Orthogonal Graph Embeddings with Minimal Number of Bends

Based on a result by

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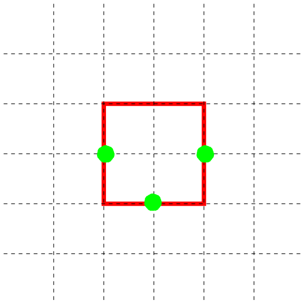
Orthogonal Embeddings

What is an Orthogonal Grid Embedding?

A mapping that maps the vertices of a (planar) graph $G = (V, E)$ into the grid points and the edges into interior disjoint paths on the grid.



- A triangle embedded on a grid.
- The 3 vertices are shown as circles.
- Embedding has 4 bends. This can be reduced.

Introduction

Our Aim

Given a planar representation G, compute an orthogonal embedding with the minimum number of bends.

- We consider only graphs with max degree 4.
- Among orthogonal embeddings with the minimal number of bends, we want the the one with the smallest area, (and/or width, length *etc.*)
- Two sources for all the material:
  - Graph Drawing: Algorithms for the Visualization of Graphs by Di Battista, Eades, Tamassia and Tollis (Book).
  - On embedding a graph in the grid with the minimum number of bends, Tamassia, *SIAM J. Computing Vol 16, No. 3, 1987*.
- The book is a more complete source.
- All material can be found at: `www.cs.nyu.edu/~raghavan/gd` (for now).

A Few Definitions

- Planar Representation: A planar graph G together with the set of faces of G and the order of edges around each face.
- 90° bend: A bend in a face f of an orthogonal embedding such that the 90° is inside f.
- Orthogonal Representation: A planar representation together with the the following information for every face f:
  - The angle between every pair of consecutive edges in f. This can be one of $90^\circ, 180^\circ, 270^\circ, 360^\circ$.
  - For every edge $e \in f$, a list of all 90° and 270° bends in e. Note that e can have only these two kinds of bends.

  In a nutshell, an Orthogonal Representation (Ortho-rep) is an orthogonal embedding but without any information about the lengths of the edges.
- Normalized Ortho-Rep: An ortho-rep every one of whose faces is rectangular in shape.

Summary of Results

- $\mathcal{NP}$-hard to find minimal embedding of a 4-planar graph G over all possible planar representations (Garg-Tamassia 95).
- For a fixed planar representation this can be done in $O(n^2 \log n)$ time (Tamassia 1987).
- This was improved to $O(n^{\frac{7}{4}} \log n)$ (Garg-Tamassia 1997).
- If G has max degree 3 then a bend minimal embedding over all possible planar representations can be found in polynomial time (Di Battista, Liotta and Vargiu 93).
- **We Consider:** Planar graphs with max degree 4 which have a fixed planar representation.

Overview of the Algorithm

Given a planar representation G,

- We first compute a bend minimal ortho-rep.
- We then refine it to get a normalized ortho-rep.
- This is then embedded into the grid.
- The fictitious edges added during the normalization are then deleted to obtain and embedding of G.

Each of the above steps can be performed in $O(n^2 \log n)$ time.

Pretty Simple, Ain't it?

Outline

- A (very brief) review of the Minimum-cost-flow problem.
- Sums of the angles of a polygon.
- Grid embedding of a normalized ortho-rep.
- Normalizing an ortho-rep.
- Computing a bend minimal ortho-rep.
- Characterizing bend minimal embeddings.
- Homework ;-).

Review of Minimum Cost Flow

- A network $N = (V, E, low, capacity, cost, demand)$ consists of :
  - A finite set of vertices V.
  - A set E of ordered pairs of vertices.
  - $low, capacity, cost$ and $demand$ are functions such that: $low : E \longrightarrow \mathcal{R}, capacity : E \longrightarrow \mathcal{R}$, $cost : E \longrightarrow \mathcal{R}$ and $demand : V \longrightarrow \mathcal{R}$.
  - For any $v \in V$ if $demand(v) > 0$ then v is called a source and if $demand(v) < 0$ then v is called a sink.
- A flow in N is a function $x : E \longrightarrow \mathcal{R}$ such that:
  - $low(e) \leq x(e) \leq capacity(e) \forall e \in E$.
  - $\Sigma_w x(v, w) - \Sigma_u x(u, v) = demand(v) \forall v \in V$.
- The value of the flow is $\Sigma_{u \in Sources} demand(u)$.
- The cost of a flow is $\Sigma_{e \in E} cost(e) x(e)$.
- The Minimum cost flow problem asks to compute the flow with the minimum cost among those with a given value.

