Gerstenhaber and Batalin-Vilkovisky structures on Lagrangian intersecions

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To Yuri I. Manin

on the occasion of his 70th birthday

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Abstract

Let M and N be Lagrangian submanifolds of a complex symplectic manifold S. We construct a Gerstenhaber algebra structure on $\mathcal{T}or_*^{\mathcal{O}_S}(\mathcal{O}_M, \mathcal{O}_N)$ and a compatible Batalin-Vilkovisky module structure on $\mathcal{E}xt^*_{\mathcal{O}_S}(\mathcal{O}_M, \mathcal{O}_N)$. This gives rise to a de Rham type cohomology theory for Lagrangian intersections.

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Introduction

We are interested in intersections of Lagrangian submanifolds of holomorphic symplectic manifolds. Thus we work over the complex numbers in the analytic category.

Lagrangian intersection numbers: smooth case

Let S be a (complex) symplectic manifold and L, M Lagrangian submanifolds. Since L and M are half-dimensional, the expected dimension of their intersection is zero. Intersection theory therefore gives us the intersection number

$$\#(L \cap M)$$

if the intersection is compact. In the general case, we get a class

$$[L \cap M]^{\mathrm{vir}} \in A_0(L \cap M)$$

in degree zero Borel-Moore homology, such that in the compact case

$$\#(L \cap M) = \deg[L \cap M]^{\mathrm{vir}} .$$

If the intersection $X = L \cap M$ is smooth,

$$[X]^{\mathrm{vir}} = c_{\mathrm{top}}(E) \cap [X] ,$$

where E is the excess bundle of the intersection, which fits into the exact sequence

$$0 \longrightarrow T_X \longrightarrow T_L|_X \oplus T_M|_X \longrightarrow T_S|_X \longrightarrow E \longrightarrow 0$$

of vector bundles on X. The symplectic form σ defines an isomorphism $T_S|_X = \Omega_S|_X$. Under this isomorphism, the subbundle $T_L|_X$ corresponds to the conormal bundle $N^\vee_{L/S}$. Thus we can rewrite our exact sequence as

$$0 \longrightarrow E^\vee \longrightarrow N^\vee_{L/S} \oplus N^\vee_{M/S} \longrightarrow \Omega_S|_X \longrightarrow \Omega_X \longrightarrow 0 ,$$

which shows that the excess bundle E is equal to the cotangent bundle Ω_X. Thus, in the smooth case

$$[X]^{\mathrm{vir}} = c_{\mathrm{top}}(E) = c_{\mathrm{top}}(\Omega_X) \cap [X] = (-1)^n c_{\mathrm{top}}(T_X) \cap [X] ,$$

and in the smooth and compact case

$$\#(L \cap M) = \deg[X]^{\mathrm{vir}} = (-1)^n \int_X c_{\mathrm{top}}(T_X) = (-1)^n \chi(X) ,$$

where $2n$ is the dimension of S and $\chi(X)$ is the topological Euler characteristic of X. This shows that we can make sense of the intersection number even if the intersection is not compact: define the intersection number to be signed Euler characteristic.

Intersection numbers: singular case

In [1], is was shown how to make sense of the statement that Lagrangian intersection numbers are signed Euler characteristics in the case that the intersection X is singular. We introduced an integer invariant $\nu_X(P) \in \mathbb{Z}$ of the singularity of the analytic space X at the point $P \in X$. (Essentially, $\nu_X(P)$ is MacPherson's local Euler obstruction applied the the signed support of the intrinsic normal cone of X at P.) The basic properties of $\nu_X(P)$ are

(i) $\nu_X : X \to \mathbb{Z}$ is a constructible function with respect to the Zariski topology on X,

(ii) if X is smooth at P, then $\nu_X(P) = (-1)^{\dim X}$,

(iii) if $X = Z(df)$ is the critical set of a holomorphic function f on an n-dimensional manifold M, then $\nu_X(P) = (-1)^n\big(1 - \chi(F_P)\big)$, where F_P is the Milnor fibre of f at P.

If f is a holomorphic function on the manifold M, we consider the cotangent bundle Ω_M as a symplectic manifold. The zero section M, and the graph of the closed 1-form df are Lagrangian submanifolds, and $X = Z(df)$ is their intersection.

The main theorem of [1] implies that if L and M are Lagrangian submanifolds of the symplectic manifold S, with intersection X, then

$$\#X = \deg[X]^{\mathrm{vir}} = \chi(X, \nu_X)\,,$$

the weighted Euler characteristic of X with respect to the constructible function ν_X, which is defined as

$$\chi(X, \nu_X) = \sum_{i \in \mathbb{Z}} i \cdot \chi(\{\nu_X = i\})\,.$$

In particular, arbitrary Lagrangian intersection numbers are always well-defined: the intersection need not be smooth or compact. The integer $\nu_X(P)$ may be considered as the contribution of the point P to the intersection $X = L \cap M$.

Categorifying intersection numbers: smooth case

To categorify the intersection number means to construct a cohomology theory such that the intersection number is equal to the alternating sum of Betti numbers. If X is smooth (not necessarily compact) a natural candidate is (shifted) holomorphic de Rham cohomology

$$\#(X) = (-1)^n \chi(X) = \sum (-1)^{i-n} \dim_{\mathbb{C}} \mathbb{H}^i\big(X, (\Omega_X^\bullet, d)\big)\,.$$

Here $(\Omega_X^\bullet, d)$ is the holomorphic de Rham complex of X and $\mathbb{H}^i$ its hypercohomology. Of course, by the holomorphic Poincaré lemma, hypercohomology reduces to cohomology.

Categorification: compact case

If the intersection $X = L \cap M$ is compact, but not necessarily smooth, we have

$$\begin{aligned} \#X &= \sum_i (-1)^{i-n} \dim_{\mathbb{C}} \operatorname{Ext}^i_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M) \\ &= \sum_{i,j} (-1)^i (-1)^{j-n} \dim_{\mathbb{C}} H^i\big(X, \mathcal{E}xt^j_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)\big) \, . \end{aligned}$$

If X is smooth, $\mathcal{E}xt^j_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M) = \Omega^j_X$, so this reduces to Hodge cohomology

$$\#X = \sum_{i,j} (-1)^i (-1)^{j-n} \dim_{\mathbb{C}} H^i(X, \Omega^j_X) \, .$$

This justifies using the sheaves $\mathcal{E}xt^j_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)$ as replacements for the sheaves Ω^j_X if X is not smooth any longer. To get finite-dimensional cohomology groups, we will construct de Rham type differentials

$$d : \mathcal{E}xt^j_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M) \longrightarrow \mathcal{E}xt^{j+1}_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)$$

so that the hypercohomology groups

$$\mathbb{H}^i\big(X, (\mathcal{E}xt^\bullet_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M), d)\big)$$

are finite dimensional, even if X is not compact. Returning to the compact case, for any such d, we would necessarily have

$$\#X = \sum_i (-1)^{i-n} \dim_{\mathbb{C}} \mathbb{H}^i\big(X, (\mathcal{E}xt^\bullet_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M), d)\big) \, .$$

Categorification: local case

Every symplectic manifold S is locally isomorphic to the cotangent bundle Ω_N of a manifold N. The fibres of the induced vector bundle structure on S are Lagrangian submanifolds, and thus we have defined (locally on S) a foliation by Lagrangian submanifolds, i.e., a *Lagrangian foliation*. (Lagrangian foliations are also called *polarizations*.) We may assume that the leaves or our Lagrangian foliation of S are transverse to the two Lagrangians L and M whose intersection we wish to study. Then L and M turn into the graphs of 1-forms on N. The Lagrangian condition implies that these 1-forms on N are closed. Without loss of generality, we may assume that one of these 1-forms is the zero section of Ω_N and hence identify M with N. By making $M = N$ smaller if necessary, we may assume that the closed 1-form defined by L is exact. Then L is the graph of the 1-form df, for a holomorphic function f on M. Thus the intersection $L \cap M$ is now the zero locus of the 1-form df:

$$X = Z(df) \, .$$

This is the *local case*.

Multiplying by df defines a differential

$$\begin{aligned} s : \Omega_M^j &\longrightarrow \Omega_M^{j+1} \\ \omega &\longmapsto df \wedge \omega \,. \end{aligned}$$

Because df is closed, the differential s commutes with the de Rham differential $d : \Omega_M^j \to \Omega_M^{j+1}$. Thus the de Rham differential passes to cohomology with respect to s:

$$d : h^j(\Omega_M^\bullet, s) \longrightarrow h^{j+1}(\Omega_M^\bullet, s) \,,$$

where h^j denotes the cohomology sheaves, which are coherent sheaves of $\mathcal{O}_X$-modules. Let us denote these cohomology sheaves by

$$\mathcal{E}^j = h^j(\Omega_M^\bullet, s) \,.$$

We have thus defined a complex of sheaves on X

$$(\mathcal{E}^\bullet, d) \,, \tag{1}$$

where the $\mathcal{E}^i$ are coherent sheaves of $\mathcal{O}_X$-modules, and the differential d is $\mathbb{C}$-linear. It is a theorem of Kapranov [2], that the cohomology sheaves $h^i(\mathcal{E}^\bullet, d)$ are constructible sheaves on X and thus have finite dimensional cohomology groups. It follows that the hypercohomology groups

$$\mathbb{H}^i\big(X, (\mathcal{E}^\bullet, d)\big)$$

are finite-dimensional as well.

Kapranov [2] also examines the relationship of $(\mathcal{E}^\bullet, d)$ with the perverse sheaf of vanishing cycles on X. In fact, he proves that there is a spectral sequence from the former to the latter. This implies that the constructible function

$$P \mapsto \sum_i (-1)^{i-n} \dim_{\mathbb{C}} \mathbb{H}^i_{\{P\}}\big(X, (\mathcal{E}, d)\big) \,,$$

of fiberwise Euler characteristic of $(\mathcal{E}, d)$ is equal to ν_X. This achieves the categorification in the local case. In particular, for the non-compact intersection numbers we have

$$\chi(X, \nu_X) = \sum_i (-1)^{i-n} \dim_{\mathbb{C}} \mathbb{H}^i\big(X, (\mathcal{E}, d)\big) \,.$$

To make the connection with the compact case (and because this construction is of central importance to the paper), let us explain why

$$\mathcal{E}^i = \mathcal{E}xt^i_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M) \,.$$

Denote the projection $S = \Omega_M \to M$ by π. The 1-form on Ω_M which corresponds to the vector field generating the natural $\mathbb{C}^*$-action on the fibres we shall call α. Then $d\alpha = \sigma$ is the symplectic form on S. We consider the 1-form $s = \alpha - \pi^* df$ on S. Its zero locus in S is equal to the graph of df. Let us denote the subbundle of Ω_S annihilating vector fields tangent to the fibres of π by E. Then $e \in \Omega_S$ is a section of E and we obtain a resolution of the structure sheaf of $\mathcal{O}_L$ over $\mathcal{O}_S$:

$$\cdots \longrightarrow \Lambda^2 E^\vee \xrightarrow{\widetilde{s}} E^\vee \xrightarrow{\widetilde{s}} \mathcal{O}_S \ ,$$

where $\widetilde{s}$ denotes the derivation of the differential graded $\mathcal{O}_S$-algebra $\Lambda^\bullet E^\vee$ given by contraction with s. Taking duals and tensoring with $\mathcal{O}_M$, we obtain a complex of vector bundles $(\Lambda E^\vee|_M, s|_M)$ which computes $\mathcal{E}xt^i_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)$. One checks that $(\Lambda E^\vee|_M, s|_M) = (\Omega_M, s)$.

