Lyapunov Exponents For Some Quasi-Periodic Cocycles

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Abstract. We consider $SL(2,\mathbb{R})$-valued cocycles over rotations of the circle and prove that they are likely to have Lyapunov exponents $\approx \pm\log\lambda$ if the norms of all of the matrices are $\approx \lambda$. This is proved for λ sufficiently large. The ubiquity of elliptic behavior is also observed.

Consider an area preserving diffeomorphism f of a compact surface. Assume that f is not uniformly hyperbolic, but that it has obvious hyperbolicity properties on a large part of phase space. We are interested in whether or not f has positive Lyapunov exponents on a positive measure set. A widely shared belief among workers in the subject is that positive exponents are quite prevalent, and numerical evidence seems to substantiate that view. Yet so far little has been proved beyond systems with continuous families of invariant cones.

One reason why it is hard to obtain a lower bound for $\|Df_x^n\|$ is that even when both Df_x^n and $Df_{f^nx}^m$ are strongly hyperbolic matrices, it can happen that $\|Df_x^{n+m}\| \ll \|Df_{f^nx}^m\| \cdot \|Df_x^n\|$. This phenomenon has been used to prove the C^1 genericity of zero exponents (see [M]). It is also related to the fact that small perturbations near homoclinic tangencies can produce elliptic periodic orbits (see [N]). Elliptic behavior, when present, further complicates the task of proving positive exponents.

In this paper we consider a model problem. Let $T : (X,m) \circlearrowleft$ be a measure preserving transformation, and let $A : X \to GL(n,\mathbb{R})$ be an arbitrary mapping. We are interested in the Lyapunov exponents of $\cdots A(T^2x)\cdot A(Tx)\cdot A(x)$ for $m-$a.e. x. Abusing language slightly we call $(T;A)$ a *cocycle*. Sometimes we will also refer to A as a cocycle over T. Clearly, the cocycle setting is more general; it includes among other things random matrices and diffeomorphisms. It is easier, however, to work with the space of cocycles than the space of diffeomorphisms, because for cocycles one can vary the dynamics and matrix maps independently.

Our theorems imply the following picture. *Let $\lambda \in \mathbb{R}$ be a large number. We consider a 2-parameter family of cocycles $(T_\alpha; A_t)$ where $T_\alpha : S^1 \circlearrowleft$ is rotation by $2\pi\alpha$ and $\{A_t\}$ is a generic C^1 family of maps from S^1 to $SL(2,\mathbb{R})$ satisfying $||A_t(x)|| \approx \lambda$ for all x, t. For simplicity let us rule out the possibility that for some open set in t-space $(T_\alpha; A_t)$ is uniformly hyperbolic for every α. Then*

(1) *the set of parameters (α, t) for which the Lyapunov exponents of $(T_\alpha; A_t)$ are $\approx \pm\log\lambda$ has nearly full measure;*

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(2) *the closure of the set of (α, t) where $\alpha \in \mathbb{Q}$ and $(T_\alpha; A_t)$ has some elliptic behavior also has nearly full measure.*

Both sets tend to full measure as $\lambda \to \infty$. Precise formulations of these results are given in Section 1.

In broad outline there is much in common between our proofs of (1) and those in [J] and [BC], particularly the latter. We consider a 1-parameter family of cocycles, inductively identify certain regions of criticality, study orbit segments that begin and end near these regions, and try to concatenate long blocks of matrices that have been shown to be hyperbolic. Parameters are deleted to ensure the hyperbolicity of the concatenated blocks, and the induction moves forward. The idea of inductively constructing a "critical set" was first used in [BC].

Among the many known results on the positivity of Lyapunov exponents, we mention in particular those for random matrices (see e.g. [F]), the Schrödinger operator in 1-dimension (see e.g. [FSW], [Ko] and [S]), results using the analytic techniques of Herman (see [H] and e.g. [Kn], [SS]), and those for dynamical systems for which invariant cones have been identified (see e.g. [W]). See also [You] for some examples of nonuniformly hyperbolic cocycles.

§1 Precise statement of results

This paper is about cocycles in which the norms of the matrices are uniformly large. More precisely, for C, $\lambda \geq 1$, let

$$\begin{aligned} \mathcal{A}_{C,\lambda} := \{ A : S^1 \to SL(2,\mathbb{R}) \text{ s.t. } & A \text{ is a } C^1 \text{ map and} \\ & \text{(i)} \quad C^{-1}\lambda \leq \|A(x)\| \leq C\lambda \qquad \forall x \in S^1 , \\ & \text{(ii)} \quad \|\frac{dA}{dx}\| , \quad \|\frac{dA^{-1}}{dx}\| \leq C\lambda \,\} . \end{aligned}$$

We consider $A \in \mathcal{A}_{C,\lambda}$ where C is thought of as $\mathcal{O}(1)$ and λ is as large as need be. It will be shown in Section (2.1) that if $T : S^1 \circlearrowleft$ is a rotation and $A \in \mathcal{A}_{C,\lambda}$, then $(T; A)$ is equivalent to another cocycle $(T; A')$ where $A' = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) \circ B$ and $\|B^{\pm}\|$, $\|\frac{dB^{\pm}}{dx}\| \leq \mathcal{O}(1)$. Our theorems deal exclusively with cocycles in this canonical form.

For $v \neq 0 \in \mathbb{R}^2$, let $\bar{v} \in \mathbb{P}^1$ denote the projectivization of v. Similarly, if $A : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map with $detA \neq 0$, let $\bar{A} : \mathbb{P}^1 \to \mathbb{P}^1$ be the projectivization of A. We coordinatize $\mathbb{P}^1$ as $\mathbb{P}^1 = [0, \pi]/\{0, \pi\}$ by positively orienting S^1 and letting $\theta = 0$ correspond to $\overline{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$, $\theta = \frac{\pi}{2}$ correspond to $\overline{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$ etc. All the maps in question are assumed to be C^1. Theorems 1 and 2 below give two sets of "typical" conditions under which we have the "correct" Lyapunov exponents.

Theorem 1. *Let $T_\alpha : S^1 \circlearrowleft$ be rotation by $2\pi\alpha$, and consider $B : S^1 \to SL(2,\mathbb{R})$. We define $\beta : S^1 \to \mathbb{P}^1$ by $\beta(x) := \overline{B(x)}^{-1}(\frac{\pi}{2})$ and assume that*
(T1) *$x \mapsto (x, \beta(x))$ is transversal to $\{\theta \equiv 0\}$.*
For $\varepsilon_0 > 0$, let

$$\Delta(\lambda) = \{\alpha \in [0,1] : (T_\alpha; \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) \circ B) \text{ has an exponent } > (1-\varepsilon_0)\log\lambda\}\,.$$

Then $\mathrm{Leb}(\Delta(\lambda)) \to 1$ *as $\lambda \to \infty$.*

Instead of varying the dynamics we can also fix $T : S^1 \circlearrowleft$ and consider a 1-parameter family of maps $\{B_t\}$ from S^1 to $SL(2,\mathbb{R})$. We write $\beta(x,t) = \overline{B_t(x)}^{-1}(\frac{\pi}{2})$. Let $\rho(T)$ denote the rotation number of T, and let $\{p_n/q_n\}$ be the convergents of $\rho(T)$ in its continued fraction expansion.

Theorem 2. *Let $T : S^1 \circlearrowleft$ be s.t. its rotation number satisfies*

$$\sum \frac{\log q_{n+1}}{q_n} < \infty\,,^*$$

and let $\{B_t\}$ be s.t. for each t,
(T1) *$x \mapsto (x, \beta(x,t))$ is transversal to $\{\theta \equiv 0\}$,*
(T2) *$\frac{\partial\beta}{\partial t}/\frac{\partial\beta}{\partial x}$ takes on distinct values at different points in $\beta(\cdot,t)^{-1}(0)$.*
For $\varepsilon_0 > 0$, let

$$\Delta(\lambda) := \{t \in [0,1] : (T; \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) \circ B_t) \text{ has an exponent } > (1-\varepsilon_0)\log\lambda\}\,.$$

Then $\mathrm{Leb}(\Delta(\lambda)) \to 1$ as $\lambda \to \infty$.

In a generic family $\{B_t\}$, we must allow **(T1)** to be violated at some points. This does not concern us because our assertion does not hold on a small measure set of parameters. A similar comment applies to **(T2)**.

Next we turn to the existence of elliptic behavior. Let us say for brevity that $(T;A)$ *has elliptic periodic orbits* if $T^q x = x$ for some q and there is an interval $J \subset [0,2\pi]$ s.t. for every $\theta \in J, \exists x \in S^1$ s.t. $A(T^{q-1}x)\cdots A(Tx)A(x)$ has eigenvalues $e^{\pm 2\pi i\theta}$.

Theorem 3. *Let $\{T_\alpha\}$ be as in Theorem 1 and $\{B_t\}$ be as in Theorem 2. We assume that $\beta(x,t) = 0$ for some (x,t). Let*

$$\Gamma(\lambda) := \{(\alpha;t) : (T_\alpha; \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) \circ B_t) \text{ has elliptic periodic orbits}\}\,.$$

Then Leb (closure $\Gamma(\lambda)) \to 1$ as $\lambda \to \infty$.

We conclude this section with the following remarks.

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*This is the same condition used by Brjuno in [B]. The author is grateful to L. Carleson for pointing out that the Brjuno condition is exactly what is used in her proof of Theorem 2.

Remark 1. If $\beta(x) \neq 0 \ \forall x$, then $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) \circ B$ is uniformly hyperbolic for all large λ. This is true for any $(T;B)$ and is an easy exercise. If $\beta(x)=0$ for some x, then it will follow from our proofs that $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) \circ B$ *cannot* be uniformly hyperbolic for any of the parameters we pick out in Theorems 1 and 2.

Remark 2. Parameter deletions are made only to avoid undesirable interactions between different critical points. They are not needed if there is essentially only one critical point. For instance,

Corollary 1. *Let T be as in Theorem 2, and let $B(x) = R_{\varphi(x)}$, where $R_\theta = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$, and $\varphi : S^1 \to \mathbb{R}/2\pi\mathbb{Z}$ is a diffeomorphism satisfying $\varphi(x)+\pi = \varphi(x+\pi)$ (e.g. $\varphi(x)=x$). Then for all sufficiently large λ, $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) \circ B$ is nonuniformly hyperbolic and has exponents $\approx \pm\log\lambda$.*

The rest of this paper is organized as follows. Section 2 contains preliminaries on 2×2 matrices and a quick review on the dynamics of rotations. Theorem 2 and Corollary 1 are proved in Sections 3 and 4. Theorem 3 and is proved in Section 5. In Section 6 we indicate how our arguments in earlier sections can be modified to give a proof of Theorem 1.

