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From Invariant Curves to Strange Attractors

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Abstract: We prove that simple mechanical systems, when subjected to external periodic forcing, can exhibit a surprisingly rich array of dynamical behaviors as parameters are varied. In particular, the existence of global strange attractors with fully stochastic properties is proved for a class of second order ODEs.

Introduction

In the history of classical mechanics, dissipative systems received only limited attention, in part because it was believed that in these systems all orbits eventually tended toward stable equilibria (fixed points or periodic cycles). Evidence that second order equations with a periodic forcing term can have interesting behavior first appeared in the study of van der Pol's equation, which describes an oscillator with nonlinear damping. The first observations were due to van der Pol and van der Mark. Cartwright and Littlewood proved later that in certain parameter ranges, this equation had periodic orbits of different periods [CL]. Their results pointed to an attracting set more complicated than a fixed point or an invariant curve. Levinson obtained detailed information for a simplified model [Ln]. His work inspired Smale, who introduced the general idea of a horseshoe [Sm], which Levi used later to explain the observed phenomena [Li1].

A number of other differential equations with chaotic behavior have been studied in the last few decades, both numerically and analytically. Examples from the dissipative category include the equations of Lorenz [Lo, G, Ro, Ry, Sp, T, W], Duffing's equation [D, Ho], Lorentz gases acted on by external forces [CELS], and modified van der Pol type systems [Li2]. For a systematic treatment of the Lorenz and Duffing equations, see [GH]. While some progress has been made, the number of equations for which a rigorous global description of the dynamics is available has remained small.

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In this paper, we consider an equation of the form

$$\frac{d^2\theta}{dt^2} + \lambda(\frac{d\theta}{dt} - 1) = \Phi(\theta)P_T(t),$$

where $\theta \in S^1$ and $\lambda > 0$. If the right side is set identically equal to zero, this equation represents the motion of a particle subjected to a constant external force which causes it to decelerate when its velocity exceeds one and to accelerate when it is below one. Independent of the initial condition, the particle approaches uniform motion in which it moves with velocity equal to one. To this extremely simple dynamical system, we add another external force in the form of a *pulse*: Φ is an arbitrary function, P_T is time-periodic with period T, and for $t \in [0, T)$, it is equal to 1 on a short interval and 0 otherwise. We learned after this work was completed that a similar equation has been studied numerically in the physics literature by G. Zaslavsky.[1]

We prove that the system above exhibits, for different values of λ and T, a very rich array of dynamical phenomena, including

(a) *invariant curves with quasi-periodic behavior,*
(b) *gradient-like dynamics with stable and unstable equilibria,*
(c) *transient chaos caused by the presence of horseshoes, with almost every trajectory eventually tending to a stable equilibrium, and*
(d) *strange attractors with SRB measures and fully stochastic behavior.*

These results are new for the equation in question. As abstract dynamical phenomena, (a)–(c) are fairly well understood, and their occurrences in concrete models have been noted; see [GH]. The situation with regard to (d) is very different. The analysis that allows us to handle attractors of this type was not available until recently. To our knowledge, this is the first time a concrete differential equation has been proved analytically to have a global nonuniformly hyperbolic attractor with an SRB measure.[2] We regard Theorem 3, which discusses the strange attractor case, as the main result of this paper.

Our proof of Theorem 3 is based on [WY], in which we built a dynamical theory for a (general) class of attractors with one direction of instability and strong dissipation. In [WY], we identified a set of conditions which guarantees the existence of strange attractors with strong stochastic properties. The properties in question include most of the standard mathematical notions associated with chaos: positive Lyapunov exponents, positive entropy, SRB measures, exponential decay of correlations, symbolic coding of orbits, fractal geometry, etc. The occurrence of scenario (d) above is proved by checking the conditions in [WY]. For the convenience of the reader, we will recall these conditions as well as the package of results that follows once these conditions are checked.

Our purpose in writing this paper is not only to point out the range of phenomena that can occur when simple second order equations are periodically forced, but to bring to the foreground the techniques that have allowed us to reach these conclusions in a relatively straightforward manner. These techniques are clearly not limited to the systems considered here. It is our hope that they will find applications in other dynamical systems, particularly those that arise naturally from mechanics or physics.

[1] Zaslavsky produced in [Z1] numerical evidence of strange attractors. He also discussed in [Z2] how this model can be viewed as a strong idealization of the turbulence problem.

[2] Levi proved in [Li1] the occurrence of scenario (c) for his modified van der Pol systems, not scenario (d) as is sometimes incorrectly reported.

1. Statement of Results

1.1. Setting and assumptions. Consider the differential equation

$$\frac{d^2\theta}{dt^2} + \lambda \frac{d\theta}{dt} = \mu + \Phi(\theta) P_T(t), \tag{1}$$

where $\theta \in S^1$, $\lambda, \mu > 0$ are constants, $\Phi : S^1 \to \mathbb{R}$ is a smooth function, and P_T has the following form: for some $t_0 < T$, P_T satisfies

$$P_T(t) = P_T(t+T) \quad \text{for all } \; t$$

and

$$P_T(t) = \begin{cases} 1 & \text{for } \; t \in [0, t_0], \\ 0 & \text{for } \; t \in (t_0, T). \end{cases}$$

As discussed in the introduction, (1) describes a simple mechanical system consisting of a particle moving in a circle subjected to an external time-periodic force. With $r = \frac{d\theta}{dt} - \frac{\mu}{\lambda}$, (1) is equivalent to

$$\begin{aligned} \frac{d\theta}{dt} &= r + \frac{\mu}{\lambda}, \\ \frac{dr}{dt} &= -\lambda r + \Phi(\theta) P_T(t). \end{aligned} \tag{2}$$

Let F_T denote the time-T-map of (2), that is, the map that transforms the phase space $S^1 \times \mathbb{R}$ from time 0 to time T. Unless explicitly stated otherwise, when we write F_T, it will be assumed that T is the period of the forcing.

We set $\mu = \lambda$ for simplicity, and normalize the forcing term as follows: Given a function $\Phi_0 : S^1 \to \mathbb{R}$, we let $\Phi = \frac{1}{t_0}\Phi_0$, that is to say, the magnitude of this part of the force is taken to be inversely proportional to the duration of its action, and the proportionality constant is taken to be 1 for simplicity. Our analysis will proceed as follows:

* The function Φ_0 is fixed throughout. With the exception of Theorem 2(b) (where more is assumed), the only requirements are that Φ_0 is of class C^4 and all of its critical points are nondegenerate.
* We assume $t_0 < \frac{1}{10}\min\{\lambda^{-1}, K_0^{-2}\}$, where $K_0 = \max\{\|\Phi_0\|_{C^4}, 1\}$. Further restrictions on t_0 are imposed in each case as needed. (We do not regard t_0 as an important parameter and will assume it is as small as the arguments require.)
* The two important parameters are λ and T. We will prove that (i) the properties of (1) are intrinsically different for λ small and for λ large, and (ii) for fixed λ, the properties of (1) depend quite delicately on the value of T.

To interpret our results correctly, the reader should keep in mind that the dynamical pictures described below are not the only ones that can occur, and it is possible to have combinations of them, such as sinks and strange attractors, on different parts of the phase space. Our aim here is to identify several important *pure dynamics types*, to indicate the nature and approximate locations of the parameter sets on which they occur, and to convey a sense of *prevalence*, meaning that these phenomena occur naturally and not as a result of mere coincidence.

1.2. Statements of theorems. The setting of Sect. 1.1 is assumed throughout. We consider the discrete-time system defined by the Poincaré map F_T. Precise meanings of some of the technical terms are given after the statements of the theorems. Theorem 3 is our main result. The scenarios presented in Theorems 1 and 2 are also integral parts of the picture.

Theorem 1 (Existence of invariant curves). *Let $\lambda \geq 4K_0$ and $T \geq t_0 + \frac{3}{2}$. Then there is a simple closed curve Ω of class C^4 to which all the orbits of F_T converge. Moreover, we have the following dichotomy:*

(a) **(Quasi-periodic attractors)** *Let $\Delta_0 = \{T : \rho(T) \in \mathbb{R} \setminus \mathbb{Q}\}$, where $\rho(T)$ is the rotation number of $F_T|\Omega$. Then (i) Δ_0 intersects every unit interval in $[\frac{3}{2}, \infty)$ in a set of positive Lebesgue measure, and (ii) the following hold for $T \in \Delta_0$: $F_T|\Omega$ is topologically conjugate to an irrational rotation, and for every $z \in S^1 \times \mathbb{R}$, $\frac{1}{n}\sum_0^{n-1} \delta_{F_T^i z}$ converges weakly to μ where μ is the unique invariant probability measure on Ω.*

(b) **(Periodic sinks and saddles)** *There is an open and dense subset Δ_1 of $[t_0+\frac{3}{2}, \infty)\setminus\bar{\Delta}_0$ such that for $T \in \Delta_1$, F_T has a finite number of periodic sinks and saddles on Ω. Every orbit of F_T converges to one of these periodic orbits.*

Theorem 2 is elementary; it uses standard techniques, and Φ_0 is required only to be C^2. We include this result because the dynamical pictures described occur for a nontrivial set of parameters.

Theorem 2 (Convergence to stable equilibria).

(a) **(Gradient-like dynamics)** $\exists\lambda_0 < \max|\Phi_0'|$ *such that $\forall\lambda > \lambda_0$, if t_0 is sufficiently small, then there are open intervals of T for which F_T has a finite number of periodic points all of which are saddles or sinks, and every orbit not on the stable manifold of a saddle tends to a sink.*

(b) **(Transient chaos)** *Assume Φ_0 has exactly two critical points. Then there exist intervals of λ accumulating at* 0 *such that for each of these λ, if t_0 is sufficiently small, then there are open intervals of T for which F_T has a periodic sink and a "horseshoe", i.e. a uniformly hyperbolic invariant set Λ such that $F_T|\Lambda$ is conjugate to a shift of finite type with positive topological entropy. Lebesgue-a.e. $z \in S^1 \times \mathbb{R}$ is attracted to the sink as $n \to \infty$.*

Remarks. (i) The picture in Theorem 2(a) is more general than that in Theorem 1(b): there are no simple closed invariant curves in general (see Proposition 4.1).
(ii) We describe the scenario in Theorem 2(b) as "transient chaos" for the following reasons: Λ being an invariant set, points near it tend to stay near it for some period of time, mimicking the dynamics on Λ. This chaotic behavior, however, is transient, because Λ has Lebesgue measure zero, and for a typical initial condition, the orbit eventually leaves Λ behind and heads for a sink.

Our next result deals with a notion of chaos that is sustained through time. A compact, F_T-invariant set $\Omega \subset S^1 \times \mathbb{R}$ is called a *global attractor* for F_T if for every $z \in S^1 \times \mathbb{R}$, $\text{dist}(F_T^n z, \Omega) \to 0$ as $n \to \infty$. In order not to interrupt the flow of ideas, we postpone the technical definitions of some of the terms used in Theorems 2 and 3 to after the statements of both results.

Here is our main result:

Theorem 3 (Strange attractors). *For the parameters specified below, $F = F_T$ has a strange attractor, a description of which follows:*

Relevant parameter set. *There exist $\bar{\lambda}, \bar{t}_0 > 0$ such that for every $\lambda < \bar{\lambda}$ and $t_0 < \bar{t}_0$, there is a positive Lebesgue measure set $\Delta = \Delta(\lambda, t_0)$ in T-space for which the results of this theorem hold; $\Delta \subset [T_0, \infty)$ for some large T_0, and meets every subinterval of $[T_0, \infty)$ of length $\mathcal{O}(\lambda)$ in a set of positive Lebesgue measure.*

Dynamical characteristics. *Let $\lambda < \bar{\lambda}$, $t_0 < \bar{t}_0$, and $T \in \Delta(\lambda, t_0)$. Then $F = F_T$ has a global attractor Ω with the following dynamical properties:*

(1) **Hyperbolic behavior.** *$F|\Omega$ is nonuniformly hyperbolic with an identifiable set $\mathcal{C} \subset \Omega$ which is the source of all nonhyperbolic behavior. More precisely:*
  (a) *$\mathcal{C} = \cup_i \mathcal{C}_i$ where $\mathcal{C}_i$ is a Cantor set located near $(\theta, r) = (c_i, 0)$, c_i being the critical points of Φ_0; at each $z \in \mathcal{C}$, stable and unstable directions coincide, i.e. there is a vector v with $\|DF^n(z)v\| \to 0$ exponentially fast as $n \to \pm\infty$.*
  (b) *Away from $\mathcal{C}$ the dynamics is uniformly hyperbolic. More precisely, let*

$$\Omega_\varepsilon := \{z \in \Omega : d_{\mathcal{C}}(F^n z) \geq \varepsilon \forall n \in \mathbb{Z}\},$$

  where $d_{\mathcal{C}}(\cdot)$ is a notion of distance to $\mathcal{C}$. Then Ω is the closure of $\cup_{\varepsilon>0}\Omega_\varepsilon$, Ω_ε is a uniformly hyperbolic invariant set for each $\varepsilon > 0$, and the hyperbolicity of $F|\Omega_\varepsilon$ deteriorates (e.g. minimum $\angle(E^u, E^s) \to 0$) as $\varepsilon \to 0$.