Review of Minimum Cost Flow II

- When the $low, capacity$ and $demand$ functions are integral and $cost$ is non-negative, the Min-Cost-Flow problem can be solved in time $O(|x|(V+E)\log V)$ where x is the value of the flow.
- Consider a cycle C, with respect to a flow x, in the underlying undirected graph G of the network such that:
  - For every edge e traversed by C in the direction of e we have $x(e) < capacity(e)$.
  - For every edge e traversed in the opposite direction, we have $x(e) > low(e)$.
- We can some additional flow along C to obtain a new flow function x' with value same as that of x. How?
- The cycle C is called a **Flow Augmenting Cycle** if the cost of x' is less than that of x.
- The flow x is a Min-cost-flow **iff** there is no Flow Augmenting Cycle with respect to it (Ahuja, Magnanti and Orlin).

Sums of Angles in a Polygon

- Sum of internal angles of a simple, convex, not necessarily orthogonal, polygon with n sides is $\pi(n-2)$.
- Sum of external angles $\pi(n+2)$.
- What if the polygon is non-convex (but still simple)? Same result holds.
- Let n_{90° and n_{270° be the number of convex and reflex vertices of an orthogonal polygon, then $n_{90^\circ} - n_{270^\circ} = 4$ (Homework!).
- Given an orthogonal polygon with vertex set V, let $V' \subset V$ and let n_{d°, n'_{d° and n''_{d° be the number of convex angles in V and V' and $V - V'$ where $d = \{90^\circ, 270^\circ\}$. Then

Equation

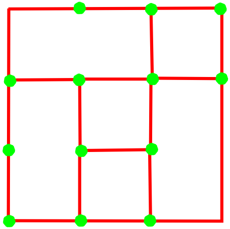
$$\frac{2\left(\Sigma_{v \in V'} angle(v)\right)}{\pi} + n''_{270^\circ} - n''_{90^\circ} = 2|V'| - 4$$

Grid Embedding for a Normalized Ortho-Rep

- We are given an planar representation G and H an ortho-rep of G such that every face of H is *rectangular* in shape (though not necessarily a rectangle).
- More formally, we have the following information:
  - The set of faces of G and the list of edges forming each face are given.
  - The angle between any two consecutive edges in a face is fixed at 90° or 180° and all but the four "corner" edges have no bends.
  - The number of bends in each corner edge is fixed.

What we need to get a Grid Embedding

We only need to compute the length of the horizontal and vertical segments. All other information is already present.



- **Note:** The lengths of the horizontal and vertical segments can be computed independently.

Grid Embedding for a Normalized Ortho-Rep

The Algorithm outline:

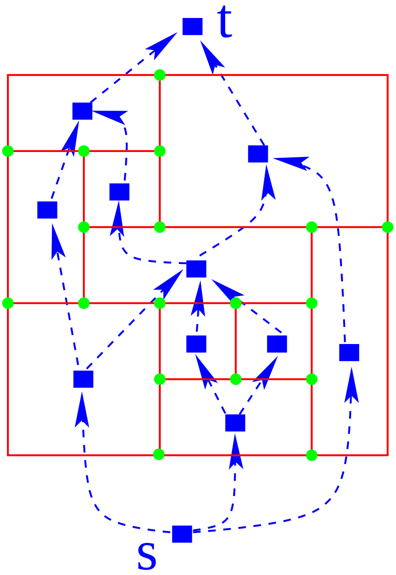
- First, we compute the lengths of the horizontal segments:
  - Construct a flow network N_{hor} associated with H.
  - Compute the min cost flow in N_{hor}.
  - Compute the length of each horizontal edge from the min cost flow.
- Lengths of the vertical segments are computed in a similar manner.
- Putting these together, we have a grid embedding of H.
- The embedding obtained has minimum width, height, area and total edge length.