§2 Preliminaries

We declare once and for all that *all the matrices in this paper belong in $SL(2,\mathbb{R})$.*

The following notations will be used throughout. If $\cdots A_{-1}, A_0, A_1, A_2, \cdots$ are matrices, we write $A^n = A_{n-1}\circ\cdots\circ A_0$ and $A^{-n} = A_{-n}^{-1}\circ\cdots\circ A_{-1}^{-1}$. Similarly, if $A : S^1 \to SL(2,\mathbb{R})$ is a cocycle, we write $A^n(x) = A(T^{n-1}x)\cdots A(x)$. If A is a matrix with $\|A\| > 1$, let $s(A)$ and $u(A)$ denote unit vectors in the most contracted and the most expanded directions of A. Note that $s \perp u$ and $As \perp Au$. We will use μ to denote a large number, assumed always to be as large as necessary.

(2.1) Canonical form for a certain class of cocycles

We justify the assertion made in the first paragraph of Section 1.

Proposition 1. *Let $T : S^1 \circlearrowleft$ be a rotation, and let $A : S^1 \to SL(2,\mathbb{R})$ be a C^1 map s.t. for some constants $\lambda > C \geq 1$,*

(i) $\frac{\lambda}{C} \leq \|A(x)\| \leq C\lambda \qquad \forall x \in S^1$;

(ii) $\|\frac{dA}{dx}\|, \|\frac{dA^{-1}}{dx}\| \leq C\lambda$.

Then there exist C^1 maps A', B and $U : S^1 \to SL(2,\mathbb{R})$ s.t.

(a) $A(x) = U(x)A'(x)U^{-1}(T^{-1}x)$,

(b) $A' = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) \circ B$

and (c) $\|B^\pm\|, \|\frac{dB^\pm}{dx}\| \leq C'$ *for some C' depending only on C.*

It follows immediately from (a) that $(T;A)$ and $(T;A')$ have the same Lyapunov exponents.

Proof. Write $A(x) = O_2(x)\left(\begin{smallmatrix} \mu(x) & 0 \\ 0 & \mu(x)^{-1} \end{smallmatrix}\right)O_1(x)$, where $\mu > 1$ and O_1 and O_2 are orthogonal matrices. If we take $B(x) = \left(\begin{smallmatrix} \lambda^{-1}\mu(x) & 0 \\ 0 & \lambda\mu(x)^{-1} \end{smallmatrix}\right)\circ O_1(x)\circ O_2(T^{-1}x)$, and let A' be defined by (b), then (a) will follow with $U(x) = O_2(x)$. The only part of (c) that requires checking is the following. Let $r_i(x)$ be the amount rotated by $O_i(x)$, $i = 1, 2$. We need to argue that $|\frac{dr_i}{dx}| \leq$ some constant that depends only on C.

Fix $x \in S^1$. Then Δr_1 is the angle between $u(A(x + \Delta x))$ and $u(A(x))$, and Δr_2 is the angle between $u(A(x + \Delta x)^{-1})$ and $u(A(x)^{-1})$. Let us assume for definiteness that $\Delta r_2 \geq \Delta r_1$. Let

$$(*) \quad := \quad A(x + \Delta x)u(A(x + \Delta x)) - A(x)u(A(x + \Delta x))\,.$$

Then $|(*)| \leq C\lambda\Delta x$ by (ii). We also have

$$\begin{aligned} \langle (*), u(A(x)^{-1})\rangle &= \mu(x + \Delta x)\sin(\Delta r_2) - \frac{1}{\mu(x)}\sin(\Delta r_1) \\ &\approx \mu(x + \Delta x)\sin(\Delta r_2)\,. \end{aligned}$$

So

$$\frac{\Delta r_2}{\Delta x} \leq \frac{2C\lambda}{\mu(x + \Delta x)} \leq 2C^2\,.$$

□

(2.2) Lemmas

Lemma 1. *Let $A_0, A_1, \cdots, A_{n-1}$ be s.t. $\|A_k\| \leq \mu^{3/2}$ and $\|A^k\| \geq \mu^k$ $\forall k$. Let $s_k = s(A^k)$ and let $\sphericalangle$ denote angle. Then*

(a) $\sphericalangle(s_k, s_n) \leq \mu^{-2k+1}$;

(b) $|A^k s_n| \leq \mu^{-\frac{1}{2}k+1}$.

Proof. Let $u_k = u(A^k)$ and $\theta_k = \sphericalangle(s_k, s_{k+1})$.

(a) We write $s_k = v_1 \oplus v_2$ respecting $s_{k+1} \oplus u_{k+1}$. Then

$$|\sin\theta_k| \cdot |A^{k+1}u_{k+1}| = |A^{k+1}v_2| \leq |A^{k+1}s_k| \leq \mu^{3/2}|A^k s_k|\,.$$

Since s_{k+1} is very near s_k, we have

$$\theta_k \approx |\sin\theta_k| \leq \mu^{3/2}\cdot\mu^{-k}\cdot\mu^{-(k+1)} = \mu^{-2k+\frac{1}{2}}\,.$$

This gives

$$\sphericalangle(s_k, s_n) \leq \sum_{j=k}^{n-1}\theta_j \leq \mu^{-2k+1}$$

for μ sufficiently large.

(b) Write $s_n = v_1 \oplus v_2$ respecting $s_k \oplus u_k$. Then

$$A^k s_n = A^k v_1 + A^k v_2 \,.$$

We have

$$|A^k v_1| \le |A^k s_k| \le \mu^{-k}$$

and

$$|A^k v_2| \le \sphericalangle(s_k, s_n) \cdot |A^k u_k| \le \mu^{-2k+1} \mu^{\frac{3}{2}\,k} \,.$$

□

The next lemma is purely combinatorial.

Lemma 2. *Let $M \ll \mu$ be 2 real numbers. Let $v_0 \in \mathbb{R}$, and define $v_1, \ldots, v_n$ recursively using the formula*

$$v_j = a_{j-1}(v_{j-1} + b_{j-1})$$

where $|b_j| \le M$, and

$$|a_{n-j} \cdots a_{n-1}| \le \begin{cases} \mu^{-2} & \text{for} \quad j = 1, 2, 3 \\ \mu^{-j+2} & \text{for} \quad j > 3 \,. \end{cases}$$

Then

$$\begin{cases} \textit{either} & |v_n| \le 10 M \mu^{-2} \\ \textit{or} & |v_0| \ge \left(\frac{1}{2}\right)^n \mu^{n-2} |v_n| \,. \end{cases}$$

Proof. If $|v_{n-j}| \le M$ for some $j \ge 1$, then

$$\begin{aligned} |v_n| &\le |a_{n-1}| \cdot (|v_{n-1}| + M) \\ &\le |a_{n-1}| \cdot (|a_{n-2}| \cdot (\cdots (|a_{n-j}| \cdot (|v_{n-j}| + M) + M) \cdots) + M) \\ &\le 2M |a_{n-j} \cdots a_{n-1}| + M |a_{n-j+1} \cdots a_{n-1}| + \cdots + M |a_{n-1}| \\ &\le 10 M \mu^{-2} \,. \end{aligned}$$

If $|v_{n-j}| > M \ \forall j \ge 1$, then

$$\begin{aligned} |v_{n-j+1}| &\le |a_{n-j}| \cdot (|v_{n-j}| + M) \\ &= |a_{n-j}| \cdot |v_{n-j}| \cdot \left(1 + \frac{M}{|v_{n-j}|}\right) \le 2 |a_{n-j}| \cdot |v_{n-j}| \,, \end{aligned}$$

giving

$$|v_n| \le 2^n |a_0 \cdots a_{n-1}| \cdot |v_0| \,.$$

□

We will also need the following dual version of Lemma 2.

Lemma 2$'$. *Let M, μ, a_j, b_j be as in Lemma 2. We consider v_0', $v_1', \ldots, v_n'$ where v_j' is given by*

$$v_j' = a_{j-1}v_{j-1}' + b_{j-1}\,.$$

Then

$$\begin{cases} \textit{either} & |v_n' - b_{n-1}| \le 10M\mu^{-2} \\ \textit{or} & |v_0'| \ge \left(\frac{1}{2}\right)^n \mu^{n-2}|v_n' - b_{n-1}|\,. \end{cases}$$

Proof. Let $v_0 = v_0'$, and $v_j = v_j' - b_{j-1}$ for $j \ge 1$. Apply Lemma 2. $\square$

(2.3) Curves of most contracted directions for hyperbolic matrices

Let $B : S^1 \to SL(2,\mathbb{R})$ be a cocycle, and consider $A = \Lambda \circ B$ when Λ is the matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. The projective actions corresponding to these cocycles are given by

$$\Phi = \Phi_\Lambda \circ \Phi_B : S^1 \times \mathbb{P}^1 \circlearrowleft$$

where

$$\Phi_B(x,\theta) = (Tx, \quad \overline{B(x)}\theta)$$

and

$$\Phi_\Lambda(x,\theta) = (x, \bar{\Lambda}\theta)\,.$$

(Recall that $\bar{v}, \bar{\Lambda}$ denote the projectivization of v, Λ etc.) Now suppose that on some interval $I \subset S^1$, $A^n(x)$ is hyperbolic for all $x \in I$. Let $s, u : I \to \mathbb{P}^1$ be the functions

$$s(x) := \overline{s(A^n(x))}\,, \qquad u(x) := \overline{u(A^n(x))}\,.$$

We also define $s', u' : T^n(I) \to \mathbb{P}^1$ by

$$s'(x) := \overline{s(A^{-n}(x))}\,, \qquad u'(x) := \overline{u(A^{-n}(x))}\,.$$

Note that

$$\text{graph(s}') = \Phi^n(\text{graph}(u))$$

and

$$\text{graph(u}') = \Phi^n(\text{graph(s)})\,.$$

The differentiability of the functions s, u, s' and u' follows from the Implicit Function Theorem. More explicitly, let $\hat{\theta}$ be the unit vector corresponding to $\theta \in \mathbb{P}^1$. Then the mapping $g : (x,\theta) \mapsto \frac{\partial}{\partial\theta}|A^n(x)\hat{\theta}| = \frac{1}{|A^n(x)\hat{\theta}|^2}$ is C^1, and $\frac{\partial g}{\partial \theta} \ne 0$ at $g^{-1}(0)$.