(2) **Statistical properties.**
  (a) *F admits a unique SRB measure μ supported on Ω.*
  (b) *With the exception of a Lebesgue measure zero set of initial conditions, the asymptotic behavior of every orbit of F is governed by μ. More precisely, for Lebesgue-a.e. $z \in S^1 \times \mathbb{R}$, if $\varphi : S^1 \times \mathbb{R} \to \mathbb{R}$ is a continuous function, then $\frac{1}{n}\sum_0^{n-1} \varphi(F^i z) \to \int \varphi d\mu$ as $n \to \infty$.*
  (c) *(F, μ) is ergodic, mixing, and Bernoulli.*
  (d) *For every observable $\varphi : \Omega \to \mathbb{R}$ of Hölder class, the sequence*

$$\varphi, \varphi \circ F, \varphi \circ F^2, \cdots, \varphi \circ F^n, \cdots$$

  viewed as a stochastic process with underlying probability space (Ω, μ) has exponential decay of correlations and obeys the Central Limit Theorem.

(3) **Symbolic coding and other geometric properties.**
  (a) *Kneading sequences are well defined for all critical orbits, i.e. all orbits emanating from $\mathcal{C}$.*
  (b) *With respect to the partition defined by the fractal sets $\mathcal{C}_i$, the coding of orbits in Ω is well defined and essentially one-to-one. More precisely, if σ is the shift operator, then there is a closed subset $\Sigma \subset \Pi_{-\infty}^{\infty}\{1, \cdots, s\}$ with $\sigma(\Sigma) \subset \Sigma$ and a continuous surjection $\pi : \Sigma \to \Omega$ such that $\pi \circ \sigma = F \circ \pi$; moreover, π is one-to-one except over $\cup_{-\infty}^{\infty} F^i \mathcal{C}$, where it is two-to-one. (In general, (Σ, σ) is not a shift of finite type.)*
  (c) *Let $h_{\text{top}}(F)$ denote the topological entropy of F, N_n the number of cylinder sets of length n in Σ above, and P_n the number of fixed points of F^n. Then*

$$h_{\text{top}}(F) = \lim_{n\to\infty} \frac{1}{n} \log N_n = \lim_{n\to\infty} \frac{1}{n} \log P_n.$$

  Moreover, F has an invariant measure of maximal entropy.

For a more detailed description of the dynamics on these strange attractors, see [WY]. We review below the definitions and related background information for some of the technical terms used in the theorems. For more information on this material, see [KH] and [Y1].

A compact F-invariant set Λ is called *uniformly hyperbolic* if the following hold: (1) The tangent space at every $x \in \Lambda$ splits into $E^u(x)+E^s(x)$ with $\min_{x\in\Lambda} \angle(E^u, E^s) > 0$; (2) this splitting is DF-invariant; and (3) there exist $C \geq 1$ and $\sigma < 1$ such that for all $x \in \Lambda$ and $n \geq 0$, $\|DF^n(x)v\| \leq C\sigma^n\|v\|$ for all $v \in E^s(x)$, $\|DF^{-n}(x)v\| \leq C\sigma^n\|v\|$ for all $v \in E^u(x)$.

In Theorem 3(1)(b), not only does $\min\angle(E^u, E^s) \to 0$ as $\varepsilon \to 0$, we have $C \to \infty$ as well. This means the smaller ε, the longer it takes for the geometry of hyperbolic behavior to take hold.

An F-invariant Borel probability measure μ is called an *SRB measure* if F has a positive Lyapunov exponent μ-a.e. and the conditional measures of μ on unstable manifolds are equivalent to the Riemannian volume on these leaves. SRB measures are of physical relevance because they can be observed: in dissipative dynamical systems, all invariant probability measures are necessarily singular, but ergodic SRB measures with nonzero Lyapunov exponents have the property that there is a positive Lebesgue measure set of points z for which $\frac{1}{n}\sum_0^{n-1}\varphi(F^i z) \to \int \varphi d\mu$ as $n \to \infty$ for every continuous function φ.

Referring to the set of points z above as the *measure-theoretic basin* of μ, Theorem 3(2)(b) says that the measure-theoretic basin here is not just a positive Lebesgue measure set, it is, modulo a set of Lebesgue measure zero, the entire phase space.

By a decomposition theorem for SRB measures with no zero exponents ([Le]), the uniqueness of μ implies that it is ergodic, and the mixing and Bernoulli properties are equivalent to (F^n, μ) being ergodic for all $n \geq 1$.

We say the dynamical system (F, μ) has *exponential decay of correlations* for Hölder continuous observables if given a Hölder exponent η, there exists $\tau = \tau(\eta) < 1$ such that for all $\varphi \in L^\infty(\mu)$ and $\psi : \Omega \to \mathbb{R}$ Hölder with exponent η, there exists $K = K(\varphi, \psi)$ such that

$$\left|\int (\varphi \circ F^n)\psi d\mu - \int \varphi d\mu \int \psi d\mu\right| \leq K(\varphi, \psi)\tau^n$$

for all $n \geq 1$. Finally, we say the *Central Limit Theorem* holds for φ with $\int \varphi d\mu = 0$ if $\frac{1}{\sqrt{n}}\sum_0^{n-1}\varphi \circ F^i$ converges in distribution to the normal distribution, and the variance is strictly positive unless $\varphi \circ F = \psi \circ F - \psi$ for some ψ.

1.3. Illustrations. Figure 1 below shows the approximate location and shape of the invariant curve or strange attractor (corresponding to different values of λ and T) for the time-T-map $F_T : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$.

Figure 2 explains the mechanisms behind the changes in the dynamical picture as λ decreases. The straight line in (a) represents $\{r = 0\}$ in (θ, r)-coordinates, and the subsequent pictures show the images of this line (or circle) at various times under the flow. Figure 2(b) shows the effect of the forcing; observe that it need not constitute a large perturbation. For $t \in (t_0, T]$, the forcing is turned off, and the system relaxes to a limit cycle with contraction rate $e^{-\lambda}$. Figure 2(d) shows the image of $\{r = 0\}$ for $\lambda > 1$ and $e^{-\lambda T}$ reasonably contractive; these parameters correspond to the existence of invariant

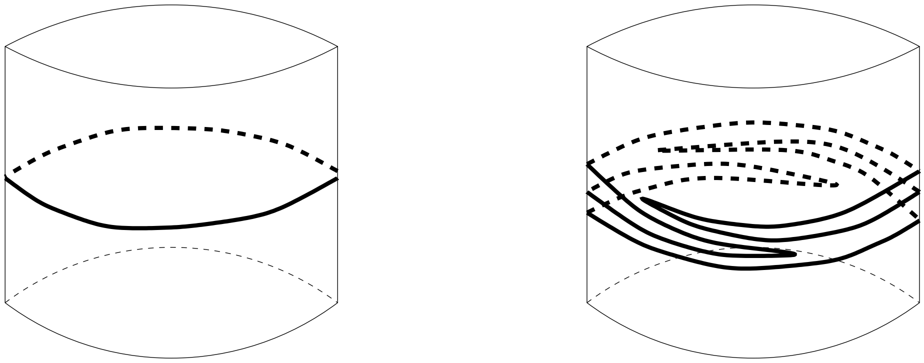

Fig. 1. *Left:* Invariant curves $\lambda > 1$; *right:* Strange attractor $\lambda \ll 1$

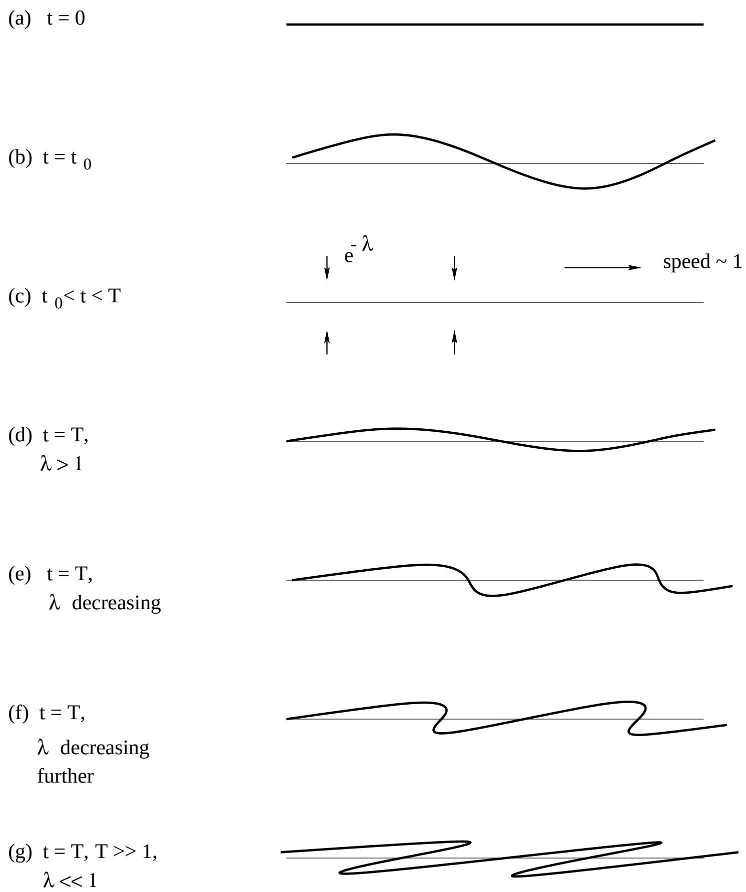

Fig. 2 a–g. Image of $\{r = 0\}$ at time t

curves. As λ decreases, the effect of the shear term in (2) becomes more prominent, as shown in (e). As λ decreases further, one sees a phenomenon resembling "the breaking of the wave" which accompanies the break-up of the invariant circle. Finally, in Figure 2(g), a tubular neighborhood of $\{r = 0\}$ is folded and mapped into itself, leading to the formation of horseshoes and/or strange attractors.

2. Preliminary Information on the ODE

2.1. Singular limits. Let

$$\begin{pmatrix} \theta(t) \\ r(t) \end{pmatrix} = \begin{pmatrix} \theta(\theta_0, r_0; t) \\ r(\theta_0, r_0; t) \end{pmatrix}$$

denote the solution of (2) with $\theta(0) = \theta_0$ and $r(0) = r_0$. Then a simple exercise gives

$$F_T : \begin{pmatrix} \theta_0 \\ r_0 \end{pmatrix} \mapsto \begin{pmatrix} \theta(T) \\ r(T) \end{pmatrix} = \begin{pmatrix} \theta(t_0) + (T - t_0) + \frac{r(t_0)}{\lambda}(1 - e^{-\lambda(T-t_0)}) \\ r(t_0)e^{-\lambda(T-t_0)} \end{pmatrix},$$

where the value of $\theta(T)$ above is to be interpreted as mod 1 or on S^1. We let $a = \{T - t_0\}$ be the fractional part of $T - t_0$, $b = e^{-\lambda n}$, where $n = [T - t_0]$ is the integer part of $T - t_0$, and let $T_{a,b} = F_T$. Then

$$T_{a,b} : \begin{pmatrix} \theta_0 \\ r_0 \end{pmatrix} \mapsto \begin{pmatrix} \theta(t_0) + a + \frac{r(t_0)}{\lambda} - be^{-\lambda a}\frac{r(t_0)}{\lambda} \\ be^{-\lambda a} r(t_0) \end{pmatrix}. \tag{3}$$

(The appearance of "T" in both F_T and $T_{a,b}$ is unfortunate; we hope it is not confusing. We wish eventually to make a connection to [WY] and this notation is used there.)

We first fix t_0 and λ, and let $T \to \infty$. Clearly, $b \to 0$ as $T \to \infty$. The limit of F_T as $T \to \infty$ does not exist. However, $T_{a,b}$ has the following well defined *singular limit* as $b \to 0$:

$$T_{a,0} : \begin{pmatrix} \theta_0 \\ r_0 \end{pmatrix} \mapsto \begin{pmatrix} \theta(t_0) + \frac{r(t_0)}{\lambda} + a \\ 0 \end{pmatrix}. \tag{4}$$

Let $T^1_{a,0}$ denote the first component of $T_{a,0}$. We will show in Sect. 2.3 that as $t_0 \to 0$, $T^1_{a,0} \to \hat{T}^1_{a,0}$, where

$$\hat{T}^1_{a,0} : \begin{pmatrix} \theta_0 \\ r_0 \end{pmatrix} \mapsto \theta_0 + \frac{r_0}{\lambda} + \frac{1}{\lambda}\Phi_0(\theta_0) + a. \tag{5}$$

In later sections, we will also work with two families of circle maps f_a and $\hat{f}_a$ obtained by restricting $T^1_{a,0}$ and $\hat{T}^1_{a,0}$ respectively to $\{r_0 = 0\}$, i.e.

$$f_a(\theta_0) = \theta(\theta_0, 0; t_0) + \frac{r(\theta_0, 0; t_0)}{\lambda} + a;$$
$$\hat{f}_a(\theta_0) = \theta_0 + \frac{1}{\lambda}\Phi_0(\theta_0) + a.$$

While our results are not confined to these limiting situations, the relation between F_T and the objects that appear in Eq. (1), namely, Φ_0, λ and t_0, can be made transparent by comparing first $T_{a,b}$ and $T_{a,0}$ and then $T^1_{a,0}$ and $\hat{T}^1_{a,0}$. This is how we will go about obtaining information on F_T.