Categorification: global case

We now come to the contents of this paper. let S be a symplectic manifold and L, M Lagrangian submanifolds with intersection X. Let us use the abbreviation $\mathcal{E}^i = \mathcal{E}xt^i_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)$. The $\mathcal{E}^i$ are coherent sheaves of $\mathcal{O}_X$-modules. The main theorem of this paper is that the locally defined de Rham differentials (1) do not depend on the way we write S as a cotangent bundle, or, in other words, that d is independent of the chosen polarization of S. Thus the locally defined d glue and we obtain a globally defined canonical de Rham type differential

$$d : \mathcal{E}^i \to \mathcal{E}^{i+1} \, .$$

In the case that X is smooth, $\mathcal{E}^i = \Omega^i_X$, and d is the usual de Rham differential. We may call $(\mathcal{E}^\bullet, d)$ the *virtual de Rham complex* of the Lagrangian intersection X. It categorifies Lagrangian intersection numbers in the sense that for the local contribution of the point $P \in X$ to the Lagrangian intersection we have

$$\nu_X(P) = \sum_i (-1)^{i-n} \dim_{\mathbb{C}} \mathbb{H}^i_{\{P\}}\big(X, (\mathcal{E}, d)\big) \, .$$

Hence, for the non-compact intersection numbers we have

$$\chi(X, \nu_X) = \sum_i (-1)^{i-n} \dim_{\mathbb{C}} \mathbb{H}^i\big(X, (\mathcal{E}, d)\big) \, ,$$

In particular, if the intersection is compact, we have written $\#X = \chi(X, \nu_X)$ as alternating sum of the Betti numbers of the hypercohomology groups of the virtual de Rham complex.

Gerstenhaber and Batalin-Vilkovisky structures

The virtual de Rham complex $(\mathcal{E}^\bullet, d)$ is just one half of the story. There is also the graded sheaf of $\mathcal{O}_X$-algebras $\mathcal{A}^\bullet$ given by

$$\mathcal{A}^i = \mathcal{T}or^{\mathcal{O}_S}_{-i}(\mathcal{O}_L, \mathcal{O}_M) \, ,$$

Locally, $\mathcal{A}^\bullet$ is given as the cohomology of $(\Lambda T_M, \widetilde{s})$, in the above notation. The Lie-Schouten bracket induces a $\mathbb{C}$-linear bracket operation

$$[\,,\,] : \mathcal{A}\bullet \otimes_{\mathbb{C}} \mathcal{A}^\bullet \longrightarrow \mathcal{A}^\bullet$$

of degree $+1$. We show that these locally defined brackets glue to give a globally defined bracket making $(\mathcal{A}^\bullet, \wedge, [\,,\,])$ a Gerstenhaber algebra.

Then $\mathcal{E}^\bullet$ is a sheaf of modules over $\mathcal{A}^\bullet$. The bracket on $\mathcal{A}^\bullet$ and the differential on $\mathcal{E}^\bullet$ satisfy a compatibility condition, see (6). We say that $(\mathcal{E}, d)$ is a *Batalin-Vilkovisky module* over the Gerstenhaber algebra $(\mathcal{A}^\bullet, [\,,\,])$.

In the case that L and M are oriented submanifolds, i.e., the highest exterior powers of the normal bundles have been trivialized, we have an identification

$$\mathcal{A}^i = \mathcal{E}^{n+i}\,.$$

Transporting the differential from $\mathcal{E}^\bullet$ to $\mathcal{A}^\bullet$ via this identification turns $(\mathcal{A}^\bullet, \wedge, [\,,\,], d)$ into a Batalin-Vilkovisky algebra.

To prove these facts we have to study differential Gerstenhaber algebras and differential Batalin-Vilkovisky modules over them. We will prove that locally defined Gerstenhaber algebras and their Batalin-Vilkovisky modules are quasi-isomorphic, making their cohomologies isomorphic and hence yielding the well-definedness of the bracket and the differential.

DG category of Lagrangian submanifolds

Choose an affine open cover $\mathfrak{U} = (U_\alpha)$ of S. Then we associate to Lagrangians L and M the Čech double complex

$$\check{C}^\bullet\big(\mathfrak{U}, (\mathcal{E}xt^\bullet_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M), d)\big) \tag{2}$$

which computes $\mathbb{H}^i\big(X, (\mathcal{E}xt^\bullet_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M), d)\big)$. Define a differential graded category with objects the Lagrangian submanifolds of S and morphism spaces (2). One can also enlarge this category to include d-branes (local systems on Lagrangian submanifolds).

Donaldson-Thomas invariants

Our original motivation for this research was a better understanding of Donaldson-Thomas invariants. It is to be hoped that the moduli spaces giving rise to Donaldson-Thomas invariants (spaces of stable sheaves of fixed determinant on Calabi-Yau threefolds) are Lagrangian intersections, at least locally. We have two reasons for believing this: first of all, the obstruction theory giving rise to the virtual fundamental class is *symmetric*, a property shared by the obstruction theories of Lagrangian intersections. Secondly, at least heuristically, these moduli spaces are equal to the critical set of the holomorphic Chern-Simons functional.

Our exchange property should be useful for gluing virtual de Rham complexes if the moduli spaces are only local Lagrangian intersections.

In this way we hope to construct a virtual de Rham complex on the Donaldson-Thomas moduli spaces and thus categorify Donaldson-Thomas invariants.

1 Algebraic Preliminaries

1.1 The algebra $K(E,s)$ and its dual $L(E,s)$

Let S be a manifold and E be a vector bundle on S. By $K(E)$, we will always mean the sheaf of graded $\mathcal{O}_S$-algebras $\Lambda^\bullet(E^\vee)$, where $E^\vee$ is placed in degree -1. By $L(E)$, we will denote $\Lambda^\bullet E$, where E is placed in degree $+1$. We will denote the natural pairing of $E^\vee$ and E by $X \lrcorner\, \omega \in \mathcal{O}_S$, for $X \in E^\vee$ and $\omega \in E$.

Lemma 1.1 *There exists a unique extension of* $\lrcorner$ *to an action of the sheaf of graded* $\mathcal{O}_S$*-algebras* $K(E)$ *on the sheaf of graded* $\mathcal{O}_S$*-modules* $L(E)$*, which satisfies*

(i) $f \lrcorner\, \omega = f\omega$, *for* $f \in \mathcal{O}_S$ *and* $\omega \in L(E)$ *(linearity over* $\mathcal{O}_S$*),*

(ii) $X \lrcorner (\omega_1 \wedge \omega_2) = (X \lrcorner\, \omega_1) \wedge \omega_2 + (-1)^{\overline{\omega}_1} \omega_1 \wedge (X \lrcorner\, \omega_2)$, *for* $X \in E^\vee$ *and* $\omega_1, \omega_2 \in L(E)$*, (the degree* -1 *part acts by derivations)*

(iii) $(X \wedge Y) \lrcorner\, \omega = X \lrcorner (Y \lrcorner\, \omega)$, *for* $X, Y \in K(E)$, $\omega \in L(E)$ *(action property)*

Proof. Standard. The homomorphism $X \lrcorner : E \to \mathcal{O}_S$ extends in a unique way to a derivation of degree -1 on $L(E)$. For different X, Y these derivations anticommute, which defines a morphism of $\mathcal{O}_S$-algebras $K(E) \to \mathrm{Hom}_{\mathcal{O}_S}\big(L(E), L(E)\big)$. $\square$

Remark 1.2 Set $\langle X, \omega \rangle$ equal to the degree zero part of $X \lrcorner\, \omega$. This defines a perfect pairing $K(E) \otimes_{\mathcal{O}_S} L(E) \to \mathcal{O}_S$, identifying $L(E)$ as the $\mathcal{O}_S$-dual of $K(E)$. One checks that

$$\langle X_1 \wedge \ldots \wedge X_p, \omega_1 \wedge \ldots \wedge \omega_p \rangle = (-1)^{\frac{p(p-1)}{2}} \sum_{\sigma \in S_p} \mathrm{sign}(\sigma)(X_{\sigma(1)} \lrcorner\, s_1) \ldots (X_{\sigma(p)} \lrcorner\, s_p)\,.$$

Lemma 1.3 *For* $X_1, \ldots, X_p \in E^\vee$*, for* $s \in E$ *and* $\omega \in L(E)$ *we have*

$$X_1 \wedge \ldots \wedge X_p \lrcorner (s \wedge \omega) =$$
$$(-1)^p s \wedge (X_1 \wedge \ldots \wedge X_p \lrcorner\, \omega) + \sum_{i=1}^{p} (-1)^{p-i} \langle X_i, s \rangle\, X_1 \wedge \ldots \widehat{X_i} \ldots \wedge X_p \lrcorner\, \omega \quad (3)$$

Proof.Straightforward calculation.$\square$

Now turn things around and note that any section $s \in E$ defines a derivation of degree $+1$ on $K(E)$, which we shall denote by $\widetilde{s}$. It is the unique derivation which extends the map $E^\vee \to \mathcal{O}_S$ given by $\widetilde{s}(X) = \langle X, s \rangle$, for all $X \in E^\vee$. (Note that this is not a violation of the universal sign convention, see Remark 1.5.)