The goal of this subsection is to study the slopes of the s- and s'- curves when the sequence $\{A(x), A(Tx), \ldots, A(T^{n-1}x)\}$ is sufficiently hyperbolic. Let us first fix a notion of hyperbolicity that is sufficient for our purposes:

Definition. *Let μ be a large number, and let $n \geq 10$. We say that the sequence of matrices $\{A_0, A_1, \ldots, A_{n-1}\}$ is μ-hyperbolic if*

(i) $\|A_i\| \leq \mu^{3/2} \quad \forall i,$

(ii) $\|A^i\| \geq \mu^i \quad \forall i,$

(iii) $|A^i(s(A^n))| \leq \mu^{-1}$ *for* $i = 1, 2, 3,$

and (i)-(iii) *hold if* $\{A_0, \ldots, A_{n-1}\}$ *is replaced by* $\{A_{n-1}^{-1}, \ldots, A_0^{-1}\}$.

The next lemma plays a central role in our analysis. Let $\beta(x) := \overline{B(x)}^{-1}\left(\frac{\pi}{2}\right)$.

Lemma 3. *Let $I \subset S^1$ be an interval, and let μ be between $\lambda^{2/3}$ and λ, λ sufficiently large. We assume that $\{A(x), A(Tx), \ldots, A(T^{n-1}x)\}$ is μ-hyperbolic $\forall x \in I$. Then*

(a) $|\dfrac{ds}{dx} - \dfrac{d\beta}{dx}| < \mu^{-1} \qquad \forall x \in I;$

(b) $|\dfrac{ds'}{dx}| < \mu^{-1} \qquad \forall x \in T^n(I).$

In what follows, let $D(\cdot)$ denote the derivative of the mapping $(\cdot)$. First we calculate $D\Phi^j$. If $D\Phi_B := \left(\begin{smallmatrix} 1 & 0 \\ c & d \end{smallmatrix}\right)$, then

$$\begin{aligned}(D\Phi)_{(x,\theta)} &= \begin{pmatrix} 1 & 0 \\ 0 & (D\bar{\Lambda})_{\overline{Bx}\theta} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c(x,\theta) & d(x,\theta) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ (D\bar{\Lambda})_{\overline{Bx}\theta} c(x,\theta) & (D\bar{\Lambda})_{\overline{Bx}\theta} d(x,\theta) \end{pmatrix}.\end{aligned}$$

We introduce the following notation. For $(x_0, \theta_0) \in S^1 \times \mathbb{P}^1$, let

$$\begin{aligned} (x_j, \theta_j) &:= \Phi^j(x_0, \theta_0), \quad j = 1, 2, \cdots; \\ c_j &:= c(x_j, \theta_j); \quad d_j := d(x_j, \theta_j); \\ \text{and} \quad e_j &:= (D\overline{Ax_j})_{\theta_j} = (D\bar{\Lambda})_{\overline{Bx_j}\,\theta_j} d_j . \end{aligned}$$

It is easy to verify that for any $v_0 \in \mathbb{R}$,

$$(D\Phi^j)_{(x_0,\theta_0)} \begin{pmatrix} 1 \\ v_0 \end{pmatrix} = \begin{pmatrix} 1 \\ v_j \end{pmatrix}$$

where v_j is defined recursively by

$$v_j = e_{j-1}\left(v_{j-1} + \frac{c_{j-1}}{d_{j-1}}\right).$$

An analogous computation shows that if we iterate backwards, letting $(x_j', \theta_j') = \Phi^{-j}(x_0', \theta_0')$, $D\Phi_B^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ c' & d' \end{smallmatrix}\right)$, $c_j' = c'(x_j', \bar{\Lambda}^{-1}\theta_j'), d_j' = d'(x_j', \bar{\Lambda}^{-1}\theta_j')$, and

$$(D\Phi^{-j})_{(x_0',\theta_0')} \begin{pmatrix} 1 \\ v_0' \end{pmatrix} = \begin{pmatrix} 1 \\ v_j' \end{pmatrix},$$

then v'_j is given by the recursive formula

$$v'_j = e'_{j-1} v'_{j-1} + c'_{j-1}$$

where $e'_{j-1} = D\overline{(A(x'_j)^{-1})}\theta'_{j-1}$.

Proof of Lemma 3. We let $x_0 \in I$, $\theta_0 = u(x_0)$, $v_0 = \frac{du}{dx}(x_0)$, and let v_j be given by $(D\Phi^j)_{(x_0,\theta_0)} \begin{pmatrix} 1 \\ v_0 \end{pmatrix} = \begin{pmatrix} 1 \\ v_j \end{pmatrix}$. We wish to apply Lemma 2, so let us first verify its hypotheses. The recursive formula for v_j as computed above clearly has the right form. Since c_j and d_j depend only on the cocycle B, and $d_j \neq 0$, we may assume that $c_j/d_j \leq$ some M which is very small compared to λ or μ. Recall that for any linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$ and $w \in \mathbb{R}^2$ with $|w| = 1$, $|(D\bar{L})_{\bar{w}}| = \frac{1}{|Lw|^2}$. This together with Lemma 1(b) give $|a_{n-j} \cdots a_{n-1}| \leq \mu^{-j+2}$. The μ^{-2} estimate is part of the definition of μ-hyperbolicty. Lemma 2 therefore tells us that

$$\left.\begin{array}{rl} \text{either} & \dfrac{ds'}{dx}(x_n) \approx 0 \\ \text{or} & \left|\dfrac{du}{dx}(x_0)\right| \gg \left|\dfrac{ds'}{dx}(x_n)\right| \end{array}\right\} . \qquad (*)$$

Next we consider $x'_0 \in T^n I$, $\theta'_0 = u'(x'_0)$, $v'_0 = \frac{du'}{dx}(x'_0)$, and apply Lemma 2$'$ to $v'_0, v'_1, \ldots, v'_n$. We claim that b_{n-1} in Lemma 2$'$ satisfies $|b_{n-1} - \frac{d\beta}{dx}(x'_n)| \leq \mathcal{O}(\mu^{-2})$. To see this, first recall that by definition, $b_{n-1} = c'_{n-1}$ above, i.e., b_{n-1} is defined by

$$(D\Phi_B^{-1})_{(x'_{n-1}, \bar{\Lambda}^{-1}\theta'_{n-1})} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ b_{n-1} \end{pmatrix} .$$

On the other hand,

$$(D\Phi_B^{-1})_{(x'_{n-1}, \frac{\pi}{2})} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ \dfrac{d\beta}{dx}(x'_n) \end{pmatrix} .$$

To estimate $|b_{n-1} - \frac{d\beta}{dx}(x'_n)|$, first note that $|\bar{\Lambda}^{-1}\theta'_{n-1} - \frac{\pi}{2}| <$ const $\cdot\, \mu^{-2}$. This is because $\theta'_n = \overline{s(A^n(x'_n))}$, and by hypothesis, $|A(x'_n)s(A^n(x'_n))| < \mu^{-1}$, which forces $\overline{B(x'_n)}\theta'_n$ to be very near $\frac{\pi}{2}$. Next observe that by the Lipschitzness of $x \mapsto B(x)^{-1}$, we have $\forall x \in S^1$ and $\forall v$ with $|v| = 1$,

$$\begin{aligned} \left| \left(\frac{d}{d\theta} \overline{B(x)^{-1}} \right)_{\bar{v}} - \left(\frac{d}{d\theta} \overline{B(x+\Delta x)^{-1}} \right)_{\bar{v}} \right| &= \left| |B(x)^{-1}v|^{-2} - |B(x+\Delta x)^{-1}v|^{-2} \right| \\ &< \text{const } \cdot \Delta x \, . \end{aligned}$$

This gives $|b_{n-1} - \frac{d\beta}{dx}(x_n')| < \text{const } \cdot |\bar{\Lambda}^{-1}\theta_{n-1}' - \frac{\pi}{2}|$. We may therefore confuse b_{n-1} with $\frac{d\beta}{dx}(x_n')$ in Lemma 2$'$ and conclude that

$$\left.\begin{array}{ll} \text{either} & \left|\dfrac{ds}{dx}(x_n') - \dfrac{d\beta}{dx}(x_n')\right| < \mu^{-1}, \\ \text{or} & \left|\dfrac{du'}{dx}(x_0')\right| \gg \left|\dfrac{ds}{dx}(x_n') - \dfrac{d\beta}{dx}(x_n')\right| \end{array}\right\} . \qquad (**)$$

Suppose now that $\exists\, x_0 \in I$ s.t. $|\frac{ds'}{dx}(x_n)| > \mu^{-1}$. Since $\mu^{-1} > 10M\mu^{-2}$, it follows from the precise statement of (*) that

$$\left|\frac{du}{dx}(x_0)\right| > \frac{1}{2^n}\mu^{n-2}\left|\frac{ds'}{dx}(x_n)\right| .$$

Since u-curves and s-curves are exactly $\frac{\pi}{2}$ apart, we have

$$\left|\frac{ds}{dx}(x_0)\right| = \left|\frac{du}{dx}(x_0)\right| > \frac{1}{2^n}\mu^{n-3} .$$

Assuming that $\frac{1}{2^n}\mu^{n-3} \gg |\frac{d\beta}{dx}(x_0)|$, we have by (**)

$$\left|\frac{du'}{dx}(x_n)\right| \gg \left|\frac{ds}{dx}(x_0) - \frac{d\beta}{dx}(x_0)\right| \gg \left|\frac{ds'}{dx}(x_n)\right| ,$$

contradicting $\frac{du'}{dx}(x_n) = \frac{ds'}{dx}(x_n)$. This proves that $|\frac{ds'}{dx}(x_n)| \le \mu^{-1}$.

