $\mathcal{B}$ which preserves the set $\{0, \infty\}$ of zeroes of T_2T_3. Obviously $\mathcal{B} = D : 2$, where $D \cong \mathbb{C}^{*2}$ is the subgroups of diagonal matrices and $2 = \langle (t_3, t_2) \rangle$. Let $\chi_{\mathcal{B}} : \mathcal{B} \to \mathbb{C}^*$ be the corresponding character. Its restriction to D is the determinant character and its restriction to 2 is trivial. The kernel $\mathcal{B}'$ of $\chi_{\mathcal{B}}$ is $(D \cap SL(2, \mathbb{C})) : 2 \cong \mathbb{C}^* : 2$, where the group 2 acts by $c \mapsto c^{-1}$.

We have natural homomorphism

$$\varphi : \mathrm{Aut}(Y_g) \to \mathcal{A} \times \mathcal{B}, \quad \sigma \mapsto (A_\sigma, B_\sigma).$$

Its image is the subgroup of pairs (A_σ, B_σ) such that $\chi_{\mathcal{A}}(A_\sigma) = \chi_{\mathcal{B}}(B_\sigma)$. Its kernel consists of pairs $(cI_2, c^{g+1}I_2)$, where $c \in \mathbb{C}^*$. By Goursat's Lemma, we obtain an isomorphism

$$\mathrm{Aut}(Y_g)/\mathrm{Ker}(\varphi) \cong (\mathcal{A}, \mathcal{A}', \mathcal{B}, \mathcal{B}')_\phi$$

where we identify $\mathcal{A}/\mathcal{A}'$ with $\mathbb{C}^*/\mu_{2g+2}$ by means of $\chi_{\mathcal{A}}$, we also identify $\mathcal{B}/\mathcal{B}'$ with $\mathbb{C}^*/\mu_2$ by means of $\chi_{\mathcal{B}}$ and take ϕ to be the map induced by $z \mapsto z^{g+1}$.

Now let us consider the images P, P' (resp. Q, Q') of $\mathcal{A}, \mathcal{A}'$ (resp. $\mathcal{B}_0, \mathcal{B}'$) in $\mathrm{PGL}_{\mathbb{C}}(2)$. This defines a homomorphism

$$f' : \mathrm{Aut}(Y_g) \to P \times Q.$$

Its kernel is generated by the coset of the pair $(I_2, -I_2)$ modulo $\mathrm{Ker}(f)$ equal to the coset of $(-I_2, I_2)$ if g is even. Since the images of $\mathcal{A}$ and $\mathcal{A}'$ (resp. $\mathcal{B}$ and $\mathcal{B}'$) in P (resp. Q) coincide we obtain an isomorphism

$$\mathrm{Aut}(Y_g)/(\langle I_2, -I_2\rangle) \cong (P, P, Q, Q) \cong P \times Q = P \times (\mathbb{C}^* : 2).$$

Now let G be a finite minimal subgroup of $\mathrm{Aut}(Y_g)$ and $G' = G/G \cap (\langle I_2, -I_2\rangle)$. Let Q' be the projection of G' to Q and P' be the projection of G' to P. Since G is minimal, its image G_Q in Q contains a transformation which switches the points 0 and ∞. Thus $G_Q \cong D_{2k}$ (where $D_2 = 2$). There is no restriction on the image G_P of G in P. Applying Goursat's Lemma, we get

$$G' = (G_P, G'_P, G_Q, G'_Q)_\phi \cong G'_P.G_Q \cong G'_Q.G_P.$$

If $G = G'$, i.e. $(I_2, -I'_2) \notin G$, then both extensions split and $G \cong G_Q \times G_P$. This happens only if $G_Q = D_{2k}$ with k odd since only in this case D_{2k} admits a lift in $\mathcal{B}' \subset \mathrm{SL}(2, \mathbb{C})$. If $(I_2, -I_2) = (-I_2, (-1)^{g+1}I_2) \in G$, and g is odd, we have $(-I_2, I_2)$ is the identity automorphism, so the second extension splits. If g is even, neither extension splits. However, passing to the binary extensions of groups G_P, G'_P, G_Q, G'_Q, we always have

$$G \cong 2.(G'_Q.G_P) \cong \overline{G'_Q}.G_P \cong G'_Q : \overline{G_P}. \tag{5.2}$$

Theorem 5.3. *Let $Y_g, g \geq 1$, be an exceptional conic bundle and G be its finite minimal group of automorphisms. Then G is conjugate to one of the following groups*

$$D_{4k+2} \times P, \quad \overline{D}_{2k} : P \ (g \text{ is odd}), \quad D_{2k} : \overline{P} \ (g \ \text{ is even}),$$

where P is a polyhedral group leaving $V(F_{2g+2}(T_0, T_1))$ invariant.

Example 5.4. It is easy to find all abelian groups acting on an exceptional conic bundle Y_n. The group $D_{4k+2} \times P$ is abelian if and only if $k = 0$ and $P \cong m$ or $P \cong 2^2$. This gives abelian groups $2m, 2 \times m$ (m is even), 2^3.

If the group $\bar{D}_{2k} : P$ is abelian then $k = 1$, $P = m, 2^2$. If $P = m$, we get the groups $G = 2^2 \times m$ or 2^4.

If the group $D_{2k} : \bar{P}$ is abelian, we get the groups $2 \times 2m, 2^2 \times 2m$.

5.4. **Minimal conic bundles G-surfaces.** Now assume that a finite group G acts minimally on a conic bundle $\phi : S \to \mathbb{P}^1$. As we had noticed already S always contains two sections E, E' with $E^2 = E'^2 = -n < 0$. So we can apply Lemma 5.1 to get $2n \leq k - r$. If $k = 2$, then $n = 1$ and $r = 0$. Thus the G-orbit of a section E with self-intersection -1 consists of disjoint (-1)-curves. We can blow them down G-equivariantly. So, the group G is not minimal. This gives

$$k > 2, \quad K_S^2 = 8 - k \leq 5. \tag{5.3}$$

Let (S, G) be a rational G-surface and

$$a : G \to \mathrm{O}(\mathrm{Pic}(X)), \quad g \mapsto g^* \tag{5.4}$$

be the natural representation of G in the orthogonal group of the Picard group. We denote by G_0 the kernel of this representation. Since $k > 2$ and G_0 fixes any component of a singular fibre, it acts identically on the base of the conic bundle fibration. Since G_0 fixes the divisor class of a section, and sections with negative self-intersection do not move in a linear system, we see that G_0 fixes pointwisely any section with negative self-intersection. If we consider a section as a point of degree 1 on the generic fibre, we see that G_0 must be is a cyclic group.

Theorem 5.5. *Assume $G_0 \neq \{1\}$. The surface S is an exceptional conic bundle.*

Proof. Let g_0 be a non-trivial element from G_0. Let E be a section with $E^2 = -n < 0$. Take an element $g \in G$ such that $E' = g(E) \neq E$. Since g_0 has two fixed points on each component we obtain that E and E' do not intersect the same component. By Lemma 5.1, we obtain that $k = 2n$. Now we blow down n components in n fibres intersecting E and n components in the remaining n fibres intersecting E' to get a minimal ruled surface with two disjoint sections with self-intersection 0. It must be isomorphic to $\mathbf{F}_0$. So, we see that S is an exceptional conic bundle (Construction 1) with $n = g + 1$. We use the description of finite automorphism groups of exceptional conic bundles. It remains only to explain why the projection G to the group of linear transformations of the variables t_2, t_3 is a dihedral group but not a cyclic group. This follows from the minimality of G. Since G must contain an element which switches the sections, its image in the automorphism group of the pencil $\mathcal{P}$ switches the points 0 and ∞. Since elements from G_0 fix 0 and ∞ we see that the image of G generates a dihedral group. □

In notation of Proposition 5.3, the group G_0 is the cyclic subgroup of order m in D_{2m}.

From now on in this section, we assume that $G_0 = \{1\}$.

Let S_η be the general fibre of ϕ. By Tsen's theorem it is isomorphic to $\mathbb{P}^1_K$, where $K = \mathbb{C}(t)$ is the field of rational functions of the base. Consider S_η as a scheme over $\mathbb{C}$. Then

$$\mathrm{Aut}_{\mathbb{C}}(S_\eta) \cong \mathrm{Aut}_K(S_\eta) : \mathrm{PGL}_{\mathbb{C}}(2) \cong \mathrm{dJ}(2),$$

where $\mathrm{dJ}(2)$ is a de Jonquiéres subgroup of $\mathrm{Cr}(2)$ and $\mathrm{Aut}_K(S_\eta) \cong \mathrm{PGL}(2, K)$. A finite minimal group G of automorphisms of a conic bundle is isomorphic to a subgroup of $\mathrm{Aut}_{\mathbb{C}}(S_\eta)$. Let $G_K = G \cap \mathrm{Aut}_K(S_\eta)$ and $G_B \cong G/G_K$ be the image of G in $\mathrm{PGL}_{\mathbb{C}}(2)$. We have an extension of groups

$$1 \to G_K \to G \to G_B \to 1 \tag{5.5}$$

Let $\mathcal{R}$ be the subgroup of $\mathrm{Pic}(S)$ spanned by the divisor classes of $R_i - R_i', i = 1, \ldots, k$. It is obviously G-invariant and $\mathcal{R}_{\mathbb{Q}}$ is equal to the orthogonal complement of $\mathrm{Pic}(S)^G_{\mathbb{Q}}$ in $\mathrm{Pic}(S)_{\mathbb{Q}}$. The orthogonal group of the quadratic lattice $\mathcal{R}$ is isomorphic to the wreath product $2 \wr S_k$. The normal subgroup 2^k consists of switching R_i with R_i'. A subgroup isomorphic to S_k permutes the classes $R_i - R_i'$.

Lemma 5.6. *Let G be a minimal group of automorphisms of S. There exists an element $g \in G_K$ of order 2 which switches the components of some singular fibre.*

Proof. Since G is minimal, the G-orbit of any R_i cannot consist of disjoint components of fibres (since in this case we can G-equivariantly blow it down). Thus it contains a pair R_j, R'_j and hence there exists an element $g \in G$ such that $g(R_j) = R'_j$. If g is of odd order $2k+1$, then g^{2k} and g^{2k+1} fix R_j, hence g fixes R_j. This contradiction shows that g is of even order $2m$. Replacing g by an odd power, we may assume that g is of order $m = 2^a$.

Assume $a = 1$. Obviously the singular point $p = R_j \cap R'_j$ of the fibre belongs to the fixed locus S^g of g. Suppose p is an isolated fixed point. Then we can choose local coordinates at p such that g acts by $(z_1, z_2) \mapsto (-z_1, -z_2)$, and hence acts identically on the tangent directions. So it cannot switch the components. Thus S^g contains a curve not contained in fibres which passes through p. This implies that $g \in G_K$.

Suppose $a > 1$. Replacing g by $g' = g^{m/2}$ we get an automorphism of order 2 which fixes the point x_j and the components R_j, R'_j. Suppose $S^{g'}$ contains one of the components, say R_j. Take a general point $y \in R_j$. We have $g'(g(y)) = g(g'(y)) = g(y)$. This shows that g' fixes R'_j pointwisely. Since $S^{g'}$ is smooth this is impossible. Thus g' has 3 fixed points y, y', p on F_j, two on each component. Suppose y is an isolated fixed point lying on R_j. Let $\pi : S \to S'$ be the blowing down of R_j. The element g' descends to an automorphism of order 2 of S' which has an isolated fixed point at $q = \pi(R_j)$. Then it acts identically on the tangent directions at q, hence on R_j. This contradiction shows that $S^{g'}$ contains a curve intersecting F_j at y or at p, and hence $g' \in G_K$. Since g' is an even power, it cannot switch any components of fibres. Hence g' acts identically on $\mathcal{R}$ and hence on all $\mathrm{Pic}(S)$. Thus $g' = g^{m/2} = 1$ (recall that we assume that G_0 is trivial), contradicting the definition of order of g. □

The restriction of the homomorphism $G \to \mathrm{O}(\mathcal{R}) \cong 2^k : S_k$ to G_K defines a surjective homomorphism

$$\rho : G_K \to 2^s, \ \ s \le k.$$

An element from $\mathrm{Ker}(\rho)$ acts identically on $\mathcal{R}$ and hence on $\mathrm{Pic}(S)$. By Lemma 5.6, G_K is not trivial and $s > 0$. A finite subgroup of $\mathrm{PGL}(2, K)$ does not admit a surjective homomorphism to 2^s for $s > 2$. Thus $s = 1$ or 2.

Case 1: $s = 1$.

In this case G_K is of order 2. Let Σ' be the non-empty subset of Σ such that G_K switches the components of fibres over Σ'. Since G_K is a normal subgroup of G, the set Σ' is a G-invariant set. If $\Sigma \ne \Sigma'$, we repeat the proof of Lemma 5.6 starting from some component R_i of some fibre $F_x, x \notin \Sigma'$ and find an element in G_K of even order which switches components of some fibre F_x, where $x \notin \Sigma'$. Since $G_K = 2$, we get a contradiction.

Let $G_K = (h)$. The element h fixes two points on each nonsingular fibre. The closure of these points is a one-dimensional component C of S^h. It is a smooth bi-section of the fibration. Since we know that h switches all components, its trace on the subgroup $\mathcal{R}$ generated by the divisor classes $R_i - R'_i$ is equal to $-k$. Thus its trace on $H^2(S, \mathbb{Q})$ is equal to $2 - k$. Applying the Lefschetz fixed-point-formula, we get $e(S^h) = 4 - k$. If C is the disjoint union of two components, then S^h consists of k isolated fixed points (the singular points of fibres) and C. We get $e(S^h) = 4 + k$. This contradiction shows that C is irreducible and $e(C) = 4 - k$. Since h fixes pointwisely C and switches the components R_i and R'_i, the intersection point $R_i \cap R_i'$ must be on C. Thus the projection $C \to \mathbb{P}^1$ has $\geq k$ ramification points. Hence $4-k = e(C) = 4-(2+2g(C)) \leq 4-k$. This shows that $k = 2g(C)+2$, i.e. the singular points of fibres are the ramification points of the g_2^1.

Case 2: $s = 2$.

Let g_1, g_2 be two elements from G_K which are mapped to generators of the image of G_K in 2^k. Let C_1 and C_2 be one-dimensional components of the sets S^{g_1} and S^{g_2}. As in the previous case we show that C_1 and C_2 are smooth hyperelliptic curves of genera $g(C_1)$ and $g(C_2)$. Let Σ_1 and Σ_2 be the branch points of the corresponding double covers. As in the previous case we show that $\Sigma = \Sigma_1 \cup \Sigma_2$. For any point $x \in \Sigma_1 \cap \Sigma_2$ the transformation $g_3 = g_1g_2$ fixes the components of the fibre F_x. For any point $x \in \Sigma_1 \setminus \Sigma_2$, g_3 switches the components of F_x. Let C_3 be the one-dimensional component of S^{g_3} and Σ_3 be the set of branch points of g_3. We see that $\Sigma_i = \Sigma_j + \Sigma_k$ for distinct i, j, k, where $\Sigma_j + \Sigma_k = (\Sigma_j \cup \Sigma_k) \setminus (\Sigma_j \cap \Sigma_k)$. This implies that there exist three binary forms $p_1(t_0, t_1), p_2(t_0, t_1), p_3(t_0, t_1)$, no two of which have a common root, such that $\Sigma_1 = V(p_2p_3), \Sigma_2 = V(p_1p_3), \Sigma_3 = V(p_1p_2)$, where $V(p_1p_2)$ denotes the solution of the equation $p_1p_2 = 0$. If $d_i = \deg p_i$, we have

$$2g(C_i) + 2 = d_j + d_k.$$

As in Case 1, we show that any element in $G_K \setminus G_0$ is of order 2. Thus $G \cong G_0 : 2^2$.

Remark 5.7. In the case when $G_0 \neq \{1\}$, the group G_K contains G_0 and its image under the homomorphism $\rho : G_K \to 2^k$ is of order 2. In fact, it follows from the description of G in this case that G_K is isomorphic to a dihedral group equal to the kernel of the projection of G to the group P of automorphisms of a hyperelliptic curve C modulo its hyperelliptic involution. The homomorphism ρ is just the natural homomorphism from the diehedral group to the group of order 2.

Let us summarize what we have learnt.

Theorem 5.8. *Let G be a minimal finite group of automorphisms of a conic bundle $f : S \to \mathbb{P}^1$ with a set Σ of singular fibres. Assume $G_0 = \{1\}$. Then $k = \#\Sigma > 2$ and one of the following cases occurs.*

(1) *$G = 2.P$, where the central involution h fixes pointwisely an irreducible smooth bisection C of π and switches the components in all fibres. The curve C is a curve of genus $g = (k-2)/2$. The conic bundle projection defines a g_2^1 on C with ramification points equal to singular points of fibres. The group P is isomorphic to the group of automorphisms of C modulo its central involution.*

(2) *$G \cong 2^2.P$, each nontrivial element g_i of the subgroup 2^2 fixes pointwisely an irreducible smooth bisection C_i. The set Σ is partitioned in 3 subsets $\Sigma_1, \Sigma_2, \Sigma_3$ such that the projection $f : C_i \to \mathbb{P}^1$ ramifies over $\Sigma_j + \Sigma_k, i \neq j \neq k$. The group P is subgroup of* $\mathrm{Aut}(\mathbb{P}^1)$ *leaving the set Σ and its partition into 3 subsets Σ_i invariant.*

Example 5.9. Comparing with the previous description of finite abelian groups of automorphisms of exceptional conic bundles we get new possible abelian groups. Note that the polyhedral group P must be either cyclic or isomorphic to 2^2.

If $2.P$ does not split, we get possible groups $2m$ and 4×2. If the extension splits we get $4m \times 2, 2 \times 2m, 2^3$.

If $2^2.P$ does not split, we get possible groups $2^2 : 2m \cong 4m \times 2, 2^2 : 2^2 \cong 4 \times 2^2$. If the extension splits, we get $2^2 \times m, 2^4$.

5.5. **Automorphisms of hyperelliptic curves.** We consider a hyperelliptic curve of genus g (or an elliptic curve) as a curve C of degree $2g+2$ in $\mathbb{P}(1,1,g+1)$ given by an equation

$$T_2^2 + F_{2g+2}(T_0,T_1) = 0.$$

An automorphism g of a hyperelliptic curve is defined by a transformation

$$(t_0,t_1,t_2) \mapsto (at_1+bt_0, ct_1+dt_0, \alpha t_2),$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2)$ and $F(aT_0+bT_1, cT_0+dT_1) = \alpha^2 F(T_0,T_1)$. So to find the group of automorphisms of C we need to know relative invariants $\Phi(T_0,T_1)$ for binary polyhedral subgroups $\bar{P}$ of $\mathrm{SL}(2,\mathbb{C})$ (see [48]). The set of relative invariants is a finitely generated $\mathbb{C}$-algebra. Its generators are called Grundformen. We will list the Grunformen. Ww will also use the list later for the description of automorphism groups of Del Pezzo surfaces of degree 1.

- *$\bar{P}$ is a cyclic group of order n.*

A generator is given by the matrix

$$g = \begin{pmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n^{-1} \end{pmatrix}.$$

The Grundformen are t_0 and t_1 with characters determined by

$$\chi_1(g) = \epsilon_n, \quad \chi_2(g) = \epsilon_n^{-1}.$$

- *$\bar{P}$ is a binary dihedral group of order $4n$.*

Its generators are given by the matrices

$$g_1 = \begin{pmatrix} \epsilon_{2n} & 0 \\ 0 & \epsilon_{2n}^{-1} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The Grundformen are

$$\Phi_1 = t_0^n + t_1^n, \quad \Phi_2 = t_0^n - t_1^n, \quad \Phi_3 = t_0 t_1. \tag{5.6}$$

The generators g_1 and g_2 act on the Grundformen with characters

$$\chi_1(g_1) = \chi_2(g_1) = -1, \chi_1(g_2) = \chi_2(g_2) = i^n,$$
$$\chi_3(g_1) = 1, \chi_3(g_2) = -1,$$

- *$\bar{P}$ is a binary tetrahedral group of order 24.*

Its generators are given by the matrices

$$g_1 = \begin{pmatrix} \epsilon_4 & 0 \\ 0 & \epsilon_4^{-1} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad g_3 = \frac{1}{i-1}\begin{pmatrix} i & i \\ 1 & -1 \end{pmatrix}.$$

The Grundformen are

$$\Phi_1 = t_0 t_1 (t_0^4 - t_1^4), \Phi_2, \Phi_3 = t_0^4 \pm 2\sqrt{-3} t_0^2 t_1^2 + t_1^4.$$

The generators g_1, g_2, g_3 act on the Grundformen with characters

$$\chi_1(g_1) = \chi_1(g_2) = \chi_1(g_3) = 1,$$
$$\chi_2(g_1) = \chi_2(g_2) = \chi_3(g_1) = \chi_3(g_2) = 1,$$
$$\chi_2(g_3) = \epsilon_3, \quad \chi_3(g_3) = \epsilon_3^2.$$

- *$\bar{P}$ is a binary octahedral group of order 48.*

Its generators are

$$g_1 = \begin{pmatrix} \epsilon_8 & 0 \\ 0 & \epsilon_8^{-1} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad g_3 = \frac{1}{i-1}\begin{pmatrix} i & i \\ 1 & -1 \end{pmatrix}.$$

The Grundformen are

$$\Phi_1 = t_0t_1(t_0^4 - t_1^4),\ \Phi_2 = t_0^8 + 14t_0^4t_1^4 + t_1^8,\ \Phi_3 = (t_0^4 + t_1^4)((t_0^4 + t_1^4)^2 - 36t_0^4t_1^4).$$

The generators g_1, g_2, g_3 act on the Grundformen with characters

$$\chi_1(g_1) = -1, \chi_1(g_2) = \chi_1(g_3) = 1,$$
$$\chi_2(g_1) = \chi_2(g_2) = \chi_2(g_3) = 1,$$
$$\chi_3(g_1) = -1, \quad \chi_3(g_2) = \chi(g_3) = 1.$$

- *$\bar{P}$ is a binary icosahedral group of order 120.*

Its generators are

$$g_1 = \begin{pmatrix} \epsilon_{10} & 0 \\ 0 & \epsilon_{10}^{-1} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad g_3 = \frac{1}{\sqrt{5}}\begin{pmatrix} \epsilon_5 - \epsilon_5^4 & \epsilon_5^2 - \epsilon_5^3 \\ \epsilon_5^2 - \epsilon_5^3 & -\epsilon_5 + \epsilon_5^4 \end{pmatrix}.$$

The Gründformen are

$$\Phi_1 = t_0^{30} + t_1^{30} + 522(t_0^{25}t_1^5 - t_0^5t_1^{25}) - 10005(t_0^{20}t_1^{10} + t_0^{10}t_1^{20}),$$
$$\Phi_2 = -(t_0^{20} + t_1^{20}) + 228(t_0^{15}t_1^5 - t_0^5t_1^{15}) - 494t_0^{10}t_1^{10},$$
$$\Phi_3 = t_0t_1(t_0^{10} + 11t_0^5t_1^5 - t_1^{10}).$$

Since $P/(\pm 1) \cong A_5$ is a simple group and all Grundformen are of even degree, we easily see that the characters are trivial.

5.6. **Commuting de Jonquiéres involutions.** Recall that a de Jonquiéres involution IH_{g+2} is regularized by an automorphism of the surface S which is obtained from $\mathbf{F}_1$ by blowing up $2g+2$ points. Their images on $\mathbb{P}^2$ are the $2g+2$ fixed points of the involution of H_{g+2}. Let $\pi : S \to X = S/IH_{g+2}$. Since the fixed locus of the involution is a smooth hyperelliptic curve of genus g, the quotient surface X is a nonsingular surface. Since the components of singular fibres of the conic bundle on S are switched by IH_{g+2}, their images on X are isomorphic to $\mathbb{P}^1$. Thus X is a minimal ruled surface $\mathbf{F}_e$. What is e?

Let $\bar{C} = \pi(C)$ and $\bar{E} = \pi(E)$, where E is the exceptional section on S. The curve $\bar{E}$ is a section on X which is split under the cover π. It must be everywhere tangent to $\bar{C}$ at g points (since $IH_{g+2}(E) \cdot E = g$) or be disjoint from $\bar{C}$ if $g = 0$. Let s be the divisor class of a section on $\mathbf{F}_e$ with self-intersection $-e$ and f be the class of a fibre. It is easy to see that

$$\bar{C} = (g+1+e)f + 2s, \quad \bar{E} = \frac{g+e-1}{2}f + s.$$

Let $\bar{R}$ be a section from the class s. Suppose $\bar{R} = \bar{E}$, then $\bar{R} \cdot \bar{C} = g + 1 - e = 2g$ implies $g = 1 - e$, so $(g, e) = (1, 0)$ or $(0, 1)$. In the first case, we get an elliptic curve on $\mathbf{F}_0$ with divisor class $2f + 2s$ and S is non-exceptional conic bundle with $k = 4$. In the second case S is the conic bundle (non-minimal) with $k = 2$.

Assume that $(g, e) \neq (1, 0)$. Let $R = \pi^{-1}(\bar{R})$ be the pre-image of $\bar{R}$. We have $R^2 = -2e$. If it splits into two sections $R_1 + R_2$, then $R_1 \cdot R_2 = \bar{C} \cdot \bar{R} = g + 1 - e$, hence $-2e = 2(g + 1 - e) + 2R_1^2$ gives $R_1^2 = -g - 1$. Applying Lemma (5.1), we get $R_1 \cdot R_2 = g+1-e = g-1+(2g+2-a)/2 = -a/2$, where $a \geq 0$. This gives $e = g+1$, but intersecting $\bar{E}$ with $\bar{R}$ we get $e \leq g-1$. This contradiction shows that $\bar{R}$ does not split, and hence R is an irreducible bisection of the conic bundle with $R^2 = -2e$. We have $R \cdot E = (g-e-1)/2, R \cdot R_i = R \cdot R_i' = 1$, where $R_i + R_i'$ are reducible fibres of the conic fibration.

This shows that the image of R in the plane is a hyperelliptic curve $H'_{g'+2}$ of degree $d = (g-e+3)/2$ and genus $g' = d-2 = (g-e-1)/2$ with the point q of multiplicity g'. It also passes through the points $p_1, \ldots, p_{2g+2}$. Its Weierstrass points $p_1', \ldots, p'_{2g'+2}$ lie on H_{g+2}. Here we use the notation from 2.3. The curve $H'_{g'+2}$ is also invariant with respect to the de Jonquiéres involution.

Write the equation of $H'_{g'+2}$ in the form

$$A_{g'}(T_0,T_1)T_2^2 + 2A_{g'+1}(T_0,T_1)T_2 + A_{g'+2}(T_0,T_1) = 0. \tag{5.7}$$

It follows from the geometric definition of the de Jonquiéres involution that we have the following relation between the equation of $H'_{g'+2}$ and H_{g+2} (cf. [17], p.126)

$$F_g A_{g'+2} - 2F_{g+1}A_{g'+1} + F_{g+2}A_{g'} = 0. \tag{5.8}$$

Consider this as a system of linear equations with coefficients of $A_{g'+2}, A_{g'+1}, A_{g'}$ considered as unknowns. The number of unknowns is $(3g-3e+9)/2$. The number of equations is $(3g-e+5)/2$. So, for a general H_{g+2} we can solve these equations only if $g = 2k+1, e = 0, d = k+2$ or $g = 2k, e = 1, d = k+1$. Moreover, in the first case we get a pencil of curves R satisfying these properties, and in the second case we have a unique such curve (as expected). The first case also covers our exceptional case $(g,e) = (1,0)$.

For example, if we take $g = 2$ we obtain that the six Weierstrass points $p_1, \ldots, p_6$ of H_{g+2} must be on a conic. Or, if $g = 3$, the eight Weierstrass points together with the point q must be the base points of a pencil of cubics. All these properties are of course not expected for a general set of 6 or 8 points in the plane.

To sum up, we have proved the following.

Theorem 5.10. *Let H_{g+2} be a hyperelliptic curve of degree $g+2$ and genus g defining a de Jonquiéres involution IH_{g+2}. View this involution as an automorphism τ of order 2 of the surface S obtained by blowing up the singular point q of H_{g+2} and its $2g+2$ Weierstrass points $p_1, \ldots, p_{2g+2}$. Then*

- (i) *the quotient surface $X = S/(\tau)$ is isomorphic to $\mathbf{F}_e$ and the ramification curve is $C = S^\tau$;*
- (ii) *if H_{g+2} is a general hyperelliptic curve then $e = 0$ if g is odd and $e = 1$ if g is even;*
- (iii) *the branch curve $\bar{C}$ of the double cover $S \to \mathbf{F}_e$ is a curve from the divisor class $(g+1+e)f + 2s$;*
- (iv) *there exists a section from the divisor class $\frac{g+e-1}{2}f + s$ which is tangent to $\bar{C}$ at each g intersection points unless $g = 0, e = 1$ in which case it is disjoint from $\bar{C}$;*
- (v) *the reducible fibres of the conic bundle on S are the pre-images of the $2g+2$ fibres from the pencil $|f|$ which are tangent to $\bar{C}$;*
- (vi) *the pre-image of a section from the divisor class s either splits if $(g,e) = (1,0)$ or a curve of genus $g = 0$ or a hyperelliptic curve C' of genus $g' = (g-e-1)/2 \geq 1$ which is invariant with respect to τ. It intersects the hyperelliptic curve C at its $2g'+2$ Weierstrass points.*

(vii) *the curve C' is uniquely defined if $e > 0$ and varies in a pencil if $e = 0$.*

Let $IH'_{g'+2}$ be the de Jonquiéres involution defined by the curve $H'_{g'+2}$ from equation (5.7). Then it can be given in affine coordinates by formulas (2.7), where F_i is replaced with A_i. Thus we have two involutions defined by the formula

$$
\begin{aligned}
IH_{g+2}:\ (x',y') &= \big(x, \frac{-yP_{g+1}(x)-P_{g+2}(x)}{P_g(x)y+P_{g+1}(x)}\big),\\
IH'_{g'+2}:(x',y') &= \big(x, \frac{-yQ_{g'+1}(x)-Q_{g'+2}(x)}{Q_g(x)y+Q_{g'+1}(x)}\big),
\end{aligned}
\tag{5.9}
$$

where P_i are the dehomogenizations of the F_i's and Q_i are the dehomogenizations of the A_i's. Composing them in both ways we see that the relation (5.8) is satisfied if and only if the two involutions commute. Thus a de Jonquiéres involution can be always included in a group of de Jonquiéres transformations isomorphic to 2^2. In fact, for a general IH_{g+2} there exists a unique such group if g is even and there is a ∞^1 such groups when g is odd. The composing formula shows that the involution $IH_{g+2}\circ H'_{g'+2}$ is the de Jonquiéres involution defined by the third hyperelliptic curve with equation

$$
\det\begin{pmatrix} F_g & F_{g+1} & F_{g+2}\\ A_{g'} & A_{g'+1} & A_{g''+2}\\ 1 & -T_2 & T_2^2\end{pmatrix} = B_{g''}T_2^2+2B_{g''-1}T_2+B_{g''+2}=0,
\tag{5.10}
$$

(cf. [17], p.126).