Lemma 1.4 *The pair $\big(K(E), \widetilde{s}\big)$ is a sheaf of differential graded algebras. Left multiplication by s defines a differential on $L(E)$ and the pair $(L(E), s)$ is a sheaf of differential graded modules over $\big(K(E), \widetilde{s}\big)$.*

PROOF. We need to check that

$$s \wedge (X \lrcorner\, \omega) = \widetilde{s}(X) \lrcorner\, \omega + (-1)^{\overline{X}} X \lrcorner\, (s \wedge \omega) \tag{4}$$

for all $\omega \in L(E)$ and $X \in K(E)$. We may assume that $X = X_1 \wedge \ldots \wedge X_p$, where $X_1, \ldots, X_p \in E^\vee$. Then

$$\widetilde{s}(X) = \widetilde{s}(X_1 \wedge \ldots \wedge X_p) = -\sum_{i=1}^{p} (-1)^i \widetilde{s}(X_i)\, X_1 \wedge \ldots \widehat{X_i} \ldots \wedge X_p \,.$$

Hence (4) follows from (3). $\square$

Remark 1.5 According to Formula (4), we have, if $\deg X + \deg \omega + 1 = 0$,

$$\langle \widetilde{s}(X), \omega \rangle + (-1)^{\overline{X}} \langle X, s \wedge \omega \rangle = 0 \,.$$

This means that the derivation $\widetilde{s}$ and left multiplication by s are $\mathcal{O}_S$-duals of one another. To explain the signs, not that we think of $\widetilde{s}$ and s as differentials on the graded sheaves $K(E)$ and $L(E)$, and for differentials of degree $+1$ the sign convention is

$$0 = D\langle X, \omega \rangle = \langle DX, \omega \rangle + (-1)^{\overline{X}} \langle X, D\omega \rangle \,.$$

In particular, for $\deg X = 1$ and $\omega = 1$ we get $\widetilde{s}(X) = \langle X, s \rangle$.

Remark 1.6 We can summarize Formula (4) more succinctly as

$$[s, i_X] = i_{\widetilde{s}X} \,.$$

Definition 1.7 We will use the notation $K(E, s) = \big(K(E), \widetilde{s}\big)$ and $L(E, s) = \big(L(E), s\big)$. We will usually think of $L(E, s)$ as a differential graded module over the differential graded algebra $K(E, s)$. We will keep in mind that $L(E, s)$ is the $\mathcal{O}_S$-dual of $K(E, s)$.

Finally, let us note that there is a canonical isomorphism of complexes of $\mathcal{O}_S$-modules

$$\begin{aligned} \Lambda^n E[-n] \otimes K(E, s) &\longrightarrow L(E, s) \\ \omega_1 \wedge \ldots \wedge \omega_n \otimes X &\longmapsto (-1)^{n\overline{X}} X \lrcorner\, (\omega_1 \wedge \ldots \wedge \omega_n)\,, \end{aligned} \tag{5}$$

where n is the rank of E. The sign is necessitated by the sign change in the differential when shifting a complex. For that reason it seems more natural to write $\Lambda^n E[-n]$ on the left rather than on the right.

1.2 Brackets vs. derivations

Definition 1.8 A **bracket** on $K(E)$ of degree $+1$ is a homomorphism

$$[\,]: K(E) \otimes_{\mathbb{C}} K(E) \longrightarrow K(E)$$

of degree $+1$ satisfying:

(i) $[\,]$ is a graded $\mathbb{C}$-linear derivation in each of its two arguments,

(ii) $[\,]$ is graded commutative (not anti-commutative).

If $[\,]$ satisfies in addition the Jacobi identity, we shall call $[\,]$ a **Lie bracket**.

The sign convention for brackets of degree $+1$ is that the comma is treated as carrying the degree $+1$, the opening and closing bracket as having degree 0. Thus, when passing an odd element past the comma, the sign changes. For example, the graded commutativity reads:

$$[Y,X] = (-1)^{\overline{XY}+\overline{X}+\overline{Y}}[X,Y]\,.$$

Proposition 1.9 *Suppose we are given a derivation $d \in \operatorname{Der}^1_{\mathbb{C}}\big(L(E),L(E)\big)$. Then there exists a unique bracket $[\,]$ of degree $+1$ on $K(E)$ such that*

$$d(X\wedge Y \lrcorner\, \omega) + (-1)^{\overline{X}+\overline{Y}} X\wedge Y \lrcorner\, d\omega + (-1)^{\overline{X}}[X,Y]\lrcorner\,\omega = (-1)^{\overline{X}} X \lrcorner\, d(Y\lrcorner\,\omega) + (-1)^{\overline{XY}+\overline{Y}} Y\lrcorner\, d(X\lrcorner\,\omega)\,, \quad (6)$$

for all $X,Y \in K(E)$ and $\omega \in L(E)$.

Moreover, every bracket of degree $+1$ comes about in this way from a unique $d \in \operatorname{Der}^1_{\mathbb{C}}\big(L(E),L(E)\big)$.

Finally, $[\,]$ is Lie bracket if and only if d is a differential, i.e., $[d,d] = d^2 = 0$.

Before we proceed with the proof, let us remark that Formula (6) can be rewritten as

$$[X,Y]\lrcorner\,\omega = \big[[i_X,d],i_Y\big](\omega)$$

or simply

$$i_{[X,Y]} = \big[[i_X,d],i_Y\big]\,. \quad (7)$$

Note also that $\big[[i_X,d],i_Y\big] = \big[i_X,[d,i_Y]\big]$.

Proof. We define the bracket $[X,Y] \in \Lambda^{-\overline{X}-\overline{Y}-1}E^{\vee}$ to be the $\mathcal{O}_S$-linear map $\Lambda^{-\overline{X}-\overline{Y}-1}E \to \mathcal{O}_S$ defined by

$$[X,Y]\lrcorner\,\omega = X\lrcorner\, d(Y\lrcorner\,\omega) + (-1)^{\overline{XY}+\overline{X}+\overline{Y}} Y \lrcorner\, d(X\lrcorner\,\omega) - (-1)^{\overline{Y}} X\wedge Y\lrcorner\, d\omega\,.$$

To check that this expression for $[X,Y]\lrcorner\,\omega$ is $\mathcal{O}_S$-linear in ω, amounts to proving that for every $f \in \mathcal{O}_S$ we have $\big[[i_X,df],i_Y\big] = 0$. But this is easy: $\big[[i_X,df],i_Y\big] = (-1)^{\overline{X}}[i_{\widetilde{df}(X)},i_Y] = 0$, by Remark 1.6.

With this definition, Formula (6) automatically holds, whenever $\deg\omega = -\deg X - \deg Y - 1$, i.e., if the expression in (6) is of degree 0.

Fix $X \in K(E)$ and let us prove that $[X, \cdot\,]$ is a derivation of degree $\deg X + 1$ on $K(E)$. The claim is that

$$[X, Y\wedge Z] = [X,Y]\wedge Z - (-1)^{\overline{X}} Y\wedge [X,Z]\,.$$

This is equivalent to checking that

$$i_{[X,Y\wedge Z]} = i_{[X,Y]}\circ i_Z - (-1)^{\overline{X}} i_Y \circ i_{[X,Z]}$$

holds after evaluating on ω of degree $-\deg X - \deg Y - \deg Z - 1$. But in this case we already know that (6) or rather (7) holds. (Note that one can commute i_Y and $i_{[X,Z]}$.) Thus we may instead prove that

$$\big[[i_X, d], i_Y\circ i_Z\big] = \big[[i_X,d], i_Y\big]\circ i_Z - (-1)^{\overline{X}} i_Y \circ \big[[i_X,d], i_Z\big]$$

holds on elements ω of the correct degree. But this latter formula is a general property of composition of functions, so is always true.

In the same way we can prove that $[\,]$ is a derivation in the first argument, and that it is symmetric.

Let us now prove that (6) holds without restrictions on the degree of ω. Again, we may instead prove (7). Note that (6) is symmetric in X and Y. If X and Y are of degree zero, the equation follows directly from the Leibnitz rule for d. If X is of degree -1 and Y of degree zero, the equation is also tautological. If X and Y are of degree -1, then i_X and i_Y are derivations, so the iterated commutator of derivations on the right hand side of (7), as well as the left hand side are derivations. Hence it is enough to check (7) after evaluation on elements $\omega \in E$, for which it holds by definition. Finally, not that both sides of (7) are derivations in the argument Y. So the general case reduces to case of degree -1.

We leave the 'Moreover' and the 'Finally' to the reader. We will not use them in the paper. □

Remark 1.10 For example, if $E = \Omega_M$, for a manifold $M = S$ and d is the exterior derivative, then $[\,]$ is the Schouten-Nijenhuis bracket on polyvector fields.

Proposition 1.11 *Given a derivation $d \in \operatorname{Der}^1\big(L(E), L(E)\big)$ and the corresponding bracket $[\,]$ of degree $+1$ on $K(E)$. Let $s \in E$ be a section and $\widetilde{s} \in \operatorname{Der}^1\big(K(E), K(E)\big)$ the corresponding derivation. Then $\widetilde{s}$ is a derivation with respect to the bracket $[\,]$, if $ds = 0 \in \Lambda^2 E$.*

Proof. Note that $[d,s]$ is equal to multiplication from the left by ds. Thus we may write $[d,s] = ds$. Also, note that $[s, i_X] = i_{\widetilde{s}(X)}$, for all $X \in K(E)$ and so we have $\big[[i_X, s], i_Y\big] = \pm[i_{\widetilde{s}(X)}, i_Y] = 0$, for all $X, Y \in K(E)$. Using these facts, a formal calculation with iterated commutators yields the following result:

$$(-1)^{\overline{X}}\big[[i_X, ds], i_Y\big] = i_{\widetilde{s}[X,Y]} - i_{[\widetilde{s}X,Y]} + (-1)^{\overline{X}} i_{[X,\widetilde{s}Y]}$$

Thus, $ds = 0$ implies

$$\widetilde{s}[X,Y] = [\widetilde{s}X,Y] - (-1)^{\overline{X}}[X,\widetilde{s}Y]\,,$$

which is the condition for $\widetilde{s}$ to be a derivation with respect to $[\,]$. □

1.3 Differential Gerstenhaber algebras and differential Batalin-Vilkovisky modules

We will formalize some of the previous considerations. We are always working on a topological space S, with a 'structure sheaf' $\mathcal{O}_S$, i.e., a sheaf of commutative $\mathbb{C}$-algebras with unit (non-graded). Examples are manifolds with their structure sheaf, or manifolds with the (pushforward of the) structure sheaf of a submanifold.

Differential Gerstenhaber algebras

Definition 1.12 A **Gerstenhaber algebra** over $\mathcal{O}_S$ is a sheaf of graded $\mathcal{O}_S$-modules A, concentrated in non-positive degrees, $A^0 = \mathcal{O}_S$, endowed with

(i) a commutative (associative, of course) product $\wedge$ of degree 0 with unit, making A an $\mathcal{O}_S$-algebra,

(ii) a Lie bracket $[\,]$ of degree $+1$ (see Definition 1.8).

In our cases, the underlying $\mathcal{O}_S$-module of A will usually be coherent.