To finish let us suppose that for some $x_0' \in T^n(I)$, $|\frac{ds}{dx}(x_n') - \frac{d\beta}{dx}(x_n')| > \mu^{-1}$. This implies by (**) that

$$\left|\frac{du'}{dx}(x_0')\right| \gg \left|\frac{ds}{dx}(x_n') - \frac{d\beta}{dx}(x_n')\right| > \mu^{-1} ,$$

contradicting the conclusion of the last paragraph. $\square$

(2.4) Concatenation of hyperbolic blocks

The purpose of this subsection is to study the following question: Let $\{A_{-m}, \dots, A_{-1}\}$ and $\{A_0 \dots, A_{n-1}\}$ be two μ-hyperbolic blocks. What can we say about $\{A_{-m}, \dots, A_{n-1}\}$?

Lemma 4. *Let $A \in SL(2,\mathbb{R})$ be hyperbolic, and let $w \in \mathbb{R}^2$ be s.t. $\sphericalangle(w, s(A^{-1})) = \theta, 0 < \theta \ll 1$. Then there is a vector v with $Av = \text{const}\cdot w$ and $\frac{|Av|}{|v|} \lesssim \frac{1}{\theta}\cdot||A||^{-1}$.*

Proof. Let $v = \varepsilon u(A) + s(A)$, where $\varepsilon = \frac{1}{\theta}\cdot ||A||^{-2}$. $\square$

Lemma 5. *Let $C \in SL(2,\mathbb{R})$ be s.t. $\|C\| \ge \mu^m$, and let $\{A_0, \dots, A_{n-1}\}$ be a μ-hyperbolic sequence. Suppose that $\sphericalangle(s(C^{-1}), s(A^n)) = 2\theta \ll 1$. Then*

(a) $\|A^n C\| \ge \mu^{m+n} \cdot \theta$;

(b) *if $\theta > \mu^{-1}$, then $||A^k C|| \ge \mu^{m+k} \cdot \theta \;\forall\, 0 < k \le n$;*
if $\theta \approx \mu^{-k_0}$ for $k_0 \ge 1$, then $\forall\, 0 < k \le n$,

$$\|A^k C\| \ge \hat{\mu}^{m+k}$$

for some $\hat{\mu}$ satisfying

$$\log \hat{\mu} \ge (1 - \frac{3k_0}{m}) \log \mu \, .$$

Proof. We give a proof of the second assertion in (b). The other assertions will be proved along the way. We claim that

(i) $\|A^k C\| \ge \mu^{m - \frac{3}{2}k}$ for $0 < k \le k_0$;
(ii) $\|A^k C\| \ge \theta \cdot \mu^{m+k} \approx \mu^{m+k-k_0}$ for $k_0 \le k \le n$.

(i) is obvious from the definition of μ-hyperbolicity. To see (ii) we first use Lemma 1(a) to conclude that $\sphericalangle(s(A^k), s(A^n)) < \mu^{-2k+1} < \theta$, so that $\sphericalangle(s(C^{-1}), s(A^k)) > \theta$. Then we use Lemma 4 to obtain a unit vector v with

$$Cv = \text{const} \cdot s(A^k) \qquad \text{and} \qquad |Cv| \le \frac{1}{\theta} \mu^{-m} \, .$$

We are interested in choosing $\hat{\mu}$ as large as possible subject to the constraint $||A^k C|| \ge \hat{\mu}^{m+k}$. Observe that in both (i) and (ii) above, our worst estimate for $\hat{\mu}$ occurs when $k = k_0$, and that in both cases $\|A^{k_0} C\| \ge \mu^{m - \frac{3}{2} k_0}$. Thus $\hat{\mu}$ defined by

$$\hat{\mu}^{m+k_0} = \mu^{m - \frac{3}{2} k_0}$$

has the desired properties. $\square$

(2.5) Dynamics of rotations

We recall here some elementary facts about rotation numbers and their arithmetic properties. Let $T : S^1 \circlearrowleft$ be rotation by angle $2\pi\alpha$, and let p_n/q_n be the rational approximations to α in its continued fraction expansion. Without loss of generality consider the orbit of $0 \in S^1$. Note that

i) $q_1, q_2, \cdots$ are the times when $T^i 0$ returns closer to 0 than ever before;

ii) if J is the interval with end points 0 and $T^{q_n} 0$, then $J, T(J), \dots, T^{q_{n+1}-1}(J)$ are pairwise disjoint and their union covers more than half of S^1; hence

$$\frac{1}{2q_{n+1}} < |T^{q_n} 0 - 0| < \frac{1}{q_{n+1}} \, .$$

In our proofs, we will need to require that the "bad points" do not return too closely to themselves too soon. A well known condition on α that will guarantee this is the *Diophantine condition*, which says that $\exists\ c, \gamma > 0$ s.t.

$$\left|\frac{p}{q} - \alpha\right| > \frac{c}{q^{2+\gamma}} \qquad \forall\ \frac{p}{q} \in \mathbb{Q}\,.$$

The set of numbers satisfying a Diophantine condition with any γ has full Lebesgue measure.

The condition on α that appears naturally in our proof of Theorem 2 is

$$\sum \frac{\log q_{n+1}}{q_n} < \infty\,.$$

This condition was first used by Brjuno in [B]; see also [Yoc]. Clearly, Brjuno's condition is considerably weaker than the Diophantine conditions, for it allows $T^j 0$ to return to 0 at nearly – though not quite – exponential rates.

§3 Analysis of cocycles with certain "good" properties

In this section we let $T : S^1 \circlearrowleft$ be a rigid rotation whose rotation number $\rho(T)$ satisfies the Brjuno condition

$$\sum \frac{\log q_{n+1}}{q_n} < \infty\,,$$

and let $B : S^1 \to SL(2, \mathbb{R})$ be an arbitrary C^1 map satisfying transversality condition **(T1)**. Let $\varepsilon_0 > 0$ be arbitrarily small but fixed, and let λ be a very large number. We will add an infinite number of conditions labeled **(Pn)** as we go along. The aim of this section is to prove that under these conditions the cocycle $A := \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1}\end{smallmatrix}\right) \circ B$ over T has a Lyapunov exponent $> (1-\varepsilon_0)\log\lambda$. We are not concerned here with how often these conditions hold in any given family of cocycles. That will be the topic of the next section.

(3.1) Getting started

Let $N \in \mathbb{Z}^+$ be sufficiently large depending on $\rho(T)$. The minimum size of λ depends on ε_0, $\rho(T)$ and N, and is assumed to be considerably greater than the C^1 norms of B and B^{-1}.

Recall that $\beta(x) := \overline{B(x)}^{-1}\left(\frac{\pi}{2}\right)$. Let $\beta^{-1}\{0\} = \{c_1, \ldots, c_k\} \subset S^1$. We will denote this set by $\mathcal{C}$ or $\mathcal{C}^{(N)}$ and call it the *initial approximation* to our *critical set*. We assume the following about $\mathcal{C}^{(N)}$:

$$\textbf{(PN)}\ \forall\, a, b \in \mathcal{C}^{(N)}, \quad |T^i a - b| > \frac{1}{q_{_N}^2} \quad \textit{for } i = 1, 2, \cdots, q_{_N} - 1.$$

Note that **(PN)** is automatic when $a = b$, because $|T^{q_{N-1}}a - a| > \frac{1}{2q_N} \gg \frac{1}{q_N^2}$, and the next closest return to a is at time q_N.

Let $I_{N,j}$ be the interval of length $\frac{1}{q_N^2}$ centered at c_j, and let $I_N = \cup_j I_{N,j}$. For $x \in I_N$, we let $r_N^+(x)$ be the smallest positive integer j with $T^j x \in I_N$, and let $r_N^-(x)$ be the smallest positive j with $T^{-j}x \in I_N$.

Claim # 1. $r_N^{\pm} \geq q_N$. (Obvious.)

Claim # 2. *$\exists \lambda_N$ with $\lambda_N > \lambda^{1-\frac{1}{2}\varepsilon_0}$ s.t. $\forall x \in I_N$, the sequence $\{A(x), A(Tx), \ldots, A(T^{r_N^+(x)-1}x)\}$ is λ_N-hyperbolic.*

See (2.3) for the definition of λ_N-hyperbolicity. We first prove a simple geometric lemma.

Lemma 6. *Let $\varepsilon > 0$ be small, and let λ be sufficiently large relative to this choice of ε. Let*

$$V = \{\theta \in \mathbb{P}^1 : |\theta - \frac{\pi}{2}| < \lambda^{-2+\varepsilon}\},$$
$$H = \{\theta \in \mathbb{P}^1 : |\theta| < \lambda^{-\frac{1}{2}\varepsilon}\},$$

and

$$I = \{x \in S^1 : \overline{B(x)}H \cap V \neq \phi\}.$$

Then for all $x \in S^1$,

$$Tx, \ldots, T^{n-1}x \notin I \Rightarrow \|A^n x\| \gtrsim \lambda^{(1-\varepsilon)n}.$$

Proof. Let $\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, and let $v = \begin{pmatrix} \lambda^{-2+\varepsilon} \\ 1 \end{pmatrix}$. Then $\Lambda v = \begin{pmatrix} \lambda^{-1+\varepsilon} \\ \lambda^{-1} \end{pmatrix}$, and $\frac{|\Lambda v|}{|v|} \approx \lambda^{-1+\varepsilon}$. This shows that for all v with $\bar{v} \in V$, $|\Lambda v| \lesssim \lambda^{-1+\varepsilon}|v|$. Note also that $\bar{\Lambda}(V) \supset (\mathbb{P}^1 - H)$.