2.2. The time-t_0-map. In this subsection we consider the solution of (2) for $t \in [0, t_0]$ and record some derivative estimates. We first write

$$\begin{aligned} r(t) &= u(t)e^{-\lambda t}, \\ \theta(t) &= v(t) + t - \frac{1}{\lambda}u(t)e^{-\lambda t}. \end{aligned} \tag{6}$$

Differentiating (6) and plugging into (2), we obtain

$$\begin{aligned} u(t) &= u_0 + \frac{1}{t_0}\int_0^t \Phi_0(\theta(\tau))e^{\lambda\tau}d\tau, \\ v(t) &= v_0 + \frac{1}{\lambda t_0}\int_0^t \Phi_0(\theta(\tau))d\tau. \end{aligned} \tag{7}$$

Substituted back into (6), this gives $u_0 = r_0$, $v_0 = \theta_0 + \frac{r_0}{\lambda}$, and

$$\begin{aligned} r(t) &= \left(r_0 + \frac{1}{t_0}\int_0^t \Phi_0(\theta(\tau))e^{\lambda\tau}d\tau\right)e^{-\lambda t}, \\ \theta(t) &= \theta_0 + t + \frac{r_0}{\lambda}(1 - e^{-\lambda t}) \\ &\quad + \frac{1}{\lambda t_0}\left\{\int_0^t \Phi_0(\theta(\tau))d\tau - e^{-\lambda t}\int_0^t \Phi_0(\theta(\tau))e^{\lambda\tau}d\tau\right\}. \end{aligned} \tag{8}$$

We assume in the rest of this subsection that $|r_0| < 1$.

Recall that $K_0 = \max\{\|\Phi_0\|_{C^4}, 1\}$. We remark that in most of our bounds involving K_0, it is in fact not necessary to use the C^4-norm. For example, K_0 in Lemmas 2.1 and 2.4 can be replaced by the C^1-norm of Φ_0.

Lemma 2.1. (i) $|\theta(t_0) - \theta_0| < 5K_0t_0$ *;*
(ii)

$$\max_{t\le t_0}\left|\frac{\partial\theta(t)}{\partial\theta_0} - 1\right| < 4K_0t_0; \quad \max_{t\le t_0}\left|\frac{\partial\theta(t)}{\partial r_0}\right| < 2t_0;$$

(iii)

$$\left|\frac{\partial r(t_0)}{\partial\theta_0}\right| < 2K_0; \quad \left|\frac{\partial r(t_0)}{\partial r_0}\right| \le 2.$$

Proof. (i) From (8),

$$\begin{aligned} |\theta(t) - \theta_0| &< t + \frac{|r_0|}{\lambda}(1 - e^{-\lambda t}) + \frac{1}{\lambda t_0}\left|\int_0^t \Phi_0(1 - e^{\lambda\tau})d\tau\right| \\ &\quad + \frac{(1 - e^{-\lambda t})}{\lambda t_0}\left|\int_0^t \Phi_0 e^{\lambda\tau}d\tau\right|. \end{aligned}$$

Using the inequalities $1 - e^{-x} \le x$ and $e^x - 1 \le xe^x$ for $x > 0$ and the fact that $\lambda t_0 < \frac{1}{10}$, we see immediately that the four terms above add up to $< 5K_0t_0$.
(ii)

$$\frac{\partial\theta}{\partial\theta_0} = 1 + A + B,$$

where

$$A = \frac{1}{\lambda t_0} \int_0^t \Phi_0' \frac{\partial \theta}{\partial \theta_0} (1 - e^{\lambda \tau}) d\tau,$$
$$B = \frac{1}{\lambda t_0} (1 - e^{-\lambda t}) \int_0^t \Phi_0' \frac{\partial \theta}{\partial \theta_0} e^{\lambda \tau} d\tau. \tag{9}$$

Letting $\Theta_1 = \max_{t \le t_0} |\frac{\partial \theta(t)}{\partial \theta_0} - 1|$ and recalling that $t_0 K_0 < \frac{1}{10} K_0^{-1} \le \frac{1}{10}$, we obtain

$$\Theta_1 \le |A| + |B| \le \frac{5}{2} K_0 t_0 (\Theta_1 + 1) \le \frac{1}{4} \Theta_1 + \frac{5}{2} K_0 t_0,$$

which implies that $\Theta_1 < 4 K_0 t_0$. Similarly, writing

$$\frac{\partial \theta}{\partial r_0} = \frac{1}{\lambda} (1 - e^{-\lambda t}) + \tilde{A} + \tilde{B},$$

where

$$\tilde{A} = \frac{1}{\lambda t_0} \int_0^t \Phi_0' \frac{\partial \theta}{\partial r_0} (1 - e^{\lambda \tau}) d\tau,$$
$$\tilde{B} = \frac{1}{\lambda t_0} (1 - e^{-\lambda t}) \int_0^t \Phi_0' \frac{\partial \theta}{\partial r_0} e^{\lambda \tau} d\tau,$$

and reasoning as above, we arrive at the desired bound for $\frac{\partial \theta}{\partial r_0}$.
(iii) follows from (ii) and a straightforward computation. □

We will also need estimates on higher derivatives.

Lemma 2.2. *For* $i = 2$ *and* 3,

$$\max_{0 \le t \le t_0} \left| \frac{\partial^i \theta(t)}{\partial \theta_0^i} \right| \le 20 K_0 t_0.$$

Proof. Letting A and B be as in (9), we have

$$\frac{\partial^2 \theta}{\partial^2 \theta_0} = \frac{\partial A}{\partial \theta_0} + \frac{\partial B}{\partial \theta_0},$$

where

$$\frac{\partial A}{\partial \theta_0} = \frac{1}{\lambda t_0} \int_0^t \left(\Phi_0'' \left(\frac{\partial \theta}{\partial \theta_0} \right)^2 + \Phi_0' \frac{\partial^2 \theta}{\partial^2 \theta_0} \right) (1 - e^{\lambda \tau}) d\tau,$$
$$\frac{\partial B}{\partial \theta_0} = \frac{1}{\lambda t_0} (1 - e^{-\lambda t}) \int_0^t \left(\Phi_0'' \left(\frac{\partial \theta}{\partial \theta_0} \right)^2 + \Phi_0' \frac{\partial^2 \theta}{\partial^2 \theta_0} \right) e^{\lambda \tau} d\tau.$$

Let $\Theta_2 = \max_{0 \le t \le t_0} \left| \frac{\partial^2 \theta(t)}{\partial \theta_0^2} \right|$. Similar reasoning as before gives

$$\Theta_2 \le \frac{5}{2} K_0 t_0 (1 + \Theta_1)^2 + \frac{1}{4} \Theta_2,$$

and thus $\Theta_2 < 8 K_0 t_0$. The proof for $i = 3$ is similar. □

2.3. Relating F_T to Φ_0, λ and t_0. Let $T^1_{a,0}$ and $\hat{T}^1_{a,0}$ be as in Sect. 2.1. Reading off $\theta(t_0)$ and $r(t_0)$ from Eq. (8), we have

$$T^1_{a,0}(\theta_0, r_0) = \theta_0 + \frac{r_0}{\lambda} + \frac{1}{\lambda t_0}\int_0^{t_0} \Phi_0(\theta(t))dt + a.$$

From this, it follows immediately that $|T^1_{a,0} - \hat{T}^1_{a,0}| \le \frac{K_0 t_0}{\lambda}$. We record the following estimates for future use.

Lemma 2.3. (i)

$$\frac{4}{5\lambda} < \frac{\partial T^1_{a,0}}{\partial r_0} < \frac{6}{5\lambda};$$

(ii) *There is a numerical constant M_0 such that for $i = 1, 2, 3$,*

$$\left|\frac{\partial^i T^1_{a,0}}{\partial \theta_0^i} - \frac{\partial^i \hat{T}^1_{a,0}}{\partial \theta_0^i}\right| \le \frac{M_0 K_0^2 t_0}{\lambda}.$$

Proof. (i) Since

$$\frac{\partial T^1_{a,0}}{\partial r_0} = \frac{1}{\lambda} + \frac{1}{\lambda t_0}\int_0^{t_0} \Phi_0' \frac{\partial \theta}{\partial r_0} dt,$$

it suffices to observe that the second term on the right has absolute value bounded above by $\frac{1}{\lambda} K_0(2t_0)$ (Lemma 2.1(ii)), which is $< \frac{1}{5\lambda}$.
(ii)

$$\begin{aligned}\left|\frac{\partial T^1_{a,0}}{\partial \theta_0} - \frac{\partial \hat{T}^1_{a,0}}{\partial \theta_0}\right| &= \frac{1}{\lambda t_0}\left|\int_0^{t_0} \Phi_0'(\theta(t))(\frac{\partial \theta}{\partial \theta_0} - 1)dt + \int_0^{t_0} (\Phi_0'(\theta(t)) - \Phi_0'(\theta_0))dt\right| \\ &\le \frac{K_0}{\lambda}(4K_0 t_0 + 5K_0 t_0) = \frac{9K_0^2 t_0}{\lambda}\end{aligned}$$

by Lemma 2.1(ii) and (i). For the second derivative,

$$\begin{aligned}\frac{\partial^2 T^1_{a,0}}{\partial \theta_0^2} - \frac{\partial^2 \hat{T}^1_{a,0}}{\partial \theta_0^2} &= \frac{1}{\lambda t_0}\int_0^{t_0}\left(\Phi_0''\left(\frac{\partial \theta}{\partial \theta_0}\right)^2 + \Phi_0' \frac{\partial^2 \theta}{\partial^2 \theta_0}\right)dt \\ &\quad - \frac{1}{\lambda}\Phi_0''(\theta_0) \\ &= \frac{1}{\lambda t_0}\int_0^{t_0} \Phi_0''\left(\left(\frac{\partial \theta}{\partial \theta_0}\right)^2 - 1\right)dt \\ &\quad + \frac{1}{\lambda t_0}\int_0^{t_0} \Phi_0' \frac{\partial^2 \theta}{\partial^2 \theta_0} dt \\ &\quad + \frac{1}{\lambda t_0}\int_0^{t_0} (\Phi_0''(\theta(t)) - \Phi_0''(\theta_0))dt.\end{aligned}$$

Observe that each of the functions of the integrals is bounded by constant $\cdot K_0^2 t_0$. The third derivative is estimated similarly. □

We are now ready to estimate the derivative of F_T. Let $\hat{T}_{a,0} = (\hat{T}^1_{a,0}, 0)$. Then

$$D\hat{T}_{a,0} = \begin{pmatrix} 1 + \frac{1}{\lambda}\Phi_0' & \frac{1}{\lambda} \\ 0 & 0 \end{pmatrix}.$$

Lemma 2.4.

$$DF_T = \begin{pmatrix} 1 + \frac{1}{\lambda}\Phi_0' & \frac{1}{\lambda} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix},$$

where

$$|A_1| \le \frac{9K_0^2 t_0}{\lambda} + \frac{2K_0}{\lambda} e^{-\lambda(T-t_0)}, \quad |B_1| \le \frac{2K_0 t_0}{\lambda} + \frac{2}{\lambda} e^{-\lambda(T-t_0)},$$
$$|C_1| \le 2K_0 e^{-\lambda(T-t_0)}, \qquad |D_1| \le 2e^{-\lambda(T-t_0)}.$$

Proof. Comparing (3) and (4), and using Lemma 2.1, we see that

$$DT_{a,b} = DT_{a,0} + \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix},$$

where $|A_2| \le \frac{2K_0}{\lambda} e^{-\lambda(T-t_0)}$, $|B_2| \le \frac{2}{\lambda} e^{-\lambda(T-t_0)}$, $|C_2| \le 2K_0 e^{-\lambda(T-t_0)}$, and $|D_2| \le 2e^{-\lambda(T-t_0)}$. By Lemma 2.3,

$$DT_{a,0} = D\hat{T}_{a,0} + \begin{pmatrix} A_3 & B_3 \\ C_3 & D_3 \end{pmatrix},$$

where $|A_3| \le \frac{9K_0^2 t_0}{\lambda}$, $|B_3| \le \frac{2K_0 t_0}{\lambda}$ and $C_3 = D_3 = 0$. □

2.4. Absorbing sets. For dynamical systems with noncompact phase spaces, it is convenient to know that the action takes place in compact regions. An *absorbing set* for F_T is an open set A with compact closure such that $F_T(A) \subset A$ and for all $z \in S^1 \times \mathbb{R}$, there exists $n = n(z)$ such that $F_T^n z \in A$.

Lemma 2.5. *Assume* $\lambda(T - t_0) \ge 1$. *Then*

$$A := \{(\theta, r) \in S^1 \times \mathbb{R} : |r| < 4K_0 e^{-\lambda(T-t_0)}\}$$

is an absorbing set for F_T.

