A Tverberg-type result on multicolored simplices

János Pach*
City College, CUNY and Courant Institute, NYU

Abstract
Let $P_1, P_2, \ldots, P_{d+1}$ be pairwise disjoint $n$-element point sets in general position in $d$-space. It is shown that there exist a point $O$ and suitable subsets $Q_i \subseteq P_i$ ($i = 1, 2, \ldots, d + 1$) such that $|Q_i| \geq c_d|P_i|$, and every $d$-dimensional simplex with exactly one vertex in each $Q_i$ contains $O$ in its interior. Here $c_d$ is a positive constant depending only on $d$.

1 Introduction
Let $P_1, P_2, \ldots, P_{d+1}$ be pairwise disjoint $n$-element point sets in general position in Euclidean $d$-space $\mathbb{R}^d$. If two points belong to the same $P_i$, then we say that they are of the same color. A $d$-dimensional simplex is called multicolored, if it has exactly one vertex in each $P_i$ ($i = 1, 2, \ldots, d + 1$). Answering a question of Bárány, Füredi, and Lovász [BFL90], Vrecica and Živaljević [ZV92], proved the following Tverberg-type result. For every $k$, there exists an integer $n(k,d)$ such that if $n \geq n(k,d)$, then any pairwise disjoint $n$-element point sets $P_1, P_2, \ldots, P_{d+1} \subset \mathbb{R}^d$ in general position induce at least $k$ multicolored vertex disjoint simplices with an interior point in common. (For some special cases, see [BL92], [JS91], [VZ94].) This theorem can be

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used to derive a nontrivial upper bound on the number of different ways one can cut a finite point set into two (roughly) equal halves by a hyperplane.

The aim of this note is to strengthen the above result by showing that there exist “large” subsets of the sets $P_i$ such that all multicolored simplices induced by them have an interior point in common.

**Theorem.** There exists $c_d > 0$ with the property that for any disjoint $n$-element point sets $P_1, P_2, \ldots, P_{d+1} \subset \mathbb{R}^d$ in general position, one can find a point $O$ and suitable subsets $Q_i \subseteq P_i$, $|Q_i| \geq c_d |P_i|$ ($i = 1, 2, \ldots, d+1$) such that every $d$-dimensional simplex with exactly one vertex in each $Q_i$ contains $O$ in its interior.

The proof is based on the $k = d + 1$ special case of the Vrecica-Živaljević theorem (see Theorem 2.1). It uses three auxiliary results, each of them interesting on its own right. The first is Kalai’s fractional Helly theorem [K84], which sharpens and generalizes some earlier results of Katchalski and Liu [KL79] (see Theorem 2.2). The second is a variation of Szemerédi’s regularity lemma for hypergraphs [S78] (Theorem 2.3), and the third is a corollary of Radon’s theorem [R21], discovered and applied by Goodman and Pollack [GPW96] (Theorem 2.4).

In the next section, we state the above mentioned results and also include a short proof of Theorem 2.3, because in its present form it cannot be found in the literature. Our argument is an adaptation of the approach of Komlös and Sós [KS96]. For some similar results, see [C91],[FR92],[KS95]. The proof of the Theorem is given in Section 3. It shows that the statement is true for a constant $c_d > 0$ whose value is triple-exponentially decreasing in $d$.

## 2 Auxiliary results

**Theorem 2.1 [ZV92]** Let $A_1, A_2, \ldots, A_{d+1}$ be disjoint $4d$-element sets in general position in $d$-space. Then one can find $d + 1$ vertex disjoint simplices with a common interior point such that each of them has exactly one vertex in every $A_i$, $1 \leq i \leq d + 1$.

A family of sets is called **intersecting** if they have an element in common.
Theorem 2.2 [K84] For any $\alpha > 0$, there exists $\beta = \beta(\alpha, d) > 0$ satisfying the following condition. Any family of $N$ convex sets in $d$-space, which contains at least $\alpha \binom{N}{d+1}$ intersecting $(d+1)$-tuples, has an intersecting subfamily with at least $\beta N$ members.