If we blow up the Weierstrass point of the curve C' (the proper inverse transform of $H'_{g'+2}$ in S), then we get a conic bundle surface S' from case (2) of Theorem 5.8.

5.7. **A question on extensions.** The only question about minimal groups of automorphisms of conic bundles which remains unsolved is to decide which extensions

$$
1\to G_K\to G\to G_B\to 1
\tag{5.11}
$$

occur. In the case of exceptional conic bundles this extension corresponds to the extension $1\to G'_Q\to G\to G_P\to 1$ from (5.2). The question of splitting of this extension has been already settled in this case.

Example 5.11. Consider a de Jonquiéres transformation

$$\mathrm{dj}_P:(x,y)\mapsto(x,P(x)/y),$$

where $P(T_1/T_0)=T_0^{-2g}F_{2g+2}(T_0,T_1)$ for some homogeneous polynomial $F_{2g+2}(T_0,T_1)$ of degree $2g+2$ defining a hyperelliptic curve of genus g. Choose F_{2g+2} to be a relative invariant of a binary polyhedral group $\bar{P}=2.P$ with character $\chi:\bar{P}\to\mathbb{C}^*$. We assume that $\chi=\alpha^2$ for some character $\alpha:\bar{P}\to\mathbb{C}^*$. For any $g=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\bar{P}$ define the transformation

$$g:(x,y)\mapsto(\frac{ax+b}{cx+d},\alpha(g)(cx+d)^{-g-1}y).$$

We have

$$P(\frac{ax+b}{cx+d})=\alpha^2(g)(cx+d)^{-2g-2}P(x).$$

It is immediate to check that g and dj_p commute. The matrix $-I_2$ defines the transformation $g_0:(x,y)\mapsto(x,\alpha(-I_2)(-1)^{g+1}y)$. So, if

$$\alpha(-I_2)=(-1)^{g+1},$$

the action of $\bar{P}$ factors through P and together with dj_P generate the group $2 \times P$. On the other hand, if $\alpha(-I_2) = (-1)^{g+2}$, we get the group $G = 2 \times \bar{P}$. In this case the group G is regularized on an exceptional conic bundle with $G_0 \cong 2$. The generator corresponds to the transformation g_0.

Our first general observation is that the extension $G = 2.P$ always splits if g is even, and of course, if P is a cyclic group of odd order. In fact, suppose G does not split. We can always find an element $g \in G$ which is mapped to an element $\bar{g}$ in P of even order $2d$ such that $g^{2d} = g_0 \in G_K$. Now $g_1 = g^d$ defines an automorphism of order 2 of the hyperelliptic curve $C = S^{g_0}$ with fixed points lying over two fixed points of $\bar{g}$ in $\mathbb{P}^1$. None of these points belong to Σ, since otherwise g_0 being a square of g_1 cannot switch the components of the corresponding fibre. Since g_1 has two fixed points on the invariant fibre and both of them must lie on C, we see that g_1 has 4 fixed points. However this contradicts the Hurwitz formula.

Recall that a double cover $f : X \to Y$ of nonsingular varieties with branch divisor $W \subset Y$ is given by an invertible sheaf $\mathcal{L}$ together with a section $s_W \in \Gamma(Y, \mathcal{L}^2)$ whose zero divisor is W. Suppose a group G acts on Y leaving invariant W. A *lift* of G is a group $\tilde{G}$ of automorphisms of X such that it commutes with the covering involution τ of X and the corresponding homomorphism $\tilde{G} \to \mathrm{Aut}(Y)$ is an isomorphism onto the group G.

The following lemma is well-known and is left to the reader.

Lemma 5.12. *A subgroup $G \subset \mathrm{Aut}(Y)$ admits a lift if and only if $\mathcal{L}$ admits a G-linearization and in the corresponding representation of G in $\Gamma(Y, \mathcal{L}^2)$ the section s_W is G-invariant.*

Example 5.13. Let $p_i(t_0, t_1), i = 0, 1, 2$, be binary forms of degree d. Consider a curve C in $\mathbf{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ given by an equation

$$F = p_0(t_0, t_1)x_0^2 + 2p_1(t_0, t_1)x_0x_1 + p_2(t_0, t_1)x_1^2 = 0.$$

Assume that the binary form $D = p_1^2 - p_0p_2$ does not have multiple roots. Then C is a nonsingular hyperelliptic curve of genus $d - 1$. Suppose $d = 2a$ is even so that the genus of the curve is odd. Let P be a polyhedral group not isomorphic to a cyclic group of odd order. Let $V = \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ and $\rho : P \to \mathrm{GL}(S^{2a}V \otimes S^2V)$ be its natural representation, the tensor product of the two natural representations of P in the space of binary forms of even degree. Suppose that $F \in S^{2a}V \otimes S^2V$ is an invariant. Consider the double cover $S \to \mathbf{F}_0$ defined by the section F and the invertible sheaf $\mathcal{L} = \mathcal{O}_{\mathbf{F}_0}(a, 1)$. Now assume additionally that P does not have a linear representation in $S^aV \otimes V$ whose tensor square is equal to ρ. Thus $\mathcal{L}$ does not admit a P-linearization and we cannot lift P to a group of automorphisms of the double cover. However, the binary polyhedral group $\bar{P}$ lifts. It s central involution acts identically on $\mathbb{F}_0$, hence lifts to the covering involution of S. It follows from the discussion in the previous subsection that S is a non-exceptional conic bundle, and the group $\bar{P}$ is a minimal group of automorphisms of S with $G_K \cong 2$ and $G_B \cong P$.

Here is a concrete example. Take

$$p_0 = t_0t_1(t_0^2 + t_1^2), \quad p_1 = t_0^4 + t_1^4, \quad p_2 = t_0t_1(t_0^2 - t_1^2).$$

Take $P = C_2$ which acts via $\bar{C}_2 \cong C_4$ on the variables t_0, t_1 by the transformation $[it_0, -it_1]$ and on the variables x_0, x_1 by the transformation $[ix_0, -ix_1]$. Then $\bar{P}$ acts on $S^2V \otimes V$ via $[-1, 1, -1] \otimes [i, -i]$. The matrix $-I_2$ acts as $1 \otimes -1$ and hence

P does not act on $S^2V \otimes V$. This realizes the cyclic group C_4 as a minimal group of automorphims of a conic bundle with $k = 2g + 2 = 8$.

The previous example shows that for any $g \equiv 1 \mod 4$ one can realize a binary polyhedral group $\bar{P} = 2.P$ as a minimal group of automorphisms of a conic bundle with $2g + 2$ singular fibres. We do not know whether the same is trie for $g \equiv 3 \mod 4$.

Example 5.14. Let $p_i(t_0, t_1), i = 1, 2, 3$, be three binary forms of even degree d with no multiple roots. Assume no two have common zeroes. Consider a surface S in $\mathbb{P}^1 \times \mathbb{P}^2$ given by a bihomogeneous form of degree $(d, 2)$

$$p_1(t_0, t_1)z_0^2 + p_2(t_0, t_1)z_1^2 + p_3(t_0, t_1)z_2^2 = 0, \tag{5.12}$$

The surface is nonsingular. The projection to $\mathbb{P}^1$ defines a conic bundle structure on S with singular fibres over the zeroes of the polynomials p_i. The curves C_i equal to the pre-images of the lines $z_i = 0$ under the second projection are hyperelliptic curves of genus $g = d-1$. The automorphisms σ_1, σ_2 defined by the negation of one of the first two coordinates z_0, z_1, z_2 form a subgroup of $\mathrm{Aut}(S)$ isomorphic to 2^2. Let P be a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$ and $g \mapsto g^*$ be its natural action on the space of binary forms. Assume that p_1, p_2, p_3 are relative invariants of P with characters χ_1, χ_2, χ_3 such that we can write them in the form η_i^2 for some characters η_1, η_2, η_3 of P. Then P acts on S by the formula

$$g((t_0, t_1), (z_0, z_1, z_2)) = ((g^*(t_0), g^*(t_1)), (\eta_1(g)^{-1}z_0, \eta_2(g)^{-1}z_1, \eta_3(g)^{-1}z_2)).$$

For example, let $P = (g)$ be a cyclic group of order 4. We take $p_1 = t_0^2 + t_1^2, p_2 = t_0^2 - t_1^2, p_3 = t_0t_1$. It acts on S by the formula

$$g : ((t_0, t_1), (z_0, z_1, z_2)) \mapsto ((it_1, it_0), (iz_0, z_1, iz_2)).$$

Thus g^2 acts identically on t_0, t_1, z_1 and multiplies z_0, z_2 by -1. We see that $G_K = \langle g^2 \rangle$ and the extension $1 \to G_K \to G \to G_B \to 1$ does not split. If we add to the group the transformation $(t_0, t_1, z_0, z_1, z_2) \mapsto (t_0, t_1, z_0, -z_1, z_2)$ we get a non-split extension $2^2.2$

On the other hand, let us replace p_2 with $t_0^2 + t_1^2 + t_0t_1$,. Define g_1 as acting only on t_0, t_1 by $[it_1, it_0]$, g_2 acts only on z_0 by $z_0 \mapsto -z_0$ and g_3 acts only on z_1 by $z_1 \mapsto -z_1$. We get the groups $\langle g_1, g_2 \rangle = 2^2$ and $\langle g_1, g_2, g_3 \rangle = 2^3$.

In another example we take P to be the dihedral group D_8. We take $p_1 = t_0^2 + t_1^2, p_2 = t_0^2 - t_1^2, p_3 = t_0t_1$. It acts on S by the formula

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} : ((t_0, t_1), (z_0, z_1, z_2)) \mapsto ((it_0, -it_1), (iz_0, iz_1, z_2)),$$

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} : ((t_0, t_1), (z_0, z_1, z_2)) \mapsto ((it_1, it_0), (z_0, iz_1, z_2)),$$

The scalar matrix $c = -I_2$ belongs to $G_K \cong 2^2$ and the quotient $P/(c) \cong 2^2$ acts faithfully on the base. This gives a non-split extension $(2^2) \cdot (2^2)$.

Finally, let us take

$$p_1 = t_0^4 + t_1^4, p_2 = t_0^4 + t_1^4 + t_0^2t_1^2, p_3 = t_0^4 + t_1^4 - t_0^2t_1^2.$$

These are invariants for the group D_4 acting via $g_1 : (t_0, t_1) \mapsto (t_0, -t_1)$, $g_2 : (t_0, t_1) \mapsto (t_1, t_0)$. Together with transformations σ_1, σ_2 this generates the group 2^4 (see another realization of this group in [7]).

6. Automorphisms of Del Pezzo surfaces

6.1. **The Weyl group.** Let S be a Del Pezzo surface. We have already considered the case when $S \cong \mathbb{P}^2$ or $S \cong \mathbb{P}^1 \times \mathbb{P}^1$. Other Del Pezzo surfaces are isomorphic to the blow-up of $N = 9 - d \leq 8$ points in $\mathbb{P}^2$ satisfying the conditions of generality from sectin 3.4. The blow-up of one or 2 points is obviously non-minimal (since the exceptional curve in the first case and the proper inverse transform of the line through the two points is G-invariant). So we may assume that S is a Del Pezzo surface of degree $d \leq 6$.

Let $\pi : S \to \mathbb{P}^2$ be the blowing-up map. Consider the decomposition (3.2) of π into a composition of blow-ups of $N = 9-d$ points. Because of generality condition, we may assume that none of the points $p_1, \ldots, p_N$ is infinitely near, or, equivalently, all exceptional configurations $\mathcal{E}_i$ are irreducible curves. We identify them with curves $E_i = \pi^{-1}(p_i)$. The divisor classes $e_0 = [\pi^*(\text{line}], e_i = [E_i], i = 1, \ldots, N$, form a basis of $\text{Pic}(S)$. It is called a *geometric basis.*

Let

$$\alpha_1 = e_0 - e_1 - e_2 - e_3, \alpha_2 = e_1 - e_2, \ldots, \alpha_N = e_{N-1} - e_N.$$

For any $i = 1, \ldots, N$ define a *reflection* automorphism s_i of the abelian group $\text{Pic}(S)$

$$s_i : x \mapsto x + (x \cdot \alpha_i)\alpha_i.$$

Obviously, $s_i^2 = 1$ and s_i acts identically on the orthogonal complement of α_i. Let W_S be the group of automorphisms of $\text{Pic}(S)$ generated by the transformations $s_1, \ldots, s_N$. It is called the *Weyl group of* S. Using the basis $(e_0, \ldots, e_N)$ we identify W_S with a group of isometries of the odd unimodular quadratic form $q : \mathbb{Z}^{N+1} \to \mathbb{Z}$ of signature $(1, N)$ defined by

$$q_N(m_0, \ldots, m_N) = m_0^2 - m_1^2 - \ldots - m_N^2.$$

Since $K_S = -3e_0 + e_1 + \ldots + e_N$ is orthogonal to all α_i's, the image of W_S in $\text{O}(q_N)$ fixes the vector $k_N = (-3, 1, \ldots, 1)$. The subgroup of $\text{O}(q_N)$ fixing k_N is denoted by W_N and is called the *Weyl group of type* E_N.

We denote by $\mathcal{R}_S$ the sublattice of $\text{Pic}(S)$ equal to the orthogonal complement of the vector K_S. The vectors $\alpha_1, \ldots, \alpha_N$ forma a $\mathbb{Z}$-basis of $\mathcal{R}_S$. By restriction the Weyl group W_S is isomorphic to a subgroup of $\text{O}(\mathcal{R}_S)$. A choice of a basis $\alpha_1, \ldots, \alpha_N$ defines an isomorphism from $\mathcal{R}_S$ to the root lattice Q of a finite root system of type $E_N (N = 6, 7, 8), D_5 (N = 5), A_4 (N = 4)$ and $A_2 + A_1 (N = 3)$. The group W_S becomes isomorphic to the corresponding Weyl group $W(E_N)$.

The next lemma is well-known and its proof goes back to Kantor [41] and Du Val [26]. We refer for modern proofs to [1] or [23].

Lemma 6.1. *Let $(e_0', e_1', \ldots, e_N')$ be another geometric basis in* $\text{Pic}(S)$ *defined by a birational morphism $\pi' : S \to \mathbb{P}^2$ and a choice of a decomposition of π' into a composition of blow-ups of a point. Then the transition matrix is an element of W_N. Conversely, any element of W_N is a transition matrix of two geometric bases in* $\text{Pic}(S)$.

The next lemma is also well-known but we had a problem to find a reference.

Lemma 6.2. *Let $\pi : S \to \mathbb{P}^2$ be a rational surface obtained by blowing up N points $x_1, \ldots, x_N$ in $(\mathbb{P}^2)^{\text{bb}}$. Assume that one can find 4 points x_i, x_j, x_k, x_m of height 0* *such that no three among them are collinear. The natural homomorphism*

$$\rho : \mathrm{Aut}(S) \to W_S, \quad g \mapsto g^*$$

is injective. In particular, ρ is injective if S is a Del Pezzo surface of degree $d \le 5$.

Proof. Let $\mathcal{E}_1, \ldots, \mathcal{E}_N$ be the exceptional configurations defined by the points $x_1, \ldots, x_N$. We may assume that the last one $\mathcal{E}_N$ is an irreducible curve. Since $g(E) = E$ for any $g \in \mathrm{Ker}(\rho)$, we can blow down $\mathcal{E}_N$ g-equivariantly to get a surface S' with exceptional configurations $\mathcal{E}'_1, \ldots, \mathcal{E}'_{N-1}$. It is easy to see that $\mathrm{Ker}(\rho)$ is contained $\mathrm{Ker}(\rho')$, where $\rho' : \mathrm{Aut}(S') \to W_{S'}$ is the corresponding homomorphism for S'. Proceeding in this way, we see that $\mathrm{Ker}(\rho)$ fixes each point x_i. Without loss of generality we may assume that the points $x_1, \ldots, x_4$ satisfy the assumption of the lemma. Since no linear transformation of $\mathbb{C}^3$ different from the identity has 4 linear independent eigenvectors, we must get $\mathrm{Ker}(\rho) = \{1\}$. □

We will use the classification of conjugacy classes of elements g of finite order in the Weyl groups. According to [13] they are indexed by certain graphs. We call them *Carter graphs*. One writes each element w as the product of two involutions $w_1 w_2$, where each involution is the product of reflections with respect to orthogonal roots. Let $\mathcal{R}_1, \mathcal{R}_2$ be the corresponding sets of such roots. Then the graph has vertices the set $\mathcal{R}_1 \cup \mathcal{R}_2$ and two vertices α, β are joined by an edge if and only if $(\alpha, \beta) \neq 0$. A Carter graph with no cycles is a Dynkin diagram. The subscript in the notation of a Carter graph indicates the number of vertices. It is also equal to the difference between the rank of the root lattice Q and the rank of its fixed sublattice $Q^{(w)}$.

Note that the same conjugacy classes may correspond to different graphs (e.g. D_3 and A_3, or $2A_3 + A_1$ and $D_4(a_1) + 3A_1$).

The Carter graph also determines the characteristic polynomial of w. In particular, it gives the trace of g^* on the cohomology $H^*(S, \mathbb{C})$. The latter should be compared with the Euler-Poincarè characteristic of the fixed locus S^g of g by using the Lefschetz fixed-point formula.

$$2 - 2\mathrm{Tr}_1(g) + \mathrm{Tr}_2(g) = \sum_{j=1}^{t} (2 - 2g(R_j) + s, \tag{6.1}$$

where S^g consists of a disjoint union of smooth curves R_i and s isolated fixed points.

To determine whether a finite subgroup G of $\mathrm{Aut}(S)$ is minimal, we use the well-known formula from the character theory of finite groups

$$\mathrm{rankPic}(S)^G = \frac{1}{\#G} \sum_{g \in G} \mathrm{Tr}_2(g).$$

6.2. **Del Pezzo surfaces of degree** 6**.** Let S be a Del Pezzo surface of degree 6. We fix a geometric basis basis e_0, e_1, e_2, e_3 which is defined by $\pi : S \to \mathbb{P}^2$ with indeterminacy points $p_1 = (1,0,0), p_2 = (0,1,0)$ and $p_3 = (0,0,1)$. The root lattice $\mathcal{R}_S = K_S^{\perp}$ is isomorphic to the root lattice Q of type $A_2 + A_1$. A root basis in $\mathcal{R}_S$ is $\alpha_1 = e_0 - e_1 - e_2 - e_3, \alpha_2 = e_1 - e_2, \alpha_3 = e_2 - e_3$). The Weyl group

$$W_S = (s_{\alpha_1}) \times (s_{\alpha_2}, s_{\alpha_3}) \cong 2 \times S_3.$$

Graph	Order	Characteristic polynomial
A_k	$k+1$	$t^k+t^{k-1}+\ldots+1$
D_k	$2k-2$	$(t^{k-1}+1)(t+1)$
$D_k(a_1)$	l.c.m$(2k-4,4)$	$(t^{k-2}+1)(t^2+1)$
$D_k(a_2)$	l.c.m$(2k-6,6)$	$(t^{k-3}+1)(t^3+1)$
$\vdots$	$\vdots$	$\vdots$
$D_k(a_{\frac{k}{2}-1})$	even k	$(t^{\frac{k}{2}}+1)^2$
E_6	12	$(t^4-t^2+1)(t^2+t+1)$
$E_6(a_1)$	9	t^6+t^3+1
$E_6(a_2)$	6	$(t^2-t+1)^2(t^2+t+1)$
E_7	18	$(t^6-t^3+1)^2(t+1)$
$E_7(a_1)$	14	t^7+1
$E_7(a_2)$	12	$(t^4-t^2+1)(t^3+1)$
$E_7(a_3)$	30	$(t^5+1)(t^2-t+1)$
$E_7(a_4)$	6	$(t^2-t+1)^2(t^3+1)$
E_8	30	$t^8+t^7-t^5-t^4-t^3+t+1$
$E_8(a_1)$	24	t^8-t^4+1
$E_8(a_2)$	20	$t^8-t^6+t^4-t^2+1$
$E_8(a_3)$	12	$(t^4-t^2+1)^2$
$E_8(a_4)$	18	$(t^6-t^3+1)(t^2-t+1)$
$E_8(a_5)$	15	$t^8-t^7+t^5-t^4+t^3-t+1$
$E_8(a_6)$	10	$(t^4-t^3+t^2-t+1)^2$
$E_8(a_7)$	12	$(t^4-t^2+1)(t^2-t+1)^2$
$E_8(a_8)$	6	$(t^2-t+1)^4$

TABLE 1. Carter graphs and characteristic polynomials

The representation $\rho : \mathrm{Aut}(S) \to W_S$ is surjective. The reflection s_{α_1} is realized by the lift of the standard Cremona transformation. The reflection s_{α_2}(resp. s_{α_3}) is realized by the transformations (x_1, x_0, x_2) (resp. (x_0, x_2, x_1)).

The group W_S contains the following minimal subgroups

$$2 \times S_3, \quad S_3 = \langle s_{\alpha_1}s_{\alpha_2}, s_{\alpha_1}s_{\alpha_3}\rangle, \quad 6.$$

The kernel of $\mathrm{Aut}(S) \to W_S$ is the lift of the group of coordinate scaling isomorphic to $(\mathbb{C}^*)^2$. A subgroup K of W_S with $\mathcal{R}^K = \{0\}$ must contain an element of order 6 conjugate to $s_{\alpha_1} \times s_{\alpha_2}s_{\alpha_3}$. This easily gives the following.

Theorem 6.3. *Let (S, G) be a minimal Del Pezzo surface of degree $d = 6$. Then $G = \langle\tau\rangle \times H$, where τ is the lift of the standard Cremona transformation and H is the lift of an imprimitive transitive subgroup of* $\mathrm{Aut}(\mathbb{P}^2)$ *from Theorem 4.5. In particular, G is isomorphic to one of the following groups*

$$n^2 : (2 \times S_3), \quad n^2 : S_3, \quad n^2 : 6, \quad G_{n,k,s} : 6, \quad n \geq 1.$$

Note that the group $2^2 : S_3 \cong S_4$. Its action on a model of S given by an equation

$$x_0y_0z_0 - x_1y_1z_1 = 0$$

in $(\mathbb{P}^1)^3$ is given in [3].

6.3. **Automorphisms of Del Pezzo surfaces of degree** $d = 5$**.** .

In this case S is isomorphic to the blow-up of the reference points $p_1 = (1,0,0),\ p_2 = (0,1,0),\ p_3 = (0,0,1), p_4 = (1,1,1)$. The lattice $\mathcal{R}_S$ is of type A_4 and $W_S \cong S_5$. It is known that the homomorphism $\rho : \mathrm{Aut}(S) \to W_S$ is an isomorphism. We already know that it is injective. To see the surjectivity one can argue, for example, as follows.

Let τ be the standard Cremona involution with fundamental points p_1, p_2, p_3. It follows from its formula that the point p_4 is a fixed point. We know that τ regularizes on the blow-up of the first three points isomorphic to a Del Pezzo surface of degree 6. Since the pre-image of p_4 is a fixed point, τ regularizes on X. Adding an involution to S_4 generates the whole group S_5.

Another way to see the isomorphism $\mathrm{Aut}(S) \cong S_5$ is to use a well-known isomorphism between S and the moduli space $\overline{\mathcal{M}_{0,5}} \cong (\mathbb{P}^1)^5//\mathrm{SL}(2)$. The group S_5 acts by permuting the factors.

Theorem 6.4. *Let* (S, G) *be a minimal Del Pezzo surface of degree* $d = 5$*. Then* $G = S_5, A_5, 5:4, 5:2$*, or* C_5*.*

Proof. $\mathrm{Aut}(S) \cong W_4 \cong S_5$. The group S_5 acts on $\mathcal{R}_S \cong \mathbb{Z}^4$ by means of its standard irreducible 4-dimensional representation (view $\mathbb{Z}^4$ as a subgroup of $\mathbb{Z}^5$ of vectors with coordinates added up to zero and consider the representation of S_5 by switching the coordinates). It is known that a maximal proper subgroup of S_5 is equal (up to a conjugation) to one of three subgroups $S_4, S_3 \times 2, A_5, 5:4$. A maximal subgroup of A_5 is either 5×2 or S_3 or $D_{10} = 5:2$. It is easy to see that the groups S_4 and $S_3 \times 2$ have invariant elements in the lattice Q_4. It is known that an element of order 5 in S_5 is a cyclic permutation, and hence has no invariant vectors. Thus any subgroup G of S_5 containing an element of order 5 defines a minimal surface (S, G). So, if (S, G) is minimal, G must be equal to one of the groups from the assertion of the theorem. □

6.4. **Automorphisms of a Del Pezzo surface of degree** $d = 4$**.** In this case $\mathcal{R}$ is of type D_5 and $W_S \cong 2^4 : S_5$. We use the following well-known classical result.

Lemma 6.5. *Let* S *be a Del Pezzo surface of degree 4. Then* S *is isomorphic to a nonsingular surface of degree 4 in* $\mathbb{P}^4$ *given by equations*

$$F_1 = \sum_{i=0}^{4} t_i^2 = 0, \quad F_2 = \sum_{i=0}^{4} a_i t_i^2 = 0, \tag{6.2}$$

where all a_i*'s are distinct.*

Proof. It is known that a Del Pezzo surface in its anti-canonical embedding is projectively normal. Using Riemann-Roch, one obtains that S is a complete intersection $Q_1 \cap Q_2$ of two quadrics. Let $\mathcal{P} = \lambda Q_1 + \mu Q_2$ be the pencil spanned by these quadrics. The locus of singular quadrics in the pencil is a homogeneous equation of degree 5 in the coordinates λ, μ. Since S is nonsingular, it is not hard to see that the equation has no multiple roots (otherwise $\mathcal{P}$ contains a reducible quadric or there exists a quadric in the pencil with singular point at S, both cases imply that S is singular). Let $p_1, \ldots, p_5$ be the singular points of singular quadrics from the pencil. Suppose they are contained in hyperplane H. Since no quadrics in the pencil contains H, the restriction $\mathcal{P}|H$ of the pencil of quadrics to H contains ≥ 5 singular members. This implies that all the quadrics in $\mathcal{P}|H$ are singular.

By Bertini's theorem, there exists a point $p \in H$ which is a singular point of all quadrics in the pencil. This point is a base point of $\mathcal{P}$ where all quadrics in $\mathcal{P}$ are tangent to the same hyperplane. One of the quadrics must be singular at p, and hence S is singular at p. This contradiction shows that $p_1, \ldots, p_5$ span $\mathbb{P}^4$. Choose coordinates in $\mathbb{P}^4$ such that the singular points of singular quadrics from $\mathcal{P}$ are the points $(1,0,0,0,0), (0,1,0,0,0)$, and so on. Then each hyperplane $V(t_i) = (t_i = 0)$ is a common tangent hyperplane of quadrics from $\mathcal{P}$ at the point p_i. This easily implies that the equations of quadrics are given by (6.2). □

Let $Q_i = V(a_iF_1 - F_2), i = 0, \ldots, 4$, be one of singular quadrics in the pencil of quadrics containing S. It is a cone over a nonsingular quadric in $\mathbb{P}^3$, hence contains two families of planes. The intersection of a plane with any other quadric in the pencil is a conic contained in S. Thus any Q_i defines a pair of pencils of conics $|C_i|$ and $|C_i'|$, and it is easy to see that $|C_i + C_i'| = |-K_S|$.

Proposition 6.6. *Let S be a Del Pezzo surface given by equations* (6.2). *The divisor classes $c_i = [C_i]$ together with K_S form a basis of* $\mathrm{Pic}(S) \otimes \mathbb{Q}$. *The Weyl group W_S acts on this basis by permuting the c_i's and sending c_i to $c_i' = [C_i'] = -K_S - c_i$.*

Proof. If we choose a geometric basis $(e_0, e_1, \ldots, e_5)$ in $\mathrm{Pic}(S)$, then the 5 pairs of pencils of conics are defined by the classes $e_0 - e_i, 2e_0 - e_1 - \ldots - e_5 + e_i$. It is easy to check that the classes $[C_i]$'s and K_S is a basis in $\mathrm{Pic}(S) \otimes \mathbb{Q}$. The group W_S contains a subgroup isomorphic to S_5 generated by the reflections in vectors $e_1 - e_2, \ldots, e_4 - e_5$., It acts by permuting $e_1, \ldots, e_5$, hence permuting the pencils $|C_i|$. It is equal to the semi-direct product of S_5 and the subgroup isomorphic to 2^4 which is generated by the conjugates of the product s of two commuting reflections with respect to the vectors $e_0 - e_1 - e_2 - e_3$ and $e_4 - e_5$. It is easy to see that $s([C_4]) = [C_4'], s([C_5]) = [C_5']$ and $s([C_i]) = [C_i]$ for $i \neq 4, 5$. This easily implies that W_S acts by permuting the classes $[C_i]$ and switching even number of them to $[C_i']$. □

Corollary 6.7. *Let $W(D_5)$ act in $\mathbb{C}^5$ by permuting the coordinates and switching the signs of even number of coordinates. This linear representation of $W(D_5)$ is isomorphic to the representation of $W(D_5)$ on $\mathcal{R}_S \otimes \mathbb{C}$.*

The group of projective automorphisms generated by the transformations which switch x_i to $-x_i$ generates a subgroup H of $\mathrm{Aut}(S)$ isomorphic to 2^4. We identify the group H with the group of subsets of even cardinality of the set $\{0, 1, 2, 3, 4\}$. The addition is of course the symmetric sum of subsets $A + B = (A \cup B) \setminus (A \cap B)$.