Definition 1.13 A **differential Gerstenhaber algebra** is a Gerstenhaber algebra A over $\mathcal{O}_S$ endowed with an additional $\mathbb{C}$-linear map $\widetilde{s} : A \to A$ of degree $+1$ which satisfies

(i) $[\widetilde{s},\widetilde{s}] = \widetilde{s}^2 = 0$,

(ii) $\widetilde{s}$ is a derivation with respect to $\wedge$, in particular it is $\mathcal{O}_S$-linear,

(iii) $\widetilde{s}$ is a derivation with respect to $[\,]$.

Thus, neglecting the bracket, a differential Gerstenhaber algebra is a sheaf of differential graded algebras over $\mathcal{O}_S$.

Lemma 1.14 *Let $(A,\widetilde{s})$ be a differential Gerstenhaber algebra. Let $I \subset \mathcal{O}_S$ be the image of $\widetilde{s} : A^{-1} \to A^0$. This is a sheaf of ideals in $\mathcal{O}_S$. Then the cohomology $h^*(A,\widetilde{s})$ is a Gerstenhaber algebra over $\mathcal{O}_S/I$.*

Proof. This is clear: the fact that $\widetilde{s}$ is a derivation with respect to both products on A implies that the two products pass to $h^*(A,\widetilde{s})$. Then all the properties of the products pass to cohomology. □

Morphisms of differential Gerstenhaber algebras

Definition 1.15 Let A and B be Gerstenhaber algebras over $\mathcal{O}_S$. A **morphism** of Gerstenhaber algebras is a homomorphism $\phi : A \to B$ of graded $\mathcal{O}_S$-modules (of degree zero) which is compatible with both $\wedge$ and $[\,]$:

(i) $\phi(X \wedge Y) = \phi(X) \wedge \phi(Y)$,

(ii) $\phi([X, Y]) = [\phi(X), \phi(Y)]$.

Definition 1.16 Let $(A, \widetilde{s})$ and $(B, \widetilde{t})$ be differential Gerstenhaber algebras. A **morphism** of differential Gerstenhaber algebras is a pair $(\phi, \{\,\})$, where $\phi : A \to B$ is a degree zero homomorphism of graded $\mathcal{O}_S$-modules, and $\{\,\} : A \otimes_{\mathbb{C}} A \to B$ is a degree zero $\mathbb{C}$-bilinear map , such that

(i) $\phi(X \wedge Y) = \phi(X) \wedge \phi(Y)$ and $\phi(\widetilde{s}X) = \widetilde{t}\phi(X)$, so that $\phi : A \to B$ is a morphisms of differential graded $\mathcal{O}_S$-algebras,

(ii) $\{\,\}$ is symmetric, i.e., $\{Y, X\} = (-1)^{\overline{XY}}\{X, Y\}$,

(iii) $\{\,\}$ is a $\mathbb{C}$-linear derivation with respect to $\wedge$ in each of its arguments, where the A-module structure on B is given by ϕ, in other words,

$$\{X \wedge Y, Z\} = \phi(X) \wedge \{Y, Z\} + (-1)^{\overline{XY}}\phi(Y) \wedge \{X, Z\}\,,$$

and

$$\{X, Y \wedge Z\} = \{X, Y\} \wedge \phi(Z) + (-1)^{\overline{YZ}}\{X, Z\} \wedge \phi(Y)\,,$$

(iv) the default of ϕ to commute with $[\,]$ is equal to the default of the $\mathcal{O}_S$-linear differentials to behave as derivations with respect to $\{\,\}$,

$$\phi[X, Y] - [\phi(X), \phi(Y)] = (-1)^{\overline{X}}\widetilde{t}\{X, Y\} - (-1)^{\overline{X}}\{\widetilde{s}X, Y\} - \{X, \widetilde{s}Y\}\,. \quad (8)$$

Remark 1.17 Suppose all conditions in Definition 1.16 except the last are satisfied. Then both sides of the equation in Condition (iv) are symmetric of degree one and $\mathbb{C}$-linear derivations with respect to $\wedge$ in each of the two arguments. Thus, to check Condition (iv), it suffices to check on $\mathbb{C}$-algebra generators for A.

Lemma 1.18 *A morphism of differential Gerstenhaber algebras*

$$(\phi, \{\,\}) : (A, \widetilde{s}) \longrightarrow (B, \widetilde{t})$$

induces a morphism of Gerstenhaber algebras on cohomology. In other words,

$$h^*(\phi) : h^*(A, \widetilde{s}) \longrightarrow h^*(B, \widetilde{t})$$

respects both $\wedge$ and $[\,]$.

PROOF. Any morphism of differential graded $\mathcal{O}_S$-algebras induces a morphism of graded algebras when passing to cohomology. Thus $h^*(\phi)$ respects $\wedge$. The fact that $h^*(\phi)$ respects the Lie brackets, follows form Property (iv) of Definition 1.16. All three terms on the right hand side of said equation vanish in cohomology. □

Definition 1.19 A **quasi-isomorphism** of differential Gerstenhaber algebras is a morphism of differential Gerstenhaber algebras which induces an isomorphism of Gerstenhaber algebras on cohomology.

Differential Batalin-Vilkovisky modules

Definition 1.20 Let A be a Gerstenhaber algebra. A sheaf of graded $\mathcal{O}_S$-modules L, with an action $\lrcorner$ of A, making L a graded A-module, is called a **Batalin-Vilkovisky module** over A, if it is endowed with a $\mathbb{C}$-linear map $d : L \to L$ of degree $+1$ satisfying

(i) $[d, d] = d^2 = 0$,

(ii) Formula (6) holds: $i_{[X,Y]} = \big[[i_X, d], i_Y\big]$, for all $X, Y \in A$.

Here i_X is the endomorphism $\omega \mapsto X \lrcorner \omega$ of L. The action property $(X \wedge Y) \lrcorner \omega = X \lrcorner (Y \lrcorner \omega)$ translates into $i_{X\wedge Y} = i_X \circ i_Y$.

In our applications, Batalin-Vilkovisky modules will always be coherent over $\mathcal{O}_S$. Note that there is no multiplicative structure on L, so there is no requirement for the differential d to be a derivation.

Definition 1.21 A **differential Batalin-Vilkovisky module** over the differential Gerstenhaber algebra $(A, \widetilde{s})$ is a Batalin-Vilkovisky module L for the underlying Gerstenhaber algebra A, endowed with an additional $\mathbb{C}$-linear map $s : L \to L$ of degree $+1$ satisfying:

(i) $[s, s] = s^2 = 0$,

(ii) (M, s) is a differential graded module over the differential graded algebra $(A, \widetilde{s})$, i.e., we have

$$s(X \lrcorner \omega) = \widetilde{s}(X) \lrcorner \omega + (-1)^{\overline{X}} X \lrcorner s(\omega)\,,$$

for all $X \in A$, $\omega \in L$. More succinctly: $[s, i_X] = i_{\widetilde{s}(X)}$.

(iii) $[d, s] = 0$.

Note that the differential s is necessarily $\mathcal{O}_S$-linear. This distinguishes it from d.

Lemma 1.22 *Let (L, s) be a differential Batalin-Vilkovisky module over the differential Gerstenhaber algebra $(A, \widetilde{s})$. Then $h^*(L, s)$ is a Batalin-Vilkovisky module for the Gerstenhaber algebra $h^*(A, \widetilde{s})$.*

Proof. First, $h^*(M, s)$ is a graded $h^*(A, \widetilde{s})$-module. The condition $[d, s] = 0$ implies that d passes to cohomology. Then the properties of d pass to cohomology as well. □

Homomorphisms of differential Batalin-Vilkovisky modules

Definition 1.23 Let A and B be Gerstenhaber algebras and $\phi : A \to B$ a morphism of Gerstenhaber algebras. Let L be a Batalin-Vilkovisky module over A and M a Batalin-Vilkovisky module over B. A **homomorphism** of Batalin-Vilkovisky modules of degree n (covering ϕ) is a degree n homomorphism of graded A-modules $\psi : L \to M$ (where the A-module structure on M is defined via ϕ), which commutes with d:

(i) $\psi(X \lrcorner \omega) = (-1)^{n\overline{X}} \phi(X) \lrcorner \psi(\omega)$,

(ii) $\psi d_L(\omega) = (-1)^n d_M \psi(\omega)$.

We write the latter condition as $[\psi, d] = 0$.

Definition 1.24 Let $(A, \widetilde{s})$ and $(B, \widetilde{t})$ be differential Gerstenhaber algebras and $(\phi, \{\,\}) : (A, \widetilde{s}) \to (B, \widetilde{t})$ a morphism of differential Gerstenhaber algebras. Let (L, s) be a differential Batalin-Vilkovisky module over $(A, \widetilde{s})$ and (M, t) a differential Batalin-Vilkovisky module over $(B, \widetilde{t})$. A **homomorphism** of differential Batalin-Vilkovisky modules of degree n covering $(\phi, \{\,\})$ is a pair (ψ, δ), where $\psi : (L, s) \to (M, t)$ is a degree n homomorphism of differential graded $(A, \widetilde{s})$-modules, where the $(A, \widetilde{s})$-module structure on (M, t) is through ϕ. Moreover, $\delta : L \to M$ is a $\mathbb{C}$-linear map, also of degree n, satisfying

(i) the commutator property

$$\psi \circ d - (-1)^n d \circ \psi = (-1)^n t \circ \delta - \delta \circ s \,, \tag{9}$$

(ii) compatibility with the bracket $\{\,\}$ property

$$\begin{aligned} \delta(X \wedge Y \lrcorner \omega) + (-1)^{n(\overline{X}+\overline{Y})} \phi(X) \wedge \phi(Y) \lrcorner \delta\omega + (-1)^{n(\overline{X}+\overline{Y})} \{X, Y\} \lrcorner \psi(\omega) \\ = (-1)^{n\overline{X}} \phi(X) \lrcorner \delta(Y \lrcorner \omega) + (-1)^{\overline{X}\overline{Y}+n\overline{Y}} \phi(Y) \lrcorner \delta(X \lrcorner \omega) \,. \end{aligned} \tag{10}$$

Remark 1.25 If we use the same letter s to denote the $\mathcal{O}_S$-linear differentials on L and M, we can rewrite the commutator conditions of Definition 1.24 more succinctly as

$$[\psi, s] = 0 \qquad [\psi, d] + [\delta, s] = 0 \qquad [\delta, d] = ???\,.$$

Lemma 1.26 *Let $(\psi, \delta) : (L, s) \to (M, t)$ be a homomorphism of differential Batalin-Vilkovisky modules over the morphism $(\phi, \{\,\}) : (A, \widetilde{s}) \to (B, \widetilde{t})$ of differential Gerstenhaber algebras. Then $h^*(\psi) : h^*(L, s) \to h^*(M, t)$ is a homomorphism of Batalin-Vilkovisky modules over the morphism of Gerstenhaber algebras $h^*(\phi) : h^*(A, \widetilde{s}) \to h^*(B, \widetilde{t})$.*

Proof. Evaluating the right hand side of Equation 9 on s-cocycles in L, yields t-boundaries in M. □

Invertible differential Batalin-Vilkovisky modules

Definition 1.27 We call the Batalin-Vilkovisky module L over the Gerstenhaber algebra A **invertible**, if, locally in S, there exists a section ω° of L such that the evaluation homomorphism

$$\begin{aligned} \Psi^\circ : A &\longrightarrow L \\ X &\longmapsto (-1)^{\overline{X}\overline{\omega}^\circ} X \lrcorner\, \omega^\circ \end{aligned}$$

is an isomorphism of sheaves of $\mathcal{O}_S$-modules. Any such ω° will be called a (local) **orientation** for L over A.