In fact, if $N$ is sufficiently large, then Theorem 2.2 is true for any $\beta < 1 - (1 - \alpha)^{1/(d+1)}$. In particular, it holds for $\beta = \alpha/(d + 1)$.

Let $\mathcal{H}$ be a $(d + 1)$-partite hypergraph whose vertex set is the union of $d + 1$ pairwise disjoint $n$-element sets, $P_1, P_2, \ldots, P_{d+1}$, and whose edges are $(d + 1)$-tuples containing precisely one element from each $P_i$. For any subsets $S_i \subseteq P_i$ ($1 \leq i \leq d + 1$), let $e(S_1, \ldots, S_{d+1})$ denote the number of edges of $\mathcal{H}$ induced by $S_1 \cup \ldots \cup S_{d+1}$. In this notation, the total number of edges of $\mathcal{H}$ is equal to $e(P_1, \ldots, P_{d+1})$.

It is not hard to see that for any sets $S_i$ and for any integers $t_i \leq |S_i|$, $1 \leq i \leq d + 1$,

$$
e(S_1, \ldots, S_{d+1}) = \sum \frac{e(T_1, \ldots, T_{d+1})}{|T_1| \ldots |T_{d+1}|} \binom{|S_i|}{t_i} \binom{|S_{d+1}|}{t_{d+1}},$$

where the sum is taken over all $t_i$-element subsets $T_i \subseteq S_i$, $1 \leq i \leq d + 1$.

**Theorem 2.3** Let $\mathcal{H}$ be a $(d + 1)$-partite hypergraph on the vertex set $P_1 \cup \ldots \cup P_{d+1}$, $|P_i| = n$ ($1 \leq i \leq d + 1$), and assume that $\mathcal{H}$ has at least $\beta n^{d+1}$ edges for some $\beta > 0$. Let $0 \leq \varepsilon < 1/2$.

Then there exist subsets $S_i \subseteq P_i$ of equal size $|S_i| = s \geq \beta^{1/2d} n$ ($1 \leq i \leq d + 1$) such that

(i) $e(S_1, \ldots, S_{d+1}) \geq \beta s^{d+1}$,

(ii) $e(Q_1, \ldots, Q_{d+1}) > 0$ for any $Q_i \subseteq S_i$ with $|Q_i| \geq \varepsilon s$ ($1 \leq i \leq d + 1$).

**Proof:** Let $S_i \subseteq P_i$ ($1 \leq i \leq d + 1$) be sets of equal size such that

$$
e(S_1, \ldots, S_{d+1})
\frac{|S_1|^{d+1 - \varepsilon 2d}}{|S_i|^{d+1 - \varepsilon 2d}}$$

is maximum, and denote $|S_1| = \ldots = |S_{d+1}|$ by $s$.

For this choice of $S_i$, condition (i) in the theorem is obviously satisfied, because

$$
e(S_1, \ldots, S_{d+1})
\frac{|S_1|^{d+1 - \varepsilon 2d}}{|S_i|^{d+1 - \varepsilon 2d}}
\geq
\frac{e(P_1, \ldots, P_{d+1})}{n^{d+1 - \varepsilon 2d}}
= \frac{\beta}{n^{-\varepsilon 2d}}
\geq \frac{\beta}{s^{-\varepsilon 2d}}.$$
Taking into account the trivial relation

\[ \frac{e(S_1, \ldots, S_{d+1})}{|S_1|^{d+1-\varepsilon^{2d}}} \leq s^{\varepsilon^{2d}}, \]

the above inequalities also yield that \( s \geq \beta^{1/\varepsilon^{2d}} n \).

