SOME EXAMPLES OF NONUNIFORMLY HYPERBOLIC COCYCLES

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Abstract. We consider some very simple examples of $SL(2,\mathbb{R})$-cocycles and prove that they have positive Lyapunov exponents. These cocycles form an open set in the C^1 topology.

Let $f : (X, m) \circlearrowleft$ be a measure preserving transformation of a probability space, and let $A : X \to SL(2,\mathbb{R})$. With a slight abuse of language we call A an $SL(2,\mathbb{R})$-*cocycle* over the dynamical system (f, m). Let $\lambda_1 \geq \lambda_2$ denote the Lyapunov exponents of $(f, m; A)$. We are interested in whether or not $(f, m; A)$ has nonzero Lyapunov exponents, or equivalently, whether or not $\lambda_1 > 0$ a.e. Because norms of matrices are sub-multiplicative, the problem of estimating λ_1 from below is in general a rather difficult one.

This note is an attempt to add to the existing pool of techniques for proving positive exponents. We consider some very simple cocycles defined over $z \mapsto z^N,\ z \in S^1$, or automorphisms of the 2-torus, and give a positive lower bound for λ_1. These examples can be made C^r for any $r \leq \omega$. If $C^1(X, SL(2,\mathbb{R}))$ denotes the space of C^1 maps from S^1 or $\mathbb{T}^2$ to $SL(2,\mathbb{R})$ endowed with the C^1 topology, then our examples fill up an open set in $C^1(X, SL(2,\mathbb{R}))$ – although none of them is uniformly hyperbolic.

This openness part of our assertion should probably be contrasted with a theorem of Mañé [M], in which he proves that away from Anosov components, the generic C^1 area-preserving diffeomorphism of a compact 2-dimensional surface has zero exponents a.e. Since $x \mapsto Df_x$ is a C^0 cocycle if f is C^1, Mañé's methods suggest that in our setting, the set of non-uniformly hyperbolic A's form a first category set in $C^0(X, SL(2,\mathbb{R}))$.

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Other known examples of nonuniformly hyperbolic cocycles include iid sequences of random matrices (see [F]), matrices arising from Schrödinger operators (see e.g. [S]), examples of Herman [H] (see also [K] and [SS]), derivative cocycles of systems with invariant cones (see e.g. [W]), and derivative cocycles of mappings such as the Hénon mappings (see [BC] and [BY]), etc. Our methods in this paper have some similarity with those in [LY], where we show that Lyapunov exponents are robust under certain stochastic perturbations.

§1 Description of cocycles and statements of results

Perhaps the simplest dynamical system over which to carry out our construction is an expanding map of the circle, so let us first discuss that case. We identify $S^1 \cong \mathbb{R}/\mathbb{Z}$. Let $f : S^1 \circlearrowleft$ be defined by $fx = Nx \mod 1$ where N is a fixed integer ≥ 2, and let m denote the Lebesgue measure on S^1.

We fix $\lambda \geq \sqrt{N+1}$, and let $\beta = \beta(\lambda)$ be a small positive number to be specified later. For $\varepsilon > 0$ we define a cocycle $A_\varepsilon : S^1 \to SL(2,\mathbb{R})$ as follows: Let $J_\varepsilon \subset S^1$ be an interval, and let $\varphi_\varepsilon : S^1 \to \mathbb{R}/2\pi\mathbb{R}$ be a C^1 function such that

(1) $\varphi_\varepsilon \equiv 0$ outside of J_ε;

(2) on J_ε, φ_ε increases monotonically from 0 to 2π;

and (3) on $\varphi_\varepsilon^{-1}[\beta, 2\pi - \beta]$, $\varphi'_\varepsilon \geq \frac{1}{\varepsilon}$.

Our cocyle A_ε is then defined to be

$$A_\varepsilon(x) = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \circ R_{\varphi_\varepsilon(x)}$$

where R_θ denotes rotation by angle θ.