Proof. Write $(\theta_n, r_n) = F_T^n(z)$ for $z = (\theta_0, r_0)$. By (3) and (8) we have

$$|r_n| < e^{-\lambda(T-t_0)}|r_{n-1}| + 2K_0 e^{-\lambda(T-t_0)}.$$

With $e^{-\lambda(T-t_0)} < \frac{1}{2}$, this proves $F_T(A) \subset A$. The other condition follows since inductively we have

$$|r_n| < e^{-n\lambda(T-t_0)}|r_0| + 2K_0 \sum_{i=1}^{n} e^{-i\lambda(T-t_0)} < 4K_0 e^{-\lambda(T-t_0)}.$$

□

3. A View from the Singular Limit

Recall that f_a is the restriction of $T^1_{a,0}$ to $\{r_0 = 0\}$ (see Sect. 2.1). We have thus defined, for each choice of Φ_0, λ and t_0, a one-parameter family of circle maps which represents the behavior of Eq. (1) as $T \to \infty$. Conversely, one can recover information on the system defined by (1) if its singular limit $\{f_a\}$ is known: for large T or, equivalently, small $b > 0$, $F_T = T_{a,b}$ can be thought of as a perturbation of f_a or an "unfolding" of f_a to a small neighborhood of $\{r_0 = 0\}$.

In this section, we look at the problem from the point of view of the singular limit. Forgetting temporarily their connection to Eq. (1), we think of f_a as *abstract circle maps*. The following is a brief review of several types of behaviors that are known to be "typical" and a general discussion of existing methods for transporting these one-dimensional behaviors to two dimensions.

The invertible case: circle diffeomorphisms. The classical theory of Poincaré and Denjoy is well known (see any elementary text). We point out a striking resemblance between $\hat{f}_a$ and the well known family of circle maps first studied by Arnold [A]:

$$g_{\mu,\varepsilon} : x \mapsto x + \mu + \varepsilon \cos(2\pi x), \quad \varepsilon \geq 0.$$

A dichotomy of behavior was observed for this family: "resonant wedges" in the (μ, ε)-plane corresponding to rational rotation numbers, and the "devil's staircase" defined by $\mu \mapsto \rho(g_{\mu,\varepsilon})$. These ideas are very much behind our results in Theorem 1.

For us, an important question is how to bring these results for f_a to $T_{a,b}$ for $b > 0$. KAM techniques (using the intersection property) come to mind for the persistence of invariant circles with Diophantine rotation numbers, but they will not be used here. Because of strong normal contraction, invariant curves are shown to exist independent of rotation number using techniques from hyperbolic theory. The situation is then reduced to one dimension.

Smooth non-invertible circle maps. For general information on one-dimensional maps, see e.g. [dMvS]. Two types of dynamical behaviors are known to be prevalent. They are (i) maps with attractive periodic cycles, and (ii) maps with absolutely continuous invariant measures. There is some evidence that these are the only observable *pure* dynamics types. For the quadratic family $Q_a : x \mapsto 1 - ax^2$, (i) and (ii) together account for a set of full Lebesgue measure in parameter space[3] [Lyu2]. We discuss these two cases separately.

(i) Periodic sinks. Continuing to use the quadratic family as a paradigm, we see that period doubling occurs for a below some a_0 (see e.g. [CE]). In this regime, Q_a has a periodic sink which attracts all points in the interval except for a finite number of unstable periodic orbits and their pre-images. Above a_0, there is an open set of a for which Q_a has an attractive periodic orbit, but the set of points not attracted to the sink is now a complicated invariant set on which the map is uniformly expanding. The set of parameters with this property has been shown to be dense ([GS] and [Lyu1]).

When the dynamical picture of a one-dimensional map is as above, it "unfolds" into a two-dimensional diffeomorphism satisfying Smale's Axiom A [Sm]. The passage of uniform expansion in one dimension to uniform hyperbolicity in two dimensions is relatively simple due to the robustness or stability of uniform hyperbolic behavior (see e.g. [Sh]).

[3] (i) and (ii) can easily occur on different parts of the phase space for multimodal maps.

(ii) Absolutely continuous invariant measures. There is another type of dynamics that is prevalent in the *probabilistic* sense.

Theorem ([J]). *Let $Q_a : x \mapsto 1 - ax^2$, $a \in [0, 2]$. Then there exists a positive measure set of a for which Q_a has an absolutely continuous invariant probability measure with a positive Lyapunov exponent.*

The question here is: *what does the existence of absolutely continuous invariant measures for f_a tell us about $T_{a,b}$ for b small?* The answer to this question is far from simple, and the situation was only resolved quite recently. It came as a result of two different sets of developments. The first is an "abstract" theory of nonuniformly hyperbolic systems, which provides a general framework for studying chaotic behavior. The most important idea that has come out of this theory is probably the notion of an *SRB measure* ([Si, R1, B, P, R2, Le, LY, PS, Y2]; see also [Y1] for an exposition). The other set of developments is more directly related to small perturbations of one-dimensional maps. Pioneering work in this direction was carried out by Benedicks and Carleson [BC], who achieved an important breakthrough on the Hénon maps. The utility of [BC] in applications, however, is limited by the fact that it relied on computations using explicitly the formulas of the Hénon maps. For other results related to the Hénon maps, see e.g. [BY, MV].

In a recent paper [WY], we extended the analysis in [BC] to a more general class of attractors, namely those with strong dissipation, one direction of instability, and well defined singular limits. We also developed the geometric and dynamical pictures of these attractors more fully, merging some of the ideas from [BC] with those from general nonuniform hyperbolic theory. Checkable conditions were given for the first time that guarantee the existence of SRB measures and their stochastic behavior. The properties in the statement of Theorem 3 is a summary of the results in [WY], and this entire package is guaranteed once certain fairly simple conditions are satisfied. These conditions are stated and checked for our equation in Sect. 5.

4. Proofs of Theorems 1 and 2

4.1. Proof of Theorem 1. We assume throughout this subsection that λ and T satisfy the hypotheses of Theorem 1, i.e. $\lambda \geq 4K_0$ and $T - t_0 \geq \frac{3}{2}$. This implies in particular that $e^{-\lambda(T-t_0)} \leq e^{-6} < \frac{1}{100}$. Recall also that a standing assumption throughout is $t_0 < \frac{1}{10}K_0^{-2} \leq \frac{1}{10}$ (see Sect. 1.1). Let $F = F_T$.

4.1.1. Existence of invariant circles. Identifying the tangent space of $z \in S^1 \times \mathbb{R}$ with the (θ, r)-plane, we introduce the following cones:

$$\mathcal{K}^c := \{(\theta, r) : |r| < \frac{1}{4}|\theta|\},$$
$$\mathcal{K}^s := \{(\theta, r) : |r| > |\theta|\}.$$

Lemma 4.1. (a) *For $z \in \{|r| < 1\}$, $v \in \mathcal{K}^c \implies DF_z(v) \in \mathcal{K}^c$ and $|DF_z(v)| > \frac{1}{3}|v|$.*
(b) *For z with $F^{-1}z \in \{|r| < 1\}$, $v \in \mathcal{K}^s \implies DF_z^{-1}(v) \in \mathcal{K}^s$ and $|DF_z^{-1}(v)| > 10|v|$.*

Proof. Write

$$DF_z = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Substituting the admissible values of λ, T and t_0 into Lemma 2.4, we obtain

$$|A-1| < \frac{1}{2}, \quad |B| < \frac{1}{3}, \quad \text{and} \quad |C|, |D| < \frac{1}{50}. \tag{10}$$

(The estimate for $|C|$ uses $K_0 e^{-6K_0} < e^{-6}$.) Let $s(v)$ denote the slope of a vector $v \in \mathbb{R}^2$. Then

$$s(DF(v)) = \frac{C + Ds(v)}{A + Bs(v)}.$$

To verify $DF(\mathcal{K}^c) \subset \mathcal{K}^c$, for example, we choose v with $|s(v)| < \frac{1}{4}$, and substituting in the numbers from (10), we obtain

$$|s(DF(v))| < \frac{\frac{1}{50} + \frac{1}{50}\frac{1}{4}}{\frac{1}{2} - \frac{1}{3}\frac{1}{4}} < \frac{1}{4}.$$

The other claims are checked similarly. □

We have thus identified a family of *stable cones* $\mathcal{K}^s$ and a family of *center cones* $\mathcal{K}^c$. We call $\mathcal{K}^c$ "center cones" because while vectors in $\mathcal{K}^c$ may be expanded or contracted by DF, they are not contracted as strongly as vectors in $DF^{-1}(\mathcal{K}^s)$. This domination implies uniform hyperbolicity on the projective level, a property relied upon heavily in the proof of the next lemma.

Recall that $A = \{|r| < 4K_0 e^{-\lambda(T-t_0)}\}$ is an absorbing set of F (Lemma 2.5).

Lemma 4.2. *There is an F-invariant curve Ω in A such that*

(a) *Ω is the graph of a C^4 function $g : S^1 \to \mathbb{R}$ with $|g'| < 1/4$;*
(b) *for every $z \in A$, $dist(F^n z, \Omega) \to 0$ as $n \to \infty$.*

Proof. By standard arguments from hyperbolic theory, it follows from Lemma 4.1 that there is a stable foliation W^s defined everywhere on A. Tangent vectors to the leaves of W^s satisfy $|s(v)| > 1$, so that each W^s-leaf is a C^1 segment joining the two boundary components of A. Moreover, F maps each W^s-leaf strictly into a W^s-leaf, contracting length by a factor $< \frac{1}{10}$. It follows from this that $\Omega := \cap_{n>0} F^n(A)$ is a compact set which meets each W^s-leaf in exactly one point. Part (b) of Lemma 4.2 follows immediately.

Let γ_0 be the curve $\{r = 0\}$. Then the images $\gamma_n := F^n \gamma_0$ converge in the Hausdorff metric to Ω, the center manifold of F. By Lemma 4.1(a), the tangent vectors to γ_n have slopes between $\pm 1/4$ for all n. This proves that Ω is the graph of a Lipschitz function g with Lipschitz constant $\leq 1/4$. That g is C^4 follows from the fact that F is C^4 and standard graph transform arguments involving the Fiber Contraction Theorem. We refer the reader to [HPS]. □

4.1.2. Dynamics on invariant circles. For each T, let Ω_T be the simple closed curve left invariant by F_T. We introduce a family of maps $h_T : S^1 \to S^1$ as follows: For $\theta_0 \in S^1$, let z be the unique point in Ω_T whose θ-coordinate is θ_0. Then $h_T(\theta_0) = \theta_1$, where θ_1 is the θ-coordinate of $F_T(z)$. Let $\rho(h_T)$ denote the rotation number of h_T. Since

$$\frac{d\theta_1}{dT} > 1 - e^{-\lambda(T-t_0)}|r(t_0)| > \frac{99}{100}, \tag{11}$$

it is an easy exercise to see that $T \mapsto \rho(h_T)$ is a continuous nondecreasing function with $\rho(h_{T+1}) \approx \rho(h_T) + 1$.

Case 1. $\rho(h_T) \in \mathbb{R} \setminus \mathbb{Q}$. By Denjoy theory, h_T is topologically conjugate to the rigid rotation by $\rho(h_T)$, which is well known to admit only one invariant probability measure. This together with Lemmas 2.5 and 4.2(b) imply immediately the unique ergodicity of F_T. To prove that Δ_0 in Theorem 1 has positive Lebesgue measure, we appeal to the following theorem of Herman:

Theorem ([He]). *Let* $\mathrm{Diff}^r_+(S^1)$ *denote the space of* C^r *orientation-preserving diffeomorphisms of* S^1. *Let* $s \mapsto h_s \in \mathrm{Diff}^3_+(S^1)$ *be* C^1 *and suppose that for some* $s_0 < s_1$, $\rho(h_{s_0}) \neq \rho(h_{s_1})$. *Then* $\{s \in [s_0, s_1] : \rho(h_s) \in \mathbb{R} \setminus \mathbb{Q}\}$ *has positive Lebesgue measure.*

Case 2. $\rho(h_T) \in \mathbb{Q}$. We fix $p, q \in \mathbb{Z}^+$, p, q relatively prime, and let I be a connected component of $\{T : \rho(h_T) = \frac{p}{q}\}$ with nonempty interior. From (11), it follows that $\frac{d}{dT}(h^q_T(\theta_0)) > \frac{99}{100}$ for every θ_0. Standard transversality arguments give an open and dense subset $\tilde{I}$ of I such that for $T \in \tilde{I}$, the graph of h^q_T is transversal to the diagonal of $S^1 \times S^1$. For $T \in \tilde{I}$, the fixed points of h^q_T (in the order in which they appear on S^1) are alternately strictly repelling and strictly contracting. With the contraction normal to Ω_T, they correspond to saddles and sinks respectively for F_T.

This completes the proof of Theorem 1.