Let x be s.t. $Tx, \ldots, T^{n-1}x \notin I$. We will show that $\exists v$ s.t. $\overline{B(T^j x)A^j(x)v} \in V \quad \forall 0 \leq j < n$. Let $V_0 = \overline{B(x)}^{-1}V$. Then $\overline{A(x)}V_0 \supset (\mathbb{P}^1 - H)$. Since $Tx \notin I$, we are guaranteed that $\overline{B(Tx)}(\overline{A(x)}V_0) \supset V$. Let $V_1 = \overline{A(x)}^{-1}\overline{B(Tx)}^{-1}V$. Then $\phi \neq V_1 \subset V_0$ and $\overline{A^2(x)}V_1 \supset (\mathbb{P}^1 - H)$ etc. $\square$

Proof of Claim # 2. Let $\lambda^{1-\frac{1}{2}\varepsilon_0} < \lambda_N < \lambda^{1-\frac{1}{3}\varepsilon_0}$. We assume λ is large enough that $I \subset I_N$. Condition (i) in the definition of λ_N-hyperbolicity is obvious. For forward iterates, Lemma 6 with $\varepsilon = \frac{1}{4}\varepsilon_0$ gives condition (ii). We contend that (iii) is also obvious: Let $n = r_N^+$, $u = u(A^n(x))$, $s = s(A^n(x))$, and let v be the unit vector we produced in Lemma 6. Then $|A^n(x)v| \leq \lambda_N^{-n}|v|$. Let $v = au + bs$. Since $|A^n(x)u| \geq \lambda_N^n$ and $|A^n(x)v| \leq \lambda_N^{-n}$, we must have $|a| \leq \lambda_N^{-2n}$, i.e. $v \approx s$. Now $\overline{B(T^i x)A^i(x)v}$ is in V for $0 \leq i < n$. Assuming that $n \geq 10$, $\overline{B(T^ix)A^i(x)s}$ is virtually in V for $i = 0, 1, 2$. The arguments for backward iterates are completely analogous.

$\square$

We conclude this start-up step with the following observation. Define $r^{\pm}_{N,j} = \min_{x \in I_{N,j}} r^{\pm}_N(x)$, and for $x \in I_{N,j}$, let

$$s_N(x) := \overline{s(A^{r^+_{N,j}}(x))}\,, \qquad s'_N(x) := \overline{s(A^{-r^-_{N,j}}(x))}\,.$$

Also, let $\varphi_j := \min_{x \in I_{N,j}} |\frac{d\beta}{dx}|$. By **(T1)**, $\varphi_j > 0$.

Claim # 3. *$s_N, s'_N : I_{N,j} \to \mathbb{P}^1$ are C^1 curves with the following properties*:

(a) $|s_N - \beta|$, $|s'_N| < \lambda_N^{-1}$;
(b) $|\frac{ds_N}{dx}| > \varphi_j - \lambda_N^{-1}$, $\quad |\frac{ds'_N}{dx}| < \lambda_N^{-1}$;
(c) graph(s_N) intersects graph(s'_N) in exactly one point; this point sits over some $c_j^{(N+1)} \in S^1$ with $|c_j^{(N+1)} - c_j| < \text{const} \cdot \lambda_N^{-1}$.

Proof. Since $\overline{s(A(x))} \approx \beta(x)$ and $\overline{s(A(x)^{-1})} \approx 0$, (a) is a consequence of Lemma 1(a) and Lemma 6. (b) follows from Lemma 3 and Claim # 2. (c) follows from (a) and (b); the constant depends only on φ_j. $\square$

We will assume that $|c_j^{(N+1)} - c_j| \ll \frac{1}{q_N^2} \quad \forall j$.

(3.2) The induction

We assume that we have inherited from previous steps the following picture: for each i with $N \leq i < n$, there is a set $\mathcal{C}^{(i)} := \{c_1^{(i)}, \dots, c_k^{(i)}\}$, which we regard as the ith approximation to the critical set. Each $\mathcal{C}^{(i)}$ is assumed to satisfy a condition called **(Pi)**. Centered at each $c_j^{(i)}$ is an interval $I_{i,j}$ of length $\frac{1}{q_i^2}$. For $x \in I_i := \cup_j I_{i,j}$, the first return time to I_i, denoted $r_i^+(x)$, is $\geq q_i$ and the sequence of matrices $\{A(x), \dots, A(T^{r_i^+(x)-1}(x))\}$ is λ_i-hyperbolic for some $\lambda_i > \lambda^{1-\varepsilon_0}$. The same is true for backward iterates. Moreover, if $r^{\pm}_{i,j} = \min_{x \in I_{i,j}} r_i^{\pm}(x)$, then on each $I_{i,j}$, the functions $s_i(x) := \overline{s(A^{r^+_{i,j}}(x))}$ and $s'_i(x) =: \overline{s(A^{-r^-_{i,j}}(x))}$ have the property that $|\frac{ds_i}{dx}| \gtrsim \varphi_j$ and $\frac{ds'_i}{dx} \approx 0$. Their graphs intersect over some point very near the middle of the interval $I_{i,j}$.

We now try to push the induction forward to $i = n$. Let $c_j^{(n)} \in I_{n-1,j}$ be the point over which the graphs of s_{n-1} and s'_{n-1} meet. We shall impose the following condition on $\mathcal{C}^{(n)} := \{c_1^{(n)}, \dots, c_k^{(n)}\}$.

$$\textbf{(Pn)}\ \forall\, a, b \in \mathcal{C}^{(n)}, |T^i a - b| > \frac{1}{q_n^2} \text{ for all } i \text{ with } q_{n-1} \leq i < q_n.$$

We observe again that **(Pn)** is automatic for $a = b$. Let $I_{n,j}$ be the interval of length $\frac{1}{q_n^2}$ centered at $c_j^{(n)}$, and define I_n and $r_n^{\pm}$ as before.

Claim # 1′. $r_n^{\pm} \geq q_n$.

Claim # 2′. *Let $x_0, \ldots, x_m$ be a T-orbit with $x_0, x_m \in I_{n-1}$ and $x_i \notin I_n$ $\forall$ $0 < i < m$. Then $\{A(x), \ldots, A(x_{m-1})\}$ is λ_n-hyperbolic where $\lambda_n > \lambda^{1-\varepsilon_0}$ is given by*

$$\log \lambda_n = \log \lambda_{n-1} - 10 \frac{\log q_n}{q_{n-1}} .$$

Proof. Let $\hat{n}$ be the largest number s.t. $x_i \in I_{\hat{n}}$ for some $0 < i < m$. If $\hat{n} < n-1$, then the claim has been proved in a previous step. We may therefore assume that $0 = j_0 < j_1 < \cdots < j_\ell = m$ are the return times of x_0 to I_{n-1}.

We first write down a lower estimate for $\|A^{j_i}(x_0)\|$, $i = 1, \ldots, \ell$. Since

$$\begin{aligned}(*) := \sphericalangle(s(A^{-j_i}(x_{j_i}))\,,\ s(A^{j_{i+1}-j_i}(x_{j_i}))) &\approx |(s'_{n-1}(x_{j_i}) - s_{n-1}(x_{j_i})| \\ &> \varphi_{\min} \cdot d(x_{j_i}, \mathcal{C}^{(n)}) \\ &> \frac{\varphi_{\min}}{q_n^2} ,\end{aligned}$$

where $\varphi_{\min} = \frac{99}{100} \min \varphi_j$, we conclude using our induction hypotheses and Lemma 5(a) that

$$\|A^{j_i}(x_0)\| \geq \hat{\lambda}_n^{j_i} , \qquad i = 1, \ldots, \ell ,$$

where $\hat{\lambda}_n$ is defined by

$$\log \hat{\lambda}_n = \log \lambda_{n-1} - \frac{3 \log\, q_n}{q_{n-1}} .$$

(We have assumed that $\varphi_{\min} \gg \frac{1}{q_N}$. Also, "$\approx$" above requires some justification; we will return to this later.)

To ensure that $\|A^j(x_0)\| \geq \lambda_n^j$ $\forall j$ (condition (ii) in the definition of λ_n-hyperbolicity), we must take into account the dips in the exponent immediately following each return to I_{n-1}. This is estimated using Lemma 5(b). If $(*) > \hat{\lambda}_n^{-1}$, we may take $\lambda_n = \hat{\lambda}_n$. Otherwise k_0 in Lemma 5(b) is $\lesssim \log(q_n^2/\varphi_{\min})/\log \lambda_{n-1}$, and it suffices to choose λ_n s.t.

$$\begin{aligned}\log \lambda_n &\geq \left(1 - \frac{3k_0}{q_{n-1}}\right) \log \hat{\lambda}_n \\ &\geq \log \hat{\lambda}_n - 7 \frac{\log q_n}{q_{n-1}} \geq \log \lambda_{n-1} - 10 \frac{\log q_n}{q_{n-1}} .\end{aligned}$$

Since $\lambda_N > \lambda^{1-\frac{1}{2}\varepsilon_0}$ and $\sum \frac{\log q_n}{q_{n-1}} < \infty$ by assumption, we clearly have $\lambda_n > \lambda^{1-\varepsilon_0}$ if λ is sufficiently large. Condition (i) in the definition of λ_n-hyperbolicity is thus trivial. Condition (iii) is also easy since the difference between $s(A^m(x_0))$ and $s(A^{r_N^+(x_0)}(x_0))$ is insignificant for this purpose.

Returning now to "$\approx$", it should be clear from what we have said that Lemma 1(a) is applicable to give

$$\sphericalangle(s(A^{-j_i}(x_{j_i})),\ s(A^{-r^-_{n-1},\cdot}(x_{j_i})) < \lambda_n^{-2q_{n-1}+1}\,.$$

We also have

$$\sphericalangle(s(A^{j_{i+1}-j_i}(x_{j_i}))\,, s(A^{r^+_{n-1},\cdot}(x_{j_i})) < \lambda_{n-1}^{-2q_{n-1}+1}\,.$$

Our use of "$\approx$" is thus legitimate provided

$$\frac{\log q_n}{q_{n-1}} \ll \log \lambda_n\,.$$

$\square$

As before, let $r^{\pm}_{n,j} = \min_{x\in I_{n,j}} r^{\pm}_n(x)$, and for $x \in I_{n,j}$ define

$$s_n(x) := \overline{s(A^{r^+_{n,j}}(x))}\,, \qquad s'_n(x) := \overline{s(A^{-r^-_{n,j}}(x))}\,.$$

Claim # 3. *On each $I_{n,j}$, we have $|\frac{ds_n}{dx}| \gtrsim \varphi_j$ and $\frac{ds'_n}{dx} \approx 0$, and the graphs of these two functions meet over some point $c_j^{(n+1)}$ with $|c_j^{(n+1)} - c_j^{(n)}| <$* const. $\lambda_n^{-2q_{n-1}+1} \ll |I_{n,j}|$.

Proof. Again use Lemma 1(a) and Lemma 3, both of which depend of course on the conclusion of Claim # 2′ with $m = r^+_{n,j}$. $\square$

The induction process is now complete.