There are two kinds of involutions i_A. Involutions of the first kind correspond to a subset A consisting of 4 elements. The fixed point set of such involution is a hyperplane section of S, an elliptic curve. The trace formula gives that the the trace of i_A in $\mathrm{Pic}(S)$ is equal to -2. The corresponding conjugacy class in W_5 is of type $4A_1$. There are 5 involutions of the first kind. The quotient surface $S/(i_A) = Q$ is isomorphic to a nonsingular quadric. The map $S \to Q$ coincides with the map $S \to \mathbb{P}^1 \times \mathbb{P}^1$ that is given by two pencils $|C_i|$ and $|C_i'|$.

Involutions of the second type correspond to subsets A of cardinality 2. The fixed-point set of such involution consists of 4 isolated points. This gives that the trace is equal to 2, and the conjugacy class is of type $2A_1$. The quotient $S/(i_A)$ is isomorphic to the double cover of $\mathbb{P}^2$ branched along the union of two conics.

The subgroup of the Weyl group $W(D_5)$ generated by involutions from the conjugacy class of type $2A_1$ is the normal subgroup 2^4 in the decomposition $W(D_5) \cong 2^4 : S_5$. The product of two commuting involutions from this conjugacy class is an involution of type $4A_1$. Thus the image of H in W_S is a normal subgroup isomorphic to 2^4.

All groups (i_A) are not minimal.

There are three kinds of subgroups H of order 4 in 2^4. A subgroup of the first kind does not contain an involution of the first kind. An example is the group generated by i_{01}, i_{12}. By the trace formula

$$\text{rank } (K_S^{\perp})^H = \frac{1}{4}\sum_{g\in H} \text{Tr}(g) = \frac{1}{4}(5+1+1+1) = 2. \tag{6.3}$$

So this group is not minimal.

A subgroup of the second type contains only one involution of the first kind. An example is the group generated by i_{01}, i_{23}. The trace formula gives rank $\text{Pic}(S)^H = 2$. So it is also non-minimal.

A subgroup of the third kind contains two involutions of the first kind. For, example a group generated by i_{1234}, i_{0234}. The trace formula gives rank $\text{Pic}(S)^H = 1$. So it is a minimal group. It is easy to see that S^H consists of 4 isolated points.

Now let us consider subgroups of order 8 of 2^4. They are parametrized by the same sets which parametrize involutions. A subgroup H_A corresponding to a subset A consists of involutions i_B such that $\#A \cap B$ is even. The subsets A correspond to linear functions on 2^4. If $\#A = 2$, say $A = \{0,1\}$, we see that H_A contains the involutions $i_{01}, i_{01ab}, i_{cd}, c, d \neq 0, 1$. The trace formula gives rank $\text{Pic}(S)^{H_A} = 1$, so these subgroups are minimal.

If $\#A = 4$, say $A = \{1,2,3,4\}$, the subgroup H_A consists of i_{1234} and i_{ab}, where $a, b \neq 0$. The trace formula shows that rank $\text{Pic}(S)^{H_A} = 2$. So this H_A is not minimal.

Since 2^4 contains a minimal subgroup, it is minimal itself.

Now suppose that the image G' of G in S_5 is non-trivial. The subgroup S_5 of $\text{Aut}(S)$ can be realized as the stabilizer of a set of 5 skew lines on S (corresponding to the basis vectors $e_1, \ldots, e_5$). Thus any subgroup H of S_5 realized as a group of automorphisms of S is isomorphic to a group of projective transformations of $\mathbb{P}^2$ leaving invariant a set of 5 points. Since there is a unique conic through these points, the group is isomorphic to a finite group of PGL(2) leaving invariant a set of 5 distinct points. In section 4, we listed all possible subgroups of $\text{GL}(2,\mathbb{C})$ and in section 5 we described their relative invariants. It follows that a subgroup leaves invariant a set of 5 distinct points if and only if is one of the following groups $C_2, C_3, C_4, C_5, S_3, D_{10}$. The corresponding binary forms of degree 5 are projectively equivalent to the following binary forms:

- $C_2 : t_0(t_0^2 - t_1^2)(t_0^2 + at_1^2),\ a \neq -1, 0, 1;$
- $C_4 : t_0(t_0^2 - t_1^2)(t_0^2 + t_1^2);$
- $C_3, S_3 : t_0t_1(t_0 - t_1)(t_0 - \epsilon_3 t_1)(t_0 - \epsilon_3^2 t_1);$
- $C_5, D_{10} : (t_0 - t_1)(t_0 - \epsilon_5 t_1)(t_0 - \epsilon_5^2 t_1)(t_0 - \epsilon_5^3 t_1)(t_0 - \epsilon_5^4 t_1).$

The corresponding surfaces are projectively equivalent to the following surfaces

(6.4) C_2 : $x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_0^2 + ax_1^2 - x_2^2 - ax_3^2 = 0,\ a \neq -1, 0, 1$

(6.5) C_4 : $x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_0^2 + ix_1^2 - x_2^2 - ix_3^2 = 0$

(6.6) S_3 : $x_0^2 + \epsilon_3 x_1^2 + \epsilon_3^2 x_2^2 + x_3^2 = x_0^2 + \epsilon_3^2 x_1^2 + \epsilon_3 x_2^2 + x_4^2 = 0$

(6.7) D_{10} : $\sum_{i=0}^{4} \epsilon_5^i x_i^2 = \sum_{i=0}^{4} \epsilon_5^{4-i} x_i^2 = 0$

Remark 6.8. Note that equations (6.6), (6.7), (6.7) are specializations of equation (6.5). It is obvious for equation (6.6) where we have to take $a = i$. Equation (6.5) specializes to equation (6.7) when we take $a = \pm\frac{1}{\sqrt{-3}}$ (use that the Moebius transformation of order 3 $x \mapsto \frac{\sqrt{a}x+1}{x+\sqrt{a}}$ permutes cyclically $\infty, \sqrt{a}, -\sqrt{-a}$ and fixes $1, -1$). Equation (6.5) specializes to equation (6.7) if we take $a = -2 \pm \sqrt{5}$ (use that the Moebius transformation $x \mapsto \frac{x+2a-1}{x+1}$ permutes cyclically $(\infty, 1, a, -a, -1)$). We thank J. Blanc for this observation.

Since the subgroup S_5 leaves the class e_0 invariant, it remains to consider subgroups G of $2^4 : S_5$ which are not contained in 2^4 and not conjugate to a subgroup of S_5. We use the following facts.

1) Suppose G contains a minimal subgroup of 2^4. Then G is minimal.

2) Let $\bar{G}$ be the image of G in S_5. Then it is a subgroup of one of the groups listed above.

3) The group $W(D_5)$ is isomorphic to the group of transformations of $\mathbb{R}^5$ which consists of permutations of coordinates and changing even number of signs of the coordinates. Each element $w \in W(D_5)$ defines a permutation of the coordinate lines which can be written as a composition of cycles $(i_1 \dots i_k)$. If w changes even number of the coordinates $x_{i_1}, \dots, x_{i_k}$, the cycle is called positive. Otherwise it is called a negative cycle. The conjugacy class of w is determined by the lengths of positive and negative cycles, except when all cycles of even length and positive in which case there are two conjugacy classes. The latter case does not occur in the case when n is odd. Assign to a positive cycle of length k the Carter graph A_{k-1}. Assign to a pair of negative cycles of lengths $i \geq j$ the Carter graph of type D_{i+1} if $j = 1$ and $D_{i+j}(a_{j-1})$ if $j > 1$. Each conjugacy class is defined by the sum of the graphs. We identify D_2 with $2A_1$, and D_3 with A_3. In Table 2 below we give the conjugacy classes of elements in $W(D_5)$, their characteristic polynomials and the traces in the root lattice of type D_5.

In the following $\bar{G}$ denotes the image of G in $K = W(D_5)/2^4 \cong S_5$.

Case 1. $\bar{G} \cong C_2$.

It follows from the description of the image of $\mathrm{Aut}(S)$ in $W(D_5)$ given in Corollaryact, that $\bar{G}$ is generated by the permutation $s = (02)(13) \in K$. Let $g \notin G \cap 2^4$. Then $g = s$ or $g = si_A$ for some A.

Assume first that $g = s$. Let $H = G \cap 2^4 \cong 2^a$. If $a = 1$, then $H = (i_A)$, where A is invariant under the permutation s. Without loss of generality we may assume that $A = \{0, 2\}$ or $\{0123\}$. In both cases, adding the traces we see that the group is not minimal. Assume $a = 2$ and H is a subgroup of the first kind corresponding to a subset of 3 elements invariant with respect to s. Without loss of generality we may assume that $H = (i_{04}, i_{24})$. Adding the traces we see that G is not minimal.

Order	Notation	Characteristic polynomial	Trace	Representatives
2	A_1	$t+1$	3	(ab)
2	$2A_1$	$(t+1)^2$	1	$(ab)(cd),(ab)(cd)i_{abcd}$
2	$2A_1^*$	$(t+1)^2$	1	i_{ab}
2	$3A_1$	$(t+1)^3$	-1	$(ab)i_{cd}$
2	$4A_1$	$(t+1)^4$	-3	i_{abcd}
3	A_2	t^2+t+1	2	$(abc),(abc)i_{ab}$
4	A_3	t^3+t^2+t+1	1	$(abcd),(abcd)i_{ab}$
4	A_1+A_3	$(t^3+t^2+t+1)(t+1)$	-1	$(ab)(cd)i_{ae}$
4	$D_4(a_1)$	$(t^2+1)^2$	1	$(ab)(cd)i_{ac}$
5	A_4	$(t^4+t^3+t^2+t+1)$	0	$(abcde),(abcde)i_A$
6	A_2+A_1	$(t^2+t+1)(t+1)$	0	$(ab)(cde)$
6	A_2+2A_1	$(t^2+t+1)(t+1)^2$	-2	$(abc)i_{de}$
6	D_4	$(t^3+1)(t+1)$	0	$(abc)i_{abce}$
8	D_5	$(t^4+1)(t+1)$	-1	$(abcd)i_{abce},(abcd)i_{de}$
12	$D_5(a_1)$	$(t^3+1)(t^2+1)$	0	$(abc)(de)i_{ac}$

TABLE 2. Conjugacy classes in $W(D_5)$

If H is of the second kind, we may assume that $H=(i_{02},i_{13})$. Again we check that G is not minimal. If H is a minimal subgroup of order 4 generated by i_{0134},i_{1234}, then G is minimal. It is easy to check that $G\cong 2^2:2\cong D_8$.

Assume $a=3$. Then H corresponds to a s-invariant subset of cardinality 2 or 4. In the first case H is minimal, hence G is minimal. This gives a minimal group isomorphic to $2^3:2$. In the second case H is not minimal and contributes 8 to the sum of traces in $K_S^\perp$. Since the trace of s is equal to 1, and other elements from the coset of s have traces ≥ -1, we get that the sum of the traces is not zero, hence G is not minimal. Thus we have found two proper minimal groups isomorphic to D_8 and $2^3:2$.

Next assume that $s\notin G$. Since all elements si_A of order 2 are conjugate to s, we may assume that all elements in G of the form si_A are of order 4. We have two choices corresponding to two possible kinds of the involution $g^2=i_{A+s(A)}$. If it is of the first kind, then g belongs to the conjugacy class $D_4(a_1)$, otherwise it belongs to A_1+A_3. No element of order 4 is minimal, so there must be some element $i_B\in H$ different from $g^2=i_{A+s(A)}$. Since $G\cap 2^4$ is normal, $si_Bs^{-1}=i_{s(B)}\in H$ and $i_{B+s(B)}\in H$. Thus $H=G\cap 2^4$ contains $g^2=i_{A+s(A)},i_B,i_{s(B)},i_{B+s(B)}$.

Assume $i_{A+s(A)}\neq i_{B+s(B)}$. Then H contains a subgroup isomorphic to 2^3. It follows from the description of such subgroups that H is minimal. So G is minimal. This gives a new minimal group isomorphic to $(2^3)^{\cdot}2\cong(2^2):4$.

Assume $i_{A+s(A)}=i_{B+s(B)}$. Without loss of generality we may assume that $i_A=i_{01},i_B=i_{03}$. The subgroup H contains a subgroup generated by i_{03},i_{12}. It is a subgroup isomorphic to 2^2 of the second kind. We know that it is not minimal, and the sum of the traces in $K_S^\perp$ is equal to 4. Computing the traces of $g,gi_{03},gi_{12},gi_{0123}$ we find that the sum of traces is equal to 8. So the group $(g,i_{03},i_{12})\cong(2^2)^{\cdot}2$ is not minimal. The only minimal group containing it is the whole group $2^4:2$.

Assume next that G contains an element $g = si_A$ from the conjugacy class $A_1 + A_3$. By the above we may assume that all elements of order 4 belong to the same conjugacy class. Without loss of generality we may assume that $A = \{04\}$. As above there must be $i_B \in G \cap 2^4$ not equal to $g^2 = i_{02}$. Since $gi_B = si_{A+B}$, the involution $i_{A+s(A)+B+s(B)}$ is of the second kind. This implies that either $B = s(B)$, or $B \neq s(B), B + s(B) = i_{0123}$. In the first case G contains a subgroup $G' = (g, i_B)$ isomorphic to 2×4. The subgroup (g) contributes 4 to the sum of the traces. If $B = s(B)$, then either $B = \{1, 3\}$ or $B = \{0123\}$. In the second case G contains $(g, i_{B+s(B)})$, which as we saw is a minimal group. Thus G contains a minimal group $(g, i_B, i_{s(B)}) \cong (2^3)^{\cdot}2$. This group is not conjugate to the group we have obtained before.

Case 2. $\bar{G} \cong C_3$.

Applying Lemma 4.2, we obtain that G is a split extension $H : 3$, where $H = G \cap 2^4$. Since all elements of order 3 in $W(D_5)$ belong to the same conjugacy class, we may assume that G contains $s = (012)$. Since no element of order 3 is minimal, $H \cong 2^a$ with $a > 0$. The element s acts on H by conjugation. If $a = 1$, we have $H = (i_A)$, where $s(A) = A$. If $\#A = 2$, we get $A = \{34\}$ and $G \cong 6$ is generated by si_{34}. Consulting Table 6.4, we find that the conjugacy class of the generator is the non-minimal class $A_2 + 2A_1$.

If $\#A = 4$, we may assume that $A = \{0123\}$. Then $G \cong 6$ is generated by si_{0123} from the conjugacy class D_4 and hence again is not minimal.

Assume $a = 2$. If s acts identically on H, we have $H \setminus \{1\} = \{i_{0123}, i_{0124}, i_{34}\}$. We have seen already that H is a minimal subgroup of $\mathrm{Aut}(S)$, hence G is minimal. It is isomorphic to $2^2 \times 3$.

If H does not act identically, $H \setminus \{1\} = \{i_{01}, i_{12}, i_{23}\}$. The product $s \cdot h, h \in H, h \neq 1$ belongs to the conjugacy class A_2 with trace equal to 2. This gives

$$\text{rank } (K_S^{\perp})^G = \frac{1}{12}(5 + 1 + 1 + 1 + 2(2 + 2 + 2 + 2)) = 2.$$

The group is not minimal.

Assume $a = 3$. If H is minimal, we may assume that $A = \{34\}$ and its non-trivial elements are $i_{34}, i_{ab34}, i_{cd}, c, d \neq 3, 4$. The group G is minimal and is isomorphic to $2 \times (2^2 : 3)$.

If H is not minimal, we may assume that $A = \{0123\}$ and its non-trivial elements are $i_{0123}, i_{ab}, a, b \neq 4$. We have already computed $\sum_{h \in H} \mathrm{Tr}_2(h)$. It is equal to 16. We know that $\mathrm{Tr}_2(s \cdot i_{0123}) = 1$, and $\mathrm{Tr}_2(s \cdot i_{ab}) = 3$, if $a, b \in \{0, 1, 2\}$. The elements $s \cdot i_{a3}, a = 0, 1, 2$ belong to the conjugacy class D_4 with trace on $\mathrm{Pic}(S)$ equal to 1. This gives

$$\text{rank Pic}(S)^G = \frac{1}{24}(16 + 2(3 + 1 + 9 + 3)) = 2.$$

The group is not minimal.

Case 3. $\bar{G} \cong S_3$.

The group $\bar{G}$ is generated by the permutations of coordinates (012) and (12)(34). The only subgroup of H invariant with respect to the conjugation action of $\bar{G}$ on H is H itself. This gives the minimal group isomorphic to $2^4 : S_3$.

Case 4. $\bar{G} \cong C_4$.

The group $2^4 : 4$ contains $2^4 : 2$, so all minimal groups of the latter group are minimal subgroups of $2^4 : 4$. Without loss of generality, we may assume that the group $\bar{G}$ is generated by the permutations of coordinates $\sigma = (0123)$. The only proper subgroup of 2^4 invariant with respect to the conjugation action of $\bar{G}$ on H are the involution (i_{0123}) or the subgroup H generated by i_A, where $\#A \cap \{0,1,2,3\}$ is even. Assume $H = (i_{0123})$ and $\sigma \in G$. Using the table of conjugacy classes we check that $\mathrm{Tr}(g) = 1$ on $K_S^\perp$, if $g \notin H$. This gives

$$\mathrm{rank}\ (K_S^\perp)^G = \frac{1}{8}(5 - 3 + 6.1) = 1.$$

The group is not minimal.

Assume $H \neq (i_{0123})$ and $\sigma \in G$. Since we know from our previous computations that rank $\mathrm{Pic}(S)^H = 2$, we get $\sum_{h \in H} \mathrm{Tr}_2(h) = 16$. From the table of conjugacy classes we obtain again that $\mathrm{Tr}_2(g) = 2$ for any $g \notin H$. This gives

$$\mathrm{rank}\ \mathrm{Pic}(S)^G = \frac{1}{32}(16 + 24.2) = 2.$$

The group is not minimal.

Now let assume that $\sigma \notin G$. Let $g = \sigma \cdot i_A \in G$. If $A \subset \{0,1,2,3\}$, then σ is conjugate to $\sigma \cdot i_A$, and hence this case is reduced to a previous case. So, we may assume that $\#A \cap \{0,1,2,3\}$ is odd. Again, replacing g by its conjugate, we may assume that $A = \{34\}$. In this case g belongs to the conjugacy class D_5 of order 8. A cyclic group of order 8 is minimal. Thus we obtain three minimal groups not contained in $2^4 : 2$: a cyclic group of order 8 if $H \cong 2$, a group isomorphic to $2^3 : 4$ if $H \cong 2^3$, and a group isomorphic to $2^4 : 4$.

Case 5. $\bar{G} = C_5$ or D_{10}.

In this case, no proper subgroup of H is invariant with respect to conjugation by a permutation of order 5, or by a subgroup of S_5 generated by (012) and (12)(34). Thus we get two minimal groups isomorphic to $2^4 : 5$ or $2^4 : D_{10}$.

The following theorem summarizes what we have found.

Theorem 6.9. *Let (S, G) be a minimal Del Pezzo surface of degree $d = 4$. Then G is isomorphic to one of the following groups*

(1) $\mathrm{Aut}(S) \cong 2^4$

$$2^4, \quad 2^3, \quad 2^2.$$

(2) $\mathrm{Aut}(S) \cong 2^4 : 2$

$$2^4, \quad 2^3, \quad 2^2, \quad 2 \times 4, \quad 2^4 : 2, \quad 2^3 : 2, \quad 2^2 : 4 \text{ (2 conjugacy classes)}, \quad D_8.$$

(3) $\mathrm{Aut}(S) \cong 2^4 : S_3$.

$$2^4, \quad 2^3, \quad 2^2, \quad 2 \times 4, \quad 2^4 : 2, \quad 2^3 : 2, \quad 2^2 : 4 \text{ (2 conjugacy classes)},$$

$$D_8, \quad 2^4 : S_3, \quad 2^4 : 3, \quad 2 \times (2^2 : 3), \quad 2^2 \times 3.$$

(4) $\mathrm{Aut}(S) \cong 2^4 : 4$

$$2^4, \quad 2^3, \quad 2^2, \quad 2^4 : 2, \quad 2^3 : 2, \quad 2^2 : 4 \text{ (2 conjugacy classes)},$$

$$D_8, \quad 2 \times 4, \quad 2^4 : 4, \quad 2^3 : 4.$$

(5) $\mathrm{Aut}(S) \cong 2^4 : D_{10}$

$$2^4, \quad 2^3, \quad 2^2, \quad 2^4 : 2, \quad 2^3 : 2, \quad 2^2 : 4 \text{ (2 conjugacy classes)},$$

$$D_8, \quad 2 \times 4, \quad 2^4 : D_{10}, \quad 2^4 : 5.$$

6.5. **Cubic surfaces.** The following theorem gives the classification of cyclic subgroups of $\mathrm{Aut}(S)$ and identifies the conjugacy classes of their generators.

Theorem 6.10. *Let S be a nonsingular cubic surface admitting a non-trivial automorphism g of order n. Then S is equivariantly isomorphic to one of the following surfaces $V(F)$ with diagonal action defined by a generator of (g) via the formula*

$$(x_0, x_1, x_2, x_3) = (x_0, \epsilon_n^a x_1, \epsilon_n^b x_2, \epsilon_n^c x_3).$$

- $4A_1$ $(n = 2)$, $(a, b, c) = (0, 0, 1)$,
$$F = T_3^2 L_1(T_0, T_1, T_2) + T_0^3 + T_1^3 + T_2^3 + \alpha T_0 T_1 T_2.$$
- $2A_1$ $(n = 2)$, $(a, b, c) = (0, 1, 1)$,
$$F = T_0 T_2 (T_2 + \alpha T_3) + T_1 T_3 (T_2 + \beta T_3) + T_0^3 + T_1^3.$$
- $3A_2$ $(n = 3)$, $(a, b, c) = (0, 0, 1)$,
$$F = T_0^3 + T_1^3 + T_2^3 + T_3^3 + \alpha T_0 T_1 T_2.$$
- A_2 $(n = 3)$, $(a, b, c) = (0, 1, 1)$,
$$F = T_0^3 + T_1^3 + T_2^3 + T_3^3.$$
- $2A_2$ $(n = 3)$, $(a, b, c) = (0, 1, 2)$,
$$F = T_0^3 + T_1^3 + T_2 T_3 (T_0 + a T_1) + T_2^3 + T_3^3.$$
- $D_4(a_1)$ $(n = 4)$, $(a, b, c) = (0, 2, 1)$,
$$F = T_3^2 T_2 + L_3(T_0, T_1) + T_2^2 (T_0 + \alpha T_1).$$
- $A_3 + A_1$ $(n = 4)$, $(a, b, c) = (2, 1, 3)$,
$$F = T_0^3 + T_0 T_1^2 + T_1 T_3^2 + T_1 T_2^2.$$
- A_4 $(n = 5)$, $(a, b, c) = (4, 1, 2)$,
$$F = T_0^2 T_1 + T_1^2 T_2 + T_2^2 T_3 + T_3^2 T_0.$$
- $E_6(a_2)$ $(n = 6)$, $(a, b, c) = (0, 3, 2)$,
$$F = T_0^3 + T_1^3 + T_3^3 + T_2^2 (\alpha T_0 + T_1).$$
- D_4 $(n = 6)$, $(a, b, c) = (0, 2, 5)$,
$$F = L_3(T_0, T_1) + T_3^2 T_2 + T_2^3.$$
- $A_5 + A_1$ $(n = 6)$, $(a, b, c) = (4, 2, 1)$,
$$F = T_3^2 T_1 + T_0^3 + T_1^3 + T_2^3 + \lambda T_0 T_1 T_2.$$
- $2A_1 + A_2$ $(n = 6)$, $(a, b, c) = (4, 1, 3)$,
$$F = T_0^3 + \beta T_0 T_3^2 + T_2^2 T_1 + T_1^3.$$
- D_5 $(n = 8)$, $(a, b, c) = (4, 3, 2)$,
$$F = T_3^2 T_1 + T_2^2 T_3 + T_0 T_1^2 + T_0^3.$$
- $E_6(a_1)$ $(n = 9)$, $(a, b, c) = (4, 1, 7)$,
$$F = T_3^2 T_1 + T_1^2 T_2 + T_2^2 T_3 + T_0^3.$$
- E_6 $(n = 12)$, $(a, b, c) = (4, 1, 10)$,
$$F = T_3^2 T_1 + T_2^2 T_3 + T_0^3 + T_1^3.$$

We only sketch a proof, referring for the details to [24]. Let g be a nontrivial projective automorphism of S of order n. All possible orders are known from the classification of conjugacy classes of $W(E_6)$. First we diagonalize g and represent it by a diagonal matrix with eigenvalues $1, \epsilon_n^a, \epsilon_n^b, \epsilon_n^c, a \le b \le c$. Then g is an automorphism of a cubic surface $V(F)$ if F belongs to an eigenspace of the action of g on the space of homogeneous cubic polynomials. We list all possible eigensubspaces. Since $V(F)$ is nonsingular, the square or the cube of each variable divides some monomial entering in F. This allows one to list all possible nonsingular $V(F)$ admitting an automorphism g. Some obvious linear change of variables allows one to find normal forms. Finally, we determine the conjugacy class by using the Lefschetz formula applied to the locus of fixed points of g and its powers.

The conjugacy classes with Carter graph with 6 vertices define the minimal cyclic groups.

Corollary 6.11. *The following conjugacy classes define minimal cyclic groups of automorphisms of a cubic surface S.*

- $3A_2$ *of order 3,*
- $E_6(a_2)$ *of order 6,*
- $A_5 + A_1$ *of order 6,*
- $E_6(a_1)$ *of order 9,*
- E_6 *of order 12.*

Next we find all possible automorphism groups of nonsingular cubic surfaces. Using a normal form of a cubic admiiting a cyclic group of automorphisms from given conjugacy class, we determine all other possible symmetries of the equation. We refer for the details to [24]. The list of possible automorphism groups of cubic surfaces is given in Table 6.5.

Type	Order	Structure	$F(T_0,T_1,T_2,T_3)$	Parameters
I	648	$3^3 : S_4$	$T_0^3+T_1^3+T_2^3+T_3^3$	
II	120	S_5	$T_0^2T_1+T_0T_2^2+T_2T_3^2+T_3T_1^2$	
III	108	$3.(3^2:4)$	$T_0^3+T_1^3+T_2^3+T_3^3+6aT_1T_2T_3$	$20a^3+8a^6=1$
IV	54	$3.(3^2:2)$	$T_0^3+T_1^3+T_2^3+T_3^3+6aT_1T_2T_3$	$a-a^4\neq 0$, $8a^3\neq -1$, $20a^3+8a^6\neq 1$
V	24	S_4	$T_0^3+T_0(T_1^2+T_2^2+T_3^2)$ $+aT_1T_2T_3$	$9a^3\neq 8a$ $8a^3\neq -1$,
VI	12	$S_3\times 2$	$T_2^3+T_3^3+aT_2T_3(T_0+T_1)+T_0^3+T_1^3$	$a\neq 0$
VII	8	8	$T_3^2T_2+T_2^2T_1+T_0^3+T_0T_1^2$	
VIII	6	S_3	$T_2^3+T_3^3+aT_2T_3(T_0+bT_1)+T_0^3+T_1^3$	$a^3\neq -1$
IX	4	4	$T_3^2T_2+T_2^2T_1+T_0^3+T_0T_1^2+aT_1^3$	$a\neq 0$
X	4	2^2	$T_0^2(T_1+T_2+aT_3)+T_1^3+T_2^3$ $+T_3^3+6bT_1T_2T_3$	$8b^3\neq -1$
XI	2	2	$T_1^3+T_2^3+T_3^3+6aT_1T_2T_3$ $+T_0^2(T_1+bT_2+cT_3)$	$b^3,c^3\neq 1$ $b^3\neq c^3$ $8a^3\neq -1$,

TABLE 3. Groups of automorphisms of cubic surfaces

Remark 6.12. Note that there are various ways to write the equation of cubic surfaces from the table. For example, using the identity

$$(x+y+z)^3+\epsilon_3(x+\epsilon_3 y+\epsilon_3 z)^3+\epsilon_3^2(x+\epsilon_3^2 y+\epsilon_3 z)^3=3(x^2z+y^2x+z^2x)$$

we see that the Fermat cubic can be given by the equation

$$T_0^3+T_1^2T_3++T_2^2T_1+T_3^2T_1=0.$$

Using Lemma 6.10 this exhibits a symmetry of order 9 of the surface, whose existence is not obvious in the original equation.

Using the Hesse form of an equation of a nonsingular plane cubic curve we see that a surface with equation

$$T_0^3+F_3(T_1,T_2,T_3)=0$$

is projectively equivalent to a surface with equation

$$T_0^3+T_1^3+T_2^3+T_3^3+6aT_0T_1T_2=0.$$

The special values of the parameters $a=0,1,\epsilon_3,\epsilon_3^2$ give the Fermat cubic. The values a satisfying $20a^3+8a^6=1$ give a plane cubic with an automorphism of order 4. Using Lemma 6.10 this exhibits symmetries of order 6 from the conjugacy class $E_6(a_2)$ for surfaces of type III, IV and of order 12 for the surface of type III isomorphic to the surface

$$T_3^2T_1+T_2^2T_3+T_0^3+T_1^3=0.$$

It remains to classify minimal groups G. Note that if G is realized as a group of projective (or weighted projective) automorphisms of a family of surfaces S_t, then G is a subgroup of the group of projective automorphisms of any surface S_{t_0} corresponding to a special value t_0 of the parameters. We indicate this by writing $S'\to S$. The types of S' when it happens are

$$IV\to III,\ IV\to I,\ ,VI,VIII,IX\to I,\ XI\to X.$$

So it suffices to consider the surfaces of types I, II, III, V, VII, X.

Type I.

Let us first classify the $\mathbb{F}_3$-subspaces of the group $K=3^3$. We view 3^3 as the quotient group $\mathbb{F}_3^4/\Delta$, where Δ is the diagonal subspace generated by $(1,1,1,1)$. We denote the image of a vector (a,b,c,d) in K by $[a,b,c,d]$. In our old notations

$$[a,b,c,d]=[\epsilon^a x_0,\epsilon^b x_1,\epsilon^c x_2,\epsilon^d x_3].$$

Obviously $[a,b,c,d]$ contains at least two equal coordinates. This shows that a one-dimension subspace is conjugate to a subspace $\langle[0,0,1,a]$. It is a cyclic group from the conjugacy class

$$\begin{cases}3A_2 & \text{if, } a=0,\\ 2A_2 & \text{if } a=1,\\ A_2 & \text{if } a=2.\end{cases}$$

A 2-dimensional subspace K is conjugate in $\mathrm{Aut}(S)$ to a subspace spanned by $[0,0,1,a]$ and $[0,1,0,b]$. There are three non-conjugate subspaces of dimension 2. They differ by the types of conjugacy classes of non-trivial elements they contain. The dot-product in $\mathbb{F}_3^4$ is invariant with respect to the action of S_4. Let V be the orthogonal subspace of Δ. For any 3-dimensional subspace containing Δ, its orthogonal complement is a one-dimensional subspace in $\Delta^\perp$. For any 2-dimensional subspace K this defines its orthogonal subspace in 3^3 generated by a vector $\mathbf{n}_K = [a,b,c,d]$ with $a+b+c+d=0$. There are three types of such vectors corresponding to 3 types of 2-dimensional spaces K. An easy computation gives the following table.