THE HORIZONTAL SEGMENTS

The network N_{hor} is constructed as follows:

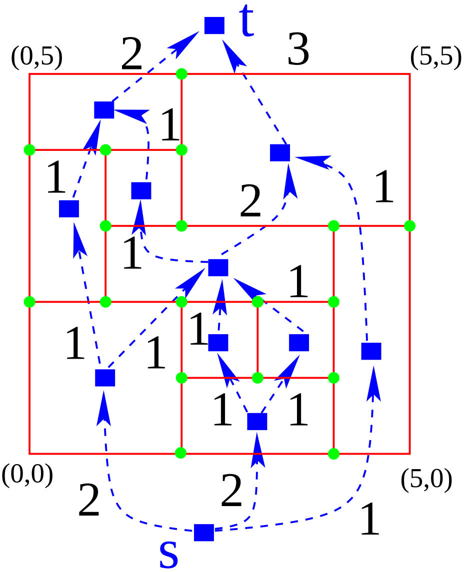
- N_{hor} has a node corresponding to every internal face of H.
- Two additional s and t representing the *lower* and *upper* regions of the outer face are also added to N_{hor}.
- Two nodes of N_{hor}, f and g are joined by an edge **iff** faces f and g of N_{hor} share a horizontal edge.
- Every edge in N_{hor} has a lower bound of 1, a capacity of $+\infty$ and a cost of 1.

AN EXAMPLE OF N_{hor}



- The original ortho-rep is shown in red (with solid lines) and the edges of N_{hor} are in blue (dashed lines).
- N_{hor} is planar with a unique source and a sink.
- **Remember:** At this point we do not have a grid embedding yet! (the figure on the right represents an ortho-rep and is not a grid-embedding yet)

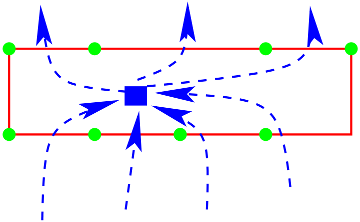
A FLOW IN N_{hor} FROM A GRID EMBEDDING OF H



- Given a grid embedding of N_{hor}, we can obtain a flow in N_{hor} by setting the flow in an edge $e \in N_{hor}$ to be the length of the corresponding horizontal edges of the grid embedding of H.
- The flow satisfies the lower bound and is conserved at the vertices of N_{hor}. Why?
- How do we compute a grid-embedding of H from a flow in N_{hor}?

FLOWS AND GRID-EMBEDDINGS: SOME INTUITION

- Consider a node v of N_{hor}:



- Since the shape is a rectangle, it follows that length of bottom segment is the same as the length of the top segment.
- This implies that flow in conserved! Why?
- Flow lower bound is 1 in the edges of N_{hor} $\iff$ Minimum length of an edge in the grid embedding is 1.
- Flow Capacity of each edge in N_{hor} is ∞ $\iff$ Edges can be of any length in the grid embedding.
- Flow has unit cost on each edge $\iff$ we want the flow with minimum total edge length.

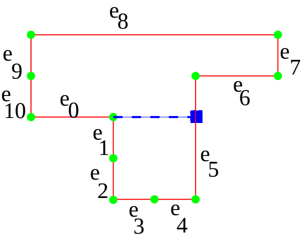
NORMALIZING AN ORTHO-REP

THE PROBLEM DEFINITION

Given a general ortho-rep F whose faces are not necessarily rectangular, how do we refine it (by adding edges and vertices) to obtain a normalized ortho-rep H?

- Add a new vertex at every bend in F.
- For each non-rectangular internal face $f \in F$:
  - For each edge e in f, let $next(e)$ be the next counterclockwise edge and let $corner(e)$ be the common vertex of e and $next(e)$.
  - Let $turn(e) = +1$ if e and $next(e)$ form a left turn, $turn(e) = 0$ if they are aligned and $turn(e) = -1$ if they form a right turn.
  - If $turn(e) = -1$ then let $front(e) = e'$ where e' follows e counterclockwise and the sum of turn values of all edges between e (included) and e' (excluded) is 1.

NORMALIZING AN ORTHO-REP II

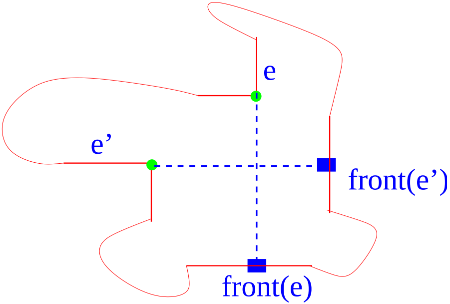


- $turn(e_0) = -1$ and $front(e_0) = e_5$.
- **Remember:** We are currently dealing with an ortho-rep, **not** with a embedding.