4.2. Proof of Theorem 2. Our analysis will proceed as follows. Referring the reader to Sect. 2.1 for definitions and notation, we will argue that uniformly expanding invariant sets of f_a translate directly into uniformly hyperbolic invariant sets of $T_{a,b}$ for b sufficiently small. That being the case, to produce the phenomena described in Theorem 2, it suffices to produce the corresponding behaviors for f_a. Furthermore, since uniformly expanding invariant sets are stable under perturbations, and f_a is a small perturbation of $\hat{f}_a$ for $t_0 << \lambda$ (Lemma 2.3), it suffices to work with $\hat{f}_a$. Recall that

$$\hat{f}_a(s) = s + \frac{1}{\lambda}\Phi_0(s) + a.$$

4.2.1. Gradient-like dynamics. Let $m_0 = -\min \Phi_0'$. Then $\hat{f}_a$ is a circle diffeomorphism if and only if $\lambda > m_0$. Fix $\lambda > m_0$. Varying a (which corresponds to moving the graph of $\hat{f}_a$ up and down), we see that there is an open set of a for which $\hat{f}_a$ has a finite number of fixed points which are alternately repelling and attracting. For these a, it is a simple exercise to show that for sufficiently small t_0 and b, $F_T = T_{a,b}$ has the gradient-like dynamics described in Theorem 2. More generally, if $\rho(\hat{f}_a) = \frac{p}{q}$, then the discussion above applies to $\hat{f}^q_a$ unless $\hat{f}^q_a = id$.

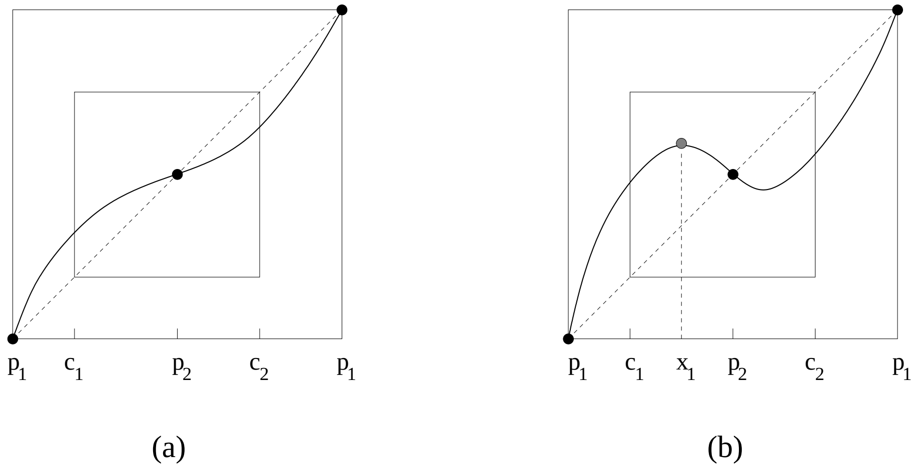

Fig. 3 a,b.

Gradient-like dynamics, in general, persist when λ drops below m_0. Intuitively, no simple closed invariant curve exists beyond this point because the unstable manifold of the saddle "turns around". We provide a rigorous proof in a restricted context.

Proposition 4.1. *Suppose Φ_0 has exactly two critical points and negative Schwarzian derivative. Then there exist intervals of λ, t_0 and T for which F_T has gradient-like dynamics but there are no smooth simple closed invariant curves.*

Proof. Let c_1 and c_2 denote the critical points of Φ_0. There is an interval of a_0 such that if $\Phi_0 = a_0 + \tilde{\Phi}_0$, then $\tilde{\Phi}_0$ has exactly two zeros, at say p_1 and p_2. Fix such an a_0. Without loss of generality, we assume $p_1 < c_1 < p_2 < c_2 < p_1 + 1 = p_1$, and $\Phi_0'(p_1) > 0$, $\Phi_0'(p_2) < 0$. In the rest of the proof, for each λ we consider, let $f = \hat{f}_a$, where $a = -\frac{a_0}{\lambda}$ mod 1, so that $f(s) = s + \frac{1}{\lambda}\tilde{\Phi}_0(s)$. Observe that p_1 is a repelling fixed point of f, p_2 is an attractive fixed point of f, and $f'(c_1) = f'(c_2) = 1$. This discussion is valid for all λ.

For large λ, f maps (c_1, c_2) strictly into itself. (See Fig. 3(a).) This continues to be the case for some interval of λ below m_0. Since $\Phi_0' < 0$ on (c_1, c_2), we have $1 - \frac{m_0}{\lambda} < f' < 1$ on (c_1, c_2), so there exist $\varepsilon, \varepsilon' > 0$ and an interval L of λ below m_0 for which $f(c_1 + \varepsilon, c_2 - \varepsilon) \subset (c_1 + 2\varepsilon, c_2 - 2\varepsilon)$ and $|f'|_{(c_1+\varepsilon, c_2-\varepsilon)}| < 1 - \varepsilon'$. (See Fig. 3(b).) Thus every point in $(c_1 + \varepsilon, c_2 - \varepsilon)$ tends to p_2, and since every point in $S^1 \setminus (c_1 + \varepsilon, c_2 - \varepsilon)$ eventually enters $(c_1 + \varepsilon, c_2 - \varepsilon)$, we conclude that f and hence $F = T_{a,b}$ have gradient-like dynamics for a as above and t_0 and b suitably small.

Let $\tilde{p}_1$ and $\tilde{p}_2$ denote the saddle and sink of F respectively. To prove the proposition, suppose F leaves invariant a smooth simple closed curve Ω. Since it is not possible for all the points in an invariant circle to converge to the same point, Ω must intersect the stable manifold of $\tilde{p}_1$. This implies $\tilde{p}_1 \in \Omega$, and hence W^u, the unstable manifold of $\tilde{p}_1$, must be contained in Ω. Fix an orientation on Ω, and let τ be a positively oriented tangent field on W^u. To derive a contradiction, we will produce, for every $\varepsilon_1 > 0$, two points $z, z' \in W^u$ such that $d(z, z') < \varepsilon_1$ and $\tau(z)$ and $\tau(z')$ point in opposite directions.

By the negative Schwarzian property of Φ_0, $f' = 0$ at exactly two points $x_1 < x_2$ in (c_1, c_2). Move λ if necessary so $x_i \neq p_2$, $i = 1, 2$. Without loss of generality, we

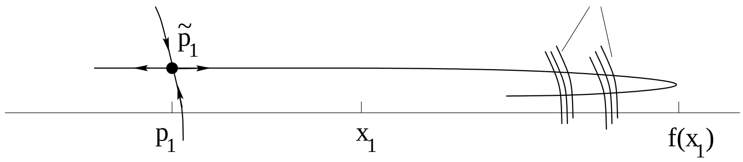

Fig. 4.

may assume $x_1 \in (c_1, p_2)$. The following two statements, which we claim are valid for suitable choices of t_0, a and b, clearly lead to the desired contradiction.

(1) The right branch of W^u is roughly horizontal until about $f(x_1)$, where it makes a sharp turn and doubles back for a definite distance, creating two roughly parallel segments with opposite orientation (see Fig. 4).

(2) There exist pairs of points on these parallel segments joined by stable curves.

Claims (1) and (2) follow from Lemma 4.3, which is a general result valid for any λ and any Φ_0 (and not just the ones considered in this subsection). It is similar in spirit to Lemma 4.2 and has the same proof, which will be omitted.

Lemma 4.3. *Given f_a and constants $\delta, \varepsilon > 0$, $\exists \bar{b} = \bar{b}(\Phi_0, \lambda, \delta, \varepsilon) << \delta$ such that the following hold for $F = T_{a,b}$ with $b < \bar{b}$. Let $z = (r, \theta) \in A$ (which depends on b) be such that $|f_a'(\theta)| > \delta$. Then:*

(a) $|s(v)| = \mathcal{O}(\frac{b}{\delta}) \implies |s(DF_z v)| = \mathcal{O}(\frac{b}{\delta})$ *and* $|DF_z v| > (1-\varepsilon)\delta |v|$*;*

(b) *there exists $C = C(\Phi_0, \lambda)$ such that $|s(DF_z v)| > C\delta \implies |s(v)| > C\delta$ and $\frac{|DF_z v|}{|v|} = \mathcal{O}(\frac{b}{\delta})$.*

Claim (1) follows immediately from Lemma 4.3(a). Part (b) of this lemma implies that if a region of A misses the two rectangles $\{(r, \theta) : |f'(\theta)| < \delta\}$ in all of its forward iterates, then it is foliated by stable curves. Since $f'(p_2) \neq 0$, Claim (2) is easily arranged by choosing δ sufficiently small. □

4.2.2. Transient chaos. We return to the family $\hat{f}_a$ where λ is now assumed to be small. Let c_1 and c_2 be the critical points of Φ_0. Then $\hat{f}_a$ has exactly two critical points s_1 and s_2 near c_1 and c_2. Let a be fixed for now. As λ is varied, the critical values $\hat{f}_a(s_1)$ and $\hat{f}_a(s_2)$ move at rates $\sim \frac{1}{\lambda}$ in opposite directions. There exists, therefore, a sequence of λ for which they coincide. Observe that this sequence is independent of a. We now fix each of these λ and adjust a so that $\hat{f}_a(s_1) = s_1$, where s_1 is the critical point with the property that $|\Phi_0''(c_1)| \leq |\Phi_0''(c_2)|$. We will show that for the (λ, a)-pairs selected above, $f = \hat{f}_a$ has the following properties: (i) it has a sink, and (ii) when restricted to the set of points that are not attracted to the sink, f is uniformly expanding.

By design, we have $f(s_1) = s_1$, which is therefore a sink, and $f(s_2) = s_1$. For $i = 1, 2$, let $\alpha_i = \frac{\sqrt{1.5}}{|\Phi_0''(c_i)|}\lambda$ and $I_i = [s_i - \alpha_i, s_i + \alpha_i]$.

Lemma 4.4. *Assume λ is sufficiently small. Then*

(a) *for $s \notin I_1 \cup I_2$, we have $|f'(s)| > \sqrt{1.4}$;*
(b) *for $s \in I_1 \cup I_2$, we have $f^n s \to s_1$ as $n \to \infty$.*

Proof. (a) We may assume for $s \notin I_1 \cup I_2$ that $|f'(s)| \geq |f'(s_i \pm \alpha_i)|$ for some i. Since this is $= \frac{1}{\lambda}|\Phi_0''(\xi_i)|\alpha_i$ for some $\xi_i \in I_i$, it is $> \sqrt{1.4}$.
(b) First we check $f(I_i) \subset I_1$, $i = 1, 2$:

$$|f(s_i \pm \alpha_i) - f(s_i)| = \frac{1}{2\lambda}|\Phi_0''(\xi_i)|\alpha_i^2 \leq \frac{1}{2\lambda}|\Phi_0''(\xi_i)| \cdot \frac{1.5}{|\Phi_0''(c_i)|^2}\lambda^2$$
$$\leq \frac{\lambda}{|\Phi_0''(c_i)|} \leq \frac{\lambda}{|\Phi_0''(c_1)|} < \alpha_1.$$

A similar computation shows that f restricted to I_1 is a contraction. □

Let $F = T_{a,b}$, where λ and a are near the ones selected above and t_0 and b are sufficiently small. Let B_i, $i = 1, 2$, be the two components of $A \setminus \{(\theta, r) : \theta \in I_1 \cup I_2\}$. With λ sufficiently small, F wraps each B_i around A (in the horizontal direction) at least once, with $F(B_i)$ crossing completely B_j every time they meet. This, on the topological level, is the standard construction of a horseshoe. Let

$$\Lambda := \{z \in A : F^n(z) \in B_1 \cup B_2 \quad \forall n \in \mathbb{Z}\}.$$

With b sufficiently small, the uniform hyperbolicity of $F|\Lambda$ follows from Lemma 4.3.
This completes the proof of Theorem 2.

5. Proof of Theorem 3

5.1. Conditions from [WY] for strange attractors. As explained in the introduction, the proof of Theorem 3 is obtained largely via a direct application of [WY] – provided the conditions in Sect. 1.1 of [WY] are verified. For the convenience of the reader, we give a self-contained discussion of these conditions here, modifying one of them to improve its checkability and adding a new one, (C4), to guarantee mixing. The notation in this section is that in [WY].

We consider a family of maps $T_{a,b} : A = S^1 \times [-1, 1] \to A$, where $a \in [a_0, a_1] \subset \mathbb{R}$ and $b \in B_0 \subset \mathbb{R}$, B_0 being any subset with 0 as an accumulation point.[4]

In this setup, b is a measure of dissipation; our results hold for b sufficiently small. We explain the role of the parameter a: For systems that are not uniformly hyperbolic, a scenario that competes with that of strange attractors and SRB measures is the presence of periodic sinks. In general, arbitrarily near systems with SRB measures, there are open sets of maps with sinks; proving directly the existence of an SRB measure for a given dynamical system requires information of arbitrarily high precision. We get around this problem by considering one-parameter families, in our case $a \mapsto T_{a,b}$, and by showing that if a family satisfies certain reasonable conditions, then a positive measure set of parameters with SRB measures is guaranteed. We now state our conditions on these families.

[4] In [WY], B_0 is taken to be an interval but the formulation here is all that is used.

(C1) Regularity conditions.

(i) *For each* $b \in B_0$*, the function* $(x, y, a) \mapsto T_{a,b}(x, y)$ *is* C^3*; and as* $b \to 0$*, these functions converge in the* C^3 *norm to* $(x, y, a) \mapsto T_{a,0}(x, y)$.