(6.8)

Type	a,b	$3A_2$	$2A_2$	A_2	$\mathbf{n}_K$	Trace
I	$a+b=0,1$	4	2	2	$2A_2$	0
II	$ab=0, a+b=2$	2	6	0	$3A_2$	0
III	$a=b=1$	0	4	4	A_2	2

Here the last column is the sum $\frac{1}{9}\sum_{g\in L}\mathrm{Tr}(g|K_S^\perp)$. This gives us two conjugacy classes of minimal subgroups isomorphic to 3^2.

Let G be a subgroup of $\mathrm{Aut}(S)$, $\bar{G}$ be its image in S_4 and $K = G\cap 3^3$. Let $k = \dim_{\mathbb{F}_3} K$.

If $k=0$, G is a section of the projection $3^3 : S_4 \to S_4$. As is well-known, the conjugacy classes of such subgroups correspond to the cohomology group $H^1(S_4, \mathbb{F}_3^3)$, where S_4 acts naturally on $\mathbb{F}_3 = \mathbb{F}_3^4/\Delta$ via permuting the factors. It is known that the cohomology space vanishes. Thus all such groups are conjugate to S_4. Let us show that it is not minimal.

The surface S contains 27 lines $\ell_{ij,ab}$ given by the equations

$$x_i - \epsilon_3^a x_j = x_k - \epsilon_3^b x_l = 0,$$

where $a,b\in\{0,1,2\}$ and $i<j, k<l, \{i,j,k,l\} = \{0,1,2,3\}$. The subgroup of $\mathrm{Aut}(S)$ isomorphic to S_4 has 3 orbits on the set of lines of cardinalities $3,12,12$. The minimal orbit consists of skew lines $\ell_{01,00}, \ell_{02,00}, \ell_{03,00}$. Thus S_4 is not a minimal subgroup. It comes from a subgroup of a Del Pezzo surface of degree 6. Suppose G does not contain minimal elements of order 3. Let $K = G\cap 3^3$. If K is trivial, G is mapped isomorphically onto a subgroup of S_4. Suppose $g\in G$ is an element of order 2 which is mapped to a an element of S_4 represented by a transposition, say $s = (01)$. Then $g : (x_0,x_1,x_2,x_3) \mapsto (\epsilon_3 x_1, \epsilon_3^2 x_0, x_2, x_3)$. It is easy to see that g is conjugate to s. Similar computation for other elements of finite order in S_4 shows that elements in G are conjugate to elements in the image of G in S_4. Since S_4 is not minimal, G is not minimal.

Here in later we will often use Lemma 4.2 to decide whether a possible extension of groups splits.

Assume now that $k=1$. By the above we may assume that $K = \langle[0,0,1,a]\rangle$. Each element in G can be written as the product gs, where $g\in 3^3$ and $s\in S_4$. In the action of G in K by conjugation, the element s should leave K invariant. If $a=0$, this implies that $\bar{G}$ is a subgroup of S_3 permuting the coordinates x_0,x_1,x_3. This gives the following minimal groups $3.S_3, 3.3, 3:2$. It is easy to see 3.3 could be isomorphic to either 9 or 3^2. The group $3:2$ must be a cyclic group 6, and the extension $3.S_3$ is isomorphic to D_{18} or $3\times S_3$. It is easy to check that the group 3^2 is conjugate to a linear subspace of 3^3 of type II. The cyclic group of order 6 is generated by a minimal element of type $E_6(2)$. The cyclic group of order 9 is generated by an element conjugate to a minimal transformation $[\epsilon_3 x_2, x_0, x_1]$ of type E_6.

If $K = \langle[0,0,1,1]\rangle$ we have possible groups $3\times 2 = 6$ or 3×2^2. One can check that none of them is minimal.

If $K = \langle[0,0,1,2]\rangle$, then G must be isomorphic to minimal 6 of type A_5+A_1, or minimal $2\times S_3$ containing minimal 6, or $3:2\cong S_3$. The latter group contains 2 elements of type $2A_2$, 3 elements of type $4A_1$ and the identity. Adding up the traces we see that the group is also minimal. This gives three groups

$$6, \quad S_3, \quad 2 \times S_3.$$

Assume $k = 2$. So the image $\bar{G}$ in S_4 is one of the previous groups $S_3, 3, 2, 2^2$. This gives us the following groups

$$G = \begin{cases} K : 2^2, K : 2 & \text{if } K \text{is of type I} \\ K : 2, K.3, K.S_3 & \text{if } K \text{is of type II} \\ K : 2, K : 2^2 & \text{if } K \text{is of type III} \end{cases}$$

Consider a group of the form $3^2 : 2^2$ or its subgroup $3^2 : 2$. The extension is classified by the restriction of a homomorphism $2^m \to \mathrm{GL}(2, \mathbb{F}_3), m = 1, 2$. In our case the homomorphism is defined as follows. We have a natural permutation representaion on $\mathbb{F}_3^4$ which defines a homomorphism $S_4 \to \mathrm{GL}(\mathbb{F}_3^4/\Delta) \cong \mathrm{GL}(3, \mathbb{F}_3)$. We restrict this homomorphism to the subgroup 2^2 of S_4 or its subgroup 2. Then we consider an invariant one-dimensional subspace spanned by the vector $\mathbf{n}_K$ and restrict the homomorphism to its orthogonal complement K. It is easy to see that an element τ of order 2 from the conjugacy class $4A_1$ (resp. $2A_1$) acts on $\mathbb{F}_3^4$ with 2 eigenvalues $-1 \in \mathbb{F}_3^*$ and two eigenvalues 1 (resp. one eigenvalue -1 and 3 eigenvalues 1). The subspace Δ is an eigensubspace with eigenvalue 1. Therefore, if $\tau(\mathbf{n}_K) = \mathbf{n}_K$, the element τ of type $4A_1$ (resp. $2A_1$) acts on K with $\dim K^\tau = 1$ (resp. acts as the negation). If $\tau(\mathbf{n}_K) = -\mathbf{n}_K$, then the element τ of type $4A_1$ (resp. $2A_1$) acts on K identically (resp. with $\dim K^\tau = 1$). We get the following groups

K	τ	$\tau(\mathbf{n}_K)$	G
I	$4A_1$	$\mathbf{n}_K$	$3 \times S_3$
I	$2A_1$	$\mathbf{n}_K$	$3^2 : 2$
II	$4A_1$	$\mathbf{n}_K$	$3 \times S_3$
III	$4A_1$	$\mathbf{n}_K$	$3 \times S_3$
III	$4A_1$	$-\mathbf{n}_K$	$3^2 \times 2$
III	$2A_1$	$-\mathbf{n}_K$	$3 \times S_3$

This gives us all possible types of extensions $K : 2$:

$$3^2 \times 2, \quad 3 \times S_3, \quad 3^2 : 2.$$

In the last case an element of order 2 acts as the negation on K. Also the first case can be realized only if K is of type I and τ is of type $4A_1$ and does not fix $\mathbf{n}_K$. So there is only one conjugacy class of such groups. There are 3 conjugacy classes of groups $3 \times S_3$ with subspaces K of all possible types. There is only one conjugacy class of groups $3^2 : 2$ with K of type III.

When K is of type I, II, the group is obviously minimal. If K is of type III, and τ is of type $2A_1$, then $G \setminus \{1\}$ consists of 4 elements of type A_2, 4 elements of type $2A_2$ and 9 elements of order 2. The group G acts transitively on the set of elements of order 2. So all of them of type $2A_1$. Adding the traces we get that the group is not minimal. Similar argument shows that the group is minimal if τ is of type $4A_1$.

There are 2 conjugacy classes of groups isomorphic to $K : 2^2$ dependent on whether its 3-Sylow subgroup K is of type I or III. Since 2^2 contains elements of type $4A_1$ and $2A_1$, the group contains a minimal group isomorphic to $K : 2$, and hence is minimal.

Finally, the groups $3^2.3$ or $3^2.S_3$ must be equal to $K.3$ or $K.S_3$, where K is of type II. So these groups are minimal. We may identify K with the subspace of elements $[0,a,b,c]$ with $a+b+c=0$,on which G acts by permutations of a,b,c. There is only one non-abelian group of order 27 isomorphic to the group of upper-triangular matrices over $\mathbb{F}_3$ with 1 at the diagonal. So the extension $3^2.3$ splits and is isomorphic to the central extension 3.3^2. By Lemma 4.2,

$$3^2.S_3 \cong (3^2.3):2 \cong 3^2:S_3.$$

Type II.

The surface is isomorphic to the surface in $\mathbb{P}^4$ given by the equations

$$\sum_{i=0}^{4} x_i^3 = \sum_{i=0}^{4} x_i = 0.$$

The group S_5 acts by permuting the coordinates. The orbit of the line $x_0 = x_1+x_2 = x_3+x_4 = 0$ consists of 15 lines. It is easy to see that the remaining 12 lines form a double-six. Their equations are as follows.

Let ω be a primitive 5th root of unity. Let $\sigma = (a_1,\ldots,a_5)$ be a permutation of $\{0,1,2,3,4\}$. Each line ℓ_σ spanned by the points $(\omega^{a_1},\ldots,\omega^{\sigma_5})$ and $(\omega^{-a_1},\ldots,\omega^{-a_5})$. This gives $12 = 5!/2.5$ different lines. One checks immediately that two lines ℓ_σ and $\ell_{\sigma'}$ intersect if and only if $\sigma' = \sigma\circ\tau$ for some odd permutation τ. Thus the orbit of the alternating subgroup A_5 of any line defines a set of 6 skew lines (a sixer). Hence A_5 is not a minimal subgroup. Since a double sixer spans $\mathrm{Pic}(S)^G$ and forms an S_5-orbit, we see that S_5 is a minimal group.

Let G be a subgroup of S_5 not contained in A_5. A maximal subgroup of S_5 not contained in A_5 is isomorphic to S_4, or $5:4$, or $2\times S_3$. Two isomorphic maximal subgroups are conjugate in S_5. So we may assume that S_4 is a subgroup leaving one letter fixed. Since any line ℓ_σ can be represented by a permutation $(0,a,b,c,d)$, we see that S_4 acts transitively on the double sixer. It has two orbits on the remaining 15 lines. One consists of three coplanar lines. None of the orbits consists of skew lines. So the group is minimal. Its maximal subgroup not contained in A_5 is the dihedral group D_8. It acts trivially on the orbit of S_4 consisting of three coplanar lines. So this group is not minimal.

The subgroup isomorphic to $5:4$ is conjugate to a subgroup generated by two cycles (01234) and (0123). Computing the traces, we find that rank $\mathrm{Pic}(S)^G = 2$. So this group is not minimal. The subgroup isomorphic to $2\times S_3$ is conjugate to a subgroup generated by $(012),(01),(34)$. Its element of order 6 belongs to the conjugacy class D_4. So this group is different from the isomorphic group in the previous case. Computing the traces we find that it is not minimal.

Type III.

The surface is a specialization of a surface of type IV. Let S be a surface of type IV. The projection from the point $(1,0,0,0)$ defines a cyclic triple cover $S\to\mathbb{P}^2$ with the branch curve $B = V(T_1^3+T_2^3+T_3^3+6aT_1T_2T_3)$. For values a with restrictions indicated in Table 3, the group of projective automorphisms of B is isomorphic to $3^2:2$. Let $c = [\epsilon_3x_0,x_1,x_2,x_3]$. It is a minimal element of type $3A_2$ which generates the center of $\mathrm{Aut}(S)$. We have

$$\mathrm{Aut}(S)/\langle c\rangle \cong \mathrm{Aut}(B) \cong 3^2:2.$$

The automorphism group of the curve B is generated by transformations

$$g_1 = [x_0, x_1, \epsilon_3, \epsilon_3^2],\ g_2 = [x_0, x_2, x_0, x_1],\ g_3 = [x_0, x_1, x_2, x_3].$$

When $a^4 = a$, the curve B acquires an additional automorphism of order 6 and the surface S specializes to a surface of type I. When $20a^4 + 8a^6 = 1$ the curve B acquires an additional automorphism of order 4 and the surface S specializes to a surface of type III.

Any minimal subgroup of the group G of automorphisms of a surface of type IV appears as a minimal group of automorphisms of a surface of type I and III. Note that all non-central elements of order 3 in G are of type $2A_2$. This implies that any subgroup isomorphic to 3^2 is of type II from (6.8). Here are subgroups of G which are also subgroups of the automorphism group of a surface of type I.

$$3, \quad 3^2, \quad 3.3^2 \cong 3^2 : 3, \quad 3^2 : 2,$$

$$6, \quad S_3, \quad 3 \times S_3, \quad (3.3^2) : 2 \cong 3^2 : S_3.$$

We can identify the subgroup $T = 3.3^2 \subset \mathrm{Aut}(S)$ with $\langle g_1, g_2, c\rangle$ and $T/(c)$ with the subgroup of 3-torsion points of the curve B. For an appropriate root of the equation $20a^4 + 8a^6 - 1 = 0$ a surface S of type III has an automorphism of order 4 of type $D_4(a_1)$

$$\tau = \frac{1}{\sqrt{3}}[\sqrt{3}x_0, x_1 + x_2 + x_3, x_1 + \epsilon_3 x_2 + \epsilon_3^2 x_2, x_1 + \epsilon_3^2 x_2 + \epsilon_3 x_3].$$

Its square is equal to g_3. The product $c\tau$ is of order 12. The element $(c\tau)^2 \in 3.3^2 : 2$ belongs to the conjugacy class $E_6(2)$ of order 6 . A new group of automorphisms of S is isomorphic to one of the following groups $3 : 4, 3^2 : 4, (3.3^2) : 4$ which contains one of the previous groups as a subgroup of index 2. This shows that $3 : 4$ must be a cyclic group of order 12. There is only one non-abelian extension $3^2 : 4 \cong 3.(3 : 4)$.

Type V.

The group S_4 acts by permuting the coordinates T_1, T_2, T_3 and multiplying them by -1 leaving the monomial $T_1T_2T_3$ unchanged. The group has 2 orbits on the set of lines, of cardinalities 3 and 24. The first orbit is contained in the plane $T_0 = 0$. Since none of the orbits consists of skew lines, we obtain that the group is minimal. The stabilizer subgroup of one of the lines is a maximal subgroup of S_4 isomorphic to the dihedral group D_8. So it is not minimal. Other maximal subgroups A_4 and S_3 are not minimal two.

Type VII.

The automorphism group of the surface of type VII is a non-minimal cyclic group of order 8.

Type X.

The automorphism group of the surface of type X consists of the identity, two involutions of type $4A_1$ and one involution of type $2A_1$. Adding up the traces, we get that the group is not minimal.

Let us summarize our result in the following.

Theorem 6.13. *Let G be a minimal subgroup of automorphisms of a nonsingular cubic surface. Then G is isomorphic to one of the following groups.*

(1) *G is a subgroup of automorphisms of a surface of type I.*

$$3,\quad 3^2\ (3),\quad 3^3,\quad 6\ (2),\quad 9,\quad 3\times S_3\ (2),\quad 3\times 6,\quad S_3,\quad 3^2:2^2\ (3),\quad 3^2:S_3,$$
$$3^2:2,\quad 3^2:3,\quad D_{18},\quad (3.3^2):2,\quad 3^3:S_4,\quad 3^3:A_4,\quad 3^3:S_3,\quad 3^3:2^2,\quad 3^3:3, 3^3:2.$$

(2) *G is a subgroup of automorphisms of a surface of type II.*

$$S_5,\quad S_4.$$

(3) *G is a subgroup of automorphisms of a surface of type III.*

$$3,\quad 6,\quad 12,\quad 3^2,\quad 3^2:3,\quad 6\times S_3,$$
$$S_3,\quad 3\times S_3,\quad (3.3^2):2\cong 3^2:S_3,\quad (3.3^2):4.$$

(4) *G is a subgroup of automorphisms of a surface of type IV.*

$$3,\quad 6,\quad 3^2,\quad 3^2:3,\quad S_3,\quad 3\times S_3,\quad 3^2:S_3.$$

(5) *G is a subgroup of automorphisms of a surface of type V.*

$$S_4.$$

(6) *G is a subgroup of automorphisms of a surface of type VI.*

$$6,\quad S_3\times 2,\quad S_3.$$

(7) *G is a subgroup of automorphisms of a surface of type VIII.*

$$S_3.$$

6.6. **Automorphisms of Del Pezzo surfaces of degree 2.** Recall that the map $S\to\mathbb{P}^2$ defined by $|-K_S|$ is a degree 2 cover. Its branch curve is a nonsingular curve of degree 4. The non-trivial automorphism of the cover defines an involution γ_0 of S which defines the conjugacy class of a Geiser involution of $\mathbb{P}^2$.

It is convenient to view a Del Pezzo surface of degree 2 as a hypersurface in the weighted projective space $\mathbb{P}(1,1,1,2)$ given by an equation of degree 4

$$T_3^2+F_4(T_0,T_1,T_2)=0 \tag{6.9}$$

It is easy to see that it belongs to the center of $\mathrm{Aut}(S)$. The corresponding element of the Weyl group acts the minus identity. The group $\mathrm{Aut}(S)'=\mathrm{Aut}(S)/\langle\gamma\rangle$ is isomorphic to the automorphism group of the branch curve of the double cover. It is a nonsingular plane quartic. The classification of automorphisms of plane quartic curves is well known (see [24]). Let G' be a group of automorphisms of the branch curve $V(F)$ given by a quartic polynomial F. Let $\chi: G'\to\mathbb{C}^*$ be the character of G' defined by $g^*(F)=\chi(g)F$. Let

$$G=\{(g',\alpha)\in G'\times\mathbb{C}^*:\chi(g')=\alpha^2\}.$$

This is a subgroup of the group $G'\times\mathbb{C}^*$. The projection to G' defines an isomorphism $G\cong 2.G'$. The extension splits if and only if χ is equal to the square of some character of G'. In this case $G\cong G'\times 2$. The group G acts on S given by equation (6.9) by

$$(g',\alpha):(t_0,t_1,t_2,t_3)\mapsto(g'^*(t_0),g'^*(t_1),g'^*(t_2),\alpha t_3).$$

Any group of automorphisms of S is equal to a group G as above.

Lemma 6.14. *Let g be an automorphism of order $n>1$ of a nonsingular plane quartic $C=V(F)$. Then one can choose coordinates in such a way that a generator of the cyclic group $G'=(g)$ is represented by a diagonal matrix $(1,\epsilon_n^a,\epsilon_n^b)$ and F is given in the following list.*

(i) $(n = 2), (a, b) = (0, 1)$,

$$F = T_2^4 + T_2^2 L_2(T_0, T_1) + L_4(T_0, T_1).$$

(ii) $(n = 3), (a, b) = (0, 1)$,

$$F = T_2^3 L_1(T_0, T_1) + L_4(T_0, T_1).$$

(iii) $(n = 3), (a, b) = (1, 2)$,

$$F = T_0^4 + \alpha T_0^2 T_1 T_2 + T_0 T_1^3 + T_0 T_2^3 + \beta T_1^2 T_2^2.$$

(iv) $(n = 4), (a, b) = (0, 1)$,

$$F = T_2^4 + L_4(T_0, T_1).$$

(v) $(n = 4), (a, b) = (1, 2)$,

$$F = T_0^4 + T_1^4 + T_2^4 + \alpha T_0^2 T_2^2 + \beta T_0 T_1^2 T_2.$$

(vi) $(n = 6), (a, b) = (3, 2)$,

$$F = T_0^4 + T_1^4 + \alpha T_0^2 T_1^2 + T_0 T_2^3.$$

(vii) $(n = 7), (a, b) = (3, 1)$,

$$F = T_0^3 T_1 + T_1^3 T_2 + T_2^3 T_0 + \alpha T_0 T_1^2 T_2.$$

(viii) $(n = 8), (a, b) = (3, 7)$,

$$F = T_0^4 + T_1^3 T_2 + T_1 T_2^3.$$

(ix) $(n = 9), (a, b) = (3, 2)$,

$$F = T_0^4 + T_0 T_1^3 + T_2^3 T_1.$$

(x) $(n = 12), (a, b) = (3, 4)$,

$$F = T_0^4 + T_1^4 + T_0 T_2^3.$$

Here the subscript in polynomial L_i indicates its degree.

The character $\chi : (g) \to \mathbb{C}^*$ is defined by $\chi(g)$. Replacing g by another generator we may assume that $\chi(g)$ is a primitive nth root of unity ϵ_n, where $n|\mathrm{ord}(g)$. Since $\chi = \eta^2$, where $\eta(g) = \epsilon_{2n}$, we see that $G = (g).2$ always splits. Thus g lifts to two automorphisms of S, one multiplies the coordinate t_3 by ϵ_{2n} another one by $-\epsilon_{2n}$.

Also observe that the diagonal matrix (t, t, t, t^2) acts identically on S.

The following lemma identifies the conjugacy class of two liftings of g in the Weyl group $W(E_7)$. The last column gives the trace of g on $K_S^\perp$.

The following is the list of elements of finite order which generate a minimal cyclic group of automorphisms.

(1) Order 2 (A_1^7) (The Geiser involution) $g = [t_0, t_1, t_2, -t_3]$

$$F = T_3^2 + F_4(T_0, T_1, T_2).$$

(2) Order 4 ($2A_3 + A_1$) $g = [t_0, t_1, it_2, t_3]$

$$F = T_3^2 + T_2^4 + L_4(T_0, T_1).$$

(3) Order 6 ($E_7(a_4)$) $[t_0, t_1, \epsilon_3 t_2, -t_3]$

$$F = T_3^2 + T_2^3 L_1(T_0, T_1) + L_4(T_0, T_1).$$

Type	Order	Notation	Trace
(0)	2	$7A_1$	-7
(i)	2	$4A_1$	-1
(i)'	2	$3A_1$	1
(ii)	3	$3A_2$	-2
(ii)'	6	$E_7(a_4)$	2
(iii)	3	$2A_2$	1
(iii)'	6	$D_6(a_2)+A_1$	-1
(iv)	4	$2A_3+A_1$	-3
(iv)'	4	$D_4(a_1)$	3
(v)	4	$D_4(a_1)+A_1$	1
(v)'	4	$2A_3$	-1
(vi)	6	$E_6(a_2)$	2
(vi)'	6	A_2+A_5	-2
(vii)	7	A_6	0
(vii)'	14	$E_7(a_1)$	0
(viii)	8	D_5	1
(viii)'	8	D_5+A_1	-1
(ix)	9	$E_6(a_1)$	1
(ix)'	18	E_7	-1
(x)	12	$E_7(a_2)$	-2
(x)'	12	E_6	0

TABLE 4. Conjugacy classes of automorphims of a Del Pezzo surface of degree 2

(4) Order 6 (A_5+A_2) $g=[t_0,-t_1,\epsilon_3t_2,-t_3]$

$$F=T_3^2+T_0^4+T_1^4+T_0T_2^3+aT_0^2T_1^2.$$

(5) Order 6 ($D_6(a_2)+A_1$) $g=[t_0,\epsilon_3x_1,\epsilon_3^2x_2,-x_3]$

$$F=T_3^2+T_0(T_0^3+T_1^3+T_2^3)+T_1T_2(\alpha T_0^2+\beta T_1T_2).$$

(6) Order 12 ($E_7(a_2)$) $[t_0,\epsilon_4t_1,\epsilon_3t_2,t_3]$

$$F=T_3^2+T_0^4+T_1^4+T_0T_2^3,\ (t_0,t_1,t_2,t_3).$$

(7) Order 14 ($E_7(a_1)$) $g=[t_0,\epsilon_4t_1,\epsilon_3t_2,t_3]$

$$F=T_3^2+T_0^3T_1+T_1^3T_2+T_2^3T_0.$$

(8) Order 18 (E_7) $[t_0,\epsilon_3t_1,\epsilon_9^2t_2,-t_3]$

$$F=T_3^2+T_0^4+T_0T_1^3+T_2^3T_1.$$

Using the information about cyclic groups of automorphisms of plane quartics, it is not hard to get the classification of possible automorphism groups (see [24]). It is given in Table 5.

Next we find minimal subgroups of automorphisms of a Del Pezzo surface of degree 2.

First notice that any group G containing the Geiser involution γ representing the conjugacy class $7A_1$ is a minimal group.

Type	Order	Structure	Equation	Parameters
I	336	$2 \times L_2(7)$	$T_3^2 + T_0^3T_1 + T_1^3T_2 + T_2^3T_0$	
II	192	$2 \times (4^2 : S_3)$	$T_3^2 + T_0^4 + T_1^4 + T_2^4$	
III	96	$2 \times 4.A_4$	$T_3^2 + T_2^4 + T_0^4 + 2\sqrt{-3}T_0^2T_1^2 + T_1^4$	
IV	48	$2 \times S_4$	$T_3^2 + T_2^4 + T_1^4 + T_0^4 + a(T_0^2T_1^2 + T_0^2T_2^2 + T_1^2T_2^2)$	$a \neq \frac{-1\pm\sqrt{-7}}{2}$
V	32	$2 \times (D_8 : 2)$	$T_3^2 + T_2^4 + T_0^4 + aT_0^2T_1^2 + T_1^4$	$a^2 \neq 0, -12, 4, 36$
VI	18	18	$T_3^2 + T_0^4 + T_0T_1^3 + T_1T_2^3$	
VII	16	$2 \times D_8$	$T_3^2 + T_2^4 + T_0^4 + T_1^4 + aT_0^2T_1^2 + bT_2^2T_0T_1$	$a, b \neq 0$
VIII	12	2×6	$T_3^2 + T_0^4 + T_1^4 + T_0T_2^3 + aT_0^2T_1^2$	
IX	12	$2 \times S_3$	$T_3^2 + T_0^4 + aT_0^2T_1T_2 + T_0(T_1^3 + T_2^3) + bT_1^2T_2^2$	
X	8	2^3	$T_3^2 + T_2^4 + T_1^4 + T_0^4 + aT_2^2T_0^2 + bT_1^2T_2^2 + cT_0^2T_1^2$	$a \neq b \neq c$
XI	6	6	$T_3^2 + T_2^3T_0 + L_4(T_0, T_1)$	
XII	4	2^2	$T_3^2 + T_2^4 + T_2^2L_2(T_0, T_1) + L_4(T_0, T_1)$	
XIII	2	2	$T_3^2 + F_4(T_0, T_1, T_2)$	

TABLE 5. Groups of automorphisms of Del Pezzo surfaces of degree 2

As in the previous case it is enough to consider surfaces S' which are not specialized to surfaces S of other types. When this happens we indicate by $S' \to S$. The following specializations may occur

$$IX \to IV \to I, II, \quad VII \to V \to II, III, \quad XII \to II,$$

$$XI \to VIII \to III, \quad VII \to III, X \to I, II, III, IV, V.$$

So it suffices to consider the surfaces of types I, II, III, VI.

Type I.

It is known that the group $L_2(7)$ is generated by an element of order 2 of type $4A_1$, of order 3 of type $3A_2$ and of order 7 of type A_6. Comparing the traces with the character table of the group $L_2(7)$ we find that the representation of $L_2(7)$ in $(K_S^{\perp}) \otimes \mathbb{C}$ is a 7-dimensional irreducible representation of $L_2(7)$. Thus the group is minimal.

Let G be a subgroup of $\mathrm{Aut}(S)$. If $\gamma \in G$, then $G = \langle\gamma\rangle \times G'$ for some subgroup of $L_2(7)$. It is minimal subgroup. If $\gamma \notin G$, then either G is a subgroup of $L_2(7)$, or it contains a subgroup G' of index 2 contained in $L_2(7)$.

Assume G is a subgroup of $L_2(7)$. It is known that maximal subgroups of $L_2(7)$ are isomorphic to S_4 or $7:3$. There are two conjugacy classes of subgroups isomorphic to S_4. It is known that $L_2(7) \cong L_3(\mathbb{F}_2)$. A subgroup isomorphic to S_4 is realized as stabilizer subgroup of a point or a line in $\mathbb{P}^2(\mathbb{F}_2)$. The permutation representation of S_4 defined by the left cosets splits into the trivial representation and 6-dimensional representation isomorphic to $V = K_S^{\perp} \otimes \mathbb{C}$ see [16]. The Frobenius Reciprocity formula shows that $\dim V^{S_4} = \langle\chi_6, \chi_6\rangle$, where χ_6 is the character of $L_2(7)$ of the 6-dimensional irreducible representation. It is easy to check that this number is equal to 1. Thus S_4 is not minimal. Similarly, we find that $\dim V^{7:3} = \langle\chi_8, \chi_6\rangle = 1$ and hence $7:3$ is not minimal too. Thus any proper subgroup of $L_2(7)$ is not minimal.

Assume that G contains a subgroup H of index 2 contained in $L_2(7)$. Let G' be its image in $L_2(7)$. Then $G = (2, 1, G', H)$. Since the group $7:3$ is of odd order, G'

must be a subgroup of S_4. So it is enough to show that $(2,1,S_4,A_4)$ is not minimal. For any subset X of a group G let

$$t(X)=\sum_{g\in X}\mathrm{Tr}(g|K_S^{\perp}). \tag{6.10}$$

The group G is minimal if and only if $t(G)=0$. Let H be a subgroup of index 2 of G and gH be its non-trivial coset. Then $t(G)=t(H)+t(gH)$. In our case $H=A_4\subset L_2(7)$ and elements $\gamma g, g\in S_4\setminus A_4$. Since $\mathrm{Tr}(\gamma g)=-\mathrm{Tr}(g)$, we obtain that

$$t(G)=t(A_4)-t(sA_4)=t(S_4)-2t(sA_4)=4!-2t(sA_4)(^2)$$

The set sA_4 consists of 6 elements of order 2 with trace -1, and 6 elements of order 4 with trace 1. Thus $t(sA_4)=0$ and we get $t(G)=4!$, hence $\dim V^G=1$. So the group is not minimal.