We claim that for sufficiently small $\varepsilon > 0$, $(f, m; A_\varepsilon)$ has a positive Lyapunov exponent m-a.e. We further claim that for each sufficiently small ε, there is a neighborhood $\mathcal{N}_\varepsilon$ of A_ε in $C^1(S^1, SL(2,\mathbb{R}))$ s.t. for all $B \in \mathcal{N}_\varepsilon$, $(f, m; B)$ also has a positive Lyapunov exponent. These claims will be proved in section 2. It will be clear from our estimates that there is in fact a uniform lower bound for the positive exponents of all the cocycles in each $\mathcal{N}_\varepsilon$.

It remains for us to argue that our cocycles can be chosen to be *not* uniformly hyperbolic. Let us define what "uniformly hyperbolic" means, for this is not completely standard for non-invertible maps. For $A : S^1 \to SL(2,\mathbb{R})$, write $A^n(x) = A(f^{n-1}x)\cdots A(x)$, and define $s_n(x) := \min_{|v|=1} |A^n(x)v|$ for $n = 1, 2, \ldots$. We say that (f, A) is *uniformly hyperbolic* if there is $C > 0$ and $\tau \in (0,1)$ s.t. $\forall x \in S^1,\ s_n(x) \le C\tau^n\ \forall n \in \mathbb{Z}^+$.

We mention two different ways to guarantee that our cocycles in $\mathcal{N}_\varepsilon$ are not uniformly hyperbolic. One is to arrange for $B(0)$ to have a complex eigenvalue for all $B \in \mathcal{N}_\varepsilon$. (Note that $0 \in S^1$ is a fixed point of f.) For given λ this is easily done by choosing $\varphi_\varepsilon(0)$ sufficiently near $\frac{\pi}{2}$.

One could also ensure nonuniform hyperbolicity by a global argument. Observe first that if a cocycle (f, A) is uniformly hyperbolic, then at each x there is a 1-dimensional subspace $E^s(x) \subset \mathbb{R}^2$ with the property that $x \mapsto E^s(x)$ is continuous and $A(x)E^s(x) = E^s(fx)$. To see this let $E^s_n(x)$ be the 1-d subspace of $\mathbb{R}^2$ most contracted by $A^n(x)$. Then $x \mapsto E^s_n(x)$ is continuous and E^s_n converges uniformly to some E^s as $n \to \infty$. (See e.g. Ruelle's proof of Oseledec's Theorem [R]). The invariance of E^s follows from the fact that $\angle(A(x)E^s_n(x),\ E^s_{n-1}(fx)) \to 0$.

Returning now to our example A_ε, let $F_\varepsilon : S^1 \times \mathbb{P}^1 \circlearrowleft$ be the projectivization of the second coordinate of the map $(x, v) \mapsto (fx, A_\varepsilon(x)v)$. If (f, A_ε) is uniformly hyperbolic, let α_ε be the curve in $S^1 \times \mathbb{P}^1$ corresponding to $x \mapsto E^s(x)$, and let k be the number of times α_ε winds around $\mathbb{P}^1$. Since $F_\varepsilon(\alpha_\varepsilon) = N \cdot \alpha_\varepsilon$, it follows that $k + 2 = Nk$, which is impossible if N is chosen > 3.

§2 Proofs of Main Results

We assume in this section that $\varepsilon > 0$ is sufficiently small and is fixed throughout, and that $B : S^1 \to SL(2,\mathbb{R})$ is sufficiently near A_ε in the C^1 metric – how small ε has to be and how near B must be to A_ε will be determined along the way. The goal of this section is to show that $(f, m; B)$ has a positive Lyapunov exponent.

2.1 Preliminaries

We view B as $B_0 \circ R_{\varphi_\varepsilon}$, where $B_0 : S^1 \to SL(2,\mathbb{R})$ is defined by

$$B_0(x) := B(x) \circ R_{-\varphi_\varepsilon(x)}.$$

Let $A_0 \equiv \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$ be the constant cocycle over (f, m). Since $A_\varepsilon(x) \circ R_{-\varphi_\varepsilon(x)} = A_0$, B_0 is C^1 very near A_0.