(ii) *For each* $b \neq 0$*,* $T_{a,b}$ *is an embedding of* A *into itself, whereas* $T_{a,0}$ *is a singular map with* $T_{a,0}(A) \subset S^1 \times \{0\}$.

(iii) *There exists* $K > 0$ *such that for all* a, b *with* $b \neq 0$,

$$\frac{|\det DT_{a,b}(z)|}{|\det DT_{a,b}(z')|} \leq K \quad \forall z, z' \in S^1 \times [-1, 1].$$

As before, we refer to $T_{a,0}$ as well as its restriction to $S^1 \times \{0\}$, i.e. the family of one-dimensional maps $f_a : S^1 \to S^1$ defined by $f_a(x) = T_{a,0}(x, 0)$, as the *singular limit* of $T_{a,b}$. The rest of our conditions are imposed on the singular limit alone.

The second condition in [WY] is:

(C2) *There exists* $a^* \in [a_0, a_1]$ *such that* $f = f_{a^*}$ *satisfies the Misiurewicz condition.*

The Misiurewicz condition (see [M]) encapsulates a number of properties some of which are hard to check or not needed in full force. We propose here to replace it by (C2'), a set of conditions that is more directly checkable (although a little cumbersome to state). That the results in [WY] are valid when (C2) is replaced by (C2') below is proved in Lemma A.1 in the Appendix.

(C2') Existence of a sufficiently expanding map from which to perturb. *There exists* $a^* \in [a_0, a_1]$ *such that* $f = f_{a^*}$ *has the following properties: There are numbers* $c_1 > 0$*,* $N_1 \in \mathbb{Z}^+$*, and a neighborhood* I *of the critical set* C *such that*

(i) f *is expanding on* $S^1 \setminus I$ *in the following sense:*
  (a) *if* $x, fx, \cdots, f^{n-1}x \notin I, n \geq N_1$*, then* $|(f^n)'x| \geq e^{c_1 n}$*;*
  (b) *if* $x, fx, \cdots, f^{n-1}x \notin I$ *and* $f^n x \in I$*, any* n*, then* $|(f^n)'x| \geq e^{c_1 n}$*;*

(ii) $f^n x \notin I \ \forall x \in C$ *and* $n > 0$*;*

(iii) *in* I*, the derivative is controlled as follows:*
  (a) $|f''|$ *is bounded away from* 0*;*
  (b) *by following the critical orbit, every* $x \in I \setminus C$ *is guaranteed a recovery time* $n(x) \geq 1$ *with the property that* $f^j x \notin I$ *for* $0 < j < n(x)$ *and* $|(f^{n(x)})'x| \geq e^{c_1 n(x)}$.

Next we introduce the notion of *smooth continuations*. Let C_a denote the critical set of f_a. For $x = x(a^*) \in C_{a^*}$, the continuation $x(a)$ of x to a near a^* is the unique critical point of f_a near x. If p is a hyperbolic periodic point of f_{a^*}, then $p(a)$ is the unique periodic point of f_a near p having the same period. It is a fact that in general, if p is a point whose f_{a^*}-orbit is bounded away from C_{a^*}, then for a sufficiently near a^*, there is a unique point $p(a)$ with the same symbolic itinerary under f_a.

(C3) Conditions on f_{a^*} **and** $T_{a^*,0}$**.**

(i) **Parameter transversality.** *For each* $x \in C_{a^*}$*, let* $p = f(x)$*, and let* $x(a)$ *and* $p(a)$ *denote the continuations of* x *and* p *respectively. Then*

$$\frac{d}{da} f_a(x(a)) \neq \frac{d}{da} p(a) \quad \text{at } a = a^*.$$

(ii) **Nondegeneracy at "turns".**

$$\frac{\partial}{\partial y} T_{a^*,0}(x,0) \neq 0 \quad \forall x \in C_{a^*}.$$

The following fact often facilitates the checking of condition (C3)(i):

Lemma 5.1 ([TTY], Sect. VII**).** *Let $f = f_{a^*}$, and suppose $\sum_{n\geq 0} \frac{1}{|(f^n)'(fx)|} < \infty$ for all $x \in C$. Then*

$$\sum_{k=0}^{\infty} \frac{[(\partial_a f_a)(f^k x)]_{a=a^*}}{(f^k)'(fx)} = \left[\frac{d}{da} f_a(x(a)) - \frac{d}{da} p(a)\right]_{a=a^*}.$$

The main conditions in [WY] are contained in (C1)–(C3) (or, equivalently, (C1), (C2') and (C3)). The conclusions of Theorem 3, however, are more specific than those of [WY], which allow the co-existence of multiple ergodic SRB measures. We now introduce a fourth condition,[5] which along with (C1)–(C3) implies the uniqueness of SRB measures and their mixing properties. This implication is proved in Lemma A.2 in the Appendix.

(C4) Conditions for mixing.

(i) *$e^{c_1} > 2$ where c_1 is in (C2').*

(ii) *Let $J_1, \cdots, J_r$ be the intervals of monotonicity of f_{a^*}, and let $P = (p_{i,j})$ be the matrix defined by*

$$p_{i,j} = \begin{cases} 1 & \text{if } f(J_i) \supset J_j, \\ 0 & \text{otherwise}. \end{cases}$$

Then there exists $N_2 > 0$ such that $P^{N_2} > 0$.

The discussion in this subsection can be summarized as follows:

Theorem 3'. *Assume $\{T_{a,b}\}$ satisfies (C1), (C2'), (C3) and (C4) above. Then for all sufficiently small $b > 0$, there is a positive measure set of a for which $T_{a,b}$ has the properties in (1), (2) and (3) of Theorem 3.*

We remark that [WY] contains a more detailed description of the dynamical picture than the statement of Theorem 3 and refer the interested reader there for more information.

In the rest of this section the discussion pertains to the differential Eq. (1) defined in Sect. 1.1. All notation is as in Sect. 2.1. To prove Theorem 3, it suffices to verify that for the parameters in question, $T_{a,b}$ satisfies the conditions above. This is carried out in the next three subsections.

[5] Condition (*) in Sect. 1.2 of [WY], the only condition in [WY] not implied by (C1)–(C3), is clearly contained in (C4).

5.2. Verification of (C2'): Expanding properties. Among the conditions to be checked, (C2'), which guarantees a suitable environment from which to perturb, is arguably the most fundamental of the four. It is also the one that requires the most work. In this subsection, we will – after placing some restrictions on λ and t_0 – show that (C2') is valid for all f_a for which (C2')(ii) is satisfied. The existence of a satisfying (C2')(ii) is the topic of the next subsection.

Let $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_{k_1}$ be the critical points of Φ_0, and let $k_2 = \min\{1, \frac{1}{2}\min_i |\Phi_0''(\bar{x}_i)|\}$. We fix $\varepsilon = \varepsilon(\Phi_0) > 0$ with the property that $|\bar{x}_i - \bar{x}_j| > 4\varepsilon$ for $i \neq j$ and $|\Phi_0''| > k_2$ on $\cup_i(\bar{x}_i - 2\varepsilon, \bar{x}_i + 2\varepsilon)$, and claim that by choosing λ and t_0 sufficiently small, we may assume the following about f_a. Let C denote the critical set of f_a, and let C_ε denote the ε-neighborhood of C. Then

(i) $C = \{x_1, \cdots, x_{k_1}\}$ with $|x_i - \bar{x}_i| < \varepsilon$;
(ii) on C_ε, $|f_a''| > \frac{k_2}{\lambda}$.

To justify these claims, observe first that by taking λ small enough, the critical set of $\hat{f}_a$ can be made arbitrarily close to that of Φ_0. Second, by choosing t_0 sufficiently small (independent of λ), we can make $\|f_a - \hat{f}_a\|_{C^3} < \frac{\varepsilon_1}{\lambda}$ for ε_1 as small as we please (Lemma 2.3). These observations together with $\hat{f}_a'' = \frac{1}{\lambda}\Phi_0''$ imply (i) and (ii).

A number of other conditions will be imposed on λ; they will be specified as we go along. Some of these conditions are determined via an auxiliary constant $K > 1$ which depends only on Φ_0 and which will be chosen to be large enough for certain purposes. Let $\sigma := 2k_2^{-1}K^3\lambda$. We assume $\frac{1}{2}\sigma < \varepsilon$, so that $|f_a'(x)| > K^3$ for $x \in C_\varepsilon \setminus C_{\frac{1}{2}\sigma}$. We also assume λ is small enough that $|f_a'| > K^3$ outside of C_ε. Together these imply

(iii) $|f_a'| > K^3$ outside of $C_{\frac{1}{2}\sigma}$.

For simplicity of notation, we write $f = f_a$ in the rest of this subsection.

Lemma 5.2. *Let $c \in C$ be such that $f^n(c) \notin C_\sigma \forall n > 0$. Consider x with $|x - c| < \frac{1}{2}\sigma$, and let $n(x)$ be the smallest n such that $|f^n(x) - f^n(c)| > \frac{1}{3K_0}K^3\lambda$. Then $n(x) > 1$ and $|(f^{n(x)})'| \geq k_3 K^{n(x)}$ for some $k_3 = k_3(K_0, k_2)$.*

Before giving the proof of this lemma, we first prove a distortion estimate.

Sublemma 5.1. *Let $x, y \in S^1$ and $n \in \mathbb{Z}^+$ be such that ω_i, the segment between $f^i x$ and $f^i y$, satisfies $|\omega_i| < \frac{1}{3K_0}K^3\lambda$ and dist$(\omega_i, C) > \frac{1}{2}\sigma$ for all i with $0 \leq i < n$. Then*

$$\frac{(f^n)'x}{(f^n)'y} \leq 2.$$

Proof.

$$\begin{aligned}
\log \frac{(f^n)'x}{(f^n)'y} &= \sum_{i=0}^{n-1} \log \frac{f'(f^i x)}{f'(f^i y)} \leq \sum_{i=0}^{n-1} \frac{|f'(f^i x) - f'(f^i y)|}{|f'(f^i y)|} \\
&\leq \sum_{i=0}^{n-1} \frac{(1 + \frac{K_0}{\lambda})|f^i x - f^i y|}{K^3} \\
&< \frac{(1 + \frac{K_0}{\lambda})}{K^3} \left(\sum_{i=0}^{n-1} \frac{1}{K^{3i}} \right) |f^{n-1}x - f^{n-1}y|.
\end{aligned}$$

Assuming that $\frac{1}{\lambda}$ and K are sufficiently large, this is $< \frac{1}{2}$. □

Proof of Lemma 5.2. First we show $n(x) > 1$. Given the location of x, we have

$$K > |f'x| = |f''(\xi)||x - c|$$

for some ξ between x and c. This implies

$$|fx - fc| = \frac{1}{2}|f''(\zeta)||x - c|^2 < \frac{1}{2}\frac{|f''(\zeta)|}{|f''(\xi)|^2}K^2$$

which we may assume is $< \frac{1}{3K_0}K^3\lambda$. For $n(x) = 2$, use

$$|(f^2)'x| \cdot |x - c| \geq \text{const}K^3\lambda \quad \text{and} \quad |x - c| < \text{const}K\lambda.$$

We assume from here on that $n = n(x) \geq 3$, and estimate $|(f^n)'(x)|$ as follows. Since $|f^n x - f^n c| > \frac{1}{3K_0}K^3\lambda$, it follows from Sublemma 5.1 that for some ξ_1,

$$\frac{1}{2}|f''(\xi_1)||x - c|^2 \cdot 2|(f^{n-1})'(fc)| > \frac{1}{3K_0}K^3\lambda. \tag{12}$$

Reversing the inequality at time $n - 1$ and using Sublemma 5.1 again, we have

$$\frac{1}{2}|f''(\xi_2)||x - c|^2 \cdot \frac{1}{2}|(f^{n-2})'(fc)| < \frac{1}{3K_0}K^3\lambda. \tag{13}$$

Substituting the estimate for $|(f^{n-1})'(fc)|$ from (12) into

$$|(f^n)'x| \geq |f''(\zeta)||x - c| \cdot \frac{1}{2}|(f^{n-1})'(fc)|,$$

we obtain

$$|(f^n)'x| \geq \frac{1}{2}\frac{|f''(\zeta)|}{|f''(\xi_1)|}\frac{1}{2K_0}K^3\lambda\frac{1}{|x - c|}.$$

Now plug the estimate for $|x - c|$ from (13) into the last inequality and use the lower bounds for $|f''(\xi_2)|$ and $|(f^{n-2})'(fc)|$ from (ii) and (iii) earlier on in this subsection. We arrive at the estimate

$$|(f^n)'x| > \frac{1}{2}\frac{|f''(\zeta)|}{|f''(\xi_1)|}\frac{1}{3K_0}K^3\lambda\sqrt{\frac{\frac{k_2}{\lambda}K^{3(n-2)}}{4\frac{1}{3K_0}K^3\lambda}} = \text{constK}^{\frac{3}{2}(n-2)+\frac{3}{2}}.$$

The power to which K is raised is $\geq n$ for $n \geq 3$. This completes the proof of Lemma 5.2. □

We have proved the following: Suppose f_a has the property that each of its critical points c satisfies $f_a^n(c) \notin C_\sigma$ for all $n > 0$. Then (C2')(i) and (iii) hold for f_a with $I = C_{\frac{1}{2}\sigma}$. This follows from properties (ii) and (iii) in the first part of this subsection and from Lemma 5.2.

