Geometry Seminar
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Room 613, Courant Institute
251 Mercer Street, New York

Hyperbolic van der Waerden and Valiant-Schrijver Conjectures

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Abstract

Let $p(x_1, ..., x_n) = p(X), X \in R^n$ be a homogeneous polynomial of degree n in n real variables, $e = (1, 1, .., 1) \in R^n$ be a vector of all ones. Such polynomial p is called e-hyperbolic if for all real vectors $X \in R^n$ the univariate polynomial equation $P(te - X) = 0$ has all real roots $\lambda_1(X) \geq ... \geq \lambda_n(X)$. The number of nonzero roots $|\{i : \lambda_i(X) \neq 0\}|$ is called $Rank_p(X)$. An e-hyperbolic polynomial p is called P-hyperbolic if roots of vectors $X \in R^n_+$ with nonnegative coordinates are also nonnegative (the orthant R^n_+ belongs to the hyperbolic cone) and $p(e) > 0$. Below $\{e_1, ..., e_n\}$ stands for the canonical orthogonal basis in R^n

Theorem 1 *1. Let $p(x_1, x_2, ..., x_n)$ be a P-hyperbolic (homogeneous) polynomial of degree n ; $Rank_p(e_i) = R_i$. Define $G_i = \min(R_i, n+1-i)$ and assume that*

$$p(x_1, x_2, ..., x_n) \geq \prod_{1 \leq i \leq n} x_i ; x_i > 0, 1 \leq i \leq n.$$

Then the following inequality holds

$$\frac{\partial^n}{\partial x_1 ... \partial x_n} p(0, ..., 0) \geq \prod_{1 \leq i \leq n} \left(\frac{G_i - 1}{G_i}\right)^{G_i - 1} \tag{1}$$

2. If $Rank_p(e_i) \leq k \leq n$ then

$$\frac{\partial^n}{\partial x_1 ... \partial x_n} p(0, ..., 0) \geq \left(\frac{k-1}{k}\right)^{(k-1)(n-k)} \frac{k!}{k^k} \tag{2}$$

This theorem is a vast generalization of as the van der Waerden conjecture on the permanents of doubly stochastic matrices as well of Schrijver-Valiant conjecture on the number of perfect matchings in k-regular bipartite graphs. These two famous powerful results correspond to the polynomials being products of linear forms. Our proof is relatively simple and "noncomputational."

For further information contact {pach,pollack}@cims.nyu.edu, or visit our website: http://www.math.nyu.edu/seminars/geometry_seminar.html