Note that if the degree of an orientation ω° is n, then $L^k = 0$, for all $k > n$, by our assumption on A. Thus orientations always live in the top degree of L. Moreover, if orientations exist everywhere locally, L^n is an invertible sheaf over $\mathcal{O}_S$.

Lemma 1.28 *Let L be an invertible Batalin-Vilkovisky module over the Gerstenhaber algebra A and assume that ω° is a (global) orientation for L over A. Then, transporting the differential d via Ψ° to A yields a $\mathbb{C}$-linear map of degree $+1$ which we will call $d^\circ : A \to A$. It is characterized by the formula*

$$d^\circ(X) \lrcorner\, \omega^\circ = d(X \lrcorner\, \omega^\circ)\,.$$

It squares to 0 *and it satisfies:*

$$(-1)^{\overline{X}}[X,Y] = d^\circ(X) \wedge Y + (-1)^{\overline{X}} X \wedge d^\circ(Y) - d^\circ(X \wedge Y)\,, \qquad (11)$$

for all $X, Y \in A$. In other words, d° is a **generator** *for the bracket* [ ]*, making A a* **Batalin-Vilkovisky algebra***.*

Proof. Follows directly from Formula (6) upon noticing that because ω° is top-dimensional, it is automatically d-closed: $d\omega^\circ = 0$. □

Corollary 1.29 *If the Gerstenhaber algebra admits an invertible Batalin-Vilkovisky module then it is locally a Batalin-Vilkovisky algebra.*

Remark 1.30 For example, let $A = K(E)$ and $L = L(E)$, for a vector bundle E on S. Let the bracket [ ] on A correspond to the differential d on L via Proposition 1.9. Then A is a Gerstenhaber algebra and L an invertible Batalin-Vilkovisky module for A. Any non-vanishing section of $\Lambda^n E$, where $n = \operatorname{rank} E$, is an orientation for L.

As a special case, the Schouten algebra $A = \Lambda^\bullet T_M$ of a manifold M is a Gerstenhaber algebra and the de Rham complex $L = \Omega^\bullet_M$ is an invertible Batalin-Vilkovisky module for A. An orientation for L is the same thing as a non-vanishing top-dimensional differential form on M. For orientable manifolds, i.e., $\Omega^n_M \cong \mathcal{O}_M$, the Schouten algebra is a Batalin-Vilkovisky algebra. For Calabi-Yau manifolds, i.e., $\Omega^n_M = \mathcal{O}_S$, a generator for the Batalin-Vilkovisky algebra is given.

Definition 1.31 Let (L, s) be a differential Batalin-Vilkovisky module over the differential Gerstenhaber algebra $(A, \widetilde{s})$. Then (L, s) is called **invertible**, if the underlying Batalin-Vilkovisky module L is invertible over the underlying Gerstenhaber algebra A.

Proposition 1.32 *Let (L, s) be an invertible differential Batalin-Vilkovisky module over the differential Gerstenhaber algebra $(A, \widetilde{s})$. Then under the isomorphism Ψ° defined by an orientation ω° of L over A, the differential $\widetilde{s}$ corresponds to the differential s. In particular, the induced differential d° on A has the property*

$$[d^\circ, \widetilde{s}] = 0\,,$$

besides satisfying (11). Hence $(A, d^\circ, \widetilde{s})$ is a **differential Batalin-Vilkovisky algebra**.

Moreover, the cohomology $h^*(L, s)$ is an invertible Batalin-Vilkovisky module over the Gerstenhaber algebra $h^*(A, \widetilde{s})$. We have $h^n(L, s) = L^n/I$ and the image of any orientation of L over A under the quotient map $L^n \to L^n/I$ gives an orientation for $h^*(L, s)$ over $h^*(A, \widetilde{s})$.

Proof. The equation $s \circ \Psi^\circ = (-1)^{\overline{\omega}^\circ} \Psi^\circ \circ \widetilde{s}$ follows immediately from $[s, i_X] = i_{\widetilde{s}(X)}$ upon noticing that $s(\omega) = 0$. As Ψ° is therefore an isomorphism of differential graded $\mathcal{O}_S$-modules, the cohomology is isomorphic: $h^*(A, \widetilde{s}) \xrightarrow{\sim} h^*(L, s)$. The rest follows from this. □

Remark 1.33 For example, let E be a vector bundle on S, d a differential on $L = L(E)$ and s a section of E such that $ds = 0$. Consider the bracket $[\,]$ induced on $A = K(E)$ from d via Proposition 1.9 and the differential $\widetilde{s}$ of A dual to s as in Remark 1.5. Then $(A, \widetilde{s})$ is a differential Gerstenhaber algebra and (L, s) an invertible differential Batalin-Vilkovisky module.

Example 1.34 As a special case, consider a closed 1-form s on a manifold M. Left multiplication by s defines the differential s on the de Rham complex $L = \Omega^\bullet_M$. We have the induced derivation $\widetilde{s}$ on the Schouten algebra $A = \Lambda^\bullet T_M$. Then $(\Omega^\bullet, s)$ is an invertible differential Batalin-Vilkovisky module over the differential Gerstenhaber algebra $(\Lambda^\bullet T_M, \widetilde{s})$.

Oriented homomorphisms of invertible Batalin-Vilkovisky modules

Definition 1.35 Let $\phi : A \to B$ be a morphism of Gerstenhaber algebras and $\psi : L \to M$ a homomorphism of invertible Batalin-Vilkovisky modules covering ϕ. Let ω_L° and ω_M° be orientations for L and M, respectively. The homomorphism $\psi : L \to M$ is said to **preserve the orientations** (or be oriented) if $\psi(\omega_L^\circ) = \omega_M^\circ$.

Lemma 1.36 *Suppose given oriented invertible Batalin-Vilkovisky modules L and M over the Gerstenhaber algebras A and B, making A and B into Batalin-Vilkovisky algebras. Suppose $\psi : L \to M$ is an oriented homomorphism of*

Batalin-Vilkovisky modules. Then under the identifications of L and M with A and B given by ω_L° and ω_M°, the map $\psi : L \to M$ corresponds to $\phi : A \to B$. Hence $\phi : A \to B$ commutes with d°. Thus ϕ is a morphism of Batalin-Vilkovisky algebras: it respects $\wedge$, $[\,]$ and d°.

Definition 1.37 Let $(\psi, \delta) : (L, s) \to (M, t)$ be a homomorphism of invertible differentiable Batalin-Vilkovisky modules over $(\phi, \{\,\}) : (A, \widetilde{s}) \to (B, \widetilde{t})$. Let ω_L° and ω_M° be orientations for L and M, respectively. We call (ψ, δ) **oriented** if $\psi(\omega_L) = \omega_M$ and $\delta(\omega_L) = 0$.

Proposition 1.38 *Suppose $(\psi, \delta) : (L, s, \omega_L^\circ) \to (M, t, \omega_M^\circ)$ is an oriented homomorphism of oriented invertible differential Batalin-Vilkovisky modules over $(\phi, \{\,\}) : (A, \widetilde{s}) \to (B, \widetilde{t})$. Then $(A, \widetilde{s}, [\,], d^\circ)$ and $(B, \widetilde{t}, [\,], d^\circ)$ are differential Batalin-Vilkovisky algebras. Transporting $\delta : L \to M$ via the identifications of L and M with A and B to a map $\delta^\circ : A \to B$, satisfying*

$$\delta^\circ(X) \lrcorner\, \omega_M = (-1)^{\overline{\delta X}} \delta(X \lrcorner\, \omega_L)\,,$$

we get a triple

$$(\phi, \{\,\}, \delta^\circ) : (A, \widetilde{s}, [\,], d^\circ) \longrightarrow (B, \widetilde{t}, [\,], d^\circ)\,,$$

which satisfies the following conditions:

(i) $\phi : (A, \widetilde{s}) \to (B, \widetilde{t})$ is a morphism of differential graded algebras,

(ii) the commutator property

$$\phi \circ d^\circ - d^\circ \circ \phi = \widetilde{t} \circ \delta^\circ - \delta^\circ \circ \widetilde{s}\,,$$

or, by abuse of notation, $[\phi, d^\circ] + [\delta^\circ, \widetilde{s}] = 0$,

(iii) the map δ° is a potential for the bracket $\{\,\}$,

$$\{X, Y\} = \delta^\circ(X) \wedge \phi(Y) + \phi(X) \wedge \delta^\circ(Y) - \delta^\circ(X \wedge Y)\,,$$

(iv) the default of ϕ to preserve $[\,]$ equals the default of $\widetilde{s}$ to be a derivation with respect to $\{\,\}$, Equation (8),

Thus $(\phi, \{\,\}, \delta^\circ)$ is a **morphism of differential Batalin-Vilkovisky algebras**.

The Lie bracket $[\,]$ is determined by its potential d^0, and the bracket $\{\,\}$ is determined by its potential δ^0. Thus, in a certain sense, the two brackets are redundant. Moreover, Condition (iv) is implied by Conditions (ii) and (iii).

Remark 1.39 A morphism of differential Batalin-Vilkovisky algebras

$$(\phi, \{\,\}, \delta^\circ) : (A, \widetilde{s}, [\,], d^\circ) \longrightarrow (B, \widetilde{t}, [\,], d^\circ)$$

induces on cohomology

$$h^*(\phi) : \big(h^*(A, \widetilde{s}), [\,], d^\circ\big) \to \big(h^*(B, \widetilde{t}), [\,], d^\circ\big)$$

a morphism of Batalin-Vilkovisky algebras.

2 Symplectic Geometry

Let S be a symplectic manifold and let σ denote the symplectic form on S. Let us choose the identification of tangent and cotangent vectors given by

$$\begin{aligned} T_S &\longrightarrow \Omega_S \\ X &\longmapsto \sigma(\,\cdot\,, X)\,. \end{aligned}$$

2.1 Lagrangian foliations and Euler sections

A **Lagrangian foliation** is an integrable distribution $H \subset T_S$, where $H \subset T_S$ is a Lagrangian subbundle. All leaves of the foliation H are then Lagrangian submanifolds of S.