5.3. Verification of (C2'): "Multiple Misiurewicz points". The goal of this section is to show that for many values of the parameter a, f_a has the property that its critical orbits (in strictly positive time) stay away from its critical set. Precise statements will be formulated later. We remark that for the quadratic family $x \mapsto 1 - ax^2$ or any other family with a single critical point, this is a trivial exercise: there are many periodic orbits or compact invariant Cantor sets Λ disjoint from the critical set, and if changes in parameter correspond to the movement of $f_a(c)$ in a reasonable way, then there would be many parameters for which $f_a(c) \in \Lambda$. We call these parameters "Misiurewicz points". For maps with more than one critical point, as circle maps necessarily are, the required condition is that *all* of the critical orbits are trapped in some invariant set away from C. This is clearly more problematic, especially with Λ having measure zero. We call parameters with these properties "multiple Misiurewicz points". Their existence and $\mathcal{O}(\lambda)$-density within the family $\{f_a\}$ is the concern of this subsection.

Recall that $\sigma = 2k_2^{-1}K^3\lambda$ and C_σ is the σ-neighborhood of C. Recall also from Sect. 5.2 that outside of C_σ, $|f_a'| > K^3$. We are looking for a parameter a^* such that $f = f_{a^*}$ has the property that for all $c \in C$, $f^n c \notin C_\sigma \forall n > 0$. Write $C = \{x_1, \cdots, x_{k_1}\}$ as before, and let Δ be a parameter interval. For $k = 1, 2, \cdots, k_1$ and $i = 1, 2, \cdots$, we introduce the *curves of critical points*

$$a \mapsto \gamma_i^{(k)}(a) := f_a^i(x_k), a \in \Delta.$$

Observe that for all k, $\frac{d}{da}\gamma_1^{(k)} = 1$, and for all i,

$$\frac{d}{da}\gamma_{i+1}^{(k)}(a) = \frac{d}{da}\gamma_i^{(k)}(a) f_a'(\gamma_i^{(k)}(a)) + 1.$$

Thus if $\gamma_j^{(k)}(a) \notin C_\sigma$ for all $j \le i$ and K is sufficiently large, then

$$\frac{d}{da}\gamma_{i+1}^{(k)}(a) \approx \frac{d}{da}\gamma_i^{(k)}(a) f_a'(\gamma_i^{(k)}(a)) \tag{14}$$

and

$$\frac{d}{da}\gamma_{i+1}^{(k)}(a) \ge \frac{1}{2}K^{3i}. \tag{15}$$

We also have the following distortion estimate:

Sublemma 5.2. *For $k = 1, 2, \cdots, k_1$ and $n \in \mathbb{Z}^+$, let $\Delta \subset [0, 1)$ be such that $\gamma_i^{(k)}(a) \notin C_\sigma$ for $i = 1, 2, \cdots, n-1$. Assume that $|\gamma_{n-1}^{(k)}| \le \frac{1}{3K_0}K^3\lambda$. Then for all $a, a' \in \Delta$, we have*

$$\left| \frac{\frac{d}{da}\gamma_n^{(k)}(a)}{\frac{d}{da}\gamma_n^{(k)}(a')} \right| \le 2.$$

Using (14) and (15), we see that the proof is entirely parallel to that of Sublemma 5.1 with slightly weaker estimates. We leave it as an exercise for the reader.

Let d be the minimum distance between critical points. Choosing λ sufficiently small, we may assume $6k_1\sigma << d$. The following is the main result of this subsection.

Lemma 5.3. *Given $\Delta_0 \subset [0, 1)$ with $|\Delta_0| = 6k_1\sigma$, there exists $a^* \in \Delta_0$ such that $\forall c \in C$, $f_{a^*}^n c \notin C_\sigma \forall n > 0$.*

Proof. We describe first an algorithm for selecting a sequence of intervals $\Delta_0 \supset \Delta_1 \supset \Delta_2 \supset \cdots$ so that $a^* \in \cap_i \Delta_i$ has the desired property:

At step n, the (k_1+1)-tuple $(\Delta_n; i_{1,n}, i_{2,n}, \cdots, i_{k_1,n})$ is called an "admissible configuration" if Δ_n is a subinterval of Δ_0, $i_{k,n} \leq n$, and the following conditions are satisfied for each k:

(A1) $\gamma_i^{(k)}|_{\Delta_n} \cap C_\sigma = \emptyset$ for all $i \leq i_{k,n}$;
(A2) for all $a, a' \in \Delta_n$,

$$\left| \frac{\frac{d}{da}\gamma_{i_{k,n}}^{(k)}(a)}{\frac{d}{da}\gamma_{i_{k,n}}^{(k)}(a')} \right| \leq 2;$$

(A3) ("minimum length condition") $|\gamma_{i_{k,n}+1}^{(k)}|_{\Delta_n}| \geq 12k_1\sigma$.

Observe that (A3) is about the length of the critical curve one iterate later.

Let us first show that we have an admissible configuration for $n = 1$. Let $i_{k,1} = 1$ for all k. The parameter interval Δ_1 is chosen as follows. Since $\frac{d}{da}\gamma_1^{(k)} = 1$, we have $|\gamma_1^{(k)}|_{\Delta_0}| = 6k_1\sigma$, so that $\gamma_1^{(k)}$ meets at most one component of C_σ and $|(\gamma_1^{(k)})^{-1}C_\sigma| \leq 2\sigma$. Even in the worst case scenario when all k_1 intervals $(\gamma_1^{(k)})^{-1}C_\sigma$ are evenly spaced, there exists an interval $\Delta_1 \subset \Delta_0$ with $|\Delta_1| = 2\sigma$ such that $\gamma_1^{(k)}|_{\Delta_1} \cap C_\sigma = \emptyset$ for all k. Equations (A1) and (A2) are trivially satisfied, as is (A3) since $|\gamma_2^{(k)}|_{\Delta_1}| > 2\sigma K^3$, and $2K^3$ is assumed to be $> 12k_1$.

We now discuss how to proceed at a generic step, i.e. step n, assuming we are handed an admissible configuration $(\Delta_n; i_{1,n}, i_{2,n}, \cdots, i_{k_1,n})$. First, we divide the set $\{1, 2, \cdots, k_1\}$ into indices k that are "ready to advance", meaning the situation is right for the k^{th} curve to progress to the next iterate, and those that are not. Say $k \in \mathcal{A}$ if

(A4) $|\gamma_{i_{k,n}}^{(k)}|_{\Delta_n}| < \frac{1}{3K_0}K^3\lambda$ (distortion estimate holds for the next iterate);
(A5) $|\gamma_{i_{k,n}+1}^{(k)}|_{\Delta_n}| < d$ (image of the next iterate meets at most one interval in C_σ).

Consider first the case where $\mathcal{A} \neq \emptyset$. We set $i_{k,n+1} = i_{k,n}+1$ for $k \in \mathcal{A}$, $i_{k,n+1} = i_{k,n}$ otherwise, and look for $\Delta_{n+1} \subset \Delta_n$ so that $(\Delta_{n+1}; i_{1,n+1}, \cdots, i_{k_1,n+1})$ is again an admissible configuration.

Let $k \in \mathcal{A}$. By virtue of (A3) and (A5), we have $12k_1\sigma < |\gamma_{i_{k,n+1}}^{(k)}|_{\Delta_n}| < d$, so that the fraction of $\gamma_{i_{k,n+1}}^{(k)}|_{\Delta_n}$ in C_σ is $\leq \frac{1}{6k_1}$. By virtue of (A4) and Sublemma 5.2, we have good control of the distortion of $a \mapsto \frac{d}{da}\gamma_{i_{k,n+1}}^{(k)}$. Together this gives

$$|(\gamma_{i_{k,n+1}}^{(k)}|_{\Delta_n})^{-1}C_\sigma| \leq \frac{1}{3k_1}|\Delta_n|. \tag{16}$$

By the same geometric argument as in the case $n = 1$, there exists a subinterval $\Delta_{n+1} \subset \Delta_n$ of length $\frac{1}{3k_1}|\Delta_n|$ with the property that $\gamma_{i_{k,n+1}}^{(k)}|_{\Delta_{n+1}} \cap C_\sigma = \emptyset$ for all $k \in \mathcal{A}$. For this choice of Δ_{n+1}, we have (A1) by design, and (A2) is given by (A4) from step n. As for (A3), observe that by the same reasoning as in (16), the pullback of any interval of S^1 of length 2σ has length $\leq \frac{1}{3k_1}|\Delta_n|$, so $|\gamma_{i_{k,n+1}}^{(k)}|_{\Delta_{n+1}}| \geq 2\sigma$, and one iterate later, it is guaranteed to have length $> 2K^3\sigma$.

Consider now $k \notin \mathcal{A}$. Conditions (A1) and (A2) are inherited from the previous step, and (A3) is checked as follows: If $k \notin \mathcal{A}$ because (A4) fails, then

$$|\gamma^{(k)}_{i_{k,n+1}}|_{\Delta_{n+1}}| \geq \frac{1}{2} \cdot \frac{1}{3k_1} |\gamma^{(k)}_{i_{k,n}}|_{\Delta_n}| \geq cK^3\lambda,$$

where c is a constant independent of K of λ. Notice that this uses only the distortion estimate from step n. One iterate later, this curve will have length $> cK^6\lambda$, which we may assume is $> 12k_1\sigma$. If (A4) holds but (A5) fails, then the distortion estimate holds for the next iterate, and

$$|\gamma^{(k)}_{i_{k,n+1}+1}|_{\Delta_{n+1}}| \geq \frac{1}{6k_1} |\gamma^{(k)}_{i_{k,n}+1}|_{\Delta_n}| \geq cd,$$

which we may also assume is $> 12k_1\sigma$. This completes the construction from step n to step $n+1$ when $\mathcal{A} \neq \emptyset$.

If $\mathcal{A} = \emptyset$, then we let Δ'_n be the left half of Δ_n, and observe that the $(n+1)$-tuple $(\Delta'_n; i_{1,n}, i_{2,n}, \cdots, i_{k_1,n})$ is again admissible. To verify (A3), we fix k, and argue separately as in the last paragraph the two cases corresponding to (i) the failure of (A4) with respect to Δ_n and (ii) the failure of (A5) but not (A4). Repeat this process if necessary until $\mathcal{A} \neq \emptyset$. □

5.4. Verification of (C1), (C3) and (C4). We now verify the remaining conditions in Sect. 5.1. Observe from the arguments below that (C1) and (C3)(ii) are quite natural for systems arising from differential equations, while (C3)(i) and (C4) are, to a large extent, consequences of the fact that the maps f_a are sufficiently expanding.

Verification of (C1): Let F_{t_0} denote the time-t_0-map of (2) (the period of the forcing continues to be T). Then (i) follows from the fact that F_{t_0} has bounded C^3 norms on $S^1 \times [-1, 1]$; (ii) is obvious, and (iii) is a consequence of the fact that $\det(DF_T) = e^{-\lambda(T-t_0)} \det(DF_{t_0})$.

Verification of (C3): For (i), since $(\partial_a f_a)(\cdot) = 1$ and $|(f^k)'(fx)| \geq K^k$, Lemma 5.1 applies, and the quantity in question has absolute value $\geq 1 - \sum_{i \geq 1} \frac{1}{K^i} > 0$. Part (ii) is Lemma 2.3(i).

Verification of (C4): (i) is proved since $e^{c_1} = K > 2$. For (ii), by choosing λ sufficiently small depending on Φ_0, it is easily arranged that $p_{i,j} = 1$ for all i, j.

This completes the proof of Theorem 3.

Appendix

We supply here the proofs of the two lemmas promised in Sect. 5.1. This appendix has to be read in conjunction with [WY].

Lemma A.4. *All the theorems in [WY] remain valid if the Misiurewicz condition in Step I, Sect.1.1, of [WY] is replaced by condition (C2') in Sect. 5.1 of this paper.*

Proof. The three most important uses of the Misiurewicz condition in [WY] are:

- the nondegeneracy of the critical points (this is guaranteed by (C2')(iii)(a));
- every critical orbit stays a fixed distance away from C (this is precisely (C2')(ii));

– there exist $c_0, c > 0$ such that for every critical point x, $|(f^n)'(fx)| > c_0 e^{cn}$ (this is guaranteed by (C2')(i) and (ii)).

These three properties aside, the only consequences of the Misiurewicz condition used in [WY] are contained in

Lemma 2.5 of [WY]. *Let C_δ denote the δ-neighborhood of C. Then there exist $\hat{c}_0, \hat{c}_1 > 0$ such that the following hold for all sufficiently small $\delta > 0$: Let $x \in S^1$ be such that $x, fx, \cdots, f^{n-1}x \notin C_\delta$, any n. Then*

(i) $|(f^n)'x| \geq \hat{c}_0 \delta e^{\hat{c}_1 n}$;
(ii) *if, in addition, $f^n x \in C_\delta$, then* $|(f^n)'x| \geq \hat{c}_0 e^{\hat{c}_1 n}$.