- For all e such that $turn(e) = -1$, add an edge $extend(e)$ from $corner(e)$ to $front(e)$ to *break* the face into two simpler pieces.
- Let $r = \{e | turn(e) = -1, e \in f\}$, then face f is broken into $r + 1$ rectangular pieces.
- The external face has to be dealt with in a slightly different way.
- Two questions remain:
  - Does $front(e)$ always exist?
  - Is the final graph planar?

NORMALIZING AN ORTHO-REP III

- For all edges $e \in f$ where f is an internal face, $front(e)$ exists since $\Sigma_{e \in f} turn(e) = 4$. Why?
- Two newly inserted edges $extend(e)$ and $extend(e')$ cannot



intersect. Why?

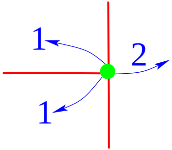
Computing a Bend Minimal Ortho-Rep

The Problem

Given a planar representation G, compute the ortho-rep with the minimum number of bends.

Some intuition about the angles in a grid embedding:

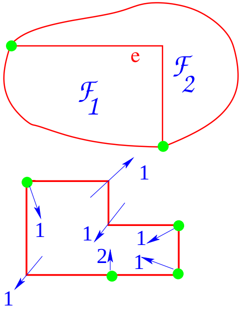
- Let one unit=$\frac{\pi}{2}$ radians.
- Each vertex *generates* 4 units worth of angles.



- A face that containing a pair of consecutive edges making an angle of α units ($1 \le \alpha \le 4$) is said to *gain* α units from the vertex.
- Each k-face of G is said to *consume* $2k - 4$ units worth of angles if it is internal and $2k + 4$ units if it is external.
- Two questions arise:
  - Why $2k - 4$ and $2k + 4$?
  - Can we say something about the bends in the edges?

First consider the bends:

- For each 90° bend we say face $\mathcal{F}_1$ *loses* one unit of angles (to face $\mathcal{F}_2$) via edge e and face $\mathcal{F}_2$ *gains* gains the same via edge e.



- Let $Bends_{90^\circ}(e, \mathcal{F})$ be the units lost by face $\mathcal{F}$ via edge e and $Bends_{270^\circ}(e, \mathcal{F})$ be the units gained.
- Let $Angle(e, \mathcal{F})$ be the angle (in units) between edge e and the next counterclockwise edge in $\mathcal{F}$.
- Let $\phi(\mathcal{F}) = \Sigma_{e \in F} Angle(e, \mathcal{F}) + Bends_{270^\circ}(e, \mathcal{F}) - Bends_{90^\circ}(e, \mathcal{F})$ and let $\mathcal{F}$ have k graph edges.

THE KEY EQUATION

RECALL

For every k-face $\mathcal{F}$, $\phi(\mathcal{F}) = 2k - 4$ if $\mathcal{F}$ is internal and $\phi(\mathcal{F}) = 2k + 4$ if $\mathcal{F}$ is external.

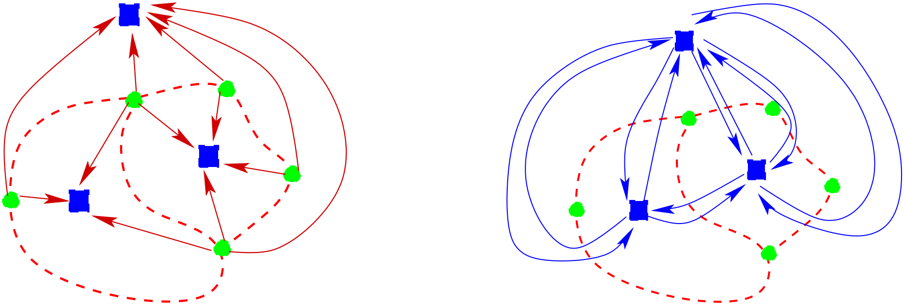
- $\phi(\mathcal{F})$ is the net units gained by $\mathcal{F}$. Hence it makes sense to say that $\mathcal{F}$ consumes $2k - 4$ (or $2k + 4$) units.