Type II.

The group $\mathrm{Aut}(S)$ is generated by the transformations

$$g_1=[t_0,it_1,t_2,t_3],\ \tau=[t_1,t_0,t_2,t_3],\ \sigma=[t_0,t_2,t_1,t_3].$$

Let $g_2=\sigma g_1\sigma^{-1}=[t_0,t_1,it_2,t_3]$. We have

$$\tau g_1\tau^{-1}=g_1^{-1}g_2^{-1}\gamma,\ \tau g_2\tau^{-1}=g_2.$$

The elements g_1,g_2,γ generate the normal subgroup isomorphic to $4^2\times 2$. The quotient group is isomorphic to S_3. Its generators of order 2 can be represented by τ and σ. Thus

$$\mathrm{Aut}(S)\cong(2\times 4^2):S_3\cong 2.(4^2:S_3)\cong 2^3.S_4. \tag{6.11}$$

Let $f:\mathrm{Aut}(S)\to S_3$ be the natural homomorphism. We will consider different cases according to the possible image of G in S_3

Case 1: $f(G)=\{1\}$.

Assume that $G\subset\mathrm{Ker}(f)=2\times 4^2$. We use Lemma 4.1 to classify subgroups of 2×4^2. The group S_3 acts on the set of subgroups by conjugation. We start with cyclic subgroups of order 4. It is generated by an element $g_1^ag_2^b$ or $g_1^ag_2^b\gamma$, where a or b is equal to 1 or 3. We have

$$\sigma(g_1^ag_2^b\gamma^c)\sigma^{-1}=g_1^bg_2^a\gamma^c,\ \tau(g_1^ag_2^b\gamma^c)\tau^{-1}=g_1^{-a}g_2^{b-a}\gamma^c.$$

Thus we may assume that $a=1$. If $c=0$, the element g_1 is minimal, the elements $g_1g_2,g_1g_2^2,g_1g_2^3$ are not minimal. If $c=1$, the element $g_1g_2\gamma$ is minimal, the other ones are not minimal. Since $(g_1g_2\gamma)$ and (g_1) are conjugate by τ, we get only one conjugacy class of minimal subgroups of order 4.

To classify subgroups isomorphic to 4^2 we need to know possible surjective homomorphisms $4^2\to 2$, up to conjugacy by an element of S_3. Such a homomorphism is defined by its values on g_1,g_2. Since g_1,g_2 are conjugate, there are two non-conjugate homomorphisms defined by $g_1,g_2\mapsto 1$, or $g_1\mapsto 1,g_2\mapsto 0$. This gives 3 subgroups isomorphic to 4^2. They are $(g_1,g_2),(g_1\gamma,g_2),(g_1\gamma,g_2\gamma)$. The first two are minimal, the third one is not. This gives 2 conjugacy classes of subgroups isomorphic to 4^2 and 2×4^2.

Next we classify subgroups G isomorphic to 2×4. If it is contained in 4^2, then it is equal to the kernel of a surjective homomorphism $4^2\to 2$. This gives two subgroups (g_1^2,g_2) and (g_1g_2,g_2^2). The first is minimal, the second is not. If G

[2] Igor, maybe $s=\gamma$?

is not contained in 4^2, it follows from Goursat's Lemma that G contains a cyclic subgroup G' of order 4 contained in one of the previous subgroups H of 4^2, the coset consists of elements γg, where g is the nontrivial coset H modulo G'. This gives 4 subgroups $(g_2, \gamma g_1^2), (g_2 g_1^2, \gamma g_2), (g_1 g_2, \gamma g_2^2), (g_1 g_2^3, \gamma g_2^2)$. The first is minimal, the rest are not.

Thus we have found two conjugacy classes of minimal subgroups isomorphic to 2×4. They are $(g_1^2, g_2), (\gamma g_1^2, g_2)$. Two more are subgroups which contain γ. They are $\langle \gamma, g_2 \rangle$ and $\langle \gamma, g_1 g_2 \rangle$.

Next assume that G is a 2-group isomorphic to 2^2 or 2^3. The trace of an element of order 2 in $K_S^\perp$ different from γ is equal to 1 or -1. There are 7 elements of order 2 in $\mathrm{Ker}(f)$. They are

$$\gamma, g_1^2, g_2^2, (g_1 g_2)^2, \gamma g_1^2, \gamma g_2^2, \gamma g_1^2, g_1^2.$$

Their traces are equal to $-7, -1, -1, -1, 1, 1, 1$, respectively. A group is minimal if the sum of traces of non-trivial elements is equal to -7. This easily easily gives that, up to conjugation, the minimal groups are $(\gamma, g_1^2, g_2^2) \cong 2^3$ and $(\gamma, g_1^2) \cong 2^2$.

Case 1: $f(G) \cong 2$.

Replacing G with conjugate group we may assume that $f(G) = (\tau)$. Let

$$g = [\epsilon_4^a x_1, x_0, \epsilon_4^b x_2, (-1)^c x_3]$$

be an element of largest order in G which is mapped to τ. We have

$$g^2 = g_2^{2b-a} \gamma^a.$$

If a is odd, then we find $g_2 \gamma^{a+c}$ in (g), and hence we may assume that $b = 0$. In this case $\mathrm{ord}(g) = 8$. If a is even and $2b - a \neq 0 \mod 4$, g is of order 4. If $a = 0$, then b is odd and we may assume that $g = \tau g_2 \gamma^c$. If $a = 2$, then b is even and we may assume that $g = \tau g_1^2 \gamma^c$. Finally, if $2b - a = 0 \mod 4$, then we may assume that $g = \tau g_2^2 \gamma^c$ or $g = \tau g_1^2 g_2 \gamma^c$, or $g = \tau \gamma^c$. To sum up we list all possible cases:

(i) $\mathrm{ord}(g) = 8, g = \tau g_1 \gamma^c$,
(ii) $\mathrm{ord}(g) = 4, g = \tau g_2 \gamma^c$;
(iii) $\mathrm{ord}(g) = 4, g = \tau g_1^2 \gamma^c$;
(iv) $\mathrm{ord}(g) = 2, g = \tau \gamma^c$;
(v) $\mathrm{ord}(g) = 2, g = \tau g_2^2 \gamma^c$;
(vi) $\mathrm{ord}(g) = 2, g = \tau g_1^2 g_2 \gamma^c$.

Let $K = \mathrm{Ker}(f : G \to S_3)$.

Case (i). If $G = (g) \cong 8$, then it is minimal if and only if $c = 0$. Suppose $G \neq (g)$ and let $k = g_1^a g_2^b \gamma^\epsilon \in K$ but not in (g). Since $g^2 = g_2^{-1} \gamma \in K$, we may assume that $b = 0$ and $a = 0, 1, 2$. If $a = 0$, G contains γ and hence $G = (\gamma, \tau g_1) \cong 8 \times 2$ or $G = (\gamma, \tau, g_1) \cong (2 \times 4^2) : 2$, both minimal.

If $a = 1$, we get $g_1 \gamma^\epsilon \in K$, hence $\tau \gamma^{\epsilon'} \in G$. Thus

$$G = (\tau \gamma^\epsilon, g_1 \gamma^{\epsilon'}) \cong 4^2 : 2.$$

If G contains one of the minimal elements $g_1, \tau g_1$, then G is minimal. This shows that the only possible non-minimal group is $G = (\tau, g_1 \gamma)$. By computing the traces, we find that it is indeed not minimal. There are three non-conjugate minimal groups $(\tau\gamma, g_1\gamma), (\tau\gamma, g_1)$, and (τ, g_1).

If $a = 2$, and $\gamma \in G$, then $G = (\gamma, g_1^2, \tau g_1) \cong 2 \times D_{16}$ with D_{16} generated by τg_1 and g_1^2.

Case (ii). We have $g^2 = g_2^2$. Let $k = g_1^a g_2^b \gamma^c$ be chosen as in case (i). If $a = 1$, then multiplying g and k we find an element of order 8 mapping to τ. This is case (i). So we may assume that $a = 0, 2$. If $a = 0$ and $b = 0$, multiplying by g^2, we find that $\gamma \in G$ and $G = (\gamma, \tau g_2) \cong 2 \times 4$ or $\gamma, \tau, g_2) \cong 2^2 \times 4$, both minimal. Assume $a = 0$ and $b = 1$. Then $G = (g_2\gamma^\epsilon, \tau\gamma^{\epsilon'}) \cong 2 \times 4$, or $G = (g_2, \tau, \gamma) \cong D_8$. In the first case G is minimal if and only if $\epsilon = 0$.

Finally assume $a = 2$. If b is even, G contains $g_1^2\gamma^\epsilon, \tau g_2 \gamma^c$. If it contains γ, then it is a minimal group isomorphic to $(2^3)^{\cdot}2 \cong 2^2 : 4$ (the same group as we had in the case of Del Pezzo surfaces of degree 4) .

If G does not contain γ, then $G = (g_1^2\gamma^\epsilon, \tau g_2\gamma^c) \cong (2^2)^{\cdot}2$ or $(g_1^2\gamma^\epsilon, \tau\gamma^{\epsilon'}, g_2\gamma^{\epsilon''}) \cong (4 \times 2) : 2$. The first group is minimal if and only if $\epsilon = 0, c = 1$. The second group is minimal if and only if $\epsilon' = 0$. If $b = 1$, then we have the following possible groups G

$$G_1 = (\tau\gamma^c, g_2\gamma^{c'}) \cong 2 \times 4, G_2 = (\tau, g_2, \gamma) \cong 2^2 \times 4, G_3 = (\tau g_2 \gamma^{c'}, g_1^2 g_2 \gamma^c) \cong (2^2)^{\cdot}2,$$

The groups G_1 is minimal only if $c' = 0$. The groups $(\tau\gamma, g_2)$ and (τ, g_2) are conjugate. The group G_2 is minimal. Computing the traces we find that the group G_3 is minimal only if $c = c' = 1$.

Case (iii) Repeating the argument from the previous case we take some $k = g_1^a g_2^b \gamma^c$ and get $gk = \tau g_1^{2+a} g_2^b \gamma^{\epsilon+c} \in G$. If a is odd we get an element of order 8 mapping to τ, so a is even. If $a = 0$ and b is odd, after multiplying by $g^2 = g_2^2$, we get the previous case. If b is even, we can replace k with γ. So, the only new minimal subgroup is $G = (\gamma, \tau g_1^2) \cong 2 \times 4$. Note this group is not conjugate to the isomorphic group $(\tau\gamma, g_2)$ appeared earlier.

Case (iv) As before let $k = g_1^a g_2^b \gamma^\epsilon$ be an element from K not from (g). If the order of $gk = \tau g_1^a g_2^b \gamma^{c+\epsilon}$ is greater than 2 we are in one of the previous cases. Since $(gk)^2 = g_2^{2b-a}$, we see that $a \equiv 2b \mod 4$. So $G = (\tau, \gamma) \cong 2^2, G = (\tau\gamma^c, g_1^2 g_2 \gamma^{c'}) \cong D_8$ or $G = (\tau, g_1^2 g_2, \gamma) \cong 2 \times D_8$. The second group is obviously minimal. Computing the traces we find that the first group is minimal if and only $c + c' \equiv 0 \mod 2$. The two possible groups are conjugate.

Case (v) As before let $k = g_1^a g_2^b \gamma^\epsilon$ be an element from K. If the order of $gk = \tau g_1^a g_2^{b+2} \gamma^{c+\epsilon}$ is greater than 2 we are in one of the previous cases. Since $(gk)^2 = g_2^{2b-a}$ we obtain that $a \equiv 2b \mod 4$. If $a = 0$, we get g_2^2 or $g_2^2 \cdot \gamma$ in G, hence τ or $\tau\gamma$ belongs to G, so we are in the previous case. If $a = 2$ we get b is odd. Thus $G = (\tau g_2^2 \gamma^c, g_1^2 g_2 \gamma^{c'}) = (\tau\gamma^c, g_1^2 g_2 \gamma^{c'}$ or $G = (\tau g_2^2, g_1^2 g_2, \gamma) = (\tau, g_1 g_2^2, \gamma)$. These are the same groups as in the previous case.

Case (vi) As before let $k = g_1^a g_2^b \gamma^\epsilon$ be an element from K. If the order of $gk = \tau g_1^{2+a} g_2^{b+1} \gamma^{c+\epsilon}$ is greater than 2 we are in one of the previous cases. Since $(gk)^2 = g_2^{2b-a}$ we obtain that $a \equiv 2b \mod 4$. Hence $g_1^2 g_2 \gamma^\epsilon \in G$, and we are reduced to case (iv).

Case 3: $f(G) \cong 3$.

By Lemma 4.2, the extension $G = (G \cap \mathrm{Ker}(f)).3$ splits and all subgroups of order 3 are conjugate. So we may assume that $G = \langle \tau\sigma, H \rangle$, where $H = G \cap \mathrm{Ker}(f)$ is $\tau\sigma$-invariant subgroup of 2×4^2.

Suppose $H \cong \langle g \rangle \cong 2$. Then $G = \langle \tau\sigma g \rangle \cong 6$ is a minimal cyclic group from the conjugacy class $D_6(a_2) + A_1$.

It is easy to see that there are no cyclic groups of order 4 invariant with respect to $\tau\sigma$. So we may assume next that $H \cong 4^2$. The group H is generated by two conjugate elements of order 4. This gives two possible groups $H = \langle g_1, g_2\rangle$ or $H = \langle g_1\gamma, g_2\gamma\rangle$. The first group is minimal and gives us a minimal group $G \cong 4^2 : 3$. In the second case G contains $\tau\sigma g_1 g_2$ from the minimal conjugacy class $D_6(a_2)+A_1$ of order 6. So it is also minimal. This gives two conjugacy classes of groups $4^2 : 3$.

Next assume that $H \cong 2^3$. It contains the central element γ and hence minimal. It is easy to see that

$$2^3 : 3 \cong 2 \times (2^2 : 3) \cong 2 \times A_4.$$

Assume $H \cong 2^2$. There is only one invariant subgroups $\langle g_1^2, g_2^2\rangle$. The group contains 3 elements of order 2 of type $4A_1$ and 8 elements of order 3 of type $2A_2$. Adding the traces we find that this group is not minimal.

Obviously, a subgroup isomorphic to 2×4 is not invariant. So we got the following minimal groups

$$6 \text{ (type } D_6(a_2) + A_1), \quad 2 \times A_4, \quad 4^2 : 3 \text{ (2 conjugacy classes)}, \quad (2 \times 4^2) : 3.$$

Case 4: $f(G) = S_3$. Let $H = \mathrm{Ker}(f : G \to S_3)$. Since G normalizes H, H must be one of the subgroups $\{1\}, \langle\gamma\rangle, \langle g_1, g_2\rangle, \langle g_1, g_2, \gamma\rangle, \langle g_1^2, g_2^2, \gamma\rangle, \langle g_1^2, g_2^2\rangle$. As before we may assume that $\tau\sigma \in G$. Let $g = \tau g_1^a g_2^b \gamma^c = (x_1, i^a x_0, i^b x_2, (-1)^c x_3)$ be an element in G of smallest order $2n$ which is mapped to τ. After conjugating by σ we may assume that $a \le b \le 2$. We have

$$g^2 = (i^a x_0, i^a x_1, i^{2b} x_2, x_3) = (x_0, x_1, i^{2b-a} x_2, (-1)^a x_3) = g_2^{2b-a}\gamma^a \in H.$$

If $n = 1$, we obtain $a = b = 0$, hence S_3 splits. If $\gamma \in G$, we get one of the following minimal groups

$$2 \times S_3 = \langle\tau, \sigma, \gamma\rangle, \quad 2 \times S_4 = \langle\tau, \sigma, g_1^2, g_2^2, \gamma\rangle, \quad 2 \times (4^2 : S_3).$$

If $\gamma \notin G$, we get one of the following minimal groups

$$S_4 \cong 2^2 : S_3 = \langle\tau\gamma^c, \sigma\gamma^c, g_1^2, g_2^2\rangle, c = 0, 1, \quad 4^2 : S_3 = \langle\tau\gamma^c, \sigma\gamma^c, g_1, g_2\rangle, c = 0, 1.$$

Note that the group $\langle\tau\gamma, \sigma\rangle$ is not minimal.

If $n = 2$, we get $a = b = 1$ or $a = 0, b = 1$. In the first case, we have $g_2\gamma \in G$. Replacing g with $g(g_2^3\gamma)$ we are in the previous case. So, we get the following minimal groups

$$2 \times 2^2.S_3 = \langle\tau g_2, \sigma g_2^2, \gamma\rangle, \quad 2^2.S_3 = \langle\tau g_2, \sigma, g_1^2, g_2^2\rangle, \quad 2^2.S_3 = \langle\tau g_2\gamma, \sigma, g_1^2, g_2^2\rangle.$$

Type III.

By Lemma 4.1, G either contains γ, or is a subgroup of $D_8 : 6$, or equals $(2, 1, G', H)$ for some subgroup H of G' of index 2. In the first case G is minimal.

The group $G' = D_8 : 6 \cong 4.A_4$ is generated by a minimal transformation g_1 of order 12 $g_1 = [\frac{t_0}{1-i} + \frac{it_1}{1-i}, \frac{t_0}{1-i} - \frac{it_1}{1-i}, i\epsilon_6 t_2, \epsilon_3 t_3]$ and transformations of order 2 $g_2 = [t_1, t_0, t_2, t_3]$, and $g_3 = [t_0, -t_1, t_2, t_3]$. The center of G' is of order 4 and is generated by a minimal element $c = g_1^3 = [t_0, t_1, it_2, t_3]$. So any subgroup containing the center is minimal.

The elements g_1^2, g_2, g_3 generate a subgroup G'' of G' of index 2 isomorphic to $D_8 : 3$, where the normal subgroup is generated by g_2, g_3.

The subgroup $\langle g_1^2, g_2 g_3\rangle \cong D_{12}$. One checks that it is not minimal.

The subgroup $\langle g_2, g_3\rangle \cong D_8$ consists of 5 elements of order 2 and 2 elements of order 4, all with trace equal to -1. Adding up the traces we obtain that D_8 is a minimal subgroup. Thus $D_8:3$ is minimal too. One can check that the remaining minimal subgroups all contain the center.

Now assume that $G = (\langle\gamma\rangle, 1, G', H) \cong H.2$. The following subgroups G' (up to conjugacy) of $D_8:6$ contain subgroups of index 2:

$$\begin{aligned}
D_8:6 = \langle g_1, g_2.g_3\rangle &\supset D_8:3 = \langle g_1^2, g_2, g_3\rangle,\\
D_8:3 = \langle g_1^2, g_2.g_3\rangle &\supset D_{12} = \langle g_1^2, g_2g_3\rangle \supset 6 = \langle g_1^2\rangle\\
D_8:2 = \langle g_2, g_3, g_1^3\rangle &\supset 2\times 4 = \langle g_2, c\rangle \supset 4 = \langle c\rangle, 2^2 = \langle g_2, c^2\rangle,\\
D_8:2 &\supset D_8 = \langle g_2, g_3\rangle \supset 2^2 = \langle g_3, c^2\rangle \supset 2 = \langle c^2\rangle,\\
D_8 &\supset 4 = \langle g_2g_3\rangle,\\
12 = \langle g_1\rangle &\supset 6 = \langle g_1^2\rangle \supset \langle g_1^4\rangle,\\
4 = \langle c\rangle &\supset \langle = \langle c^2\rangle\rangle.
\end{aligned}$$

Since $D_8:3$ is minimal, the group $(\langle\gamma\rangle, 1, D_8:6, D_8:3) \cong D_8:6$ is minimal. The subgroup $(\langle\gamma\rangle, 1, 12, 6)$ is a cyclic group generated by a non-minimal element of order 12, hence it is not minimal. Therefore its subgroups are not minimal. Since $\langle g_2, g_3\rangle \cong D_8$ is minimal, the subgroup

$$(\langle\gamma\rangle, 1, \langle g_2, g_3, g_1^3\rangle, \langle g_2, g_3\rangle) \cong (2, 1, D_8:2, D_8) \cong D_8:2$$

is minimal. Since $\langle g_2, c\rangle \cong 2\times 4$ is minimal,

$$(\langle\gamma\rangle, 1, \langle g_2, g_3, g_1^3\rangle, \langle g_2, c\rangle) \cong (2, 1, D_8:2, 2\times 4) \cong D_8:2$$

is minimal. So we have found 3 non-conjugate subgroups isomorphic to $D_8:2$.

The group $(\langle\gamma\rangle, 1, D_8:3, D_{12}) \cong D_8:3$ is minimal. The sum of the traces of elements of the coset of $\gamma\times 1$ of the subgroup D_{12} is the negative of the sum of the traces of elements from the subgroup.

By adding the traces we check that the groups $(\langle\gamma\rangle, 1, D_8, (g_2)), (\langle\gamma\rangle, 1, D_8, (g_2^2, c^2))$ and there subgroups are not minimal.

All other possible minimal groups are subgroups of $\langle\gamma, c, g_2\rangle \cong 2^2\times 4$. It is easy to find them. They are

$$\langle c^2, \gamma, g_2\rangle \cong 2^3, \quad \langle c, \gamma\rangle, \quad \langle cg_2, \gamma\rangle, \langle c, \gamma g_2\rangle, \langle c, g_2\rangle \cong 2\times 4, \quad \langle c\rangle \cong 4.$$

The group $(\langle\gamma\rangle, 1, \langle g_2, c\rangle, \langle c\rangle) = \langle c, \gamma g_2\rangle \cong 2\times 4$ is minimal since it contains the minimal element c. The group $(\langle\gamma\rangle, 1, \langle g_2, c\rangle, \langle cg_2\rangle) = \langle cg_2, c\gamma\rangle \cong 2\times 4$ is not minimal.

Type VI.

This gives only one new minimal cyclic group: $\mathrm{Aut}(S) \cong 18$.

Let us summarize what we have found.

Theorem 6.15. *Let G be a minimal subgroup of automorphisms of a Del Pezzo surface of degree 2. Then G is one of the following groups.*

(1) *G is a group of automorphisms of a surface of type I:*

$$2, \quad 2^2, \quad 6, \quad 2^3, \quad 2\times 4, \quad 14,$$

$$2\times D_8, \quad 2\times S_3, \quad 2\times S_4\ (2), \quad 2\times A_4, \quad 2\times(7\times 3), \quad L_2(7), \quad 2\times L_2(7).$$

(2) *G is a group of automorphisms of a surface of type II:*

$$2,\quad 2^2,\quad 4,\quad 2^3,\quad 8,\quad 2\times 4\ (4),$$

$$2\times 8,\quad 2^2\times 4,\quad 4^2\ (2),\quad 2\times 4^2,\quad (2\times 4):2,\quad 2^2:4,$$

$$D_8,\quad 2\times D_8,\quad 2\times D_{16},\quad 4^2:2\ (3),\quad (2\times 4^2):2,$$

$$6,\quad 2\times A_4,\quad 4^2:3\ (2),\quad (2\times 4^2):3,\quad 2\times S_3,\quad S_4\ (2),$$

$$2\times S_4,\quad 4^2:S_3,\quad 2\times 2^2.S_3\ (2),\quad 2^2.S_3\ (2),\quad 2\times(4^2:S_3).$$

(3) *G is a group of automorphisms of a surface of type III:*

$$2,\quad 2^2,\quad 4,\quad 2^3,\quad 6,\quad 12,\quad 2\times 6,$$

$$2\times 4\ (4),\quad 2^2\times 4,\quad 2\times 12,\quad 2\times 4.A_4,\quad 2\times(D_8:3),\quad 2\times(D_8:2),$$

$$2\times D_8,\quad D_8:6\ (2),\quad D_8:3\ (2),\quad D_8:2\ (3),\quad D_8.$$

(4) *G is a group of automorphisms of a surface of type IV:*

$$2\times S_4,\quad 2\times D_8,\quad 2\times A_4,\quad 2\times 4,\quad 6,\quad 2^3,\quad 2^2,\quad 2.$$

(5) *G is a group of automorphisms of a surface of type V:*

$$2\times D_8:2,\quad 2^2\times 4,\quad D_8:2,\quad D_8,\quad 2\times 4\ (2),\quad 2^3,\quad 2^2,\quad 4,\quad 2.$$

(6) *G is a group of automorphisms of a surface of type VI:*

$$18,\quad 6,\quad 2.$$

(7) *G is a group of automorphisms of a surface of type VII:*

$$2\times D_8,\quad D_8,\quad 2^3,\quad 2\times 4,\quad 2^2,\quad 2.$$

(8) *G is a group of automorphisms of a surface of type VIII:*

$$2\times 6,\quad 2^2,\quad 6,\quad 2.$$

(9) *G is a group of automorphisms of a surface of type IX:*

$$2\times S_3,\quad 6,\quad 2^2,\quad 2.$$

(10) *G is a group of automorphisms of a surface of type X:*

$$2^3,\quad 2^2,\quad 2.$$

(11) *G is a group of automorphisms of a surface of type XI:*

$$6,\quad 2.$$

(12) *G is a group of automorphisms of a surface of type XII:*

$$2^2,\quad 2.$$

(13) *G is a group of automorphisms of a surface of type XIII:*

$$2.$$

Here the number in brackets indicates the number of conjugacy classes in the automorphism group.

6.7. **Automorphisms of Del Pezzo surfaces of degree 1.** Let S be a Del Pezzo surface of degree 1. We know that the linear system $|-2K_S|$ defines a finite map of degree 2 onto a quadratic cone in $\mathbb{P}^3$. Its branch locus is a nonsingular curve of genus 4 cut out by a cubic surface. Recall that a singular quadric is isomorphic to the weighted projective space $\mathbb{P}(1,1,2)$. A curve of genus 4 of degree 6 cut out in Q by a cubic surface is given by equation $F(T_0,T_1,T_2)$ of degree 6. After change of coordinates it can be given by an equation $T_2^3+F_4(T_0,T_1)T_2+F_6(T_0,T_1)=0$, where F_4 and F_6 are binary forms of degree 4 and 6. The double cover of Q branched along such curve is isomorphic to a hypersurface of degree 6 in $\mathbb{P}(1,1,2,3)$

$$T_3^2+T_2^3+F_4(T_0,T_1)T_2+F_6(T_0,T_1)=0 \tag{6.12}$$

The vertex of Q has coordinates $(0,0,1)$ and its pre-image in the cover consist of one point $(0,0,1,a)$, where $a^2+1=0$ (note that $(0,0,1,a)$ and $(0,0,1,-a)$ represent the same point on $\mathbb{P}(1,1,2,3)$. This is the base-point of $|-K_S|$. The members of $|-K_S|$ are isomorphic to genus 1 curves with equations $y^2+x^3+F_4(t_0,t_1)x+F_6(t_0,t_1)=0$. The locus of zeros of $\Delta=F_4^3+27F_6^2$ is the set of points in $\mathbb{P}^1$ such that the corresponding genus 1 curve is singular. It consists of a simple roots and b double roots. The zeros of F_4 are either common zeros with F_6 and Δ, or represent nonsingular elliptic curves with automorphism group isomorphic to $\mathbb{Z}/6$. The zeros of F_6 are either common zeros with F_4 and Δ, or represent nonsingular elliptic curves with automorphism group isomorphic to $\mathbb{Z}/4$. Our group $\bar{G}$ acts on $\mathbb{P}^1$ via a linear action on (t_0,t_1). Let P be the corresponding subgroup of $\mathrm{Aut}(\mathbb{P}^1)$. It leaves the sets $V(F_4)$ and $V(F_6)$ invariant.

The polynomials F_4 and F_6 are relative invariants of the binary polyhedral group $\bar{P}$. They are polynomials in Grundformen which were listed in section 5.5. Since ther are no Grundformen of degree ≤ 6 for the binary icosahedron group $\bar{I}$, the group $P=I$ is not realized. Similarly, we observe that $F_4=0$ for $P=O$ and F_6 is a unique (up to a constant factor) Grundform of degree 6. Let G act on the variable T_3,T_2 with characters χ_1,χ_2, then

$$\chi_1^2=\chi_2^3=\chi_3^2\chi_2=\chi_4,$$

where χ_3,χ_4 are the characters of G corresponding to the relative invariants F_4 and F_6 (if $F_4=0$ we just need $\chi_1^2=\chi_2^3=\chi_3$). We also use that the polynomial

$$\Delta=4F_4(T_0,T_1)^2+27F_6(T_0,T_1)^2$$

has no multiple roots since otherwise the surface is singular. Using this it is not difficult to list all possible group G which may act on S and possible F_4 and F_6 (up to coordinate change).

We denote the Bertini automorphism $[t_0,t_1,t_2,-t_3]$ by β and let $G'=G/(\beta$.