We fix some notation that will be used throughout. The 1-dimensional projective space $\mathbb{P}^1$ is identified with $\mathbb{R}/\pi\mathbb{R}$, with 0 corresponding to the horizontal direction. For $v \in \mathbb{R}^2$, let $\bar{v} \in \mathbb{P}^1$ denote the projectivization of v. For $A \in SL(2,\mathbb{R})$, let $\bar{A} : \mathbb{P}^1 \circlearrowleft$ be the projectivization of A. Note that if $D\bar{A}$ denotes the derivative of $\bar{A}$, then $|D\bar{A}(\bar{v})| = \left(\frac{|Av|}{|v|}\right)^{-2}$. Define $F_B : S^1 \times \mathbb{P}^1 \circlearrowleft$ by

$$F_B(x, \theta) = (fx, \overline{B(x)}\theta),$$

and think of $F_B = F_0 \circ \Phi_\varepsilon$ where $\Phi_\varepsilon(x, \theta) = (x, \theta + \varphi_\varepsilon(x) \mod \pi)$ and $F_0(x, \theta) = (fx, \overline{B_0(x)}\theta)$. For $v \in \mathbb{R}^2$ or $T_{(x,\theta)}S^1 \times \mathbb{P}^1$, let $s(v)$ denote the slope of v.

Lemma 1. *The cocycle (f, B_0) has an invariant "contractive direction" $E_0^s(x)$ at every $x \in S^1$. More precisely, at every $x \in S^1$, there is $E_0^s(x) \in \mathbb{P}^1$ s.t.*

$$\overline{B_0(x)} E_0^s(x) = E_0^s(fx), \tag{1}$$

$$\text{if } v \in \mathbb{R}^2 \text{ is s.t. } \bar{v} \in E_0^s(x), \text{ then } |B_0(x)v| \approx \frac{1}{\lambda}|v|. \tag{2}$$

Moreover, the mapping $x \mapsto E_0^s(x)$ is Lipschitz, with Lip const $<$ some preassigned small number $\gamma_0 > 0$ if B is C^1 sufficiently near A_ε.

Proof. We assume B_0 is sufficiently near A_0 that there is a small $\delta > 0$ s.t. (2) holds for all v with $|\bar{v} - \frac{\pi}{2}| < \delta$ and $\overline{B_0(x)}(\frac{\pi}{2} - \delta,\ \frac{\pi}{2} + \delta) \supset (\frac{\pi}{2} - \delta,\ \frac{\pi}{2} + \delta)$ $\forall x \in S^1$. Then $E_0^s(x) := \cap_{n\geq 0}(\overline{B_0^n(x)})^{-1}(\dfrac{\pi}{2} - \delta,\ \dfrac{\pi}{2} + \delta)$. The Lipschitzness of $x \mapsto E_0^s(x)$ follows from our assumption that $\lambda > \sqrt{N}$. More precisely, since for $\theta \in (\frac{\pi}{2} - \delta,\ \frac{\pi}{2} + \delta)$, $(x, \theta) \mapsto (fx, \bar{A}_0\theta)$ stretches more in the θ-direction than it does in the x-direction, $\exists \gamma_0 > 0$ s.t. for $v \in T_{(x,\theta)}S^1 \times \mathbb{P}^1$ with $|s(v)| > \gamma_0$, $|s(DF_0^{k_0}v)| \geq \gamma_0$ and the component of $DF_0^k v$ in the θ-direction grows exponentially – provided that the second coordinate of $F_0^j(x, \theta)$ stays within δ of $\frac{\pi}{2}$ $\forall 0 \leq j < k$. Thus if it happens that $|E_0^s(x_1) - E_0^s(x_2)| > \gamma_0|x_1 - x_2|$ for two nearby points x_1, x_2, $|E_0^s(f^n x_1) - E_0^s(f^n x_2)|$ will increase indefinitely until one of $E_0^s(f^n x_1)$ or $E_0^s(f^n x_2)$ gets outside of the δ-neighborhood of $\frac{\pi}{2}$, which is impossible. ■

2.2 Estimating the Lyapunov exponents of $(f, m; B)$

Let $\lambda_1 \geq \lambda_2$ denote the Lyapunov exponents of $(f, m; B)$. By Oseledec's Theorem,

$$\lambda_1 \geq \lim_{n\to\infty} \frac{1}{n} \int \sum_{i=0}^{n-1} \log \left| B^i(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right| m(dx).$$

The integrand above will be estimated as follows: For $x \in S^1$, let $v_0(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\hat{v}_n(x) = R_{\varphi_\varepsilon(f^n x)} v_n(x)$, and $v_{n+1}(x) = B_0(f^n x)\hat{v}_n(x)$. Let $v_n = v_n^s + v_n^u$ and $\hat{v}_n = \hat{v}_n^s + \hat{v}_n^u$ be the decompositions of v_n and $\hat{v}_n$ with respect to the basis $E_0^s(f^n x) \oplus E_0^s(f^n x)^\perp$. Let θ_n be the $\mathbb{P}^1$-coordinate of $\hat{v}_n$.