(3.3) Estimation of Lyapunov exponents

¿From the last section it is clear that for each j, $c_j^{(n)} \to$ some $c_j^{(\infty)}$ as $n \to \infty$. Let $\mathcal{C}^{(\infty)} = \{c_1^{(\infty)}, \ldots, c_k^{(\infty)}\}$. This is our *critical set* for (T; A). Also, let $\lambda_\infty = \inf_n \lambda_n$. We assume that $\lambda_\infty > \lambda^{1-\varepsilon_0}$.

We say that $x \in S^1$ has property (*) if

$$\operatorname{dist}(T^j x, \mathcal{C}^{(\infty)}) > \begin{cases} \dfrac{1}{q_N^2} & \text{for } \ 0 \le j < q_N\,, \\[2ex] \dfrac{1}{q_n^2} & \text{for } \ q_{n-1} \le j < q_n, \quad n > N\,. \end{cases}$$

The set of points with property (*) has Lebesgue measure $> 1 - \sum_{n\ge N} \frac{1}{q_n}$, which is positive if N is sufficiently large. Since we know that Lyapunov exponents exist and are constant a.e. on S^1, it suffices to estimate

$$\varlimsup_{n\to\infty} \frac{1}{n} \log \|A^n(x)\|$$

for x with property (*).

We label the trajectory in question as $x_0, x_1, \dots$. Let j_0 be the first time it is in I_N, and let n_0 be s.t. $x_{j_0} \in I_{n_0} - I_{n_0+1}$. In general, let j_i and n_i be defined so that $x_{j_i} \in I_{n_i} - I_{n_i+1}$, and $x_{j_{i+1}}$ is the next return of x_{j_i} to I_{n_i}.

Let us examine the situation at time $j_i, i \geq 1$: First, $j_{i+1} - j_i \geq q_{n_i}$. Second, $\{A(x_{j_i}), \dots, A(x_{j_{i+1}-1})\}$ is λ_∞-hyperbolic by Claim # 2′. Lemma 5(a) then tells us that

$$\|A^{j_{i+1}}(x_0)\| \geq \|A^{j_i}(x_0)\| \cdot \lambda_\infty^{j_{i+1}-j_i} \cdot \varphi_{\min} |I_{n_i+1}| \,.$$

Inductively we obtain for all $k \geq 1$:

$$\|A^{j_k}(x_0)\| \geq \|A^{j_0}(x_0)\|^{-1} \lambda_\infty^{j_k - j_0} \cdot \prod_{i=1}^{k-1} \varphi_{\min} |I_{n_i+1}| \,.$$

Suppose that $q_{n-1} \leq j_k < q_n$. Then property (*) prohibits x_{j_i} from entering I_n for $i < k$. For $m < n$, the maximum number of j_i's with $n_i = m$ is $\leq j_k/q_m$. At each one of these returns, the angle factor is $> \varphi_{\min} \cdot |I_{m+1}| \geq \frac{\varphi_{\min}}{q_{m+1}^2}$. So

$$\begin{aligned} \frac{-1}{j_k} \log \prod_{i<k} \varphi_{\min} \cdot |I_{n_i+1}| &\leq \frac{1}{j_k} \sum_{m=0}^{n-1} \frac{j_k}{q_m} \log \frac{q_{m+1}^2}{\varphi_{\min}} \\ &\leq C \sum_m \frac{\log q_{m+1}}{q_m} \end{aligned}$$

where C is a constant independent of k or λ. This completes our proof.

(3.4) Proof of Corollary 1

Clearly, B satisfies **(T1)**. Note that $\overline{A^j(x+\pi)} = \overline{A^j(x)}\, \forall\, x$ and $\forall\, j$, so that at the nth stage, if $\mathcal{C}^{(n)}$ is defined, it will consist of 2 points $\{c^{(n)}, c^{(n)} + \pi\}$. Now if $|T^j c^{(n)} - (c^{(n)} + \pi)| = \delta$ then $|T^{2j} c^{(n)} - c^{(n)}| = 2\delta$. This proves that if we define $\{I_n\}$ as above, we will automatically have $r_n^\pm \geq \frac{1}{2} q_n$, which is good enough for the rest of this section to work.

It follows from our construction that at $c \in \mathcal{C}^{(\infty)}$, $\exists v$ s.t. $|A^{\pm n}(c)v| \sim \lambda^{-n}|v|$ as $n \to \infty$. Hence $(T; A)$ cannot be uniformly hyperbolic. $\square$

§4 Proof of Theorem 2

In this section we let $\rho(T)$ and $\{B_t\}$ be as in the statement of Theorem 2, and let $\varepsilon_0 > 0$ and λ be given, λ arbitrarily large as usual. Our goal is to estimate the Lebesgue measure of the set of parameters t for which $A_t := \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} B_t$ has properties **(Pn)**, $n \geq N$.

(4.1) Velocities of critical points as parameter varies

Let us first fix $x \in S^1$ and assume that for all t in some parameter interval ω, $\{A_t(x), \ldots, A_t(T^{n-1}x)\}$ is μ-hyperbolic for some large μ. We consider the functions s, u, s', and u' introduced in (2.3), stressing here their dependence on t. That is, we define

$$s(x,t) := \overline{s(A_t^n(x))}\,,$$
$$s'(T^n x,t) := \overline{s(A_t^{-n}(T^n x))} \qquad \text{etc.}$$

Let $\Psi_i : \omega \times \mathbb{P}^1 \to \omega \times \mathbb{P}^1$ be defined by

$$\Psi_i(t,\theta) = (t,\ \overline{A_t(T^i x)}\theta)\,, \qquad i = 0, 1, \ldots,\ n-1\,.$$

Observe that if $(t,\theta_i) = \Psi_{i-1} \circ \cdots \circ \Psi_0(t,\theta_0)$, then $(D\Psi_i)_{(t,\theta_i)}$ has exactly the same form as $(D\Phi)_{(x_i,\theta_i)}$, where Φ is the map considered in (2.3), and c_j in (2.3) is replaced by $\tilde{c}_j := \frac{\partial \overline{B_t(T^i x)}\theta}{\partial t}|_{(t,\theta_j)}$.

Assuming that $\tilde{c}_j$, $\tilde{c}_j/d_j \ll \mu$, the situation here is indistinguishable from that in (2.3). We therefore have the following version of Lemma 3:

Lemma 7. *Let $x \in S^1$ be fixed, and assume that for all t in some interval ω, $\{A_t(x), \ldots, A_t(T^{n-1}x)\}$ is μ-hyperbolic for some μ between $\lambda^{2/3}$ and λ. Then $\forall t \in \omega$,*

$$\text{(a)} \qquad \left|\frac{\partial s}{\partial t}(x,t) - \frac{\partial \beta}{\partial t}(x,t)\right| < \mu^{-1}\,;$$
$$\text{(b)} \qquad \left|\frac{\partial s'}{\partial t}(T^n x,t)\right| < \mu^{-1}\,.$$

Suppose now that for every t in some parameter interval ω, A_t satisfies **(Pi)** for $N \le i < n$, so that $\mathcal{C}^{(n)}(t) := \{c_1^{(n)}(t), \ldots, c_k^{(n)}(t)\}$ is well defined. We further assume that for each $j, r^{\pm}_{n-1,j}$ is constant for all $t \in \omega$. We want to estimate $\frac{dc_j^{(n)}(t)}{dt}$ for $j = 1, \ldots, k$. For simplicity of notation let us omit mention of n and j, and assume that for $t \in \omega, c(t) \approx$ some c, $\frac{\partial \beta}{\partial t} \approx \xi$, and $\frac{\partial \beta}{\partial x} \approx \varphi$. Our first transversality condition **(T1)** guarantees that $\varphi \neq 0$.

Lemma 8. $\frac{dc}{dt} \approx \frac{\xi}{\varphi}$.

Proof. First we freeze t and consider the picture in (x,θ)-space one t at a time. Let $s = s_{n-1}$ and $s' = s'_{n-1}$ be the functions of most contracted directions defined on I_n. (See (3.2).) We then imagine this picture moving with t.

¿From Lemma 3 we know that for each t, s' is a C^1 curve with $\frac{ds'}{dx} \approx 0$. From Lemma 7 we know that as $t \uparrow$, this curve moves pointwise up or down with a speed ≈ 0.

As for the s-curve, we know that for each t, $\frac{ds}{dx} \approx \varphi \neq 0$, and that as $t \uparrow$, this curve moves pointwise up (or down) with a speed $\approx \xi$ (which may or may not be ≈ 0).

If $p(t) \in S^1$ is the point over which the graphs of $s(\cdot, t)$ and $s'(\cdot, t)$ meet, then $|s(p(t_0), t_0 + \Delta t) - s(p(t_0), t_0)| \approx \xi \cdot \Delta t$. This is approximately equal to φ times $|p(t_0 + \Delta t) - p(t_0)| = \Delta c$. Hence $\xi \cdot \Delta t \approx \varphi \cdot \Delta c$. $\square$

Our next lemma is the main result of this subsection. It will be used many times. For $a, b \in S^1$, we let $b - a$ denote the distance from a to b moving along S^1 in the positive direction (whereas $|b - a|$ denotes distance along the shorter arc).

Lemma 9. *$\exists\, \gamma_0, \gamma_1 > 0$ s.t. the following holds. Let ω be a parameter interval on which $\mathcal{C}^{(n)}$ is defined and $r^{\pm}_{n-1,j}$ is constant (i.e. it does not depend on t) for every j. Let $a \neq b \in \mathcal{C}$, and let $a^{(n)}(t), b^{(n)}(t) \in \mathcal{C}^{(n)}(t)$ denote the corresponding critical points. Then $\forall\, t \in \omega$,*

$$\gamma_0 \leq \left| \frac{d}{dt}(b^{(n)}(t) - a^{(n)}(t)) \right| \leq \gamma_1 \,.$$

Proof. This is a consequence of Lemma 8 and **(T2)**. $\square$

(4.2) Definition of the "good" parameter set

Let $\mathcal{P}_N$ be a partition of the t-parameter space $[0,1]$ into subintervals of length $\approx \frac{1}{q_N^2}$. We discard $\omega \in \mathcal{P}_N$ if $\exists\, a, b \in \mathcal{C}(t), t \in \omega$, and $j \in (0, q_{_N})$ s.t. $|T^j a - b| < \frac{1}{q_N^2}$. Let $\Delta_N = \cup\{\omega \in \mathcal{P}_N : \omega \text{ is retained}\}$.