2.2 Lagrangian foliations and transverse Lagrangians

Let S be a symplectic manifold with symplectic form σ. Let E be a Lagrangian foliation. Think of sections of E as differential forms on S.

Lemma 2.1 *Let M be a Lagrangian submanifold of S which is everywhere transverse to E. Then there exists a unique section s of E, such that $ds = \sigma$ and $M = Z(s)$. Conversely, if s is any section of E such that $ds = \sigma$, then $Z(s)$ is a Lagrangian submanifold.*

Thus we have a canonical one-to-one correspondence between sections s of E such that $ds = \sigma$ and Lagrangian submanifolds of S transverse to E.

Definition 2.2 We call s the **Euler vector field** of M with respect to E.

Lemma 2.3 *Let s be the Euler section of E defining the Lagrangian submanifold M. Let H be any Lagrangian foliation transverse to E. Then under the canonical isomorphism $E = H^\vee$, the section s maps to a closed form: $ds = 0 \in \Lambda^2 H^\vee$.*

Proof. The canonical isomorphism $E = H^\vee$ is the composition $E \to \Omega_S \to H^\vee$. The form $ds \in \Lambda^2 H^\vee$ is the projection of the symplectic form via $\Omega_S^2 \to \Lambda^2 H^\vee$. This latter projection is 0. □

2.3 The extra structures on $\mathcal{E} = \mathcal{E}xt$ and $\mathcal{A} = \mathcal{T}or$

Let M and N be Lagrangian submanifolds of the symplectic manifold S. Let $\mathcal{O}_Z = \mathcal{O}_M \otimes \mathcal{O}_N$ be the structure sheaf of the intersection $Z = M \cap N$. Write $\mathcal{A}^i = \mathcal{T}or_{-i}(\mathcal{O}_M, \mathcal{O}_N)$, so that $\mathcal{A}$ is a sheaf of graded $\mathcal{O}_S$-modules concentrated in non-positive degrees, $\mathcal{A}^0 = \mathcal{O}_Z$. Write also $\mathcal{E}^i = \mathcal{E}xt^i(\mathcal{O}_M, \mathcal{O}_N)$. Thus $\mathcal{E}$ is a sheaf of graded $\mathcal{O}_S$-modules. We will define on $\mathcal{A}$ a canonical structure of Gerstenhaber algebra over $\mathcal{O}_Z$ and on $\mathcal{E}$ a canonical structure of invertible Batalin-Vilkovisky module over $\mathcal{A}$. Any orientation on N, i.e., trivialization of

Ω^n_N, gives rise to an orientation of the invertible Batalin-Vilkovisky module $\mathcal{E}$ and hence to the structure of Batalin-Vilkovisky algebra with generator on $\mathcal{A}$.

We will define these structures on $\mathcal{A}$ and $\mathcal{E}$ locally, and then glue.

The local construction

Let E be a Lagrangian foliations of S, transverse everywhere to both M and N. Here we will show how E gives (canonically) rise to the structure of Gerstenhaber algebra on $\mathcal{A}$ and invertible Batalin-Vilkovisky module on $\mathcal{E}$.

In a later section we will prove that these structures do not depend on the choice of E and hence glue, as locally such E can always be found.

Let s be the Euler section of E defining M. Via $T_S \to \Omega_S$ think of s as a differential form. This form we can restrict to N, the restriction we will also denote by s. As $ds = \sigma$ on S, and σ restricts to 0 on N, because N is Lagrangian, we see that s is closed on N.

Now, by Example 1.34, we have the differential Gerstenhaber algebra $(\Lambda^\bullet T_N, \widetilde{s})$ and its invertible differential Batalin-Vilkovisky module $(\Omega^\bullet_N, s)$. Thus, by Proposition 1.32, $h^*(\Omega^\bullet_N, s)$ is an invertible Batalin-Vilkovisky module over the Gerstenhaber algebra $h^*(\Lambda^\bullet T_N, \widetilde{s})$.

Recall that $L(E,s) = (\Lambda^\bullet E, s)$ is the $\mathcal{O}_S$-dual of $K(E,s) = (\Lambda^\bullet E^\vee, \widetilde{s})$, and so

$$(\Omega^\bullet_N, s) = (\Lambda^\bullet E, s)|_N = L(E,s) \otimes_{\mathcal{O}_S} \mathcal{O}_N = \mathcal{H}om_{\mathcal{O}_S}\big(K(E,s), \mathcal{O}_N\big)\,.$$

Therefore,

$$h^i(\Omega^\bullet_N, s) = \mathcal{E}xt^i_{\mathcal{O}_S}(\mathcal{O}_M, \mathcal{O}_N)\,,$$

as $K(E,s) \to \mathcal{O}_M$ is an $\mathcal{O}_S$-resolution of $\mathcal{O}_M$. Thus, $h^*(\Omega^\bullet_N, s) = \mathcal{E}$.

Similarly,

$$(\Lambda^\bullet T_N, \widetilde{s}) = (\Lambda^\bullet E^\vee, \widetilde{s})|_N = K(E,s) \otimes_{\mathcal{O}_S} \mathcal{O}_N\,,$$

and hence

$$h^i(\Lambda^\bullet T_N, \widetilde{s}) = \mathcal{T}or^{\mathcal{O}_S}_{-i}(\mathcal{O}_M, \mathcal{O}_N)\,,$$

in other words, $h^*(\Lambda^\bullet T_N, \widetilde{s}) = \mathcal{A}$.

We conclude that $\mathcal{A}$ is a Gerstenhaber algebra and $\mathcal{E}$ an invertible Batalin-Vilkovisky module over $\mathcal{A}$. Note that these extra structures on $\mathcal{A}$ and $\mathcal{E}$ are determined in a canonical way by E. Note also, that non-vanishing sections of Ω^n_N define orientations of $(\Omega^\bullet_N, s)$ over $(\Lambda^\bullet T_N, \widetilde{s})$ and hence orientations for $\mathcal{E}$ over $\mathcal{A}$.

3 Derived Lagrangian intersections on polarized symplectic manifolds

Let (S, σ) be a symplectic manifold of dimension $2n$.. An **immersed Lagrangian** of S is an unramified morphism $i : L \to S$, where L is a manifold of dimension n, such that $i^*\sigma \in \Omega_L$ vanishes.

A **polarized** symplectic manifold (S, E, σ) is a symplectic manifold (S, σ) together with a Lagrangian foliation $E \subset T_S$.

Definition 3.1 Let (S, E, σ) be a polarized symplectic manifold and L, M immersed Lagrangians of S which are both transverse to E. Then the **derived intersection**

$$L \Cap_S M$$

is the sheaf of differential Gerstenhaber algebras $(\Lambda T_M, \widetilde{t})$ on M, where $\widetilde{t}$ is the derivation on ΛT_M induced by the restriction to M of the Euler section $t \in E \subset \Omega_S$ of L.

Since $dt = \sigma$, and M is Lagrangian, the restriction of t to M is closed, and so $\widetilde{t}$ is a derivation with respect to the Schouten bracket on ΛT_M, making $(\Lambda T_M, \widetilde{t})$ a differential Gerstenhaber algebra.

Remark 3.2 After passing to suitable étale neighborhoods of L in S we can assume that L is embedded (not just immersed) in S and that L admits a globally defined Euler section t on S. This defines the derived intersection étale locally in M, and the global derived intersection is defined by gluing in the étale topology on M.

Remark 3.3 The derived intersection $L \Cap_S M$ depends a priori on the polarization E. We will see later that different polarizations lead to *locally* quasi-isomorphic derived intersections. (The quasi-isomorphism is not canonical, as it depends on the choice of a third polarization transverse to both of the polarizations being compared. It is not clear that such a third polarization can necessarily be found globally.)

Remark 3.4 The derived intersection does not seem to be symmetric. We will see below that $L \Cap_S M = M \Cap_{\overline{S}} L$, where $\overline{S} = (S, -\sigma)$, but only if $\overline{S}$ is endowed with a different polarization, transverse to E. Then the issue of change of polarization of Remark 3.3 arises.

3.1 The exchange property: polarized case

Given two symplectic manifolds S', S, of dimensions $2n'$ and $2n$, a **symplectic correspondence** between S' and S is a manifold C of dimension $n+n'$, together with morphisms $\pi' : C \to S'$ and $\pi : C \to S$, such that

(i) $\pi^*\sigma = {\pi'}^*\sigma'$ (as sections of Ω_C^2),

(ii) $C \to S' \times S$ is unramified.

Thus a symplectic correspondence is an immersed Lagrangian of $\overline{S}' \times S = (S' \times S, \sigma - \sigma')$.

Let $C \to S' \times S$ be a symplectic correspondence. We say that the immersed Lagrangian $L \to S$ is **transverse** to C, if

(i) for every $(Q,P) \in C \times_S L$ we have that

$$T_C|_Q \oplus T_L|_P \longrightarrow T_S|_{\pi(Q)}$$

is surjective, hence the pullback $L' = C \times_S L$ is a manifold of dimension n'

(ii) the natural map $L' \to S'$ is unramified (and hence L' is an immersed Lagrangian of S').

By exchanging the roles of S and S' we also get the notion of transversality to C for immersed Lagrangians of S'.

Exchange property setup

Let (S, E, σ) and (S', E', σ') be polarized symplectic manifolds. Consider a **transverse** symplectic correspondence $C \to S' \times S$. This means that $C \to S' \times S$ is transverse to the foliation $E' \times E$ of $S' \times S$. In particular, the composition

$$T_C \longrightarrow \pi^* T_S \longrightarrow \pi^* E^\vee$$

is surjective (recall that $T_S/E = E^\vee$). Hence the foliation $E \subset T_S$ pulls back to a foliation $F \subset T_C$ of rank n'. We have the exact sequence of vector bundles

$$0 \longrightarrow F \longrightarrow T_C \longrightarrow \pi^* E^\vee \longrightarrow 0\,.$$

Similarly, the foliation $E' \subset T_{S'}$ pulls back to a foliation $F' \subset T_C$ of rank n with the exact sequence

$$0 \longrightarrow F' \longrightarrow T_C \longrightarrow {\pi'}^* {E'}^\vee \longrightarrow 0\,.$$

Moreover, F and F' are transverse foliations of C and so we have $F' \oplus F = T_C = {\pi'}^* {E'}^\vee \oplus {\pi'}^* E^\vee$, and canonical identifications $F = {\pi'}^* {E'}^\vee$ and $F' = \pi^* E^\vee$.