By the B_0-invariance of E_0^s and the fact that both $B_0(f^n x)\hat{v}_n^u$ and $E_0^s(f^{n+1}x)^\perp$ are roughly horizontal, we have

$$|v_{n+1}^u| \geq \frac{99}{100} |B_0(f^n x)\hat{v}_n^u|.$$

This gives

$$\begin{aligned}|v_{n+1}| \geq |v_{n+1}^u| &\geq \frac{99}{100}|B_0(f^n x)\hat{v}_n^u| \\ &\geq \frac{98}{100}\lambda|\hat{v}_n^u| \\ &= \frac{98}{100}\lambda|\sin(\theta_n - E_0^s(f^n x))| \cdot |v_n|.\end{aligned}$$

Inductively we obtain

$$|v_n| \geq \left(\frac{98}{100}\lambda\right)^n \prod_{i=0}^{n-1} \left|\sin(\theta_i - E_0^s(f^i x))\right|$$

for all n. Hence

$$\lambda_1 \geq \log \frac{98}{100}\lambda + \overline{\lim_{n\to\infty}} \frac{1}{n} \int \sum_{i=0}^{n-1} \log \frac{2}{\pi}|\theta_i(x) - E_0^s(f^i x)| m(dx).$$

2.3 Analysis of θ_n

To prove $\lambda_1 > 0$ then, we must estimate the average distance of θ_n from E_0^s, where θ_n is as defined in the last paragraph. To do that, we divide $\mathbb{P}^1$ into zones on which $\bar{B}_0$ behaves differently. Let a, b, c be the projectivizations of $\begin{pmatrix} \frac{1}{\lambda^2} \\ \lambda \end{pmatrix}$, $\begin{pmatrix} \frac{1}{\lambda} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \frac{1}{\lambda} \end{pmatrix}$ respectively. Note that $\bar{A}_0 a = b,\ \bar{A}_0 b = c,\ D\bar{A}_0(a) \approx \lambda^2$ (which we assume to be $> N$), and $D\bar{A}_0(b) = 1$. We assume that $|E_0^s(x) - \frac{\pi}{2}| << |a - \frac{\pi}{2}|\ \forall x \in S^1$. Let $\beta = \beta(\lambda)$ be a very small positive number. (This is the β mentioned at the beginning of section 1.) Let $\hat{b} \approx b$ be s.t. $\overline{B_0(x)}a + \beta < \hat{b} < a\ \forall x \in S^1$, and let $\hat{c} \approx c$ be s.t. $\overline{B_0(x)}\hat{b} + \beta < \hat{c} < \hat{b}$. We require also that $\overline{B_0(x)}\ (\pi - a) - \beta > \pi - \hat{b}$ and $\overline{B_0(x)}\ (\pi - \hat{b}) - \beta > \pi - \hat{c}$. Let $Z(a) := [a, \pi - a],\ Z(b) := [\hat{b}, \pi - \hat{b}]$ and $Z(c) := [\hat{c}, \pi - \hat{c}]$.