Then $\forall\, t \in \Delta_N$, **(PN)** is satisfied and we can in principle define $\mathcal{C}^{(N+1)}$ as in Section 3. We wish, however, to have a more consistent definition of $\mathcal{C}^{(N+1)}$ on each element of $\mathcal{P}_N$. For $\omega \in \mathcal{P}_N$, we let

$$\begin{aligned}\tilde{r}^{+}_{N,j}(\omega) &= \min\{r^{+}_{N,j}(t) : t \in \omega\} \\ &= \min\{i : T^i x \in I_N \text{ for some } x \in I_{N,j} \text{ and } t \in \omega\} \,.\end{aligned}$$

We then let

$$s_N(x) = \overline{s(A^{\tilde{r}^{+}_{N,j}}(x))} \quad \text{and} \quad s'_N(x) = \overline{s(A^{-\tilde{r}^{-}_{N,j}}(x))}$$

and use these functions to define $\mathcal{C}^{(N+1)}$.

Moving on to the next step, we let $\mathcal{P}_{N+1}$ be a refinement of $\mathcal{P}_N | \Delta_N$, subdividing Δ_N into intervals of length $\approx \frac{1}{q_{N+1}^2}$. We then discard $\omega \in \mathcal{P}_{N+1}$ if **(P(N+1))** is violated by some $t \in \omega$, and call the remaining set Δ_{N+1}. For each $t \in \Delta_{N+1}, \mathcal{C}^{(N+2)}$ is defined using $\tilde{r}^{\pm}_{N+1,j}$ as before.

This procedure gives a decreasing sequence of sets $\Delta_N \supset \Delta_{N+1} \supset \dots$ s.t. $\forall\, t \in \Delta_n, (T; A_t)$ satisfies **(Pi)** for $N \leq i \leq n$. Moreover, there is an increasing sequence of partitions $\{\mathcal{P}_n\}, \mathcal{P}_n$ defined on Δ_n, s.t. the definition of $\mathcal{C}^{(n+1)}$ is consistent on each element of $\mathcal{P}_n$.

We remark that Section 3 is not affected by our slightly modified definitions of s_n and s_n'. Suppose that for some $t_0 \in \omega \in \mathcal{P}_n$ and $x_0 \in I_{n,j}, T^{\tilde{r}^+_{n,j}} x_0 \in I_{n,j'}$. Then using Lemma 9 we conclude that $\forall\, t \in \omega$ and $\forall\, x \in I_{n,j}$, the distance between $T^{\tilde{r}^+_{n,j}} x$ and $I_{n,j'}$ is $< (|I_{n,j}| + \gamma_1 \cdot |\omega|) \approx (\gamma_1 + 1) \cdot \frac{1}{q_n^2}$. If N is sufficiently large, then $T^{\tilde{r}^+_{n,j}} x \in I_{n-1}$, so Claim # $2'$ tells us that $\{A(x), \dots, A(T^{\tilde{r}^+_{n,j-1}} x)\}$ is λ_n-hyperbolic, and hence s_n and s_n' have the properties in Claim # $3'$.

(4.3) Estimating the measure of Δ_n

Consider $a, b \in \mathcal{C}$. We say that b is in a bad position relative to a at time $j, 1 \leq j < q_{_N}$, if $|T^j a - b| < \frac{1}{q_{_N}^2}$. Suppose T is rotation by $2\pi\alpha$. Then $T^j a = b$ is equivalent to $b - a = 2\pi\alpha j \bmod 2\pi$. Let us consider the function $\tau_N : [0,1] \to \mathbb{R}$ defined by $\tau_N(t) = b(t) - a(t)$, and delete those t that get mapped to the interval $(2\pi\alpha j \bmod 2\pi) \pm \frac{1}{q_{_N}^2}$. Since $\frac{d\tau_N}{dt} \geq \gamma_0$ (let us assume it is > 0), this t-interval has measure $< \frac{1}{\gamma_0} \frac{2}{q_{_N}^2}$, and the set of $\mathcal{P}_N$-elements deleted on account of this j^{th} iterate has measure $< \frac{c_0}{q_{_N}^2}$ for some c_0. Hence $\mathrm{Leb}(\Delta_N) > 1 - q_{_N} \cdot \frac{K}{q_{_N}^2}$ where $K = 2c_0 \begin{pmatrix} k \\ 2 \end{pmatrix}$.

The same reasoning would lead us to conclude that $\mathrm{Leb}(\Delta_n - \Delta_{n-1}) < \frac{K}{q_n}$ for every n. This is essentially true — except for the following technical problem:

Consider, for instance, $n = N + 1$, and again fix $a, b \in \mathcal{C}$. Define $\tau_{N+1} : \Delta_N \to \mathbb{R}$ by $\tau_{N+1}(t) = b^{(N+1)}(t) - a^{(N+1)}(t)$. Then we again have $\frac{d\tau_{N+1}}{dt} \geq \gamma_0$ on each element of $\mathcal{P}_N$, but τ_{N+1} may (and probably does) have discontinuities caused by our inconsistent definitions of $\mathcal{C}^{(N+1)}$ on different elements of $\mathcal{P}_N$. The fact that τ_{N+1} is not necessarily injective makes it possible for one bad position between a and b at one particular time to correspond to more than one parameter.

We will argue that $\forall\, n \geq N, \tau_{n+1} : \Delta_n \to \mathbb{R}$ is at most $2^{(n+1)-N}$ to 1. To see this, consider $\omega \in \mathcal{P}_{n-1}$, and let $\omega', \omega'' \subset \omega$ be 2 non-adjacent elements of $\mathcal{P}_n | \Delta_n, \omega'$ to the left of ω''. We claim that

$$\sup\{\tau_m(t) : t \in \omega' \cap \Delta_{m-1}, m > n\} \\ < \inf\{\tau_m(t) : t \in \omega'' \cap \Delta_{m-1}, m > n\}.$$

This is because

$$|a^{(m)}(t) - a^{(n)}(t)|, |b^{(m)}(t) - b^{(n)}(t)| < \lambda_\infty^{-2q_{n-1}+1},$$

$$\inf\{\tau_n(t) : t \in \omega''\} - \sup\{\tau_n(t) : t \in \omega'\} > \gamma_0 \cdot \frac{1}{q_n^2},$$

and

$$\lambda_\infty^{-2q_{n-1}+1} \ll \frac{\gamma_0}{q_n^2}\,.$$

So for arbitrary $c \in \mathbb{R}$ and $n \geq N, \tau_{n+1}^{-1}(c)$ intersects at most two elements ω_1, ω_2 of $\mathcal{P}_N$. Inside each $\omega_i, \tau_{n+1}^{-1}(c)$ intersects at most two elements of $\mathcal{P}_{N+1}$, and so on. Finally, $\tau_{n+1}^{-1}(c)$ intersects each element of $\mathcal{P}_n$ in at most one point. This proves that the cardinality of $\tau_{n+1}^{-1}(c)$ cannot exceed $2^{(n+1)-N}$.

Assuming this worst estimate for the cardinality of $\tau_n^{-1}(c)$ (which in reality happens only for an extremely small set of c), we conclude that $\text{Leb}(\Delta_n - \Delta_{n-1}) < K \cdot 2^{n-N} \cdot q_n \cdot |I_{n,j}|$. So if we change $|I_{n,j}|$ to $\frac{1}{q_n^3}$, we will have

$$\text{Leb}(\cap\Delta_n) > 1 - K \sum_{n \geq N} \frac{2^{n-N}}{q_n^2}\,,$$

which $\to 1$ as $N \to \infty$ because $q_n^2 \geq \left(\frac{1+\sqrt{5}}{2}\right)^{2n} > 2^n$. (This change in $|I_{n,j}|$ must of course be accompanied by changing **(Pn)** to $|T^j a - b| > \frac{1}{q_n^3}$ and changing the size of the elements of $\mathcal{P}_n$ to $\approx \frac{1}{q_n^3}$ etc.)

§5 Proof of Theorem 3

For $\alpha \in (0,1)$, let $\alpha_n = \frac{p_n}{q_n}$ be the convergents of α. Let α satisfy the Brjuno condition $\sum \frac{\log q_{n+1}}{q_n} < \infty$, and let $\{B_t\}$ satisfy the hypotheses of Theorem 2. We fix n for the moment and consider the cocycles $(T_{\alpha_n}; A_t)$ where $A_t = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \circ B_t$, $t \in [0,1]$. We partition up the t-space as in Section (4.2), but with an additional stopping rule which we now describe. This stopping rule will give rise to a collection of intervals which we shall call $\mathcal{P}^{(n)}$.

Let $\mathcal{P}_N$ and Δ_N be as in (4.2), and consider one $\omega \in \mathcal{P}_N|\Delta_N$ at a time. Since $T^{q_n} = Id$, we have $\tilde{r}^{\pm}_{N,j}(\omega) \leq q_n \forall\, j$. If $\tilde{r}^{+}_{N,j}(\omega)$ (or equivalently $\tilde{r}^{-}_{N,j}(\omega)) = q_n$ for some j, then we put $\omega \in \mathcal{P}^{(n)}$ and do not partition it further. The union of the remaining ω is called Δ'_N. On Δ'_N we construct $\mathcal{P}_{N+1}$ as in (4.2), discard those elements that violate **(P(N+1))**, and consider one $\omega \in \mathcal{P}_{N+1}|\Delta_{N+1}$ at a time. Again we put $\omega \in \mathcal{P}^{(n)}$ if $\exists\, j$ s.t. $\tilde{r}^{\pm}_{N+1,j}(\omega) = q_n$. The remaining set, which we call Δ'_{N+1}, is further partitioned and so on.

This procedure must stop at or before the n^{th} step, becuase $\tilde{r}^{\pm}_{n,j}(\omega) = q_n$ for all $\omega \in \mathcal{P}_n$. (We have used implicitly the fact that $\mathcal{C} \neq \phi$ for all t. This is guaranteed by our assumption that $\beta(x,t) = 0$ for some (x,t) and condition **(T1)**.) Let $\Delta^{(n)} := \cup\{\omega : \omega \in \mathcal{P}^{(n)}\}$. We claim that $(T_{\alpha_n}; A_t)$ has elliptic periodic orbits $\forall\, t \in \Delta^{(n)}$.

