Small triangular containers for triangles
—On a problem of Nandakumar

Gergely Kiss and János Pach*

To find a minimum area ellipse (Löwner-John ellipse), triangle, rectangle, or convex k-gon enclosing a given point set are classical problems in geometry with interesting applications in packing and covering, approximation, convexity, computational geometry, robotics, and elsewhere [2], [8], [4], [1]. Recently, R. Nandakumar, a gifted programmer, amateur mathematician, and college teacher from Kochi, who runs an exciting blog on mathematical problems, raised an interesting special instance of this problem, which is not trivial even when we want to enclose a triangle by a triangle [6]: *Determine the smallest area isosceles triangle containing a given triangle ABC.*

Nandakumar defined three special isosceles triangles associated with a triangle ABC, as follows. Denote the lengths of the sides by $a = |BC|$, $b = |AC|$, and $c = |AB|$. If two sides coincide, then ABC is is the smallest enclosing isosceles triangle of itself. In the sequel, we assume without loss of generality that $a < b < c$. Let B' denote the point on the ray $\vec{BC}$, for which $|B'C| = b$. See Fig. 1. Analogously, let C' (and C'') denote the points on $\vec{AC}$ (resp., $\vec{BC}$) with $|AC'| = c$ (resp., $|BC''| = c$). Obviously, the triangles $AB'C$, ABC', and ABC'' are *isosceles*. We call them *special containers* associated with ABC. All of them share an angle with ABC. Nandakumar suggested that for every triangle ABC, one of the three special containers associated with it is a smallest area isosceles triangle. If this were true, it would be very easy to find a smallest "container", that is, a smallest area isosceles triangle containing ABC. (It turns out that for "most" triangles ABC, apart from a set of measure 0, the smallest container is uniquely determined.)

Here, we show that the situation is more delicate.

Proposition 1. *For every $\gamma > \pi/2$, there exists a triangle ABC with largest angle γ such that none of the special containers is a smallest area isosceles container for ABC.*

Proof. Let $\gamma > \pi/2$ and, using the above notation, consider an "almost isosceles" triangle ABC, such that its largest angle (at C) is γ and b is only slightly larger than a. Let R be the unique point on the line BC such that $|AR| = |BR|$ (see Fig. 2). If $b - a$ is sufficiently small, then $\sphericalangle ARB > \pi/2$. Let $AB'C$ denote the special container defined above. We have $|AR| = |BR| < |AC| = |B'C|$. The altitudes of the triangles $AB'C$ and ABR belonging to the sides $B'C$ and BR, respectively, are the same. Therefore, the area of ABR is strictly smaller than the area of $AB'C$, showing that $AB'C$ cannot be a smallest area isosceles container. On the other hand, if $b - a$ was small enough, the areas of the other two special containers, ABC' and ABC'', are even larger than the area of $AB'C$. This means that none of the special containers are minimal. $\square$

*Alfréd Rényi Institute of Mathematics, 1364 Budapest, POB 127, Hungary. Email: pach@renyi.hu, kigergo57@gmail.com

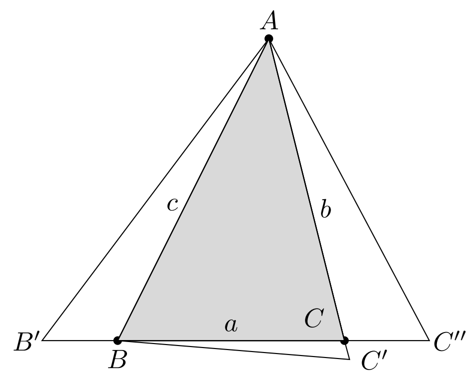

Figure 1: Special containers $AB'C$, ABC', and ABC''.

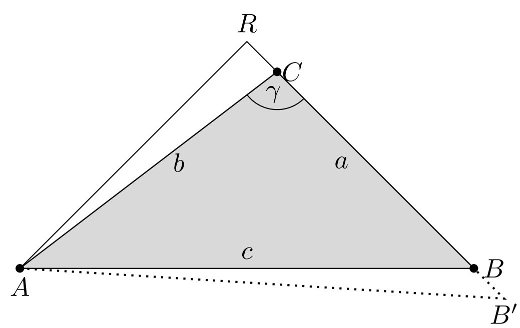

Figure 2: Obtuse triangle whose smallest area isosceles container is not special.

Clearly, all special containers of an acute triangle are acute. Next, we show that in some cases none of these special containers can be minimal.