Sometimes it is convenient to work with a lift of F_B. Let $\tilde{F}_B = \tilde{F}_0 \circ \tilde{\Phi}_\varepsilon : \mathbb{R}^2 \circlearrowleft$ be a lift of F_B. Let $p : \mathbb{R}^2 \to S^1 \times \mathbb{P}^1,\ p_1 : \mathbb{R} \to S^1$ and $p_2 : \mathbb{R} \to \mathbb{P}^1$ be the natural projections. Let $\tilde{Z}(a) = p_2^{-1}Z(a),\ \tilde{Z}(b) = p_2^{-1}Z(b),\ \tilde{Z}(c) = p_2^{-1}Z(c)$. For definiteness, assume $\tilde{F}_B([0,1] \times \mathbb{R}) = [0, N] \times \mathbb{R}$. We observe that the derivative of $\tilde{F}_0$ has the following properties:

(1) If $\theta \in \tilde{Z}(a)$ and $v \in \mathbb{R}^2$ has $s(v) \geq 1$, then $s((D\tilde{F}_0)_{(x,\theta)}v) \geq s(v)$. (This is because $\bar{B}_0$ stretches by $\approx \lambda^2$ in this region, f stretches by N, and $\lambda^2 > N$).

(2) If $\theta \in \tilde{Z}(b)$ and $s(v) \geq 1$, then $s((D\tilde{F}_0)_{(x,\theta)}v) \geq \frac{1}{2N}s(v)$. (Reason: $\bar{B}_0$ stretches by $\gtrsim 1$ in this region.)

(3) $\exists\, \gamma_1 > 0$, γ_1 small and depending on $\mathrm{dist}(A_0, B_0)$, s.t. for all $\theta \notin \tilde{Z}(b)$, $|s(v)| \leq \gamma_1 \Rightarrow |s((D\tilde{F}_0)_{(x,\theta)}v)| \leq \gamma_1$. (These invariant cones are due to the fact that $\bar{B}_0$ contracts or nearly contracts in this region whereas f expands by N).

Observe also that $s((D\tilde{\Phi}_\varepsilon)_{(x,\theta)}v) = s(v) + \varphi_\varepsilon'(p_1 x)$.

For $n = 0, 1, 2, \ldots$, we introduce functions $\tilde{\theta}_n : [0, N^n] \to \mathbb{R}$ as follows: Start with $\tilde{\theta}_0(x) := \varphi_\varepsilon(p_1 x)$, where $\varphi_\varepsilon(\cdot)$ is regarded as increasing monotonically from 0 to 2π. For $n \geq 1$, define $\tilde{\theta}_n$ by

$$\text{graph } (\tilde{\theta}_n) = \tilde{\Phi}_\varepsilon \circ \tilde{F}_0(\text{graph } (\tilde{\theta}_{n-1})).$$

Note that for $x \in [0, N^n]$, $p_2\tilde{\theta}_n(x) = \theta_n(p_1 x)$, where θ_n is as defined in section 2.3.

As we shall see, our analysis rests on the near-monotonicity of $\tilde{\theta}_n$ as a function of x. This in turn is a consequence of the monotonicity of $x \mapsto \varphi_\varepsilon(x)$.

Lemma 2. *The following hold for all n:*

(1) *if $\tilde{\theta}_n \in \tilde{Z}(b)$, then $\frac{d\tilde{\theta}_n}{dx} \geq \frac{1}{2\varepsilon}$;*

(2) *if $\tilde{\theta}_n \in \tilde{Z}(c)$, then $\frac{d\tilde{\theta}_n}{dx} \geq \frac{1}{4N\varepsilon}$;*

(3) *$\frac{d\tilde{\theta}_n}{dx} \geq -\gamma_1$ everywhere.*

Proof. Let u_n be a tangent vector to the graph of $\tilde{\theta}_n$, and use the notation $D\tilde{\Phi}_\varepsilon \circ D\tilde{F}_0\ (u_{n-1}) = u_n$. Assuming that β is sufficiently small, (1)-(3) clearly hold for $\tilde{\theta}_0$. We now assume that (1)-(3) have been proved for $\tilde{\theta}_{n-1}$. To prove (1) for $\tilde{\theta}_n$, consider the following two possibilities:

Case 1. $\tilde{\theta}_{n-1} \in \tilde{Z}(a)$. Then $s(u_{n-1}) \geq \frac{1}{2\varepsilon}$ by assumption, and from our discussion above, $s(u_n) \geq s(D\tilde{F}_0 u_{n-1}) \geq s(u_{n-1}) \geq \frac{1}{2\varepsilon}$.