(1) *Cyclic groups G'*

(i) $G'=2, G=\langle[t_0,-t_1,t_2,t_3],\beta\rangle\cong 2^2$,

$$F_4=aT_0^4+bT_0^2T_1^2+cT_1^4,\quad F_6=dT_0^6+eT_0^4T_1^2+fT_0^2T_1^4+gT_1^6.$$

(ii) $G'=2, G=\langle[t_0,-t_1,-t_2,it_3]\rangle\cong 4$,

$$F_4=aT_0^4+bT_0^2T_1^2+cT_1^4,\quad F_6=T_0T_1(dT_0^4+eT_0^2T_1^2+fT_1^4).$$

(iii) $G'=3$, $G=\langle[t_0,\epsilon_3t_1,t_2,-t_3]\rangle\cong 6$,

$$F_4=T_0(aT_0^3+bT_1^3),\quad F_6=aT_0^6+bT_0^3T_1^3+cT_1^6;$$

(iv) $G' = 3, G = \langle[t_0, \epsilon_3 t_1, \epsilon_3 t_2, -t_3]\rangle \cong 6$,
$$F_4 = T_0^2 T_1^2, \quad F_6 = aT_0^6 + bT_0^3 T_1^3 + cT_1^6;$$
(v) $G' = 3, G = \langle[t_0, t_1, \epsilon_3 t_2, -t_3]\rangle \cong 6$,
$$F_4 = 0, \quad \text{any } F_6 \text{ without multiple roots;}$$
(vi) $G' = 4, G = \langle[it_0, t_1, -t_2, it_3], \beta\rangle \cong 4 \times 2$,
$$F_4 = aT_0^4 + bT_1^4, \quad F_6 = T_0^2(cT_0^4 + dT_1^4),$$
(vii) $G' = 4, G = \langle[t_0, t_1, -it_2, -\epsilon_8 t_3]\rangle \cong 8$,
$$F_4 = aT_0^2 T_1^2, \quad F_6 = T_0 T_1 (cT_0^4 + dT_1^4),$$
(viii) $G' = 5, G = \langle[t_0, \epsilon_5 t_1, t_2, -t_3]\rangle \cong 10$,
$$F_4 = aT_0^4, \quad F_6 = T_0(bT_0^5 + T_1^5),$$
(ix) $G' = 6, G = \langle[t_0, \epsilon_6 t_1, t_2, t_3], \beta\rangle \cong 2 \times 6$,
$$F_4 = T_0^4, \quad F_6 = aT_0^6 + bT_1^6,$$
(x) $G' = 6, G = \langle[\epsilon_6 t_0, t_1, \epsilon_3^2 t_2, t_3], \beta\rangle \cong 2 \times 6$,
$$F_4 = T_0^2 T_1^2, \quad F_6 = aT_0^6 + bT_1^6,$$
(xi) $G' = 6, G = \langle[-t_0, t_1, \epsilon_3 t_2, t_3], T\rangle \cong 2 \times 6$,
$$F_4 = 0, \quad F_6 = dT_0^6 + eT_0^4 T_1^2 + fT_0^2 T_1^4 + gT_1^6,$$
(xii) $G' = 6, G = \langle[-t_0, t_1, \epsilon_6 t_2, it_3]\rangle \cong 12$,
$$F_4 = 0, \quad F_6 = T_0 T_1 (T_0^4 + aT_0^2 T_1^2 + bT_1^4),$$
(xiii) $G' = 10, G = \langle[t_0, \epsilon_{10} t_1, -t_2, it_3]\rangle \cong 20$,
$$F_4 = aT_0^4, \quad F_6 = T_0 T_1^5,$$
(xiv) $G' = 12, G = \langle[\epsilon_{12} t_0, t_1, \epsilon_3^2 t_2, -t_3], \beta\rangle \cong 2 \times 12$,
$$F_4 = aT_0^4, \quad F_6 = T_1^6,$$
(xv) $G' = 12, G = \langle[it_0, t_1, \epsilon_{12} t_2, \epsilon_8 t_3]\rangle \cong 24$,
$$F_4 = 0, \quad F_6 = T_0 T_1 (T_0^4 + bT_1^4),$$
(xvi) $G' = 15, G = \langle[t_0, \epsilon_5 t_1, \epsilon_3 t_2, \epsilon_{30} t_3]\rangle \cong 30$,
$$F_4 = 0, \quad F_6 = T_0(T_0^5 + T_1^5).$$

(2) *Dihedral groups*

(i) $G' = 2^2, G = D_8$,
$$F_4 = a(T_0^4 + T_1^4) + bT_0^2 T_1^2, \quad F_6 = T_0 T_1 [c(T_0^4 + T_1^4) + dT_0^2 T_1^2],$$
$$g_1 = [t_1, -t_0, t_2, it_3), \ g_2 = [t_1, t_0, t_2, t_3],$$
$$g_1^4 = g_2^2 = 1, g_1^2 = \beta, g_2 g_1 g_2^{-1} = g_2^{-1}.$$
(ii) $G' = 2^2, G = 2.D_4 \cong 2 \cdot 2^2$,
$$F_4 = a(T_0^4 + T_1^4) + bT_0^2 T_1^2, \quad F_6 = T_0 T_1 (T_0^4 - T_1^4),$$
$$g_1 = [t_0, -t_1, -t_2, it_3], \ g_2 = [t_1, t_0, -t_2, it_3],$$
$$g_1^2 = g^2 = (g_1 g_2)^2 = \beta.$$

(iii) $G' = D_6, G = D_{12}$,

$$F_4 = aT_0^2T_1^2, \quad F_6 = T_0^6 + T_1^6 + bT_0^3T_1^3,$$

$$g_1 = [t_0, \epsilon_3 t_1, \epsilon_3 t_2, -t_3], g_2 = [t_1, t_0, t_2, t_3],$$

$$g_1^3 = \beta, g_2^2 = 1,\ g_2g_1g_2^{-1} = g_1^{-1}.$$

(v) $G' = D_8, G = D_{16}$,

$$F_4 = aT_0^2T_1^2, \quad F_6 = T_0T_1(T_0^4 + T_1^4),$$

$$g_1 = [\epsilon_8 t_0, \epsilon_8^{-1}t_1, -t_2, it_3],\ g_2 = [t_1, t_0, t_2, t_3],$$

$$g_1^4 = \beta, g_2^2 = 1,\ g_2g_1g_2^{-1} = g_1^{-1}.$$

(vi) $G' = D_{12}, G = 2.D_{12}$,

$$F_4 = aT_0^2T_1^2, \quad F_6 = T_0^6 + T_1^6,$$

$$g_1 = [t_0, \epsilon_6 t_1, \epsilon_3^2 t_2, t_3], g_2 = [t_1, t_0, t_2, t_3], g_3 = \beta.$$

We have

$$g_1^6 = g_2^2 = g_3^3 = 1,\ g_2g_1g_2^{-1} = g_1^{-1}g_3.$$

(3) *Other groups*

(i) $G' = A_4, G = 2.A_4 \cong \bar{T}$, binar tetrahedron group,

$$F_4 = T_0^4 + 2\sqrt{-3}T_0^2T_1^2 + T_2^4, \quad F_6 = T_0T_1(T_0^4 - T_1^4),$$

$$g_1 = [it_0, -it_1, t_2, t_3],\ g_2 = [it_1, it_0, t_2, t_3],$$

$$g_3 = \frac{1}{\sqrt{2}}[\epsilon_8^{-1}t_0 + \epsilon_8^{-1}t_1, \epsilon_8^5 t_0 + \epsilon_8 t_1, \sqrt{2}\epsilon_3 t_2, \sqrt{2}t_3]$$

(ii) $G' = 3 \times D_4, G = 3 \times D_8$

$$F_4 = 0, \quad F_6 = T_0T_1(T_0^4 + aT_0^2T_1^2 + T_1^4),$$

(iii) $G' = 3 \times D_6, G = 6.D_6 \cong 2 \times 3.D_6$

$$F_4 = 0, \quad F_6 = T_0^6 + aT_0^3T_1^3 + T_1^6,$$

It is generated by

$$g_1 = [t_0, t_1, \epsilon_3 t_2, t_3],\ g_2 = [t_0, \epsilon_3 t_1, t_2, t_3],\ g_3 = ([t_1, t_0, t_2, t_3],\ \beta).$$

We have $g_3 \cdot g_2 \cdot g_3^{-1} = g_2^{-1}g_1^2$.

(iv) $G' = 3 \times D_{12}, G = 6.D_{12}$

$$F_4 = 0, \quad F_6 = T_0^6 + T_1^6,$$

It is generated by

$$g_1 = [t_0, t_1, \epsilon_3 t_2, t_3],\ g_2 = [t_0, \epsilon_6 t_1, t_2, t_3], g_3 = ([t_1, t_0, t_2, t_3], \beta).$$

We have $g_3 \cdot g_2 \cdot g_3^{-1} = g_2^{-1}g_1^2\beta$.

(v) $G' = 3 \times S_4, G = 3 \times 2.S_4$

$$F_4 = 0, \quad F_6 = T_0T_1(T_0^4 - T_1^4),$$

$$g_1 = [\epsilon_8 t_0, \epsilon_8^{-1}t_1, it_2, it_3],\ g_2 = [t_1, t_0, -t_2, it_3]$$

$$g_3 = \frac{1}{\sqrt{2}}[\epsilon_8^{-1}t_0 + \epsilon_8^{-1}t_1, \epsilon_8^5 t_0 + \epsilon_8 t_1, \sqrt{2}t_2, \sqrt{2}t_3], \quad g_4 = [t_0, t_1, \epsilon_3 t_2, t_3].$$

Type	Order	Structure	F_4	F_6	Parameters
I	144	$3 \times \bar{O}$	0	$T_0T_1(T_0^4 - T_1^4)$	
II	72	$6.D_{12}$	0	$T_0^6 + T_1^6$	
III	36	$2 \times 3.D_6$	0	$T_0^6 + aT_0^3T_1^3 + T_1^6$	$a \neq 0$
IV	30	30	0	$T_0(T_0^5 + T_1^5)$	
V	24	$3 \times D_8$	0	$T_0T_1G_2(T_0^2, T_1^2)$	
VI	24	$\bar{T}$	$T_0^4 + 2\sqrt{-3}T_0^2T_1^2 + T_1^4$	$aT_0T_1(T_0^4 - T_1^4)$	$a^2 \neq 0, 2\sqrt{-3}$
VII	24	$3 : D_8$	$aT_0^2T_1^2$	$T_0^6 + T_1^6$	$a \neq 0$
VIII	24	2×12	0	$T_0^4T_2 + T_1^6$	
IX	20	20	T_0^4	$T_0T_1^5$	
X	16	D_{16}	$T_0^2T_1^2$	$T_0T_1(T_0^4 + T_1^4)$	
XI	12	D_{12}	$T_0^2T_1^2$	$T_0^6 + aT_0^3T_1^3 + T_1^6$	$a \neq 0$
XII	12	2×6	0	$G_3(T_0^2, T_1^2)$	
XIII	12	2×6	aT_0^4	$bT_0^6 + T_1^6$	$a, b \neq 0$
XIV	10	10	aT_0^4	$T_0(bT_0^5 + T_1^5)$	$a, b \neq 0$
XV	8	$2.D_4$	$T_0^4 + T_1^4 + aT_0^2T_1^2$	$bT_0T_1(T_0^4 - T_1^4)$	$a \neq 2\sqrt{-3}, b \neq 0$
XVI	8	2×4	$aT_0^4 + T_1^4$	$T_0^2(bT_0^4 + cT_1^4)$	
XVII	8	D_8	$T_0^4 + T_1^4 + aT_0^2T_1^2$	$T_0T_1G_2(T_0^2, T_1^2)$	
XVIII	6	6	0	$F_6(T_0, T_1)$	
XIX	6	6	$T_0(aT_0^3 + bT_1^3)$	$cT_0^6 + dT_0^3T_1^3 + T_1^6$	
XX	4	4	$aT_0^4 + bT_0^2T_1^2 + cT_1^4$	$T_0T_1G_2(T_0^2, T_1^2)$	
XXI	4	2^2	$aT_0^4 + bT_0^2T_1^2 + cT_1^4$	$G_3(T_0^2, T_1^2)$	
XXII	2	2	$F_4(T_0, T_1)$	$F_6(T_0, T_1)$	

Table 6. Groups of automorphisms of Del Pezzo surfaces of degree 1

Table 6 gives the list of the full automorphism groups of Del Pezzo surfaces of degree 1.

The following is the list of cyclic minimal groups $\langle g \rangle$of automorphisms of Del Pezzo surfaces $V(F)$ of degree 1.

(1) Order 2
- A_1^8 (the Bertini involution) $g = [t_0, t_1, t_2, -t_3]$
$$F = T_3^2 + T_2^3 + F_4(T_0, T_1)T_2 + F_6(T_0, T_1),$$

(2) Order 3
- $4A_2$ $g = [t_0, t_1, \epsilon_3 t_2, t_3]$
$$F = T_3^2 + T_2^3 + F_6(T_0, T_1),$$

(3) Order 4
- $2D_4(a_1)$ $g = [t_0, -t_1, -t_2, \pm i t_3]$
$$F = T_3^2 + T_2^3 + (aT_0^4 + bT_0^2T_1^2 + cT_1^4)T_2 + T_0T_1(dT_0^4 + eT_1^4),$$

(4) Order 5
- $2A_4$ $g = [t_0, \epsilon_5 t_1, t_2, t_3]$
$$F = T_3^2 + T_2^3 + aT_0^4T_2 + T_0(bT_0^5 + T_1^5),$$

(5) Order 6
- $E_6(a_2)+A_2$ $g=[t_0,-t_1,\epsilon_3 t_2,t_3]$
$$F=T_3^2+T_2^3+G_3(T_0^2,T_1^2),$$
- $E_7(a_4)+A_1$ $g=[t_0,\epsilon_3 t_1,t_2,-t_3]$
$$F=T_3^2+T_2^3+(T_0^4+aT_0T_1^3)T_2+bT_0^6+cT_0^3T_1^3+dT_1^6,$$
- $2D_4$ $g=[\epsilon_6 t_0,\epsilon_6^{-1}t_1,t_2,t_3]$
$$F=T_3^2+aT_0^2T_1^2T_2+bT_0^6+cT_0^3T_1^3+eT_1^6,$$
- $E_8(a_8)$ $g=[t_0,t_1,\epsilon_3 t_2,-t_3]$
$$F=T_3^2+T_2^3+F_6(T_0,T_1),$$
- $A_5+A_2+A_1$ $g=[t_0,\epsilon_6 t_1,t_2,t_3]$
$$F=T_3^2+T_2^3+aT_0^4T_2+T_0^6+bT_1^6,$$

(6) Order 8
- $D_8(a_3)$ $g=[it_0,t_1,-it_2,\pm\epsilon_8 t_3]$
$$F=T_3^2+T_2^3+T_0^2T_1^2T_2+T_0T_1(bT_0^4+cT_1^4),$$

(7) Order 10
- $E_8(a_6)$ $g=[t_0,\epsilon_5 t_1,t_2,-t_3]$
$$F=T_3^2+T_2^3+aT_0^4T_2+T_0(bT_0^5+T_1^5),$$

(8) Order 12
- $E_8(a_3)$ $g=[-t_0,t_1,\epsilon_6 t_2,it_3]$
$$F=T_3^2+T_2^3+T_0T_1(T_0^4+aT_0^2T_1^2+bT_1^4),$$

(9) Order 15
- $E_8(a_5)$ $g=[t_0,\epsilon_5 t_1,\epsilon_3 t_2,t_3]$
$$F=T_3^2+T_2^3+T_0(T_0^5+T_1^5),$$

(10) Order 20
- $E_8(a_2)$ $g=[t_0,\epsilon_{10}t_1,-t_2,it_3]$
$$F=T_3^2+T_2^3+aT_0^4T_2+T_0T_1^5,$$

(11) Order 24
- $E_8(a_1)$ $g=[it_0,t_1,\epsilon_{12}t_2,\epsilon_8 t_3]$
$$F=T_3^2+T_2^3+T_0T_1(T_0^4+T_1^4),$$

(12) Order 30
- E_8 $g=[t_0,\epsilon_5 t_1,\epsilon_3 t_2,-t_3]$
$$F=T_3^2+T_2^3+T_0(T_0^5+T_1^5).$$

Let us classify minimal groups of automorphisms of a Del Pezzo surface of degree 1. As in the previous cases, to find a structure of such groups is enough to consider the types of surfaces which are not specialized to surfaces of other types. The following notation Type A→Type B indicates that a surface of type A specializes to a surface of type B.

$$III,XII,XIX,XXI\to II,\ XI\to VII,\ XIII\to VIII,\ XIV\to IV,IX,$$
$$XV,XVIII\to I,\ XVII\to V,X.$$

It remains to consider surfaces of types

$$I,\ II,\ IV,\ V,\ VI,\ VII,\ VIII,\ IX, X,\ XVI.$$

If $G \subset \mathrm{Aut}(S)$ contains a minimal cyclic group, then G is minimal. So we will be looking only for those groups which do not contain such subgroups.

Type I.

Let G be a subgroup of $3 \times 2.S_4 = 3 \times \bar{O}$. Suppose the image G' of G in $2.S_4$ contains the center generated by the Bertini involution. We have the following groups G'.

$$2.S_4,\ 2.A_4,\ 2.D_8,\ 2.D_6,\ 2.D_4,\ 8,\ 6,\ 4\ (2 \text{ conjugacy classes}),\ 2, 1.$$

Applying Goursat's Lemma we find the following groups G:

$$3 \times G',\ (3, 1, 2.A_4, 2.D_4)_\alpha,\ (3, 1, 6, 2) \cong 6.$$

It is easy to see that two different isomorphisms $\alpha : 3 \to 2.A_4/2.D_4$ and $3 \to 6/2$ give conjugate subgroups. This gives two non-conjugate subgroups isomorphic to $2.A_4$ and 2 non-conjugate cyclic groups of order 6. All these groups are minimal because they contain the Bertini involution.

Now suppose the image G' of G in $2.S_4$ does not contain the center. Then G' must be a cyclic group of order 3. Since the group 3×1 is minimal, this gives us a minimal group 3^2. It is easy to see that the other possible cyclic groups of order 3 are not minimal.

Type II.

The center is generated by a minimal element $z = [t_0, t_1, \epsilon_3 t_2, -t_3]$ of order 6 of type $E_8(a_8)$. The elements z^3 and z^2 are minimal. Thus any subgroup G which intersects the center nontrivially is minimal. If $z \in G$, we have the following possibilities.

$$6.D_{12},\ 3 \times 2.D_4,\ 2 \times 3.D_6,\ \ 6^2,\ 6 \times 3,\ 6 \times 2,\ 6.$$

Assume $G \cap (z) = (z^2)$. This gives the following groups.

$$3.D_6,\ 3 \times 6,\ 3^2,\ 6,\ 3$$

Note that the groups 3×6 and 6 are not conjugate to isomorphic subgroups from the previous case.

Assume $G \cap (z) = (z^3)$. This gives the following groups

$$2.D_4,\ 2 \times 6,\ 6,\ 3,\ 2^2,\ 2.$$

Again we are getting a new conjugacy class of elements of order 6. Now assume a minimal subgroup G intersects the center (z) trivially. We know that $D_{12} = 6.D_{12}/(z)$ is generated by the image of a minimal element τ of order 6 from the conjugacy class $A_5 + A_2 + A_1$ and the image of an element σ of order 2 from the conjugacy class $4A_1$ with the relation $\sigma\tau\sigma = \tau^{-1}$. We can replace G by a conjugate subgroup to assume that it is generated by some powers of $z^a\tau$ and $z^b\sigma$.

Assume $G = (z^a\tau, z^b\sigma) \cong D_{12}$. We have $(z^b\sigma)^2 = z^{2b}, (z^b\sigma z^a\tau)^2 = z^{2a+2b+1}$. Thus $2b \equiv 2a + 2b + 1 \equiv 0 \mod 6$, hence this case is impossible. In other words, the group $6.D_{12}$ is not a split extension.

Assume $G = (z^a\tau^2, z^b\sigma) \cong D_6$. We have $(z^b\sigma)^2 = z^{2b}, (z^a\tau^2)^3 = z^{3a} = (z^b\sigma z^a\tau^2)^2 = z^{2a+2b+2}$. This implies that $3a \equiv 2b \equiv 2a + 2b + 2 \mod 6$. The only solution is $a = 2, b = 3$. Computing the traces we find that the group is not minimal.

Assume $G = (z^a\tau^3, z^b\sigma) \cong D_4$. We have $(z^b\sigma)^2 = z^{2b}, (z^a\tau^3)^2 = z^{2a} = (z^b\sigma z^a\tau^3)^2 = z^{2a+2b+3}$. This implies that $2a \equiv 2b \equiv 2a+2b+3 \mod 6$. There are no solutions.

Assume $G = (z^a\tau^i)$. Comparing with the list of minimal elements of order 3 and 6, gives four non-conjugate minimal cyclic groups $G = (\tau), (z\tau^2), (z^3\tau^2), (z\sigma)$ of order 6. Only τ gives a new conjugacy class. Note that $z^2\tau$ is conjugate to τ^{-1} and $z^3\tau$ is not minimal. There are no minimal groups of order 3 and 2.

Type IV.

The group is generated by a minimal element g of order 30. Since the power g^{15} is the Bertini involution, all subgroups of even order are minimal. Since g^{10} is a minimal element of order 3, all cyclic groups of order divisible by 3 are minimal. Finally, g^6 is a minimal element of order 5. So all nontrivial subgroups are minimal.

Type V.

Let z be a generator of the subgroup $3 \times \{1\}$. It generates a minimal cyclic group. Write any element of G in the form $g = z^a x$, where $x \in \{1\} \times D_8$. Since the order of x is a power of 2, we see that $z \in (g)$ or $a = 0$. Thus either G contains z and hence minimal or is a subgroup of $\{1\} \times D_8$. The cyclic subgroup H of order 4 in D_8 is minimal and its subgroup of order 2 is generated by the Bertini involution. Thus the only non-minimal subgroup is a cyclic group of order 2 not contained in H.

Type VI.

The group $\mathrm{Aut}(S)/\langle\beta\rangle$ is isomorphic to A_4, the polynomials F_4 and F_6 are its Grundformens. Thus $\mathrm{Aut}(S) \cong 2.A_4 \cong \bar{T}$. If follows from the classification of finite subgroups of SL(2) that any proper subgroup $G \subset \bar{T}$ is conjugate to one of the following groups $2.D_4,\ 6, 4, 3, 2$. Any subgroup containing the center is minimal. The cyclic group of order 3 is not minimal.

Type VII.

The automorphism group is generated by the Bertini involution β, a non-minimal element $\sigma = [t_0, \epsilon_6 t_1, \epsilon_3^2, t_3]$ of order 6 and an involution $\tau = [t_1, t_0, t_2, t_3]$. We have $\sigma\tau = \sigma^{-1}\beta$. The subgroup of order 3 generated by σ^2 is normal. The quotient is isomorphic to D_8 and the extension $3.D_8 \cong 3 : D_8$ splits by the subgroup generated by τ and $\tau\sigma^3$ whose square is equal to β. The subgroup $\langle\sigma, \beta\rangle$ is isomorphic to 2×6, so $3 : D_8 \cong (2 \times 6) : 2$.

Let H be a minimal subgroup of $3 : D_8$. By Lemma 4.2, it is conjugate to a subgroup $\langle\tau, \tau\sigma^3\rangle \cong D_8$ or equal to the semi-direct product $3 : H'$, where H' is a subgroup of D_8. Any subgroup containing β or σ is minimal. This gives minimal subgroups

$$\langle\beta\rangle \cong 2,\ \langle\tau\sigma^3\rangle \cong 4,\ \langle\beta, \tau\rangle \cong 2^2,\ D_8,$$
$$\langle\sigma\rangle \cong 6,\ \langle\sigma^2\beta\rangle \cong 6,\ \langle\sigma, \beta\rangle \cong 2 \times 6,\ \langle\sigma^2,\ \tau\sigma\rangle \cong D_{12}.$$

One checks that the subgroups $\langle\sigma\beta\rangle \cong 6$ and $\langle\sigma^2, \tau\rangle \cong S_3$ are not minimal.

Type VIII.

The group contains 2 subgroups of order 12, only one is minimal. Its subgroup of order 6 is minimal. Thus we have minimal subgroups $2 \times 12, 2 \times 6, 2 \times 4, 2 \times 3, 2^2, 2, 12, 6$. Type IX.

The group is generated by a minimal element g of order 20. Since the power g^{10} is the Bertini involution, all subgroups of even order are minimal. Since g^4 is a minimal element of order 5, all cyclic groups of order divisible by 5 are minimal. So all nontrivial subgroups are minimal.

Type X.

The normal cyclic subgroup of order 8 contains the Bertini involution. So the only non-minimal subgroups are of order 2 different from the Bertini involution..

Type XVI. All cyclic subgroups of order 4 are not minimal. So the minimal subgroups are $2, 2\times 2, 2\times 4$.

Theorem 6.16. *Let G be a minimal subgroup of automorphisms of a Del Pezzo surface of degree 1. Then G is one of the following groups (always excluding the trivial group).*

(1) *G is a group of automorphisms of a surface of type I:*

$$2,\quad 3,\quad 4,\quad 6\ (2),\quad 3^2,\quad 8,\quad 12,$$

$$24,\quad 3^2\times 2,\quad \bar{O},\quad \bar{T}\ (2),\quad \bar{D}_8,\quad \bar{D}_6,\quad \bar{D}_4$$

$$3\times\bar{O},\quad 3\times\bar{T},\quad 3\times(D_8:3),\quad 3\times\bar{D}_8,\quad 3\times\bar{D}_6,\quad 3\times\bar{D}_4.$$

(2) *G is a group of automorphisms of a surface of type II:*

$$2,\quad 3\ (2),\quad 6\ (4),\quad 3^2,\quad 2\times 6\ (2),\quad 3\times 6\ (2),\quad 6^2$$

$$6.D_{12},\quad 3\times 2.D_4,\quad 2\times 3.D_6,\quad 3.D_6,\quad 2.D_4,\quad D_4\quad .$$

(3) *G is a group of automorphisms of a surface of type III:*

$$2,\quad 3,\quad 6\ (2),\quad 3^2,\quad 2\times 6,\quad 3\times 6,\quad 3.D_6,\quad 3\times 2.2^2,\quad D_8,\quad D_4$$

(4) *G is a group of automorphisms of a surface of type IV:*

$$30,\quad 15,\quad 10,\quad 6,\quad 5,\quad 4,\quad 3,\quad 2.$$

(5) *G is a group of automorphisms of a surface of type V:*

$$3\times D_8,\quad 12,\quad 3\times 2^2,\quad 6,\quad D_8,\quad 4,\quad 2^2,\quad 2.$$

(6) *G is a group of automorphisms of a surface of type VI:*

$$\bar{T},\quad 2.D_4,\quad 6,\quad 4,\quad 2.$$

(7) *G is a group of automorphisms of a surface of type VII:*

$$2,\quad 2^2,\quad 4,\quad 6,\quad 2\times 6,\quad 3:D_8,\quad D_{12},\quad D_8.$$

(8) *G is a group of automorphisms of a surface of type VIII:*

$$2\times 12,\quad 2\times 6,\quad 2\times 4,\quad 6,\quad 2^2,\quad 12,\quad 6,\quad 2.$$

(9) *G is a group of automorphisms of a surface of type IX:*

$$20,\quad 10,\quad 5,\quad 4,\quad 2.$$

(10) *G is a group of automorphisms of a surface of type X:*

$$D_{16},\quad D_8,\quad 2^2,\quad 8,\quad 4,\quad 2.$$

(11) *G is a group of automorphisms of a surface of type XI:*

$$D_{12},\quad 6,\quad 4,\quad 2.$$

(12) *G is a group of automorphisms of a surface of type XII:*

$$2 \times 6, \quad 6, \quad 2^2, \quad 3, \quad 2$$

(13) *G is a group of automorphisms of a surface of type XIII:*

$$2 \times 6, \quad 6, \quad 3, \quad 2.$$

(14) *G is a group of automorphisms of a surface of type XIV:*

$$10, \quad 5.$$

(15) *G is a group of automorphisms of a surface of type XV:*

$$\bar{D}_4, \quad 4, \quad 2.$$

(16) *G is a group of automorphisms of a surface of type XVI:*

$$2 \times 4, \quad 2^2, \quad 2.$$

.

(17) *G is a group of automorphisms of a surface of type XVII:*

$$D_8, \quad 2^2, \quad 4, \quad 2.$$

(18) *G is a group of automorphisms of a surface of type XVIII:*

$$6, \quad 3, \quad 2.$$

(19) *G is a group of automorphisms of a surface of type XIX:*

$$6, \quad 2.$$

(20) *G is a group of automorphisms of a surface of type XX:*

$$4, \quad 2.$$

(21) *G is a group of automorphisms of a surface of type XXI:*

$$2^2, \quad 2.$$

Here the number in brackets indicates the number of conjugacy classes in the automorphism group.

7. Elementary links and factorization theorem

7.1. **Noether-Fano inequality.** Let $|d\ell - m_1x_1 - \ldots - m_Nx_N|$ be a homaloidal net in $\mathbb{P}^2$. The following is a well-known classical result.

Lemma 7.1. *(Noether's inequality) Assume $d > 1, m_1 \geq \ldots \geq m_N \geq 0$. Then*

$$m_1 + m_2 + m_3 \geq d + 1,$$

and the equality holds if and only if either $m_1 = \ldots = m_N$ or $m_1 = n - 1, m_2 = \ldots = m_N$.

Proof. We have

$$m_1^2 + \ldots + m_N^2 = d^2 - 1, \quad m_1 + \ldots + m_N = 3d - 3.$$

Multiplying the second equality by m_3 and subtracting from the first one, we get

$$m_1(m_1 - m_3) + m_2(m_2 - m_3) - \sum_{i \geq 4} m_i(m_3 - m_i) = d^2 - 1 - 3m_3(d - 1).$$

¿From this we obtain

$(d-1)(m_1 + m_2 + m_3 - d - 1) = (m_1 - m_3)(d - 1 - m_1) + (m_2 - m_3)(d - 1 - m_2) +$

$$+\sum_{i\geq 4} m_i(m_3-m_i).$$

Since $d-1-m_i\geq 0$, this obviously proves the assertion. □

Corollary 7.2.

$$m_1 > d/3.$$

Let us generalize Corollary 7.2 to birational maps of any rational surfaces. The same idea works even for higher-dimensional varieties. Let $\chi : S \to S'$ be a birational map of surfaces. Let $\sigma : X \to S, \phi : X \to S'$ be its resolution. Let $|H'|$ be a linear system on S' without base points. Let

$$\phi^*(H') \sim \sigma^*(H) - \sum_i m_i\mathcal{E}_i$$

for some divisor H on S and exceptional configurations $\mathcal{E}_i$ of the map σ.

Theorem 7.3. *(Noether-Fano inequality) Assume that there exists some integer $m_0 \geq 0$ such that $|H' + mK_{S'}| = \emptyset$ for $m \geq m_0$. For any $m \geq m_0$ such that $|H + mK_S| \neq \emptyset$ there exists i such that*

$$m_i > m.$$

Proof. We know that $K_X = \sigma^*(K_S) + \sum_i \mathcal{E}_i$. Thus we have the equality in $\mathrm{Pic}(X)$

$$\phi^*(H') + mK_X = (\sigma^*(H + mK_S)) + \sum(m - m_i)\mathcal{E}_i.$$

Applying f_* to the left-hand side we get the divisor class $H' + mK_{S'}$ which, by assumption, cannot be effective. Since $|\sigma^*(H + mK_S)| \neq \emptyset$, applying ϕ_* to the right-hand side, we get the sum of an effective divisor and the image of the divisor $\sum_i(m - m_i)\mathcal{E}_i$. If all $m - m_i$ are nonnegative, it is also an effective divisor, and we get a contradiction. Thus there exists i such that $m - m_i < 0$. □

Example 7.4. Assume $S = S' = \mathbb{P}^2$, $H = d\ell$ and $H' = \ell$. We have $|H + K_{S'}| = |-2\ell| = \emptyset$. Thus we can take $m_0 = 1$. If $d \geq 3$, we have for any $1 \leq a \leq d/3$, $|H' + aK_S| = |(d-3a)\ell| \neq \emptyset$. This gives $m_i > d/3$ for some i. This is Corollary 7.2.