Attached to each $\omega \in \mathcal{P}^{(n)}$ is a stopping time i. By definition, $\mathcal{C}^{(i+1)}$ is defined $\forall\, t \in \omega$, and $\exists\, j$ s.t. $\tilde{r}^{+}_{i,j}(\omega) = q_n$. We consider one $t \in \omega$ at a time. Let $c \in$

$\mathcal{C}^{(i+1)} \cap I_{i,j}$, so that $s_i(c) = s_i'(c)$. We will show that there is a neighborhood of c on which the eigenvalues ν, $\frac{1}{\nu}$ of A^{q_n} have modulus one, and that $\frac{d}{dx}(\text{Re}\,\nu) \neq 0$.

Consider $x \in I_{i,j}$. For this paragraph only let us abuse notation and use s and s' to denote unit vectors corresponding to the projective coordinates $s_i(x)$ and $s_i'(x)$. Also, let u and u' be unit vectors perpendicular to s and s' respectively. Since $T^{q_n}x = x$, we have $A^{q_n}s =$ const. u' and $A^{q_n}u =$ const. s'. We can therefore write

$$A^{q_n}u = \mu a u + \mu b s\,,$$
$$A^{q_n}s = -\frac{1}{\mu}\,bu + \frac{1}{\mu}\,as\,,$$

where $\mu = \|A^{q_n}\|$, $a = \langle s', u\rangle$, $b = \langle s', s\rangle$. This gives $\text{Re}(\nu) = a\left(\mu + \frac{1}{\mu}\right)$. Since $a(c) = 0$, we have $\nu(c) = \pm i$, so $|\nu(x)| = 1\ \forall x$ near c. Also, since $a \approx 0$ near c while $\frac{da}{dx} \approx \frac{d\beta}{dx}$, which is bounded away from 0, we conclude that for x sufficiently near c, $\frac{d}{dx}\ \text{Re}(\nu) \approx \frac{da}{dx} \cdot \left(\mu + \frac{1}{\mu}\right) \neq 0$.

To finish we need to estimate the measure of $c\ell(\Gamma(\lambda))$, the closure of $\Gamma(\lambda)$. We saw in Section 4 that $\text{Leb}(\Delta^{(n)}) > 1 - K\sum_{n\geq N} \frac{2^n}{q_n^2}$. This lower bound passes on to the set of t s.t. $(\alpha, t) \in c\ell(\Gamma(\lambda))$. Note that N depends only on the arithmetic of α and on $\{B_t\}$. The assertion in Theorem 3 follows. $\square$

§6 Proof of Theorem 1

Since the proof of Theorem 1 is in many respects quite similar to that of Theorem 2, we will be rather sketchy here, emphasizing only the differences between the two proofs.

(6.1) Overall Scheme

Let T_α and B be as in Theorem 1, i.e. $T_\alpha : S^1 \circlearrowleft$ is rotation by $2\pi\alpha$, and $B : S^1 \to SL(2,\mathbb{R})$ satisfies transversality condition **(T1)**. We assume that $N \in \mathbb{Z}^+$ and λ are sufficiently large for our purposes, and let $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1}\end{pmatrix} \circ B$. We consider those α's s.t. with the modifications below the inductive steps in Sections (3.1) and (3.2) can be carried out for $(T_\alpha; A)$.

First, for $n = 0, 1, 2, \ldots$, the critical set at the nth stage is required to satisfy the following condition:

$$\textbf{(P'n)} \quad \forall\, a, b \in \mathcal{C}^{(n)}\,, \quad \text{possibly with} \quad a = b\,,$$
$$|T^i a - b| > \frac{1}{(2^n N)^3} \quad \text{for} \quad 2^{n-1}N \leq i < 2^n N\,.$$

This condition replaces **(Pn)** in Section 3.

Second, $I_{n,j}$, the intervals around $c_j^{(n)}$, are to have lengths $\frac{1}{(2^n N)^3}$.

With these two changes, Claim # 1 becomes $r_n^{\pm} \geq 2^n N$. Claims # 2 and 3 remain essentially unchanged.

The estimation of Lyapunov exponents for $(T; A)$ continues to be valid if property (*) in Section (3.3) is replaced by

$$\operatorname{dist}(T^j x, \mathcal{C}^{(\infty)}) > \frac{1}{(2^n N)^3} \quad \text{if} \quad 2^{n-1} N \leq j < 2^n N\,.$$

In the next two subsections we estimate the set of α for which $(T_{\alpha;} A)$ satisfies **(P′n)** $\forall n$.

(6.2) Velocities of critical points as functions of α

This subsection is parallel to (4.4), which in turn uses the notation of (2.3).

$$\text{Let} \quad \Xi : S^1 \times [0,1] \times \mathbb{P}^1 \circlearrowleft \quad \text{be defined by}$$

$$\Xi(x, \alpha, \theta) = (T_\alpha x,\ \alpha,\ \bar{\Lambda}\,\overline{B(x)}\theta)\,.$$

For $x_0 \in S^1$, let $W_0 = \{x = x_0\} \subset S^1 \times [0,1] \times \mathbb{P}^1$, and let $W_i = \Xi\ W_{i-1}$, $i = 1, \ldots, n$. On each W_i we use the coordinate system (ω, θ) where θ denotes the unit vector in $\mathbb{P}^1$ and ω is a fixed unit vector $\perp\ \theta$. In these coordinates, the maps $\Xi_i := \Xi | W_i : W_i \to W_{i+1}$ have derivatives

$$D\Xi_i = \begin{pmatrix} 1 & 0 \\ 0 & D\bar{\Lambda} \end{pmatrix} \begin{pmatrix} f_i & 0 \\ c_i & d_i \end{pmatrix}$$

where $f_i = \sqrt{1 + (1+i)^2}\,/\,\sqrt{1+i^2}$, c_i is a directional derivative of $\bar{B}$, and d_i is as before. (See (2.3).) For a given v_0, we define $v_1, v_2, \ldots$ by

$$D\Xi_{j-1} \begin{pmatrix} 1 \\ v_{j-1} \end{pmatrix} = \text{const.} \begin{pmatrix} 1 \\ v_j \end{pmatrix}.$$

Then

$$v_j = \frac{\sqrt{1+(j-1)^2}}{\sqrt{1+j^2}}\ e_{j-1} \left(v_{j-1} + \frac{c_{j-1}}{d_{j-1}} \right).$$

A similar estimate holds for Ξ_j^{-1}. All told, Lemma 3 applies and we obtain the following version of Lemma 7:

Lemma 10. *Suppose that for all (x, α) near some $(x_0, \alpha_0) \in S^1 \times [0,1]$, $\{A(x), \ldots, A(T_\alpha^n x)\}$ is μ-hyperbolic for some μ between $\lambda^{2/3}$ and λ. Then*

$$\left| \frac{\partial s}{\partial \alpha}\ (x_0, \alpha_0) \right|\ ,\ \left| \frac{\partial s'}{\partial \alpha}\ (T^n_{\alpha_0} x_0, \alpha_0) \right| < \mu^{-1}\,.$$

Note that $\frac{\partial \beta(x,\alpha)}{\partial \alpha} \equiv 0$ since B is independent of α. To prove Lemma 10, one first applies Lemma 3 to $W_0, \ldots, W_n$, obtaining $|\frac{\partial s}{\partial \alpha}| = |\frac{\partial s}{\partial w}| < \mu^{-1}$ on W_0. To prove the other inequality let $W_0' = \{x \equiv T^n_{\alpha_0} x_0\}$ and $W_j' = \Xi^{-1}\, W_{j-1}'$. Lemma 3 applied to $W_0', \ldots, W_n'$ then gives $|\frac{\partial s'}{\partial \alpha}| = |\frac{\partial s'}{\partial w}| < \mu^{-1}$ on W_0'.

The proof of Lemma 8 can now be repeated verbatim, giving

Lemma 11. *Assume that $\mathcal{C}^{(n)}(\alpha)$ is consistently defined (i.e. using the same return times) for all α in some parameter interval. Then for every $c \in \mathcal{C}^{(n)}$,*

$$\frac{dc}{d\alpha} \approx \frac{\partial \beta}{\partial \alpha} \Big/ \frac{\partial \beta}{\partial x} = 0\,.$$

(6.3) Measure of good parameters

We construct Δ_n and $\mathcal{P}_n$ as in Section (4.2), except that we are now working in α-space. The only difference is that the elements of $\mathcal{P}_n$ should be chosen to have length $\approx \frac{1}{(2^n N)^6}$. The reason for this fine division is that for each $\omega \in \mathcal{P}_n$, we want to be sure that $T_\alpha^{\tilde{r}^{\pm}_{n,j}(\omega)} x \in I_{n-1}$ $\forall x \in I_{n,j}$ and $\forall \alpha \in \omega$. (see the last paragraph of (4.2).) Our choice of $|\omega|$ is motivated by the fact that for $\alpha_1, \alpha_2 \in \omega$,

$$|T_{\alpha_1}^{\tilde{r}^+_{n,j}} x - T_{\alpha_2}^{\tilde{r}^+_{n,j}} x| \leq |\omega| \cdot \tilde{r}^+_{n,j}\,,$$

and $\tilde{r}^+_{n,j}$ could be as large as $(2^n N)^3$.

To estimate the measure deleted on account of $|T^j a - b| < \frac{1}{(2^n N)^3}$ for some $j \in [2^{n-1}N, 2^n N)$, we consider $\tau_{n,j} : \Delta_{n-1} \to \mathbb{R}$ defined by

$$\tau_{n,j}(\alpha) = a^{(n)}(\alpha) - b^{(n)}(\alpha) + 2\pi\alpha j$$

and delete those α that get mapped to $2\pi\mathbb{Z} \pm \frac{1}{(2^n N)^3}$. (c.f. (4.3).) Here, $a^{(n)}(\alpha) - b^{(n)}(\alpha) \approx a - b = \text{const}$, and by Lemma 10, we have $\frac{d\tau_{n,j}}{d\alpha} \approx 2\pi j$ on each element of $\mathcal{P}_{n-1}$. Similar reasoning as before shows that $\tau_{n,j}$ is at most 2^{n-N} to 1. Summing over all pairs of critical points and all $j \geq 1$, we obtain

$$\mathrm{Leb}(\cap \Delta_n) \geq 1 - K \sum_{n \geq N} \frac{2^n}{(2^n N)^2}\,.$$

This completes the proof of Theorem 1.

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