Now we look at things from a Network Flow Perspective by building a network $\mathcal{N}$:

- $\mathcal{N}$ has a vertex for every vertex and face of G.
- Directed edge $(v, \mathcal{F}) \in E(\mathcal{N})$ **iff** face $\mathcal{F}$ contains vertex v in G.
- If faces $\mathcal{F}_1$ and $\mathcal{F}_2$ are adjacent in G then $\mathcal{N}$ contains directed edges $(\mathcal{F}_1, \mathcal{F}_2)$ and $(\mathcal{F}_2, \mathcal{F}_1)$.
- The vertices of $\mathcal{N}$ corresponding to vertices of G are sources with each producing 4 units of flow. The vertices corresponding to the faces of G are sinks with each face $\mathcal{F}$ consuming $\phi(\mathcal{F})$ units.

The Network Flow Perspective

- Edge $(v, \mathcal{F})$ of $\mathcal{N}$ has capacity 4, cost 0 and lower bound of 1.
- Edge $(\mathcal{F}_1, \mathcal{F}_2)$ of $\mathcal{N}$ has capacity $+\infty$, cost 1 and lower bound of 0.



- The original graph is in red (dotted lines) with green (circular) vertices.
- The dual vertices are in blue.
- The left figure shows all vertex-face edges in the network.
- The right figure shows all face-face edges in the network.

Given a flow $\mathcal{X}$,

- The flow in edge $(v, \mathcal{F})$ can be thought of as the angle at vertex v of face $\mathcal{F}$.
- The flow in edge $(\mathcal{F}_1, \mathcal{F}_2)$ can be thought of as the number of 90° bends in the common edge.
- The flow is conserved at every vertex implies that every vertex has a net flow of 4 units away from it (as this is the supply).
- Flow is conserved at every face implies that every k-face gets $2k - 4$ units of flow (as this is the demand).
- Total flow out of the sources = Total flow into the sinks (by Eulers formula).

Network Flows and Bends

- Given a grid embedding of G we can compute the associated network flow in $\mathcal{N}$. How?
- The total cost of any flow in $\mathcal{N}$ is the total number of bends in the associated grid embedding. Why?
- So we can compute min-cost flow and find the number of bends of each type in each edge of G.
- Total value of flow = $O(n)$. Hence algorithm runs in $O(n^2 \log n)$ time.

Characterizing Bend Minimal Embeddings

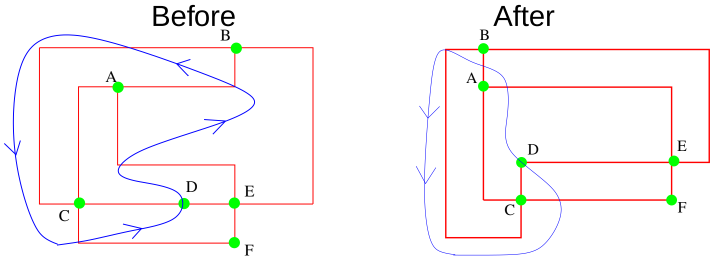
Theorem

A orthogonal embedding of planar representation G is bend-minimal **iff** there is no directed cycle J such that :(a) J intersects each edge of G at most twice, (b) J enters vertices of G only from angles of at least 180° and (c) More than half the edges crossed by J have a bend with the angle of 270° on the side from which J enters.

Proof: Augmenting Cycle in $\mathcal{N}$ implies the existence of J.

- If the embedding then a flow-augmenting cycle C must exist in the network $\mathcal{N}$
- If this curve enters a vertex of G, it must do so while traversing a vertex-face edge in the reverse direction. Why?
- If C traverses a total of k face-face edges of $\mathcal{N}$, then it must traverse at least $\frac{k}{2}$ of these in the reverse direction of the edge.
- We can also show that J implies the existence of an augmenting cycle in a similar way.

AN EXAMPLE



In the left figure:

- The curve enters edges CF, AE and AB from a 270° angle.
- CB does not have a 270° angle in the direction the curve enters.
- The curve enters vertex which has an angle of 180° in the direction the curve enters.

The right figure has two less bends.

(Simple!) Homework

- Let n_{90° and n_{270° be the number of convex and reflex vertices of an orthogonal polygon, then prove that $n_{90^\circ} - n_{270^\circ} = 4$
- Show that no edge of a planar bend-minimal orthogonal embedding has two bends with an angle of 90° on opposite sides.