Example 7.5. Let $\chi : S - \to S'$ be a birational map of Del Pezzo surfaces. Assume that S' is not a quadric or the plane. Consider the complete linear system $H' = |-K_{S'}|$. Then $|H' + mK_{S'}| = \emptyset$ for $m \geq 2$. Let $chi^{-1}(H') = |D - \eta|$ be its proper transform on S. Choose a standard basis $(e_0, \ldots, e_k)$ in $\mathrm{Pic}(S)$ corresponding to the blow-up $S \to \mathbb{P}^2$. Since $K_S = -3e_0 + e_1 + \ldots + e_k$, we can write $\chi^{-1}(H') = |-aK_S - \sum m_ix_i|$, where $a \in \frac{1}{3}\mathbb{Z}$. Assume that $\chi^*(H') = -aK_S$. Then there exists a point with multiplicity $\geq a$ if $a > 1$ that we assume.

Remark 7.6. The Noether inequality is of course well-known (see, for example, [1], [34]). We give it here to express our respect of the classical important and beautiful results. Its generalization from Theorem 7.3 is also well-known (see [38], 1.3). Note that the result can be also applied to G-equivariant maps χ provided that the linear system $|H'|$ is G-invariant. In this case the linear system $|H - \eta|$ is also G-invarian and the bubble cycle $\eta = \sum m_ix_i$ consists of the sum of G-orbits.

The existence of base points of high multiplicity in the linear system $|H - \eta| = \chi^{-1}(H')$ follows from the classical theory of termination of the adjoint system for rational surfaces which goes back to G. Castelnuovo. This theory has now an elegant interpretation in the Mori theory which we give in the next section.

7.2. **Elementary links.** We will be dealing with minimimal G-surfaces S with $\mathrm{Pic}(S)^G \cong \mathbb{Z}$ (Del Pezzo surfaces) or $\mathbb{Z}^2$ (conic bundles). In the G-equivariant version of the Mori theory they are interpreted as extremal contractions $\phi : S \to C$, where $C = \mathrm{pt}$ is a point in the first case and $C \cong \mathbb{P}^1$ in the second case. They are also two-dimensional analogs of rational Mori G-fibrations.

A birational G-map between the Mori fibrations are diagrams

$$\begin{array}{ccc} S & \overset{\chi}{\dashrightarrow} & S' \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi'} \\ C & & C' \end{array} \tag{7.1}$$

which in general do not commute with the fibrations. These maps are decomposed into *elementary links*. These links are divided into the four following types.

- Links of type I:

They are commutative diagrams of the form

$$\begin{array}{ccc} S & \overset{\sigma}{\longleftarrow} & Z = S' \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi'} \\ C = \mathrm{pt} & \overset{\alpha}{\longleftarrow} & C' = \mathbb{P}^1 \end{array} \tag{7.2}$$

Here $\sigma : Z \to S$ is the blow-up of a G-orbit, S is a minimal Del Pezzo surface, $\phi' : S' \to \mathbb{P}^1$ is a minimal conic bundle G-fibration, α is the constant map. For example, the blow-up of a G-fixed point on $\mathbb{P}^2$ defines a minimal conic G-bundle $\phi' : \mathbf{F}_1 \to \mathbb{P}^1$ with a G-invariant exceptional section.

- Links of type II:

They are commutative diagrams of the form

$$\begin{array}{ccccc} S & \overset{\sigma}{\longleftarrow} & Z & \overset{\tau}{\longrightarrow} & S' \\ {\scriptstyle\phi}\downarrow & & & & \downarrow{\scriptstyle\phi'} \\ C & & = & & C' \end{array} \tag{7.3}$$

Here $\sigma : Z \to S, \tau : Z \to S'$ are the blow-ups of G-orbits such that rank $\mathrm{Pic}(Z)^G =$ rank $\mathrm{Pic}(S)^G + 1 =$ rank $\mathrm{Pic}(S')^G + 1$, $C = C'$ is either a point or $\mathbb{P}^1$. An example of a link of type II is the Geiser (or Bertini) involution of $\mathbb{P}^2$, where one blows up 7 (or 8) points in general position which form one G-orbit. Another frequently used link of type II is an elementary transformation of minimal ruled surfaces and conic bundles.

- Links of type III:

These are the birational maps which are the inverses of links of type I.

- Links of type IV:

They exist when S has two different structures of G-equivariant conic bundles. The link is the exchange of the two conic bundle structures

$$\begin{array}{ccc} S & = & S' \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi'} \\ C & & C' \end{array} \tag{7.4}$$

One uses these links to relate elementary links with respect to one conic fibration to elementary links with respect to another conic fibration. Often the change of the conic bundle structures is realized via an involution in $\mathrm{Aut}(S)$, for example, the switch of the factors of $S = \mathbb{P}^1 \times \mathbb{P}^1$ (see the following classification of elementary links).

7.3. **The factorization theorem.** Let $\chi : S - \to S'$ be a G-equivariant birational map of minimal G-surfaces. We would like to decompose it into a composition of elementary links. This is achieved with help of G-equivariant theory of log-pairs (S, D), where D is a G-invariant $\mathbb{Q}$-divisor on S. It is chosen as follows. Let us fix a G-invariant very ample linear system H' on S'. If S' is a minimal Del Pezzo surface we take $\mathcal{H}' = |-a'K_{S'}|, a' \in \mathbb{Z}_+$. If S' is a conic bundle we take $\mathcal{H}' = |-a'K_{S'} + b'f'|$, where f' is the class of a fibre of the conic bundle, a', b' are some appropriate positive integers.

Let $\mathcal{H} = \mathcal{H}_S = \chi^{-1}(\mathcal{H}')$ be the proper inverse transform of $\mathcal{H}'$ on S. Then

$$\mathcal{H} = |-aK_S - \sum m_x x|,$$

if S is a Del Pezzo surface, $a \in \frac{1}{2}\mathbb{Z}_+ \cup \frac{1}{3}\mathbb{Z}_+$, and

$$\mathcal{H} = |-aK_S + bf - \sum m_x x|,$$

if S is a conic bundle, $a \in \frac{1}{2}\mathbb{Z}_+, b \in \frac{1}{2}\mathbb{Z}$. The linear system $\mathcal{H}$ is G-invariant, and the 0-cycle $\sum m_x x$ is a sum of G-orbits with integer multiplicities. One uses the theory of log-pairs (S, D), where D is a general divisor from the linear system $\mathcal{H}$, by applying some "untwisting links" to χ in order to decrease the number a, the algebraic degree of $\mathcal{H}$. Since a is a rational positive number with bounded denominator, this process terminates after finitely many steps (see [19],[38]).

Theorem 7.7. *Let $f : S - \to S'$ be a birational map of minimal G-surfaces. Then χ is equal to a composition of elementary links.*

The proof of this theorem is the same as in the arithmetic case ([38], Theorem 2.5). Each time one chooses a link to apply and the criterion used for termination of the process is based on the following version of Noether's inequality in the Mori theory.

Lemma 7.8. *In the notation from above, if $m_x \le a$ for all base points x of $\mathcal{H}$ and $b \ge 0$ in the case of conic bundles, then χ is an isomorphism.*

The proof of this lemma is the same as in the arithmetic case ([38], Lemma 2.4).

We will call a base points x of $\mathcal{H}$ with $m_x > a$ a *maximal singularity* of $\mathcal{H}$. It follows from 3.2 that if $\mathcal{H}$ has a maximal singularity of height > 0, then it also has a maximal singularity of height 0. We will be applying the "untwisting links" of types I-III to these points. If $\phi : S \to \mathbb{P}^1$ is a conic bundle with all its maximal singularities untwisted with helps of links of type II, then either the algorithm terminates, or $b < 0$. In the latter case the linear system $|K_S + \frac{1}{a}\mathcal{H}| = |\frac{b}{a}f|$ is not nef and has *canonical singularities* (i.e. no maximal singularities). Applying the theory of log-pairs to the pair $(S, |\frac{b}{a}f|)$ we find an extremal contraction $\phi' : S \to \mathbb{P}^1$, i.e. another conic bundle structure on S. Rewriting $\mathcal{H}$ in a new basis $-K_S, f'$ we find the new coefficient $a' < a$. Applying the link of type IV relating ϕ and ϕ', we start over the algorithm with decreased a.

It follows from the proofs of Theorem 7.7 and Lemma 7.8 that all maximal singularities of H are in general position in the following sense.

(i) If S is a minimal Del Pezzo G-surface, then the blow-up of all maximal singularities of $\mathcal{H}$ is a Del Pezzo surface (of course this agrees with the description of points in general position at the end of section 3.8).

(ii) If $\phi : S \to \mathbb{P}^1$ is a conic bundle, then none of the maximal singularities lie on a singular fibre of ϕ and no two lie on one fibre.

The meaning of these assertions is that the linear system $|H|$ has no fixed components. In the case of Del Pezzo surfaces with an orbit of maximal singular points we can find a link by blowing up this orbit to obtain a surface Z with $\mathrm{Pic}(Z)^G \cong \mathbb{Z}\oplus\mathbb{Z}$ and two extremal rays. By applying Kleiman's criterion this implies that $-K_Z$ is ample. The similar situation occurs in the case of conic bundles (see [38], Comment 2).

Let S be a minimal Del Pezzo G-surface of degree d. Let us write $\mathcal{H}_S = |-aK_S + \sum m_\kappa \kappa|$ as in (3.8).

Lemma 7.9. *Let $\kappa_1, \ldots, \kappa_n$ be the G-orbits of maximal multiplicity. Then $\sum d(\kappa_i) < d$.*

Proof. Let $D_1, D_2 \in \mathcal{H}_S$ be two general divisors from $\mathcal{H}_S$. Since $\mathcal{H}_S$ has no fixed components, we have

$$0 \le D_1 \cdot D_2 = a^2 d - \sum m_\kappa^2 d(\kappa) \le a^2 d - \sum_{i=1}^{n} m_{\kappa_i}^2 d(\kappa_i) =$$

$$a^2(d-n) - \sum_{i=1}^{n} (m_{\kappa_i}^2 - a^2) d(\kappa_i) - a^2 \sum_{i=1}^{n} d(\kappa_i).$$

This implies that $d - n < 0$. □

Definition 7.10. A minimal Del Pezzo G-surface is called *superrigid* (resp. *rigid*) if any birational G-map $\chi : S - \to S'$ is a G-isomorphism (resp. there exists a birational G-automorphism $\alpha : S - \to S$ such that $\chi \circ \alpha$ is a G-isomorphism).

A minimal conic bundle $\phi : S \to \mathbb{P}^1$ is called *superrigid* (resp. *rigid*) if for any birational G-map $\chi : S - \to S'$, where $\phi' : S' \to \mathbb{P}^1$ is a minimal conic bundle, there exists an isomorphism $\delta : \mathbb{P}^1 \to \mathbb{P}^1$ such that the following diagram is commutative

(7.5)
$$\begin{array}{ccc} S & \overset{\chi}{\dashrightarrow} & S' \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi'} \\ \mathbb{P}^1 & \xrightarrow{\delta} & \mathbb{P}^1 \end{array}$$

(resp. there exists a birational G-automorphism $\alpha : S - \to S'$ such that the diagram is commutative after we replace χ with $\chi \circ \alpha$).

Applying Lemma 7.8 and Lemma 7.9, we get the following.

Corollary 7.11. *Let S be a minimal Del Pezzo G-surface of degree $d = K_S^2$. If S has no G-orbits κ with $d(\kappa) < d$, then S is superrigid. In particular, a Del Pezzo surface of degree 1 is always superrigid and a Del Pezzo surface of degree 2 is superrigid unless G has a fixed point.*

A minimal conic G-bundle with $K_S^2 \le 0$ is superrigid.

The first assertion is clear. To prove the second one, we untwist all maximal base points of $\mathcal{H}_S$ with help of links of type II to get a conic bundle $\phi' : S' \to \mathbb{P}^1$ with $b' < 0$. Since $H_{S'}^2 = a^2K_{S'}^2 + 4ab' - \sum {m'_x}^2 \geq 0$ and $K_{S'}^2 = K_S^2 \leq 0, 4ab' < 0$, we get a contradiction with Lemma 7.8. Thus χ after untwisting maximal base points terminates at an isomorphism (see [35], [36], [38], Theorem 1.6).

7.4. **Classification of elementary links.** Here we consider an elementary link $f : S - \to S'$ defined by a resolution $(\sigma : Z \to S, \tau : Z \to S'$. We take $H_{S'}$ to be the linear system $|-aK_{S'}|$ if S' is a Del Pezzo surface and $|f|$ if S' is a conic bundle, where f is the divisor class of a fibre. It is assumed that the point which we blow up are in general position in sense of the previous subsection.

We denote by $\mathcal{D}_k$ (resp. $\mathcal{C}_k$) the set of isomorphism classes of minimal Del Pezzo surfaces (resp. conic bundles) with $k = K_S^2$ (resp. $k = 8 - K_S^2$).

Proposition 7.12. *Let S, S' be as in Link I of type I. The map $\sigma : Z = S' \to S$ is the blowing up of a G-invariant bubble cycle η with $\mathrm{ht}(\eta) = 0$ of some degree d. The proper transform of the linear system $|f|$ on S' is equal to the linear system $\mathcal{H}_S = |-aK_S - m\eta|$. Here f is the class of a fibre of the conic bundle structure on S'. The following cases are possible:*

(1) $K_S^2 = 9$
- $S = \mathbb{P}^2, S' = \mathbf{F}_1, d = 1, m = 1, a = \frac{1}{3}$.
- $S = \mathbb{P}^2, S' \in \mathcal{C}_3, d = 4, m = 1, a = \frac{2}{3}$.

(2) $K_S^2 = 8$
- $S = \mathbf{F}_0, \pi : S' \to \mathbb{P}^1$ *a conic bundle with* $k = 2$, $d = 2$, $m = 1$, $a = \frac{1}{2}$.

(3) $K_S^2 = 4$
- $S' \in \mathcal{D}_4, p : S' \to \mathbb{P}^1$ *a conic bundle with* $f = -K_{S'} - l$, *where l is a (-1)-curve,* $d = a = 1, m = 2$.

Proof. Let $\mathcal{H}_S = |-aK_S - b\eta|$, where η is a G-invariant bubble cycle of degree d. We have

$$(-aK_S - b\eta)^2 = a^2K_S^2 - b^2d = 0, \quad (-aK_S - b\eta, -K_S) = aK_S^2 - bd = 2.$$

Let $t = b/a$. We have

$$(td)^2 = dK_S^2, \quad K_S^2 - td = 2/a > 0.$$

The second inequality, gives $td < K_S^2$, hence $d < K_S^2$. Giving the possible values for K_S^2 and using that $a \in \frac{1}{3}\mathbb{Z}$, we check that the only possibilities are:

$$(K_S^2, d, t) = (9, 1, 3),\ (8, 2, 2),\ (4, 1, 2),\ (4, 2, 1).$$

This gives our cases and one extra case $(4, 2, 1)$. In this case $a = 2$ and $\mathcal{H}_S = |-2K_S - 2x_1|$ contradicting the primitivity of f. Note that this case is realized in the case when the ground field is not algebraically closed (see [38]). □

Proposition 7.13. *Let S, S' be as in Link of type II. Assume that S, S' are both minimal Del Pezzo surfaces. Then $(S \overset{\sigma}{\leftarrow} Z \overset{\tau}{\to} S')$, where σ is the blow-up of a G-invariant bubble cycle η with $\mathrm{ht}(\eta) = 0$ and some degree d. The proper inverse transform of the linear system $|-K_{S'}|$ on S is equal to $|-aK_S - m\eta|$. And similarly defined d', m', a' for τ. The following cases are possible:*

(1) $K_S^2 = 9$
- $S' \cong S = \mathbb{P}^2, d = d' = 8, m = m' = 18, a = a' = 17$ *$(S \leftarrow Z \to S')$ is a minimal resolution of a Bertini transformation).*

- $S' \cong S = \mathbb{P}^2, d = d' = 7, m = m' = 9, a = a' = 8$ *($S \leftarrow Z \rightarrow S'$) is a minimal resolution of a Geiser transformation).*
- $S' \cong S = \mathbb{P}^2, d = d' = 6, m = m' = 6, a = a' = 5$ *($S \leftarrow Z \rightarrow S'$) is a minimal resolution of a Cremona transformation given by the linear system* $|5\ell - 2p_1 - 2p_2 - 2p_3 - 2p_4 - 2p_5|$,
- $S \cong \mathbb{P}^2, S' \in \mathcal{D}_5, d = 5, m' = 6, a = \frac{5}{3}, d' = 1, m = 2, a' = 3.$
- $S \cong S' = \mathbb{P}^2, d = d' = 3, m = m' = 1, a = a' = \frac{2}{3}$, ($S \leftarrow Z \rightarrow S'$) *is a minimal resolution of a standard quadratic transformation.*
- $S = \mathbb{P}^2, S' = \mathbf{F}_0, d = 2, m = 3, a' = \frac{3}{2}, d' = 1, a = \frac{4}{3}.$

(2) $K_S^2 = 8$
- $S \cong S' \cong \mathbf{F}_0, d = d' = 7, a = a' = 15, m = m' = 16.$
- $S \cong S' \cong \mathbf{F}_0, d = d' = 6, a = a' = 7, m = m' = 8.$
- $S \cong \mathbf{F}_0, S' \in \mathcal{D}_5, d = 5, d' = 2, a = \frac{5}{2}, m = 4, a' = 4, m' = 6.$
- $S \cong \mathbf{F}_0, S' \cong \mathbf{F}_0, d = d' = 4, a = a' = 3, m = m' = 4.$
- $S \cong \mathbf{F}_0, S' \in \mathcal{D}_6, d = 3, d' = 1, a = \frac{3}{2}, m = 2, m' = 4, a' = 2.$
- $S \cong \mathbf{F}_0, S' \cong \mathbb{P}^2, d = 1, d' = 2, a = \frac{3}{2}, m = 3, a' = \frac{4}{3}, m' = 2.$ *This link is the inverse of the last case from the preceding list.*

(3) $K_S^2 = 6$
- $S \cong S' \in \mathcal{D}_6, d = d' = 5, a = 11, m = 12.$
- $S \cong S' \in \mathcal{D}_6, d = d' = 4, a = 5, m = 6.$
- $S \cong S' \in \mathcal{D}_6, d = d' = 3, a = 3, m = 4.$
- $S \cong S' \in \mathcal{D}_6, d = d' = 2, a = 2, m = 3.$
- $S \in \mathcal{D}_6, S' = \mathbf{F}_0, d = 1, d' = 3, a = \frac{3}{2}, m = 2.$ *This link is the inverse of the link from the preceding list with* $S' \in \mathcal{D}_6, d = 3.$

(4) $K_S^2 = 5$
- $S \cong S' \in \mathcal{D}_5, d = d' = 4, m = m' = 10, a = a' = 5..$
- $S = S' \in \mathcal{D}_5, d = d' = 3, m = m' = 5, a = a' = 4.$
- $S \in \mathcal{D}_5, S' = \mathbf{F}_0, d = 2, d' = 5.$ *This link is inverse of the link with* $S = \mathbf{F}_0, S' \in \mathcal{D}_5, d = 5.$
- $S \in \mathcal{D}_5, S' = \mathbb{P}^2, d = 1, d' = 5.$ *This link is inverse of the link with* $S = \mathbb{P}^2, S' \in \mathcal{D}_5, d = 5.$

(5) $K_S^2 = 4$
- $S \cong S' \in \mathcal{D}_4, d = d' = 3.$ *This is an analog of the Bertini involution.*
- $S \cong S' \in \mathcal{D}_4, d = d' = 2.$ *This is an analog of the Geiser involution.*

(6) $K_S^2 = 3$
- $S \cong S' \in \mathcal{D}_3, d = d' = 2.$ *This is an analog of the Bertini involution.*
- $S \cong S' \in \mathcal{D}_3, d = d' = 1.$ *This is an analog of the Geiser involution.*

(7) $K_S^2 = 2$
- $S = S' \in \mathcal{D}_2, d = d' = 1.$ *This is an analog of the Bertini involution.*

Proof. Similar to the proof of the previous proposition, we use that

$$\mathcal{H}_S^2 = a^2 K_S^2 - b^2 d = K_{S'}^2, \quad aK_S^2 - bd = K_{S'}^2,$$

$$H_{S'}^2 = a'^2 K_{S'}^2 - b'^2 d = K_S^2, \quad a' K_S^2 - b' d = K_S^2.$$

Since the link is not a biregular map, by Noether's inequality we have $a > 1, a' > 1, b > a, b' > a'$. This implies

$$d < K_S^2 - \frac{1}{a} K_{S'}^2, d' < K_{S'}^2 - \frac{1}{a'} K_S^2.$$

It is not difficult to list all solutions. For example, assume $K_S^2 = 1$. Since d is a positive integer, we see that there are no solutions. If $K_S^2 = 2$, we must have $d = d' = 1$. □

Proposition 7.14. *Let S, S' be as in Link of type II. Assume that S, S' are both minimal conic bundles. Then $(S \leftarrow Z \rightarrow S')$ is a composition of elementary transformations $\mathrm{elm}_{x_1} \circ \ldots \circ \mathrm{elm}_{x_s}$, where $(x_1, \ldots, x_s)$ is a G-orbit of points not lying on a singular fibre with no two points lying on the same fibre.*

We skip the classifiation of links of type III. They are the inverses of links of type I.

Proposition 7.15. *Let S, S' be as in Link of type IV. Recall that they consist of changing the conic bundle structure. The following cases are possible:*

- $K_S^2 = 8, S' = S, f' = -K_{S'} - f$, *it is represented by a switch automorphism;*
- $K_S^2 = 4, S' = S, f' = -K_{S'} - f$;
- $K_S^2 = 2, S' = S, f' = -2K_{S'} - f$; *it is represented by a Geiser involution;*
- $K_S^2 = 1, S' = S, f' = -4K_{S'} - f$; *it is represented by a Bertini involution;*

Proof. In this case S admits two extremal rays and rank $\mathrm{Pic}(S)^G = 2$ so that $-K_S$ is ample. Let $|f'|$ be the second conic bundle. Write $f' \sim -aK_S + bf$. Using that $f'^2 = 0, f \cdot K_S = f' \cdot K_S = -2$, we easily get $b = -1$ and $aK_S^2 = 4$. This gives all possible cases from the assertion. □

8. Birational classes of minimal G-surfaces

8.1. Let S be a minimal G-surface S and $d = K_S^2$. We will classify all isomorphism classes of (S, G) according to the increasing parameter d.

- $d \leq 0$.

By Corollary 7.11, S is a superrigid conic bundle with $k = 8 - d$ singular fibres. The number k is a birational invariant. The group G is of de Jonquiéres type and its conjugacy class in Cr(2) is determined uniquely by Theorem 5.8 or Theorem 5.3.

Also observe that if $\phi : S \rightarrow \mathbb{P}^1$ is an exceptional conic bundle and $G_0 = \mathrm{Ker}(G \rightarrow \mathrm{O}(\mathrm{Pic}(S))$ is non-trivial, then no links of type II is possible. Thus the conjugacy class of G is uniquely determined by the isomorphism class of S.

- $d = 1$, S is a Del Pezzo surface.

By Corollary 7.11, the surface S is superrigid. Hence the conjugacy class of G is determined uniquely by its conjugacy class in $\mathrm{Aut}(S)$. All such conjugacy clases are listed in Theorem 6.16.

- $d = 1$, S is a conic bundle.

Let $\phi : S \rightarrow \mathbb{P}^1$ be a minimal conic bundle on S. It has $k = 7$ singular fibres. If $-K_S$ is ample, i.e. S is a (non-minimal) Del Pezzo surface, then the center of $\mathrm{Aut}(S)$ contains the Bertini involution β. We know that β acts as -1 on $K_S^{\perp}$, thus β does not act identically on $\mathrm{Pic}(S)^G$, hence $\beta \notin G$. Since k is odd, the conic bundle is not exceptional, so we cam apply Theorem 5.8, Case (2). It follows that G must contain a subgroup isomorphic to 2^2, adding β we get that S is a minimal Del Pezzo 2^3-surface of degree 1. However, the classification shows that there are no such surfaces.

Thus $-K_S$ is not ample. It follows from Proposition 7.13 that the structure of a conic bundle on S is unique. Any other conic bundle birationally G-isomorphic to S is obtained from S by elementary transformations with G-invariant set of centers.

- $d = 2$, S is a Del Pezzo surface.

By Corollary 7.11, S is superrigid unless G has a fixed point on S. If $\chi : S- \to S'$ is a birational G-map, then H_S has only one maximal base point and it does not lie on a (-1)-curve. We can apply an elementary link $Z \to S, Z \to S$ of type II which together with the projections $S \to \mathbb{P}^2$ resolves the Bertini involution. These links together with the G-automorphisms (including the Geiser involution) generate the group of birational G-automorphisms of S (see [38], Theorem 4.6). Thus the surface is rigid. The conjugacy class of G in $\mathrm{Cr}(2)$ is determined uniquely by the conjugacy class of G in $\mathrm{Aut}(S)$. All such conjugacy clases are listed in Theorem 6.15.

- $d = 2$, $\phi : S \to \mathbb{P}^1$ is a conic bundle.

If $-K_S$ is ample, then ϕ is not exceptional. The center of $\mathrm{Aut}(S)$ contains the Geiser involution γ. Since γ acts non-trivially on $\mathrm{Pic}(S)^G = \mathbb{Z}^2$, we see that $\gamma \notin G$. Applying γ we obtain another conic bundle structure. In other words, γ defines an elementary link of type IV. Using the factorization theorem we show that the group of birational G-automorphisms of S is generated by links of type II, the Geiser involution, and G-automorphisms (see [36],[39], Theorem 4.9). Thus $\phi : S \to \mathbb{P}^1$ is a rigid conic bundle.

If S is not a Del Pezzo surface, ϕ could be exceptional bundle with $g = 2$. In any case the group G is determined in Theorem 5.3. We do not know whether S can be mapped to a conic bundle with $-K_S$ ample (see [36]).

Applying Proposition 5.2, we obtain that any conic bundle with $d \geq 3$ is a non-minimal Del Pezzo surface, unless $d = 4$ and S is an exceptional conic bundle. In the latter case, the group G can be found in Theorem 5.3. It is not known whether it is birationally G-isomorphic to a Del Pezzo surface. It is true in the arithmetic case.

- $d = 3$, S is a minimal Del Pezzo surface.

The classification of elementary links shows that S is rigid. Birational G-automorphisms are generated by links of type (6) from Proposition 7.12. The conjugacy class of G in $\mathrm{Cr}(2)$ is determined by the conjugacy class of G in $\mathrm{Aut}(S)$.

- $d = 3$, S is a minimal conic bundle.

Since $k = 5$ is odd, G has 3 commuting involutions, the fixed-point locus of one of them must be a rational 2-section of the conic bundle. It is easy to see that it is a (-1)-curve C from the divisor class $-K_S - f$. The other two fixed-point curves are of genus 2. The group G leaves the curve C invariant. Thus blowing it down, we obtain a minimal Del Pezzo G-surface S' of degree 4. The group G contains a subgroup isomorphic to 2^2. Thus G can be found in the list of minimal groups of degree 4 Del Pezzo surfaces with a fixed point. For example, the group 2^2 has 4 fixed points.

- $d = 4$, S is a minimal Del Pezzo surface.

If $S^G = \emptyset$, then S admits only self-links of type II, so it is rigid or superrigid. The conjugacy class of G in $\mathrm{Cr}(2)$ is determined by the conjugacy class of G in $\mathrm{Aut}(S)$ and we can apply Theorem 6.9. If x is a fixed point of G, then we can apply a link of type I, to get a minimal conic bundle with $d = 3$. So, all groups with $S^G \neq \emptyset$ are conjugate to groups of de Jonquiéres type realized on a conic bundle $S \in \mathcal{C}_5$.

- $d = 4$, S is a minimal conic bundle.

Since $k = 4$, it follows from Lemma 5.1 that either S is an exceptional conic bundle with $g = 1$, or S is a Del Pezzo surface with two sections with self-intersection -1 intersecting at one point. In the latter case, S is obtained by regularizing a de Jonquéres involution IH_3 (see section 2.3). In the case when S is exceptional, the group G is determined by Theorem 5.3. If $G_0 = \mathrm{Ker}(G \to \mathrm{O}(\mathrm{Pic}(S))$ is not trivial, then no elementary transformation is possible. So, S is not birationally isomorphic to a Del Pezzo surface.

- $d = 5$, S is a Del Pezzo surface, $G \cong 5$.

Let us show that (S, G) is birationally isomorphic to $(\mathbb{P}^2, G)$. Since rational surfaces are simply-connected, G has a fixed point x on S. The anti-canonical model of S is a surface of degrfee 5 in $\mathbb{P}^5$. Let P be the tangent plane of S at x. The projection from P defines a birational G-equivariant map $S- \to \mathbb{P}^2$ given by the linear system of anti-canonical curves with double point at x. It is an elementary link of type II.

- $d = 5$, S is a Del Pezzo surface, $G \cong 5 : 2, 5 : 4$.

Since $G \cong 5 : 2$ contains a of index 2 isomorphic to a cyclic group of order 5, S contains an orbit $\kappa = x_1 + x_2$ with $d(\kappa) = 2$. Using an elementary link of type II with $S' = \mathbf{F}_0$, we obtain that G is conjugate to a group acting on $\mathbf{F}_0$. If $G \cong 5 : 4$, then S has no orbits κ with $d(\kappa) \leq 2$, hence S is rigid, and the conjugacy class of G in $\mathrm{Aut}(S)$ determines the conjugacy class of G in $\mathrm{Cr}(2)$.

- $d = 5$, S is a Del Pezzo surface, $G \cong A_5, S_5$.

It is clear that $S^G = \emptyset$ since otherwise S admits a faithful 2-dimensional linear representation. It is known that it does not exist. Since A_5 has no index 2 subgroups S does not admit orbits κ with $d(\kappa) = 2$. The same is obviously true for $G = S_5$. It follows from the classification of links that (S, G) is superrigid.

- $d = 5$, $\phi : S \to \mathbb{P}^1$ is a minimal conic bundle.