Now assume given immersed Lagrangians L of S and M' of S'. Assume both are transverse to C. Then we obtain manifolds L' and M by the pullback diagram

$$\begin{array}{ccccc} & & L' & \longrightarrow & L \\ & & \downarrow & \square & \downarrow \\ M & \longrightarrow & C & \xrightarrow{\pi} & S \\ \downarrow & \square & \downarrow{\scriptstyle \pi'} & & \\ M' & \longrightarrow & S' & & \end{array} \qquad (12)$$

Note that L' is then an immersed Lagrangian for S' and M an immersed Lagrangian for S.

Finally, we assume that L and M are transverse to E and that M' and L' are transverse to E'. As a consequence, L' is transverse to F' and M is transverse to F.

Theorem 3.5 *Let (S, E, σ) and (S', E', σ') be polarized symplectic manifolds, and $C \to S' \times S$ a transverse symplectic correspondence. Let $M' \to S'$ and $L \to S$ be immersed Lagrangians, both transverse to C, such that L and M are transverse to E and M' and L' are transverse to E', with notation as in (12). Then there are canonical quasi-isomorphisms of differential Gerstenhaber algebras*

$$(M' \times L) \Cap_{\overline{S}' \times S} C \longrightarrow L \Cap_S M \,,$$

and

$$(M' \times L) \Cap_{\overline{S}' \times S} C \longrightarrow M' \Cap_{\overline{S}'} L' \,.$$

In particular, the derived intersections $L \times_S M$ and $M' \times_{\overline{S}'} L'$ are canonically quasi-isomorphic.

PROOF. Passing to étale neighborhoods of L in S and M' in S' will not change anything about either derived intersection $L \times_S M$ or $M' \times_{\overline{S}'} L'$, so we may assume, without loss of generality, that

(i) L is embedded (not just immersed) in S (and same for M' in S'),

(ii) L admits a global Euler section t with respect to E on S (and M' has the Euler section s' in E' on S')

Then the Euler section of M' with respect to E' on $\overline{S}'$ is $-s'$. Thus the derived intersection $L \times_S M$ is equal to $(\Lambda T_M, \widetilde{t})$ and the derived intersection $M' \times_{\overline{S}'} L'$ equals $(\Lambda T_{L'}, -\widetilde{s}')$.

Pulling back the 1-form t via π, we obtain a 1-form on C, which we shall, by abuse of notation, also denote by t. Similarly, pulling back s' via π' we get the 1-form s' on C. The difference $t - s'$ is closed on C, and thus we have the differential Gerstenhaber algebra $(\Lambda T_C, \widetilde{t} - \widetilde{s}')$. We remark that it is equal to $(M' \times L) \Cap_{\overline{S}' \times S} C$.

Recall that we have the identification $T_C = {\pi'}^* {E'}^\vee \oplus \pi^* E^\vee$. Under this direct sum decomposition $\widetilde{t} - \widetilde{s}'$ splits up into two components, $-\widetilde{s}'$ and $\widetilde{t}$. Hence we obtain the decomposition

$$(\Lambda T_C, \widetilde{t} - \widetilde{s}') = {\pi'}^*(\Lambda {E'}^\vee, -\widetilde{s}') \otimes \pi^*(\Lambda E^\vee, \widetilde{t})$$

of differential graded $\mathcal{O}_C$-algebras.

Recall that $\mathcal{O}_{S'} \to \mathcal{O}_{M'}$ induces a quasi-isomorphism of differential graded $\mathcal{O}_{S'}$-algebras $(\Lambda {E'}^\vee, -\widetilde{s}') \to \mathcal{O}_{M'}$. Because the pullback $M = M' \times_{S'} C$ is transverse, we get an induced quasi-isomorphism

$${\pi'}^*(\Lambda {E'}^\vee, -\widetilde{s}') \longrightarrow \mathcal{O}_M$$

of differential graded $\mathcal{O}_C$-algebras. Tensoring with $\pi^*(\Lambda E^\vee, \widetilde{t})$, we obtain the quasi-isomorphism

$$(\Lambda T_C, \widetilde{t} - \widetilde{s}') \longrightarrow (\Lambda E^\vee, \widetilde{t})|_M \,.$$

Noting that $E^\vee|_M = T_M$, because M is an immersed Lagrangian in S transverse to E, we see that $(\Lambda E^\vee, \widetilde{t})|_M = (\Lambda T_M, \widetilde{t})$ and so we have a quasi-isomorphism of differential graded $\mathcal{O}_C$-algebras

$$\phi : (\Lambda T_C, \widetilde{t} - \widetilde{s}') \longrightarrow (\Lambda T_M, \widetilde{t}) \,. \qquad (13)$$

For analogous reasons, we also have the quasi-isomorphism

$$\phi' : (\Lambda T_C, \widetilde{t} - \widetilde{s}') \longrightarrow (\Lambda T_{L'}, -\widetilde{s}')\,.$$

The proof will be finished, if we can enhance ϕ and ϕ' by brackets, making them morphisms of differential Gerstenhaber algebras. We will concentrate on ϕ. The case of ϕ' follows by symmetry.

Thus we shall define a bracket

$$\{\,\} : \Lambda T_C \otimes_{\mathbb{C}} \Lambda T_C \longrightarrow \Lambda T_M\,, \tag{14}$$

such that $(\phi, \{,\})$ becomes a morphism of differential Gerstenhaber algebras.

We shall need a certain symmetric bilinear form

$$\eta : N^{\vee}_{M/C} \otimes_{\mathcal{O}_M} N^{\vee}_{M/C} \longrightarrow \mathcal{O}_M\,, \tag{15}$$

where $N_{M/C}$ is the normal bundle of M in C. This bilinear map will be canonical, so we can construct it étale locally on C. Thus we shall assume that M is embedded in S and has the Euler section s with respect to E. We also assume that the closed 1-form $s - s'$ on C is exact. Let I be the ideal of M in S. Then there exists a unique regular function $f \in I^2$ such that $df = s - s'$. The fact that $f \in I^2$ follows because s and s' vanish in $\Omega_C|_M$, so df vanishes in $\Omega_C|_M$. Then the Hessian of f is a symmetric bilinear form

$$H(f) : N_{M/C} \otimes_{\mathcal{O}_M} N_{M/C} \longrightarrow \mathcal{O}_M\,.$$

Claim. The bilinear form $H(f)$ is non-degenerate.

To prove the claim, let us start by putting $H(f)$ into the composition

$$T_C|_M \longrightarrow N_{M/C} \xrightarrow{H(f)} N^{\vee}_{M/C} \longrightarrow \Omega_C|_M\,,$$

and noting that it suffices to prove that this composition has rank n' at every point of M. We also note that the above composition is equal to the restriction to M of the composition

$$T_C = \pi'^* E'^{\vee} \oplus \pi^* E^{\vee} \xrightarrow{-\widetilde{s}' \oplus \widetilde{s}} \mathcal{O}_C \xrightarrow{d} \Omega_C\,.$$

So it suffices to prove that the restriction to M of

$$\pi'^* E'^{\vee} \xrightarrow{\widetilde{s}'} \mathcal{O}_C \xrightarrow{d} \Omega_C\,,$$

which is an $\mathcal{O}_M$-linear map, has rank equal to n' everywhere. But this follows directly from the fact that $s' \in \pi'^* E'$ is a regular section of a vector bundle of rank n' whose zero locus is exactly M. This latter fact, in turn, follows from the transversality of the pullback $M = M' \times_{S'} C$. Thus the claim is proved. Note that this claim uses the transversality of C to $E' \times E$ in an essential way.

We define the form η mentioned above (15) to be the *inverse* of $H(f)$.

Now we are ready to construct the bracket (14). We use the foliation $F \subset T_C$. It defines a partial connection

$$\nabla : T_C/F \longrightarrow F^\vee \otimes_{\mathcal{O}_C} T_C/F\,,$$

by the rule

$$\begin{aligned} \nabla : T_C/F &\longrightarrow \mathcal{H}om_{\mathcal{O}_C}(F, T_C/F) \\ X + F &\longmapsto [\,\cdot\,, X+F] + F\,. \end{aligned}$$

In other words,

$$\nabla(X+F)(Y) = \nabla_Y(X+F) = [Y, X+F] + F\,.$$

Abusing notation, we also write $\nabla(X)(Y) = [Y,X]$. By the usual formulas we can transport ∇ onto the exterior powers of T_C/F. Or we can remark that $\Lambda(T_C/F)$ is equal to the quotient of ΛT_C by the ideal generated by F and the Schouten bracket respects this ideal. Hence the same rule $\nabla(X) = [\,\cdot\,, X]$ gives rise to a well-defined $\mathbb{C}$-linear map

$$\nabla : \Lambda(T_C/F) \longrightarrow \mathcal{H}om_{\mathcal{O}_C}\big(F, \Lambda(T_C/F)\big)\,,$$

which satisfies the Leibnitz rule.

In our context, we obtain

$$\nabla : \pi^*\Lambda E^\vee \longrightarrow F^\vee \otimes_{\mathcal{O}_C} \pi^*\Lambda E^\vee\,.$$

To get the signs right, we will consider the elements of the factor $F^\vee$ in this expression to have degree zero.

Let us write the projection $\Lambda T_C \to \pi^*\Lambda E^\vee$ as ρ. We identify $F^\vee|_M$ with $N^\vee_{M/C}$ and $(\pi^*\Lambda E^\vee)|_M$ with ΛT_M. Then ϕ is the composition of ρ with restriction to M. We now define for $X, Y \in \Lambda T_C$

$$\{X, Y\} = \eta\big(\nabla(\rho X)|_M \wedge \nabla(\rho Y)|_M\big)\,. \tag{16}$$

In this formula, '$\wedge$' denotes the homomorphism

$$\begin{aligned} (N^\vee_{M/C} \otimes_{\mathcal{O}_C} \Lambda T_M) \otimes_{\mathcal{O}_C} (N^\vee_{M/C} \otimes_{\mathcal{O}_C} \Lambda T_M) &\longrightarrow (N^\vee_{M/C} \otimes_{\mathcal{O}_C} N^\vee_{M/C}) \otimes_{\mathcal{O}_C} \Lambda T_M \\ v \otimes X \otimes w \otimes Y &\longmapsto v \otimes w \otimes X \wedge Y\,. \end{aligned}$$

There is no sign correction in this definition, because the elements of $N^\vee_{M/C}$ are considered to have degree zero, by our sign convention. We have also extended the map η linearly to

$$\eta : (N^\vee_{M/C} \otimes_{\mathcal{O}_C} N^\vee_{M/C}) \otimes_{\mathcal{O}_C} \Lambda T_M \longrightarrow \Lambda T_M\,.$$

Claim. The conditions of Definition 1.16 are satisfied by $(\phi, \{\,\})$.