Case 2. $\tilde{\theta}_{n-1} \notin \tilde{Z}(a)$. Then in order for $\tilde{\theta}_n$ to be in $\tilde{Z}(b)$, we must have $|\varphi_\varepsilon - \pi\mathbb{Z}| > \beta$. (See the definition of $\tilde{Z}(b)$). Our condition on φ_ε guarantees that $\varphi'_\varepsilon \geq \frac{1}{\varepsilon}$, so that $s(u_n) \geq s(D\tilde{F}_0 u_{n-1}) + \frac{1}{\varepsilon} \geq \frac{1}{2\varepsilon}$.

The proof of (2) is similar. For $\tilde{\theta}_n \in \tilde{Z}(c)$, if $\tilde{\theta}_{n-1} \in \tilde{Z}(b)$, then $s(u_n) \geq s(DF_0 u_{n-1}) \geq \frac{1}{2\varepsilon} \cdot \frac{1}{2N}$; and if $\tilde{\theta}_{n-1} \notin \tilde{Z}(b)$, then $\varphi'_\varepsilon \geq \frac{1}{\varepsilon}$ as above.

To prove (3), $\frac{d\tilde{\theta}_n}{dx} \geq 0$ if $\tilde{\theta}_{n-1} \in \tilde{Z}(b)$. If $\tilde{\theta}_{n-1} \notin \tilde{Z}(b)$, use the invariant cone property of $D\tilde{F}_0$ and the fact that $s(D\tilde{\Phi}_\varepsilon v) \geq s(v)$. ■

2.4 Completing the argument

Let α be the image of the curve $x \mapsto E_0^s(x)$ in $S^1 \times \mathbb{P}^1$, and let $U_\delta(\alpha) := \{(x,\theta) : |\theta - E_0^s(x)| < \delta\}$. Note that $p^{-1}\alpha$ is the disjoint union of a countable number of curves $\tilde{\alpha}_k$, $k \in \mathbb{Z}$, so that each $\tilde{\alpha}_k$ is C^0 near $\theta \equiv (k + \frac{1}{2})\pi$ and is the graph of a Lipschitz function with Lip constant $<$ some small γ_0. Let $\tilde{U}_\delta(\alpha) = p^{-1}U_\delta(\alpha)$.

Lemma 3. *$\exists\, C_1 = C_1(N,\lambda)$ s.t. $\forall \delta > 0$ with $U_\delta(\alpha) \subset S^1 \times Z(c)$, we have*

$$m\{x \in S^1 : (f^n x, \theta_n(x)) \in U_\delta(\alpha)\} < C_1 \varepsilon \delta$$

for all $n \geq 0$.

Proof. Since the action of $F_B : S^1 \times \mathbb{P}^1 \cong \mathbb{T}^2 \circlearrowleft$ with respect to the two usual generators is given by the matrix $\begin{pmatrix} N & 0 \\ 2 & 1 \end{pmatrix}$, $\exists\, C_0$ s.t. $\forall n \geq 0$, $\tilde{\theta}_n(N^n) - \tilde{\theta}_n(0) \leq C_0 N^n$. Now whenever $\tilde{\theta}_n$ crosses a component of $\tilde{U}_\delta(\alpha)$, it does so monotonically with derivative $\geq \frac{1}{4N\varepsilon}$. So $\tilde{\theta}_n$ crosses at most $C_0 N^n$ components of $\tilde{U}_\delta(\alpha)$. Thus

$$\begin{aligned} &m\{x \in S^1 : (f^n x, \theta_n(x)) \in U_\delta(\alpha)\} \\ &\quad = \frac{1}{N^n} \cdot m\{x \in [0, N^n] : (x, \tilde{\theta}_n(x)) \in \tilde{U}_\delta(\alpha)\} \\ &\quad \leq \frac{1}{N^n} \cdot C_0 N^n \cdot 3\delta \cdot (4\varepsilon N) := C_1 \varepsilon \delta. \quad ■ \end{aligned}$$

Lemma 4. *For $\lambda \geq 2$ and ε sufficiently small, $\exists \sigma = \sigma(\lambda, \varepsilon) > 0$ s.t. $\forall n \in \mathbb{Z}^+$,*

$$\log \frac{98}{100}\lambda + \int \log \frac{2}{\pi} |\theta_n(x) - E_0^s(f^n x)| m(dx) \geq \sigma.$$