It follows from Lemma 5.1 that S contains 2 disjoint (-1)-curves. The group cannot be isomorphic to 2 since S is minimal. Thus G contains 2^2 and hence S contains 4 disjoint (-1)-curves. Blowing them down we get a birational map $\sigma : S \to \mathbb{P}^2$. This is link of type III, the inverse to the link of type I with $K_X^2 = 9, d = 4$. The fixed loci of the three involutions in 2^2 are rational curves. The map $(\phi, \sigma) : S \to \mathbb{P}^1 \times \mathbb{P}^2$ defines an isomorphism from S and a surface in $\mathbb{P}^1 \times \mathbb{P}^2$ given by an equation

$$a_1(t_0, t_1)x_0^2 + a_2(t_0, t_1)x_1^2 + a_3(t_0, t_1)x_2^2 = 0,$$

where a_1, a_2, a_3 are linear forms. It is clear that $G \cong 2^2$.

- $d = 6$.

This case was considered in [39], [40] (the papers also discuss the relation of this problem to some questions in the theory of algebraic groups raised in [43]). It is proved there that (S, G) is birationally isomorphic to $(\mathbb{P}^2, G)$ but birationally isomorphic to minimal $(\mathbf{F}_0, G)$. The birational isomorphism is easy to describe. We know that G contains the lift of the standard Cremona involution. It has 4 fixed points in S, the lifts of the points given in affine coordinates by $(\pm 1, \pm 1)$. The group S_3 fixes $(1, 1)$ and permutes the remaining points p_1, p_2, p_3. The proper inverse transforms of the lines $\langle p_i, p_j\rangle$ in S are disjoint (-1)-curves E_i. The anticanonical model of S is a surface of degree 6 in $\mathbb{P}^6$. The projection from the tangent plane to S at the fixed point, is a link of type II with $S' = \mathbf{F}_0$. It blows-up the fixed point, and then blows down the pre-images of the curves E_i. The group G now acts on $\mathbf{F}_0$ with $\mathbf{F}_0^G = \emptyset$.

The same is true for the subgroup S_3 or 6 of G, however it acts on $\mathbf{F}_0$ with a fixed point. The projection from this point defines a birational isomorphism (S, G) and $(\mathbb{P}^2, G)$.

Finally, assume that $G = H : G'$, where H is not trivial and G' is one of the groups from above. It is easy to see that $G = H : 6$ has no fixed points on S. It follows from the classification of links that S is rigid.

- $d = 8$.

In this case $S = \mathbf{F}_0$ or $\mathbf{F}_n, n > 1$. In the first case (S, G) is birationally isomorphic to $(\mathbb{P}^2, G)$ if $S^G \neq \emptyset$ (via the projection from the fixed point). This implies that the subgroup G' of G belonging to the connected component of the identity of $\mathrm{Aut}(\mathbf{F}_0)$ is an extension of cyclic groups. As we saw in Theorem 4.10 this implies that G' is an abelian group of transformations $(x, y) \mapsto (\epsilon_{nk}^a x, \epsilon_{mk}^b y)$, where $a = sb \mod k$ for some s coprime to k. If $G \neq G'$, then we must have $m = n = 1$ and $s = \pm 1 \mod k$.

If $\mathbf{F}_0^G = \emptyset$ and $\mathrm{Pic}(\mathbf{F}_0)^G \cong \mathbb{Z}$, then the classification of links shows that links of type II with $d = d' = 7, 6, 5, d = 3, d' = 1$ map $\mathbf{F}_0$ to $\mathbb{F}_0$ or to minimal Del Pezzo surfaces of degrees 5 or 6. These cases have been already considered. If rank $\mathrm{Pic}(S)^G = 2$, then any birational G-map $S - \to S'$ is composed of elementary transformations with respect to one of the conic bundle fibrations. They do not change K_S^2 and do not give rise a fixed points. So, G is not conjugate to any subgroup of $\mathrm{Aut}(\mathbb{P}^2)$.

Assume $n > 1$. Then $G = A.B$, where $A \cong n$ acts identically on the base of the fibration and $B \subset \mathrm{PGL}_{\mathbb{C}}(2)$. The subgroup B fixes pointwisely two disjoint sections, one of them is the exceptional one. Let us consider different cases corresponding to possible groups B.

$B \cong C_n$. In this case B has two fixed points on the base, hence G has 2 fixed points on the non-exceptional section. Performing an elementary transformation with center at one of these points we descend G to a subgroup of $\mathbf{F}_{n-1}$. Proceeding in this way, we arrive to the case $n = 1$, and then obtain that G is a group of automorphisms of $\mathbb{P}^2$.

$B \cong D_n$. In this case B has an orbit of cardinality 2 in $\mathbb{P}^1$. A similar argument shows that G has an orbit of cardinality 2 on the non-exceptional section. Applying the product of the elementary transformations at these points we descend G to a subgroup of automorphisms of $\mathbf{F}_{n-2}$. Proceeding in this way we obtain that G is a conjugate to a subgroup of $\mathrm{Aut}(\mathbb{P}^2)$ or of $\mathrm{Aut}(\mathbf{F}_0)$. In the latter case it does not have fixed points, and hence is not conjugate to a linear subgroup of $\mathrm{Cr}(2)$.

$B \cong T$. The group B has an orbit of cardinality 4 on the non-exceptional section. A similar argument shows that G is conjugate to a group of automorphisms of $\mathbf{F}_2, \mathbf{F}_0$, or $\mathbb{P}^2$.

$B \cong O$. The group B has an orbit of cardinality 6. As in the previous case we show that G is conjugate to a group of automorphisms of $\mathbb{P}^2$, or $\mathbf{F}_0$, or $\mathbf{F}_2$, or $\mathbf{F}_3$.

$B \cong I$. The group B has an orbit of cardinality 12. We obtain that G is conjugate to a group of automorphisms of $\mathbb{P}^2$ or of $\mathbf{F}_n$, where $n = 0, 2, 3, 4, 5, 6$.

- $d = 9$.

In this case $S = \mathbb{P}^2$ and G is a finite subgroup of $\mathrm{PGL}_{\mathbb{C}}(3)$. The methods of the representation theory allows us to classify them up to conjugacy in the group $\mathrm{PGL}_{\mathbb{C}}(3)$. However, some of non-conjugate groups can be still conjugate inside the Cremona group.

For example all cyclic subgroups of $\mathrm{PGL}_{\mathbb{C}}(3)$ of the same order n are conjugate in $\mathrm{Cr}(2)$. Any element g of order n in $\mathrm{PGL}_{\mathbb{C}}(3)$ is conjugate to a transformation g given in affine coordinates by the formula $(x, y) \mapsto (\epsilon_n x, \epsilon_n^a y)$. Let $T \in \mathrm{dJ}(2)$ be given by the formula $(x, y) \mapsto (x, x^a/y)$. Let $g' : (x, y) \mapsto (\epsilon_n^{-1} x, y)$. We have

$$g' \circ T \circ g : (x, y) \mapsto (\epsilon_n x, \epsilon_n^a y) \mapsto (\epsilon_n x, x^a/y) \mapsto (x, x^a/y) = T.$$

This shows that g' and g are conjugate.

We do not know whether any two isomorphic non-conjugate subgroups of $\mathrm{PGL}_{\mathbb{C}}(3)$ are conjugate in $\mathrm{Cr}(2)$.

9. What is left?

Here we list some problems which have not been yet resolved.

- Find the conjugacy classes in $\mathrm{Cr}(2)$ of subgroups of $\mathrm{PGL}_{\mathbb{C}}(3)$. For example, there are two non-conjugate subgroups of $\mathrm{PGL}_{\mathbb{C}}(3)$ isomorphic to A_5 or A_6 which differ by an outer automorphism of the groups. Are they conjugate in $\mathrm{Cr}(2)$?
- Find the finer classification of the conjugacy classes of de Jonquiéres groups.

We already know that the number of singular fibres in a minimal conic bundle G-surface is an invariant. Even more, the projective equivalence class of the corresponding k points on the base of the conic fibration is an invariant. Are there other invariants? In the case when $G_K \cong 2$, we know that the quotient of the conic bundle by the involution generating G_K is a minimal rule surface $\mathbf{F}_e$. Is the number e a new invariant?

- Give a finer geometric description of the algebraic variety parametrizing conjugacy classes.

Even in the case of Del Pezzo surfaces we give only normal forms. What is precisely the moduli space of Del Pezzo surfaces with a fixed isomorphism class of a minimal automorphism group?

We know that conic bundles (S, G) with $k \geq 8$ singular fibres are superrigid, so any finite subgroup G' of $\mathrm{Cr}(2)$ conjugate to G is realized as an automorphism group of a conic bundle obtained from S by a composition of elementary transformations with G-invariant centers. If S is not exceptional and $G \cong 2.P$, then the invariant of the conjugacy class is the hyperelliptic curve of fixed points of the central involution. If $G \cong 2^2.P$, then we have three commuting involutions and their curves of fixed points are the invariants of the conjugacy class. Do they determine the conjugacy class?

When $k = 6, 7$ we do not know whether (S, G) is birationally isomorphic to (S', G), where S' is a Del Pezzo surface. This is true when $k \in \{0, 1, 2, 3, 5\}$ or $k = 4$ and S is not exceptional.

- Find more explicit description of groups G as subgroups of $\mathrm{Cr}(2)$.

This has been done in the case of abelian groups in [5]. For example one may ask to reprove and revise Autonne's classification of groups whose elements are quadratic transformations [2]. An example of such non-cyclic group is the group of automorphisms S_5 of a Del Pezzo surface of degree 5.

- Finish the clasical work on the birational classification of rational cyclic planes $z^n = f(x, y)$.

More precisely, the quotient S/G of a rational surface S by a cyclic group of automorphisms defines a cyclic extension of the fields of rational functions. Thus there exists a rational function $R(x, y)$ such that there exists an isomorphism of fields $\mathbb{C}(x, y)(\sqrt[n]{R(x, y)}) \cong \mathbb{C}(x, y)$, where n is the order of G. Obviously we may assume that $R(x, y)$ is a polynomial $f(x, y)$, hence we obtain an affine model of S in the form $z^n = f(x, y)$. A birational isomorphism of G-surfaces replaces the branch curve $f(x, y) = 0$ by a Cremona equivalent curve $g(x, y)$. The problem is to describe the Cremona equivalence classes of the branch curves which define rational cyclic planes.

For example, when (S, G) is birationally equivalent to $(\mathbb{P}^2, G)$, we may take $f(x, y) = x$ since all cyclic groups of given order are conjugate in Cr(2). When $n = 2$, the problem was solved by M. Noether [46] and later G. Castelnuovo and F. Enriques [15] had realized that the classification follows from Bertini's classification of involutions in Cr(2). When n is prime the problem was studied by A. Bottari [10]. We refer for modern work on this problem to [11], [12].

- Extend the classification to the case of non-algebraically closed fields, e.g. $\mathbb{Q}$, and algebraically closed fields of positive characteristic.

Note that there are more automorphism groups in the latter case. For example, the Fermat cubic surface $T_0^3 + T_1^3 + T_2^3 + T_3^3 = 0$ over a field of characteristic 2 has the automorphism group isomorphic to $U(4, \mathbb{F}_4)$, which is a subgroup of index 2 of the Weyl group $W(E_6)$.

10. Tables

In the following tables we give the order of a group G, its structure, the type of a surface on which the group is realized as a minimal group of automorphisms, the equation of the surface, and the number of conjugacy classes, if finite, or the dimension of the variety of conjugacy classes.

For any positive integer k we denote by $\phi(k)$ is the value of the Euler function at k and by $\psi(k)$ the number of solutions of the equation $s^2 - s + 1 \equiv 0 \mod k$.

We denote by $\mathcal{C}$ the set of isomorphism classes of conic bundles.

As we had warned in the introduction, we are not 100% sure that our tables give a complete classification of all conjugacy classes. We refer to [7], [6], [21], [50] for modern treatment of the cases of cyclic groups of prime order and p-elementary abelian groups.

Order	Structure	Surface	Conjugacy classes
mn/k	$(\frac{m}{k} \times \frac{n}{k})^{\cdot}k$	$\mathbb{P}^2$	$\phi(k)$
$8nmk$	$(2m \times \overline{D}_{2n})^{\cdot}k$	$\mathbb{P}^2$	$\phi(k)/2$
$4mn$	$m \times \overline{D}_{2n}$	$\mathbb{P}^2$	1
$2mn$	$m \times D_{2n}$	$\mathbb{P}^2$	1
$120m$	$m \times \bar{I}$	$\mathbb{P}^2$	1
$48m$	$(m \times \bar{T})^{\cdot}2$	$\mathbb{P}^2$	1
$48m$	$m \times \bar{O}$	$\mathbb{P}^2$	1
$24m$	$m \times \bar{T}$	$\mathbb{P}^2$	1
$24m$	$(m \times \overline{D}_4)^{\cdot}3$	$\mathbb{P}^2$	1
$6n^2$	$n^2 : S_3$	$\mathbb{P}^2$	1
$3n^2/k$	$(n \times \frac{n}{k}) : 3$	$\mathbb{P}^2$	$\psi(k)$
$2mn$	$m \times D_{2n}$	$\mathbf{F}_0$	1
$12m$	$m \times T$	$\mathbf{F}_0$	$e = 0, 2$
$24m$	$m \times O$	$\mathbf{F}_e$	$e = 0, 2, 3$
$60m$	$m \times I$	$\mathbf{F}_e$	$e = 0, 2 \ldots, 6$
$4mn$	$D_{2n} \times D_{2m}$	$\mathbf{F}_0$	1
$120n$	$D_{2n} \times I$	$\mathbf{F}_0$	1
$48n$	$D_{2n} \times O$	$\mathbf{F}_0$	1
$24n$	$D_{2n} \times T$	$\mathbf{F}_0$	1
$2n$	D_{2n}	$\mathbf{F}_0$	$\phi(n)/2$
$4mnk$	$(m \times n) : D_{2k}$	$\mathbf{F}_0$	1
$8mn$	$(D_{2m} \times D_{2n}) : 2$	$\mathbf{F}_0$	1
$4mn$	$(m \times D_{2n}) : 2$	$\mathbf{F}_0$	1
$12n$	$(2^2 \times n) : 3$	$\mathbf{F}_0$	$e = 0, 2$
$24n$	$(2^2 \times D_{2n}) : S_3$	$\mathbf{F}_0$	1
$24n$	$n.O \cong T.2n \cong (2n, n, O, T)$	$\mathbf{F}_e$	$e = 0, 2, 3$
$24n$	$n.O \cong T.D_{2n} \cong (D_{2n}, n, O, T)$	$\mathbf{F}_e$	$e = 0, 3$
$48n$	$(T \times D_{2m}) : 2$	$\mathbf{F}_0$	1
$4n$	$D_{2n} : 2$	$\mathbf{F}_0$	1
$16n$	$((D_{2n} \times D_{2n}) : 2) : 2$	$\mathbf{F}_0$	1
$8n^2$	$(D_{2n} \times D_{2n}) : 2$	$\mathbf{F}_0$	1
$2n$	$2n$	$\mathcal{C}$	?
$4n$	$2 \times 2n$	$\mathcal{C}$	?
$4n$	$2^2 \times n$	$\mathcal{C}$	?
$4n$	$2.D_{2n}$	$\mathcal{C}$	?
$4n$	$2^2.n$	$\mathcal{C}$	?
$8n$	$2^2.D_{2n}$	$\mathcal{C}$	?
$2n(2k+1)mn$	$D_{4k+2} \times n$	$\mathcal{C}^{ex}$	?
$4n(2k+1)mn$	$D_{4k+2} \times D_{2n}$	$\mathcal{C}^{ex}$	?
$24(2k+1)mn$	$D_{4k+2} \times T$	$\mathcal{C}^{ex}$	?
$48(2k+1)mn$	$D_{4k+2} \times O$	$\mathcal{C}^{ex}$	?
$120(2k+1)mn$	$D_{4k+2} \times I$	$\mathcal{C}^{ex}$	?
$2n(2k+1)mn$	$\bar{D}_{2k} : n$	$\mathcal{C}^{ex}$	?
$4n(2k+1)mn$	$\bar{D}_{2k} : D_{2n}$	$\mathcal{C}^{ex}$	?
$24(2k+1)mn$	$\bar{D}_{2k} : T$	$\mathcal{C}^{ex}$	?
$48(2k+1)mn$	$\bar{D}_{2k} : O$	$\mathcal{C}^{ex}$	?
$120(2k+1)mn$	$\bar{D}_{2k} : I$	$\mathcal{C}^{ex}$	?
$2n(2k+1)mn$	$D_{2k} : 2n$	$\mathcal{C}^{ex}$	?
$4n(2k+1)mn$	$D_{2k} : \bar{D}_{2n}$	$\mathcal{C}^{ex}$	?
$24(2k+1)mn$	$D_{2k} : \bar{T}$	$\mathcal{C}^{ex}$	?
$48(2k+1)mn$	$D_{2k} : \bar{O}$	$\mathcal{C}^{ex}$	?
$120(2k+1)mn$	$D_{2k} : \bar{I}$	$\mathcal{C}^{ex}$	?

TABLE 7. Infinite series

Order	Type	Surface	Equation	Conjugacy
2		$\mathcal{C}_{2g+2}$		∞^{2g-1}
2	A_1^7	$\mathcal{D}P_2$	XIII	∞^6
2	A_1^8	$\mathcal{D}P_1$	XII	
3	$3A_2$	$\mathcal{D}P_3$	I,III,IV	∞^1
3	$4A_2$	$\mathcal{D}P_1$	XVIII	∞^3
4	$2A_3 + A_1$	$\mathcal{D}P_2$	II,III,IV,V	∞^1
4	$2D_4(a_1)$	$\mathcal{D}P_1$	I,VI, X,XVII,XX	∞^5
5	$2A_4$	$\mathcal{D}P_1$	XIV	∞^2
6	$E_6(a_2)$	$\mathcal{D}P_3$	I,III,IV,VI	∞^1
6	$A_5 + A_1$	$\mathcal{D}P_3$	I,VI	∞^1
6	$E_7(a_4)$	$\mathcal{D}P_2$	XI	∞^1
6	$A_5 + A_2$	$\mathcal{D}P_2$	VIII	∞^1
6	$D_6(a_2) + A_1$	$\mathcal{D}P_2$	II,III,IV,IX	∞^1
6	$A_5 + A_2 + A_1$	$\mathcal{D}P_1$	II,VIII,XIII	∞^2
6	$E_6(a_2) + A_2$	$\mathcal{D}P_1$	II,XII	∞^2
6	$E_8(a_8)$	$\mathcal{D}P_1$	I,II,III,IV,XII,XVIII	∞^3
6	$2D_4$	$\mathcal{D}P_1$	VII,XI	∞^1
6	$E_7(a_4) + A_1$	$\mathcal{D}P_1$	II,VIII,XIX	∞^4
8	D_5	$\mathcal{D}P_4$	(6.6)	1
8	$D_8(a_3)$	$\mathcal{D}P_1$	X	1
9	$E_6(a_1)$	$\mathcal{D}P_3$	I	1
10	$E_8(a_6)$	$\mathcal{D}P_1$	IV,IX,XIV	∞^2
12	E_6	$\mathcal{D}P_3$	III	1
12	$E_7(a_2)$	$\mathcal{D}P_2$	III	1
12	$E_8(a_3)$	$\mathcal{D}P_1$	I,V	∞^2
14	$E_7(a_1)$	$\mathcal{D}P_2$	I	1
15	$E_8(a_5)$	$\mathcal{D}P_1$	IV	1
18	E_7	$\mathcal{D}P_2$	VI	1
20	$E_8(a_2)$	$\mathcal{D}P_1$	IX	1
24	$E_8(a_1)$	$\mathcal{D}P_1$	I	1
30	E_8	$\mathcal{D}P_1$	IV	1

TABLE 8. Cyclic subgroups

Order	Structure	Surface	Equation	Conjugacy classes
4	2^2	$\mathcal{DP}_4, \mathcal{C}_5$		∞^2
4	2^2	$\mathcal{DP}_2$	XII	∞^5
4	2^2	$\mathcal{DP}_1$	VII,VIII,XII,XIII,XVI,XXI	∞^5
4	2^2	$\mathcal{DP}_1$	V, VI,X,XV,XVII	∞^3
8	2×4	$\mathcal{DP}_4$	(6.5)	∞^1
8	2×4	$\mathcal{DP}_2$	II,III,IV	$2 \times \infty^1$
8	2×4	$\mathcal{DP}_2$	II,III,V	∞^1
8	2×4	$\mathcal{DP}_2$	I-V,VII	∞^2
8	2×4	$\mathcal{DP}_1$	VIII,XVI	∞^2
8	2^3	$\mathcal{DP}_4$		∞^2
8	2^3	$\mathcal{DP}_2$	I-V,X	∞^3
9	3^2	$\mathcal{DP}_3$	I	1
9	3^2	$\mathcal{DP}_3$	I,III,IV	∞^1
9	3^2	$\mathcal{DP}_1$	I	1
9	3^2	$\mathcal{DP}_1$	III	∞^1
9	$3^2 \cong G_{3,3,2}$	$\mathbb{P}^2$		1
12	2×6	$\mathcal{DP}_4$	(6.6)	1
12	2×6	$\mathcal{DP}_2$	III	∞^1
12	2×6	$\mathcal{DP}_1$	II,VIII,XIII	∞^2
12	2×6	$\mathcal{DP}_1$	III,XII	∞^2
12	2×6	$\mathcal{DP}_1$	II,VII	∞^1
16	2^4	$\mathcal{DP}_4$		∞^2
16	$2^2 \times 4$	$\mathcal{DP}_2$	V	∞^1
16	4^2	$\mathcal{DP}_2$	II	2
16	2×8	$\mathcal{DP}_2$	II	1
18	2×3^2	$\mathcal{DP}_1$	I	1
18	2×3^2	$\mathcal{DP}_1$	III	∞^1
18	3×6	$\mathcal{DP}_1$	II,III	∞^1
18	3×6	$\mathcal{DP}_1$	II	1
18	3×6	$\mathcal{DP}_3$	I	1
24	2×12	$\mathcal{DP}_1$	VIII	∞^1
24	2×12	$\mathcal{DP}_2$	III	1
27	3^3	$\mathcal{DP}_3$	I	1
32	2×4^2	$\mathcal{DP}_2$	II	1
36	6^2	$\mathcal{DP}_1$	II	1

TABLE 9. Abelian non-cyclic subgroups

Order	Structure	Surface	Equation	Conjugacy classes
6	D_6	$\mathcal{D}P_3$	I,III,IV	∞^1
8	D_8	$\mathcal{D}P_4$	(6.5)	∞^1
8	D_8	$\mathcal{D}P_2$	II,III,V,VII	∞^2
8	D_8	$\mathcal{D}P_1$	V,X,XVII	∞^3
8	$\overline{D}_4$	$\mathcal{D}P_1$	II	1
8	$\overline{D}_4$	$\mathcal{D}P_1$	I,VI,XV	∞^2
12	D_{12}	$\mathcal{D}P_1$	VII,XI	∞^1
12	D_{12}	$\mathcal{D}P_1$	VII	1
12	$2 \times D_6$	$\mathcal{D}P_3$	VI	∞^1
12	$2 \times D_6$	$\mathcal{D}P_2$	I,II,IV,IX	∞^2
12	$\overline{D}_6$	$\mathcal{D}P_1$	I	1
16	D_{16}	$\mathcal{D}P_1$	X	1
16	$2 \times D_8$	$\mathcal{D}P_2$	II,III,IV	∞^1
16	$2 \times D_8$	$\mathcal{D}P_2$	II,III,V,VII	∞^2
18	D_{18}	$\mathcal{D}P_3$	I	1
18	$3 \times D_6$	$\mathcal{D}P_3$	I	1
18	$3 \times D_6$	$\mathcal{D}P_3$	I,III,IV	∞^1
24	O	$\mathcal{D}P_3$	II	1
24	O	$\mathcal{D}P_2$	II	2
24	O	$\mathcal{D}P_3$	V	∞^1
24	$\overline{T}$	$\mathcal{D}P_1$	I,VI	∞^1
24	$\overline{T}$	$\mathcal{C}_{2g+2}$	g is odd	?
24	$2 \times T$	$\mathcal{D}P_2$	I,II,IV	∞^1
24	$2 \times T$	$\mathcal{D}P_4$	(6.7)	1
24	$2 \times T$	$\mathcal{C}_{2g+2}$		
24	$3 \times \bar{D}_4$	$\mathcal{D}P_1$	I	1
24	$3 \times \bar{D}_4$	$\mathcal{D}P_1$	II,III	∞^1
24	$3 \times D_8$	$\mathcal{D}P_1$	I,V	∞^1
32	$2 \times D_{16}$	$\mathcal{D}P_2$	II	1
36	$3 \times \bar{D}_6$	$\mathcal{D}P_1$	I	1
36	$6 \times S_3$	$\mathcal{D}P_3$	III	1
48	$3 \times \bar{D}_8$	$\mathcal{D}P_1$	I	1
48	$\overline{O}$	$\mathcal{C}_{2g+2}$	g is odd	?
48	$\overline{O}$	$\mathcal{D}P_1$	I	2
48	$2 \times O$	$\mathcal{D}P_2$	I,II,III,IV	∞^1
48	$2 \times O$	$\mathcal{D}P_2$	I	1
48	$2 \times O$	$\mathcal{D}P_1$	I	2
48	$2 \times O$	$\mathcal{C}_{2g+2}$		?
60	I	$\mathcal{D}P_5$		1
72	$3 \times \bar{T}$	$\mathcal{D}P_1$	I	1
120	$2 \times I$	$\mathcal{C}_{2g+2}$		?
120	$\bar{I}$	$\mathcal{C}_{2g+2}$		?
144	$3 \times \bar{O}$	$\mathcal{D}P_1$	I	1

TABLE 10. Products of cyclic groups and polyhedral or binary polyhedral non-cyclic group

Order	Structure	Surface	Equation	Conjugacy classes
16	$2^3 : 2$	$\mathcal{D}P_4$	(6.5)	∞^1
16	$2^2 : 4$	$\mathcal{D}P_4$	(6.5)	$2 \times \infty^1$
16	$2^2 : 4$	$\mathcal{D}P_2$	II	1
18	$3^2 : 2$	$\mathcal{D}P_3$	I	1
21	$7 : 3$	$\mathbb{P}^2$		
24	$3 : D_8$	$\mathcal{D}P_1$	VII	1
27	$3^2 : 3$	$\mathcal{D}P_3$	I,III,IV	∞^1
32	$2^4 : 2$	$\mathcal{D}P_4$	(6.5)	∞^1
32	$4^2 : 2$	$\mathcal{D}P_2$	II	3
32	$(2^2 \times 4) : 2$	$\mathcal{D}P_2$	II	1
32	$(2^2 \times 4) : 2$	$\mathcal{D}P_2$	II	1
32	$2^3 : 4$	$\mathcal{D}P_4$	(6.6)	1
36	$3^2 : 2^2$	$\mathcal{D}P_3$	I	2
36	$3^2 : 4$	$\mathcal{D}P_3$	III	1
48	$2^4 : 3$	$\mathbb{F}_0$		1
48	$2^4 : 3$	$\mathcal{D}P_4$	(6.7)	1
48	$4^2 : 3$	$\mathcal{D}P_2$	II	2
54	$3^3 : 2$	$\mathcal{D}P_3$	I	1
54	$3^2 : S_3$	$\mathcal{D}P_3$	III	1
64	$2^4 : 4$	$\mathcal{D}P_4$	(6.6)	1
64	$(2 \times 4^2) : 2$	$\mathcal{D}P_2$	II	1
72	$3^2 : D_8$	$\mathbb{P}^2$		
81	$3^3 : 3$	$\mathcal{D}P_3$	I	1
90	$2^3 : D_{10}$	$\mathcal{D}P_4$	(6.7)	1
90	$2^4 : 5$	$\mathcal{D}P_4$	(6.7)	1
96	$2^4 : S_3$	$\mathbb{F}_0$		1
96	$2^4 : S_3$	$\mathcal{D}P_4$	(6.7)	1
96	$(2 \times 4^2) : 3$	$\mathcal{D}P_2$	II	2
96	$2^2.O$	$\mathcal{C}_{2g+2}$		?
108	$3^2 : A_4$	$\mathbb{P}^2$		
108	$3^3 : 2^2$	$\mathcal{D}P_3$	I	1
160	$2^4 : D_{10}$	$\mathcal{D}P_4$	(6.7)	1
162	$3^3 : S_3$	$\mathcal{D}P_3$	I	1
192	$(2 \times 4^2) : S_3$	$\mathcal{D}P_2$	II	1
216	$3^2 : S_4$	$\mathbb{P}^2$		
240	$2^2.I$	$\mathcal{C}_{2g+2}$		?
324	$3^3 : A_4$	$\mathcal{D}P_3$	I	1
648	$3^3 : S_4$	$\mathcal{D}P_3$	I	1

TABLE 11. Conjugacy classes of semi-direct product $A : B$, where A is abelian

Order	Structure	Surface	Equation	Conjugacy classes
16	$D_8 : 2$	$\mathcal{D}P_2$	III,V	$3 \times \infty^1$
18	$3.S_3$	$\mathcal{D}P_3$	III,IV	∞^1
18	$3.S_3$	$\mathcal{D}P_1$	II	1
24	$D_8 : 3$	$\mathcal{D}P_2$	III	2
24	$2^2.S_3$	$\mathcal{D}P_2$	II	2
32	$2 \times (D_8 : 2)$	$\mathcal{D}P_2$	III	1
42	$2 \times (3 : 7)$	$\mathcal{D}P_2$	I	1
48	$D_8 : 6$	$\mathcal{D}P_2$	III	2
48	$2 \times 2^2.S_3$	$\mathcal{D}P_2$	II	2
54	$(3.3^2) : 2$	$\mathcal{D}P_3$	I,III,IV	∞^1
72	$6.D_{12}$	$\mathcal{D}P_1$	I	1
72	$6.S_3$	$\mathcal{D}P_1$	I	1
96	$2 \times D_8 : 6$	$\mathcal{D}P_2$	III	2
108	$3.(3^2 : 4)$	$\mathcal{D}P_3$	III	1
120	S_5	$\mathcal{D}P_3$	II	1
168	$L_2(7)$	$\mathcal{D}P_2$	I	1
192	$(2 \times 4^2) : S_3$	$\mathcal{D}P_2$	V	∞^1
316	$2 \times L_2(7)$	$\mathcal{D}P_2$	I	1
360	A_6	$\mathbb{P}_2$	I	1

TABLE 12. Other groups

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