Each problem is worth 10 points. Hence 30 points is full credit. But you can still get 5 more points (for a total of 35 out of 30) if you also get the bonus.

1

(i). Anything in the nullspace of the reduced form is also in the nullspace of \( A \). For example, the vector \[
\begin{pmatrix}
2 \\
-1 \\
0 \\
0
\end{pmatrix}
\].

(ii). Since it is reduced, we can’t know what any of the original columns were. The only vector which we know is in the column space is \[
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]. (This is the only possible correct answer.)

2

If \( x \in \text{nul}(B) \), then \( Bx = 0 \) so also \( ABx = 0 \) and \( x \in \text{nul}(AB) \). If \( x \in \text{nul}(AB) \), then \( ABx = 0 \Rightarrow Bx = 0 \Rightarrow x \in \text{nul}(B) \). So anything in \( \text{nul}(AB) \) is in \( \text{nul}(B) \) and vice versa. That is, \( \text{nul}(AB) = \text{nul}(B) \). Also, it is a theorem that 
\[
\dim(\text{nul}(B)) + \text{rank}(B) = \#\text{cols}(B),
\]
so \( \text{rank}(B) = \#\text{cols}(B) - \dim(\text{nul}(B)) = 3 - \dim(\text{nul}(AB)) = 3 - 2 = 1. \) (So the answer is 1.)

3

Let \( r_i \) denote the \( i \)th row of \( A \). Then we can write
\[
A = \begin{pmatrix}
r_1 \\
r_2 \\
r_3 \\
r_4
\end{pmatrix}
\quad \text{and} \quad
B = \begin{pmatrix}
r_1 \\
r_3 \\
r_2 \\
r_2 - r_4
\end{pmatrix}.
\]

So to get from \( A \) to \( B \) we can (1) switch rows \( r_2 \) and \( r_3 \) [mult the det by -1], then (2) mult \( r_4 \) by -1 [mult the det by -1 again], and finally (3) add the new third row \( (r_2) \) to the new fourth row \( (-r_4) \) [no change to det]. In sum, we end up with \( \det(B) = \det(A) = -7. \) [Grader: don’t give full credit if they just guessed the correct answer; the students must show work to justify it.]