Proof. Let $\delta = \frac{\pi}{2} - \hat{c}$, and assume in the following calculation that $|E_0^s - \frac{\pi}{2}|$ is negligible. Then

$$\begin{aligned}
&\int_{\{|\theta_n(x) - E_0^s(f^n x)| < \delta\}} \log \frac{2}{\pi}|\theta_n(x) - E_0^s(f^n x)| m(dx) \\
&= \sum_{k=0}^{\infty} \int_{\{\frac{\delta}{2^{k+1}} \le |\cdot| < \frac{\delta}{2^k}\}} \log \frac{2}{\pi} |\cdot| \\
&\ge \sum_{k \ge 0} \left(\log \frac{2}{\pi} \frac{\delta}{2^{k+1}} \right) \cdot C_1 \frac{\delta}{2^k} \cdot \varepsilon \qquad \text{by Lemma 3} \\
&:= C_2 \varepsilon.
\end{aligned}$$

Also

$$\int_{\{|\cdot| > \delta\}} \log \frac{2}{\pi} |\cdot| \ge \log \frac{2}{\pi} \delta,$$

which is $\gtrsim \log(1 - \frac{2}{\pi}\frac{1}{\lambda})$ since $\frac{\pi}{2} - \delta \lesssim \frac{1}{\lambda}$. For λ large, it is clear that $\log \frac{98}{100}\lambda + \log(1 - \frac{2}{\pi}\frac{1}{\lambda}) > 0$. It is easy to check that $\lambda \ge 2$ is more than sufficient. ■

§3. Nonuniformly hyperbolic cocycles over toral automorphisms

Let $\mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1$ be the 2-torus, and consider $T : \mathbb{T}^2 \circlearrowleft$ induced from $\tilde{T} \in SL(2, \mathbb{Z})$ with eigenvalues μ, μ^{-1} satisfying $0 < \mu^{-1} < 1 < \mu$. (Everyone's favorite example is $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$). Let m denote the Lebesgue measure on $\mathbb{T}^2$. In this section we show that cocycles similar to those considered in sections 1 and 2 can be defined over (T, m). Other generalizations are clearly possible, but we leave them to the reader.

Let (T, m) be as above, and let W^u denote the unstable manifolds of T. We can fix an orientation on the leaves of W^u. Since $\mu > 0$, T preserves this orientation. (For definiteness, say the eigen-direction of $\tilde{T}$ pointed into the first quadrant is positively oriented). Here μ will play the role of N, so we must assume that $\lambda \ge \sqrt{\mu} + 1$. Let $J_\varepsilon \subset S^1$, and let $\varphi_\varepsilon : \mathbb{T}^2 \to \mathbb{R}/2\pi\mathbb{R}$ be a C^1 function such that

(1) $\varphi_\varepsilon \equiv 0$ outside of $J_\varepsilon \times S^1$;

(2) on $J_\varepsilon \times S^1$, φ_ε increases monotonically from 0 to 2π *along the leaves of* W^u; and

(3) on $\varphi_\varepsilon^{-1}[\beta, 2\pi - \beta]$, the directional derivatives of φ_ε *along the leaves of* W^u are $\geq \frac{1}{\varepsilon}$.

("Along the leaves of W^u" means as one moves in the positive direction).

A_ε is defined as before, and the claim is that for all sufficiently small ε, there is a neighborhood $\mathcal{N}_\varepsilon$ of A_ε in $C^1(\mathbb{T}^2, SL(2,\mathbb{R}))$ s.t. $\forall B \in \mathcal{N}_\varepsilon$, $(T, m; B)$ has a positive Lyapunov exponent. The same methods as before guarantee that these cocycles are not uniformly hyperbolic.

To prove that $(T, m; B)$ has a positive exponent, we will show that on every compact piece of W^u-leaf W,

$$\int \lim_{n\to\infty} \frac{1}{n} \log \left| B^i(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right| m_W(dz) > 0$$

where m_W denotes Lebesgue measure on W. The proof (including Lemma 3) proceeds *mutatis mutandis* as in section 2, with $\frac{d\tilde{\theta}_n}{dx}$ replaced by the directional derivative of $\tilde{\theta}_n$ along W^u.

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