It remains to verify (ii). To simplify the notation, assume that \( \varepsilon s \) is an integer, and let \( Q_i \) be any \( \varepsilon s \)-element subset of \( S_i \) (1 \( \leq i \leq d + 1 \)). Then

\[
e(Q_1, \ldots, Q_{d+1}) = e(S_1, \ldots, S_{d+1})
- e(S_1 - Q_1, S_2, S_3, \ldots, S_{d+1})
- e(Q_1, S_2 - Q_2, S_3, \ldots, S_{d+1})
- e(Q_1, Q_2, S_3 - Q_3, \ldots, S_{d+1})
\ldots
- e(Q_1, Q_2, Q_3, \ldots, S_{d+1} - Q_{d+1}).
\]

In view of (1), it follows from the maximal choice of \( S_i \) that

\[
e(S_1 - Q_1, S_2, \ldots, S_{d+1})
= (1 - \varepsilon)s^{d+1}e(S_1 - Q_1, S_2, \ldots, S_{d+1})
\]

\[
= (1 - \varepsilon)s^{d+1} \sum_{T_1 \subseteq S_1, |T_1| = (1 - \varepsilon)s} \frac{e(S_1 - Q_1, T_2, \ldots, T_{d+1})}{(1 - \varepsilon)s^{d+1}} \left( \frac{s}{\varepsilon s} \right)^d
\]

\[
\leq (1 - \varepsilon)s^{d+1} \frac{e(S_1, S_2, \ldots, S_{d+1})}{s^{d+1-\varepsilon^{2d}}} [(1 - \varepsilon)s]^{-\varepsilon^{2d}}
= e(S_1, ..., S_{d+1})(1 - \varepsilon)^{1-\varepsilon^{2d}}.
\]

Similarly, for any \( i, 2 \leq i \leq d + 1 \), we have

\[
e(Q_1, \ldots, Q_{i-1}, S_i - Q_i, S_{i+1}, \ldots, S_{d+1})
\leq e(S_1, ..., S_{d+1})\varepsilon^{i-1-\varepsilon^{2d}}(1 - \varepsilon).
\]

Summing up these inequalities, we obtain
\[ e(Q_1, \ldots, Q_{d+1}) \geq e(S_1, \ldots, S_{d+1})(1 - (1 - \varepsilon)^{1-\varepsilon^{2d}} - \sum_{i=2}^{d+1} \varepsilon^{i-1-\varepsilon^{2d}}(1 - \varepsilon)) \]
\[
\geq e(S_1, \ldots, S_{d+1})(1 - (1 - \varepsilon)^{1-\varepsilon^{2d}} - \varepsilon^{1-\varepsilon^{2d}} + \varepsilon^{d+1-\varepsilon^{2d}}) > 0,
\]
as required. \qed

A \((d+1)\)-tuple of convex sets in \(d\)-space is called \textit{separated} if any \(j\) of them can be strictly separated from the remaining \(d+1-j\) by a hyperplane, \(1 \leq j \leq d\). An arbitrary family of at least \(d+1\) convex sets in \(d\)-space is \textit{separated} if every \((d+1)\)-tuple of it is separated.

**Theorem 2.4** [GPW96] A family of convex sets in \(d\)-space is separated if and only if no \(d+1\) of its members can be intersected by a hyperplane.

Let \(n \geq d+1\). Two sequences of points in \(d\)-space, \((p_1, \ldots, p_n)\) and \((q_1, \ldots, q_n)\), are said to have the same \textit{order type} if for any integers \(1 \leq i_1 < \ldots < i_{d+1} \leq n\), the simplices \(p_{i_1} \ldots p_{i_{d+1}}\) and \(q_{i_1} \ldots q_{i_{d+1}}\) have the same orientation [GPW93]. It readily follows from the last result that if \(C_1, \ldots, C_n\) form a separated family of convex sets, then the order type of \((p_1, \ldots, p_n)\) will be the same for every choice of elements \(p_i \in C_i, 1 \leq i \leq n\).

### 3 Proof of Theorem

Let \(P_1, \ldots, P_{d+1}\) be pairwise disjoint \(n\)-element point sets in general position in \(d\)-space. If a simplex has precisely one vertex in each \(P_i\), we call it \textit{multicolored}. The number of multicolored simplices is \(N = n^{d+1}\).