We claim that the conclusions of this lemma also follow from (C2'). Let $n_1 < \cdots < n_q$, $0 \leq n_1, n_q \leq n$, be the times when $f^{n_i} x \in I$. Then

– $|(f^{n_1})'x| \geq e^{c_1 n_1}$ by (C2')(i)(b);
– $|(f^{n_{i+1}-n_i})'(f^{n_i}x)| \geq e^{c_1(n_{i+1}-n_i)}$ by (C2')(iii)(b) followed by (i)(b);
– $|(f^{n-n_q})'(f^{n_q}x)| = |f'(f^{n_q}x)| \cdot |(f^{n-(n_q+1)})'(f^{n_q+1}x)|$,
where $|f'(f^{n_q}x)| \geq |f''(\xi)| d(x, C) \geq c_0' \delta$ by (C2')(iii)(a)
and $|(f^{n-(n_q+1)})'(f^{n_q+1}x)| \geq c_0'' e^{c_1(n-(n_q+1))}$ by (C2')(i)(a).

Together these inequalities prove both of the assertions in the lemma. □

Lemma A.5. *Let $\{T_{a,b}\}$ be as in Sect. 5.1 of this paper, and let Δ be the set of (a, b) such that $T = T_{a,b}$ satisfies the conclusions of Theorem 1 in [WY]. Suppose $\{T_{a,b}\}$ also satisfies (C4), and δ is smaller than a number depending on c_1. Then*

(i) *T admits at most one SRB measure μ;*
(ii) *(T, μ) is mixing.*

Proof. Let $\{x_1 < \cdots < x_r\}$ be the set of critical points of f. Consider a segment $\omega \subset \partial R_0$ corresponding to an outermost $I_{\mu j}$ at one of the components of $\mathcal{C}^{(0)}$. First we claim there exist $N \in \mathbb{Z}^+$ and $\hat{\omega} \subset \omega$ such that $T^i \hat{\omega} \cap \mathcal{C}^{(0)} = \emptyset$ for all $0 < i < N$ and $T^N \hat{\omega}$ connects two components of $\mathcal{C}^{(0)}$.

This claim is proved as follows. Let ω' denote the image of ω at the end of its bound period. Then ω' has length $> \delta^{K\beta}$. We continue to iterate, deleting all parts that fall into $\mathcal{C}^{(0)}$. Then i steps later, the undeleted part of $T^i \omega'$ is made up of finitely many segments. Suppose that for all $i \leq n$, none of these segments is long enough to connect two components of $\mathcal{C}^{(0)}$, so that the number of segments deleted up to step i is $\leq 2^i$. We estimate the *average length* of these segments at time n as follows: First, the pull-back to ω' of all the deleted parts has total measure $\leq \sum_{i \leq n} 2^i e^{-c_1 i}(2\delta)$ by (C2')(i)(b). Since $2 < e^{c_1}$ by (C4)(i), we may assume this is $< \frac{1}{2}\delta^{K\beta}$ provided δ is sufficiently small. The undeleted segments of $T^n \omega$ add up, therefore, to $> e^{c_1 n} \frac{1}{2} \delta^{K\beta}$ in length, and since there are at most 2^n of them, their average length is $> 2^{-n} e^{c_1 n} \frac{1}{2} \delta^{K\beta}$. Thus one sees that as n increases, there must come a point when our claim is fulfilled.

Next we observe that if ω is a $C^2(b)$ segment connecting two components of $\mathcal{C}^{(0)}$, then using (C4)(ii) and reasoning as with finite state Markov chains, we have that for every $n \geq N_2$ and every $k \in \{1, \cdots, r\}$, there is a subsegment $\omega_{n,k} \subset \omega$ such that for all $i < n$, $T^i \omega_{n,k} \cap \mathcal{C}^{(0)} = \emptyset$ and $T^n \omega_{n,k}$ stretches across the region between x_k and x_{k+1}, extending beyond the critical regions containing these two points.

Recall that in [WY], Sects. 8.1 and 8.2, a finite number of ergodic SRB measures $\{\mu_i, i \le r'\}$ are constructed, and it is shown in Sect. 8.3 that these are all the ergodic SRB measures T has. The discussion above shows that starting at any reference set, a segment $\omega \subset \partial R_0$ as above will spend a positive fraction of time in every reference set, proving that $r' \le 1$. Furthermore, starting from any reference set, the return time to it takes on all values greater than some N_0, proving that μ_1 is mixing. $\square$

6. Concluding Remarks

• For area-preserving maps, it is well known that when integrability first breaks down, the phase portrait is dominated by KAM curves. Farther away from integrability, one sees larger Birkhoff zones of instability interspersed with elliptic islands. Continuing to move toward the chaotic end of the spectrum, it is widely believed – though not proved – that most of the phase space is covered with ergodic regions with positive Lyapunov exponents.

This paper deals with the corresponding pictures for strongly dissipative systems. We consider a simple model consisting of a periodically forced limit cycle. Keeping the magnitude of the "kick" constant, we prove that scenarios roughly parallel to those in the last paragraph occur for our Poincaré maps, with attracting invariant circles (taking the place of KAM curves), periodic sinks (instead of elliptic islands), and as the contractive power of the cycle diminishes, we prove that the stage is shared by at least two scenarios occupying parameter sets that are delicately intertwined: horseshoes and sinks, and strange attractors.

By "**strange attractors**", we refer to attractors characterized by SRB measures, positive Lyapunov exponents, and strong mixing properties. For the differential equation in question, we prove that the system has global strange attractors of this kind for a positive measure set of parameters.

• Our second point has to do with bridging the gap between abstract theory and concrete problems. Today we have a fairly good hyperbolic theory, yet chaotic phenomena in naturally occurring dynamical systems have continued to resist analysis. One of the messages of this paper is that for certain types of strange attractors, the situation is now improved: **For attractors with strong dissipation and one direction of instability, there are now relatively simple, checkable conditions which, when satisfied, guarantee the existence of an attractor with a detailed package of statistical and geometric properties.** Our conditions are formulated to give rigorous results, but where rigorous analysis is out of reach, they can also serve as a basis for numerical work to provide justification for various mathematical statements about strange attractors.

References

[A] Arnold, V.I.: Small denominators, I: Mappings of the circumference onto itself. AMS Transl. Ser. 2 **46**, 213–284 (1965)

[BC] Benedicks, M. and Carleson, L.: The dynamics of the Hénon map. Ann. Math. **133**, 73–169 (1991)

[BY] Benedicks, M. and Young, L.-S.: Sinai-Bowen-Ruelle measure for certain Hénon maps. Invent. Math. **112**, 541–576 (1993)

[B] Bowen, R.: *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*. Lecture Notes in Math. Vol. **470**, Berlin: Springer, 1975

[CL] Cartwright, M.L. and Littlewood, J.E.: On nonlinear differential equations of the second order. J. London Math. Soc. **20**, 180–189 (1945)

[CELS] Chernov, N., Eyink, G., Lebowitz, J. and Sinai, Ya.G.: Steady-state electrical conduction in the periodic Lorentz gas. Commun. Math. Phys. **154**, 569–601 (1993)

[CE] Collet, P. and Eckmann, J.-P.: Iterated Maps on the Interval as Dynamical Systems. Progress on Physics **I** (1980)

[D] Duffing, G.: Erzwungene Schwingungen bei veränderlicher Eigenfrequenz. Braunschweig: 1918

[GS] Graczyk, J. and Swiatek, G.: Generic hyperbolicity in the logistic family. Ann. Math. **146**, 1–52 (1997)

[G] Guckenheimer, J.: A strange strange attractor. In: *Bifurcation and its Application*(J.E. Marsden and M. McCracken, ed.). Berlin–Heidelberg–New York: Springer-Verlag, [redacted] pp. 81

[GH] Guckenheimer, J. and Holmes, P.: *Nonlinear oscillators, dynamical systems and bifurcations of vector fields*. Appl. Math. Sciences **42**, Berlin–Heidelberg–New York: Springer-Verlag, 1983

[He] Herman, M.: *Mesure de Lebesgue et Nombre de Rotation*. Lecture Notes in Math. **597**, Berlin–Heidelberg–New York: Springer, 1977, pp. 271–293

[HPS] Hirsch, M., Pugh, C. and Shub, M.: *Invariant Manifolds*. Lecture Notes in Math. **583**, Berlin–Heidelberg–New York: Springer Verlag, 1977

[Ho] Holmes, P.: A nonlinear oscillator with a strange attractor. Phil. Trans. Roy. Soc. A. **292**, 419–448 (1979)

[J] Jakobson, M.: Absolutely continues invariant measures for one-parameter families of one-dimensional maps. Commun. Math. Phys. **81**, 39–88 (1981)

[KH] Katok, A. and Hasselblatt, B.: *Introduction to the modern dynamical systems*. Cambridge: Cambridge University Press, 1995

[Le] Ledrappier, F.: Propriétés ergodiques des mesures de Sinai. Publ. Math. Inst. Hautes Etud. Sci. **59**, 163–188 (1984)

[LY] Ledrappier, F. and Young, L.-S.: The metric entropy of diffeomorphisms. Ann. Math. **122**, 509–574 (1985)

[Li1] Levi, M.: Qualitative analysis of periodically forced relaxation oscillations. Mem. AMS **214**, 1–147 (1981)

[Li2] Levi, M.: A new randomness-generating mechanism in forced relaxation oscillations. Physica D **114**, 230–236 (1998)

[Ln] Levinson, N.: A second order differential equation with singular solutions. Ann. Math. **50**, No. 1, 127–153 (1949)

[Lo] Lorenz, E.N.: Deterministic nonperiodic flow. J. Atmos. Sc. **20**, No. 1, 130–141 (1963)

[Lyu1] Lyubich, M.: Dynamics of quadratic polynomials, I-II. Acta Math. **178**, 185–297 (1997)

[Lyu2] Lyubich, M.: Regular and stochastic dynamics in the real quadratic family. Proc. Natl. Acad. Sci. USA **95**, 14025–14027 (1998)

[dMvS] de Melo, W. and van Strien, S.: *One-dimensional Dynamics*. Berlin–Heidelberg–New York: Springer-Verlag, 1993

[M1] Misiurewicz, M.: Absolutely continues invariant measures for certain maps of an interval. Publ. Math. IHES. **53**, 17–51M.

[MV] Mora, L. and Viana, M.: Abundance of strange attractors. Acta. Math. **171**, 1–71 (1993)

[P] Pesin, Ja.B.: Characteristic Lyapunov exponents and smooth ergodic theory. Russ. Math. Surv. **32.4**, 55–114 (1977)

[PS] Pugh, C. and Shub, M.: Ergodic attractors. Trans. A. M. S. **312**, 1–54 (1989)

[Ro] Robinson, C.: Homoclinic bifurcation to a transitive attractor of Lorenz type. Nonlinearity **2**, 495–518 (1989)

[R1] Ruelle, D.: A measure associated with Axiom A attractors. Am. J. Math. **98**, 619–654 (1976)

[R2] Ruelle, D.: Ergodic theory of differentiable dynamical systems. Publ. Math. Inst. Hautes Étud. Sci. **50**, 27–58 (1979)

[Ry] Rychlik, M.: Lorenz attractors through Sil'nikov-type bifurcation, Part I. Ergodic Theory and Dynamical Systems **10**, 793–822 (1990)

[Sh] Shub, M.: *Global Stability of Dynamical Systems*. Berlin–Heidelberg–New York: Springer, 1987

[Si] Sinai, Y.G.: Gibbs measure in ergodic theory. Russ. Math. Surv. **27**, 21–69 (1972)

[Sp] Sparrow, C.: *The Lorenz Equations*. Berlin–Heidelberg–New York: Springer, 1982

[TTY] Thieullen, P., Tresser, C. and Young, L.-S.: Positive exponent for generic 1-parameter families of unimodal maps. C.R. Acad. Sci. Paris, t. **315**, Serie (1992), 69–72; J. d'Analyse **64**, 121–172 (1994)

[T] Tucker, W.: The Lorenz attractor exists. C. R. Acad. Sci. Paris Ser. I Math. 328 (1999), no. 12, 1197–1202

[W] Williams, R.: The structure of Lorenz attractors. In: *Turbulence Seminar Berkeley 1996/97* (P. Bernard and T. Ratiu, ed.), Berlin–Heidelberg–New York: Springer-Verlag, 1977, pp. 94–112

[WY] Wang, Q.D. and Young, L.-S.: Strange attractors with one direction of instability. Commun. Math. Phys. **218**, 1–97 (2001)

[Y1] Young, L.-S.: Ergodic theory of differentiable dynamical systems. In: *Real and Complex Dynamical Systems*, B. Branner and P. Hjorth (eds.), Dordrecht: Kluwer Acad. Press, 1995

[Y2] Young, L.-S.: Statistical properties of dynamical systems with some hyperbolicity. Ann. of Math. **147**, 585–650 (1998)

[Z1] Zaslavsky, G.: The simplest case of a strange attractor. Phys. Lett. A **69**, no. 3, 145–147 (1978)
[Z2] Zaslavsky, G.: *Chaos in Dynamic Systems*. [redacted]: Harwood Academic Publishers, first printing, 1985

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