Proposition 2. *There exists an acute triangle ABC contained in an obtuse isosceles triangle whose area is smaller than the area of any special container associated with ABC.*

Proof. Start with an almost isosceles triangle ABC such that b is only slightly larger than a, and the angle at C is $\pi/2$. Then c is close to $\sqrt{2}b$. Let D denote the point on the ray $\vec{AB}$, different from A, at distance b from C; see Fig. 3. As before, let $AB'C$ be the special container with $|B'C| = b$.

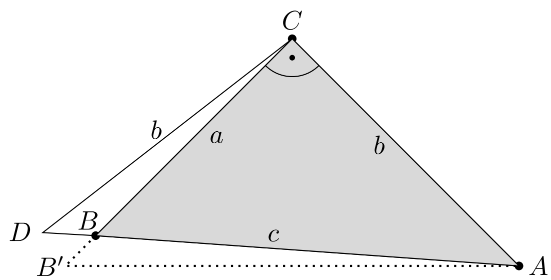

Figure 3: Acute triangle with an obtuse container smaller than the special containers.

Since $\sphericalangle ACD > \sphericalangle ACB' = \pi/2$ and $|CD| = |CB'| = b$, the area of the special container $AB'C$ is slightly larger than the area of the triangle ACD. The areas of the other two special containers, ABC' and ABC'', are even larger (roughly $\sqrt{2}$ times larger).

Keep A and B fixed, and continuously move C away from A without changing the direction of AC. Then ABC becomes an acute triangle, and the point D at distance b from C continuously moves away from B. At the beginning of the motion, $\sphericalangle ACD > \pi/2$ and the area of the isosceles triangle ACD is still smaller than the area of the special containers associated with ABC. Thus, ABC meets the requirements of the Proposition. $\square$

However, in a forthcoming paper [5], under an additional assumption, we shall verify Nandakumar's conjecture.

Theorem 3. [5] *Suppose that a triangle ABC has an acute smallest area isosceles container. Then this container must be identical with one of the special containers associated to ABC.*

Consider now a triangle ABC with an obtuse isosceles container that satisfies the conditions in Proposition 2. It follows immediately from Theorem 3 that all smallest area isosceles containers of ABC must be obtuse or right-angled.

Although, it might happen that none of the special containers is a minimal container, a slightly weaker conjecture of Nandakumar is still true.

Theorem 4. [5] *Any triangle and any of its smallest area isosceles containers share a vertex and the angle at this vertex.*

The proof of this fact requires a surprising amount of work. It builds on a result of Post [7], according to which if a triangle S contains another triangle T, then it also contains a congruent copy of T whose one side lies on a side of S. This result was also used by Jerrard and Wetzel [3] to determine the size of the smallest equilateral triangle that contains a given triangle T.

It is not difficult to construct triangles for which the *smallest area* and the *smallest perimeter* isosceles containers are not the same [6]. Nevertheless, an analogue of Theorem 4 is true for smallest perimeter containers.

Theorem 5. *Any triangle and any of its smallest perimeter isosceles containers share a vertex and the angle at this vertex.*

We do not know if the analogue of Theorem 3 is true for smallest perimeter containers.

Question 6. Is it true that if a triangle ABC has a smallest perimeter isosceles container which is acute, then this container must be identical to one of the special containers associated to ABC?

Instead of smallest area (or perimeter) isosceles containers, we could also try to find a maximum area (perimeter, resp.) isosceles triangle inscribed in a given triangle ABC.

References

[1] J. E. Boyce, D. P. Dobkin, R. L. Drysdale, and L. J. Guibas, Finding extremal polygons, *SIAM J. Computing* **14** (1985), 134–147.

[2] L. Fejes Tóth, *Lagerungen in der Ebene, auf der Kugel und im Raum, Die Grundlehren der Mathematischen Wissenschaften (in German)*, **LXV**, Springer-Verlag, Berlin, 1953.

[3] R. P. Jerrard and J. E. Wetzel, Equilateral triangles and triangles, *American Mathematical Monthly*, **109** (10) (2002), 909–915.

[4] F. John, Extremum problems with inequalities as subsidiary conditions. In: *Studies and Essays presented to R. Courant on his 60th Birthday*, Interscience Publishers, NY, 1948, 187–204.

[5] G. Kiss, J. Pach, and G. Somlai, Minimum area isosceles containers, manuscript, 2019.

[6] R. Nandakumar, Oriented convex container of polygons, https://arxiv.org/ftp/arxiv/papers/1802/1802.10447.pdf

[7] K. A. Post, Triangle in a triangle: On a problem of Steinhaus, *Geometriae Dedicata* **45** (1) (1993), 115–120.

[8] J. O'Rourke, Finding minimal enclosing boxes, *Internat. J. Computer and Information Sc.* **14** (3) (1985), 183–199.