All but the last condition follow easily from the definitions. Let us check Condition (iv). We use Remark 1.17. The $\mathbb{C}$-algebra ΛT_C is generated in degrees

0 and -1. As generators in degree -1, we may take the basic vector fields of a coordinate system for C. We choose this coordinate system such that M is cut out by a subset of the coordinates. Then, if we plug in any of these generators of degree -1 for both X and Y in Formula (8), every term vanishes. Also, if we plug in terms of degree 0 for both X and Y, both sides of (8) vanish for degree reasons. By symmetry, we thus reduce to considering the case where X is of degree -1, i.e., a vector field on C, and Y is of degree 0, i.e., a regular function on C.

Hence we need to prove that for all $X \in T_C$ and $g \in \mathcal{O}_C$ we have

$$X(g)|_M - \rho(X)|_M\big(g|_M\big) = \{(\widetilde{t} - \widetilde{s}')X, g\} - \widetilde{t}\{X, g\}\,. \tag{17}$$

We will prove that

$$X(g)|_M - \rho(X)|_M\big(g|_M\big) = \{(\widetilde{s} - \widetilde{s}')X, g\} \tag{18}$$

and

$$\{(\widetilde{t} - \widetilde{s})X, g\} = \widetilde{t}\{X, g\}\,. \tag{19}$$

Equation (18) involves only M, not L, and Equation (19) involves only E, not E'. Together, they imply Equation (17).

All terms in these three equations are $\mathcal{O}_S$-linear in X and derivations in g, and may hence be considered as $\mathcal{O}_C$-linear maps $T_C \to \mathcal{D}er(\mathcal{O}_C, \mathcal{O}_M)$. As $\mathcal{D}er(\mathcal{O}_C, \mathcal{O}_M) = \mathcal{H}om_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_M) = T_C|M$, we may also think of them as $\mathcal{O}_C$-linear maps $T_C \to T_C|_M$.

For example, the $\mathcal{O}_C$-linear map

$$\begin{aligned} T_C &\longrightarrow T_C|_M \\ X &\longmapsto \{(\widetilde{s} - \widetilde{s}')X, \,\cdot\,\} \end{aligned} \tag{20}$$

Is equal to the composition

$$T_C \xrightarrow{df} \mathcal{O}_C \xrightarrow{d} \Omega_C \longrightarrow F^\vee|_M \xrightarrow{\eta} F|_M \longrightarrow T_C|_M\ ,$$

if we choose $df = s - s'$, as above. The the commutative diagram

$$\begin{array}{ccccccccc}
T_C & \xrightarrow{df} & \mathcal{O}_C & \xrightarrow{d} & \Omega_C & & & & \\
\downarrow & & & & \downarrow & \searrow & & & \\
T_C|_M & & & & \Omega_C|_M & \longrightarrow & F^\vee|_M & \xrightarrow{\eta} F|_M \longrightarrow & T_C|_M \\
\downarrow & & & & \uparrow & \nearrow_{\mathrm{id}} & & & \\
N_{M/C} & & \xrightarrow{H(f)} & & N^\vee_{M/C} & & & &
\end{array}$$

and the fact that η is the inverse of $H(f)$, proves that (20) is equal to the composition

$$T_C \longrightarrow T_C|_M \xrightarrow{p} T_C|_M\ ,$$

where p is the projection onto the the second summand of the decomposition

$$T_C|_M = T_M \oplus N_{M/C}$$

given by the foliation F transverse to M in C. If we denote by q the projection onto the first summand, we see that the map

$$\begin{aligned} T_C &\longrightarrow T_C|_M \\ X &\longmapsto \rho(X)|_M(\,\cdot\,|_M) \end{aligned} \tag{21}$$

is equal to

$$T_C \longrightarrow T_C|_M \xrightarrow{q} T_C|_M \, .$$

Thus (20) and (21) sum up to the restriction map $T_C \to T_C|_M$, which is equal to the map given by $X \mapsto X(\,\cdot\,)|_M$. This proves (18).

Now, let us remark that for any closed 1-form u on C we have

$$\widetilde{u}[Y,X] = Y\big(\widetilde{u}(X)\big) - X\big(\widetilde{u}(Y)\big)\, .$$

If $u \in \pi^* E \subset \Omega_C$, then $\widetilde{u}(Y) = 0$, for all $Y \in F$. So if $Y \in F$ we have

$$\widetilde{u}[Y,X] = Y\big(\widetilde{u}(X)\big)\, .$$

We have $\widetilde{u}[Y,X] = \widetilde{u}\big(\nabla(X)(Y)\big)$ by definition of the partial connection ∇ and we can write $Y\big(\widetilde{u}(X)\big) = \langle Y, d\big(\widetilde{u}(X)\big)\rangle$. In other words, the diagram

$$\begin{array}{ccccc} T_C/F & & \xrightarrow{\quad\nabla\quad} & & \mathcal{H}om(F, T_C/F) \\ \Big\downarrow{\scriptstyle \widetilde{u}} & & & & \Big\downarrow{\scriptstyle \_\circ\widetilde{u}} \\ \mathcal{O}_C & \xrightarrow{d} & \Omega_C & \longrightarrow & F^\vee \end{array}$$

commutes. Thus, the larger diagram

$$\begin{array}{ccccccccc} T_C & \xrightarrow{\rho} & \pi^*E^\vee & \xrightarrow{\nabla} & F^\vee \otimes \pi^*E^\vee & \longrightarrow & F^\vee|_M \otimes T_M & \xrightarrow{\eta\otimes\mathrm{id}} & F|_M \otimes T_M \\ & {\scriptstyle\widetilde{u}}\searrow & \Big\downarrow{\scriptstyle\widetilde{u}} & & \Big\downarrow{\scriptstyle \mathrm{id}\otimes\widetilde{u}} & & \Big\downarrow{\scriptstyle \mathrm{id}\otimes\widetilde{u}|_M} & & \Big\downarrow{\scriptstyle \mathrm{id}\otimes\widetilde{u}|_M} \\ & & \mathcal{O}_C & \xrightarrow{d} & F^\vee & \longrightarrow & F^\vee|_M & \xrightarrow{\eta} & F|_M \\ & & & & & & & & \Big\downarrow \\ & & & & & & & & T_C|_M \end{array}$$

commutes as well. We can apply these considerations to $u = t - s$. Then $\widetilde{u} = \widetilde{t} - \widetilde{s}$ and $\widetilde{u}|_M = \widetilde{t}|_M$. Thus the upper composition in this diagram represents the right hand side of Equation (19), and the lower composition represents the left hand side of Equation (19). This exhibits that (19) holds and finishes the proof of the theorem. $\square$

3.2 The Batalin-Vilkovisky case

By a *local system* we mean a vector bundle (locally free sheaf of finite rank) endowed with a flat connection. Every local system P on a manifold M has an associated de Rham complex $(P \otimes_{\mathcal{O}_M} \Omega^{\bullet}_M, d)$, where d denotes the covariant derivative.

Definition 3.6 Let (S, E, σ) be a polarized symplectic manifold and L, M immersed Lagrangians, both transverse to E. Let P be a local system on M and Q a local system on S. The **derived hom** from $Q|L$ to $P|M$ is the differential Batalin-Vilkovisky module

$$\mathbb{R}\mathcal{H}om_S\big(Q|L, P|M\big) = \big(\Omega^{\bullet}_M \otimes Q^{\vee}|_M \otimes P, t\big)$$

over the differential Gerstenhaber algebra

$$L \Cap_S M = (\Lambda T_M, \widetilde{t})\,.$$

Here the tensor products are taken over $\mathcal{O}_M$. The $\mathcal{O}_M$-linear differential t is multiplication by t and the $\mathbb{C}$-linear differential d is covariant derivative with respect to the induced flat connection on $Q^{\vee}|_M \otimes P$.

Remark 3.7 If we forget about the $\mathbb{C}$-linear differential d and hence the flat connections on P and Q, the underlying complex of $\mathcal{O}_S$-modules $\mathbb{R}\mathcal{H}om_S(Q|L, P|M)$ represents the derived sheaf of homomorphisms $\mathbb{R}\mathcal{H}om_{\mathcal{O}_S}(Q|_L, P)$ in the derived category of sheaves of $\mathcal{O}_S$-modules.

For the exchange property in the Batalin-Vilkovisky case, we need the notion of orientation on a Lagrangian foliation.

Definition 3.8 Let (S, σ) be a symplectic manifold and E a Lagrangian foliation on S. An **orientation** of E is an isomorphism

$$\Lambda^n E \xrightarrow{\sim} \mathcal{O}_S$$

of $\mathcal{O}_S$-modules. The corresponding nowhere vanishing section $\theta \in \Lambda^n E^{\vee}$ will be called the **orientation field** of the orientation on E. A Lagrangian foliation (or a polarized symplectic manifold) is called **orientable**, if it admits an orientation.

Theorem 3.9 *Consider (S, E, σ), (S', E', σ'), $C \to S' \times S$, $M' \to S'$ and $L \to S$ as in Theorem 3.5. Assume E and E' are orientable. In addition, let P' be a local system on S' and Q a local system on S. Let $P = \pi'^{*} P'$ and $Q' = \pi^{*} Q$ be the pullbacks of our local systems to C. Then every orientation of E' defines a quasi-isomorphism of differential Batalin-Vilkovisky modules*

$$\mathbb{R}\mathcal{H}om_{\overline{S'} \times S}\big(\mathcal{O}|(M' \times L), (P \otimes Q'^{\vee})|C\big) \longrightarrow \mathbb{R}\mathcal{H}om_S\big(Q|L, P|M\big)$$

of degree $-n'$, covering the corresponding canonical quasi-isomorphism of differential Gerstenhaber algebras of Theorem 3.5. Every orientation of E defines a quasi-isomorphism of differential Batalin-Vilkovisky modules

$$\mathbb{R}\mathcal{H}om_{\overline{S}'\times S}\big(\mathcal{O}|(M'\times L),(P\otimes Q'^{\vee})|C\big)\longrightarrow \mathbb{R}\mathcal{H}om_{\overline{S}'}\big(P'^{\vee}|M',Q'^{\vee}|L'\big)$$

of degree $-n$, covering the other canonical quasi-isomorphism of differential Gerstenhaber algebras of Theorem 3.5. Thus, if E and E' are oriented, the derived homs $\mathbb{R}\mathcal{H}om_S(Q|L,P|M)$ and $\mathbb{R}\mathcal{H}om_{\overline{S}'}(P'^{\vee}|M',Q'^{\vee}|L')$ are canonically quasi-isomorphic, up to a degree shift $n'-n$.

4 The virtual de Rham complex

Applying the exchange property to $C=S=S'$, but different polarizations E, E', we get the independence of the bracket $[\,,\,]$ and the differential d from the polarization. Hence we get globally defined Gerstenhaber algebra structures on $\mathcal{A}$ and Batalin-Vilkovisky module structures on $\mathcal{E}$.

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