By Theorem 2.1, any collection of \(4d\)-element subsets \(A_i \subseteq P_i, 1 \leq i \leq d + 1\), induce \(d+1\) vertex disjoint multicolored simplices with a common interior point. Thus, the total number of intersecting \((d+1)\)-tuples of multicolored simplices is at least

\[
\frac{n}{3d-1}^{d+1} > \frac{1}{(5d)^d} \left( \frac{N}{d+1} \right).
\]

Hence, we can apply Theorem 2.2 with \(\alpha = 1/(5d)^d\). We obtain that there is a point \(O\) contained in the interior of at least \(\beta N = \beta (1/(5d)^d, d)n^{d+1}\) multicolored simplices.
Let $\mathcal{H}$ denote the $(d + 1)$-partite hypergraph on the vertex set $P_1 \cup \ldots \cup P_{d+1}$, whose edge set consists of all multicolored $(d + 1)$-tuples that induce a simplex containing $O$ in its interior.

Set $\varepsilon = 1/2^{dd}$, and apply Theorem 2.3 to the hypergraph $\mathcal{H}$ to find $S_i \subset P_i$, $1 \leq i \leq d + 1$, meeting the requirements. By throwing out some points from each $S_i$, but retaining a positive proportion of them, we can achieve that the convex hulls of the sets $S_i$ are separated. Indeed, assume e.g. that there is no hyperplane strictly separating $S_1 \cup \ldots \cup S_j$ from $S_{j+1} \cup \ldots \cup S_{d+1}$. By the ham-sandwich theorem [B33], one can find a hyperplane $h$ which simultaneously bisects $S_1, \ldots, S_d$ into as equal parts as possible. Assume without loss of generality that at least half of the elements of $S_{d+1}$ are “above” $h$. Then throw away all elements of $S_1 \cup \ldots \cup S_j$ that are above $h$ and all elements of $S_{j+1} \cup \ldots \cup S_{d+1}$ that are below $h$. We can repeat this procedure as long as we find a non-separated $(d + 1)$-tuple. In each step, we reduce the size of every set by a factor of at most 2.

Notice that in the same manner we can also achieve that e.g. the $(d + 1)$-tuple $\{\{O\}, \text{conv}(S_1), \ldots, \text{conv}(S_d)\}$ becomes separated. In this case, $h$ will always pass through the point $O$, therefore $O$ will never be deleted.

After at most $(d + 2)2^d$ steps we end up with $Q_i \subset S_i$, $|Q_i| > \varepsilon s$ ($1 \leq i \leq d + 1$) such that $\{\{O\}, \text{conv}(S_1), \ldots, \text{conv}(S_{d+1})\}$ is a separated family. It follows from the remark after Theorem 2.4 that there are only two possibilities: either every multicolored simplex induced by $Q_1 \cup \ldots \cup Q_{d+1}$ contains $O$ in its interior, or none of them does. However, this latter option is ruled out by part (ii) of Theorem 2.3. This completes proof. □

Instead of applying Theorem 2.2, we could have started the proof by referring to the following result of Alon, Bárány, Füredi, and Kleitman [ABFK92], which is also based on Theorem 2.1. For any $\beta > 0$ there is a $\beta_d > 0$ such that any family of $\beta n^{d+1}$ simplices induced by $n$ points in $d$-space has at least $\beta_d n^{d+1}$ members with non-empty intersection.

Our proof easily yields the following.

**Theorem 3.1** For any $\beta > 0$ there is a $\beta_d > 0$ with the property that given any family $\mathcal{S}$ of $\beta n^{d+1}$ simplices induced by an $n$-element set $P \subset \mathbb{R}^d$, one can find a point $O$ and pairwise disjoint subsets $Q_i \subset P$ ($i = 1, 2, \ldots, d + 1$) such that every $d$-dimensional simplex with exactly one vertex in each $Q_i$ contains $O$, and at least $\beta_d n^{d+1}$ of them belong to $\mathcal{S}$.
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**References**


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