Ergodic Theory of Infinite Dimensional Systems with Applications to Dissipative Parabolic PDEs

Nader Masmoudi[1] and Lai-Sang Young[2]

Courant Institute of Mathematical Sciences
251 Mercer Street, New York, NY 10012

May 12, 2001

1 Introduction

This paper concerns the ergodic theory of a class of nonlinear dissipative PDEs of parabolic type. Leaving precise statements for later, we first give an indication of the nature of our results. We view the equation in question as a semi-group or dynamical system S_t on a suitable function space H, and assume the existence of a compact attracting set (as in Temam [15], Chapter 1). To this deterministic system, we add a random force in the form of a "kick" at periodic time intervals, defining a Markov chain $\mathcal{X}$ with state space H. We assume that the combined effect of the semi-group and our kicks sends balls to compact sets. Under these conditions, the existence of invariant measures for $\mathcal{X}$ is straightforward. The goal of this paper is a better understanding of the set of invariant measures and their ergodic properties.

In a state space as large as ours, particularly when the noise is bounded and degenerate, the set of invariant measures can, in principle, be very large. In this paper, we discuss two different types of conditions that reduce the complexity of the situation. The first uses the fact that for the type of equations in question, high modes tend to be contracted. By actively driving as many of the low modes as needed, we show that the dynamics resemble those of Markov chains on $\mathbb{R}^N$ with smooth transition probabilities. In particular, the set of ergodic invariant measures is finite, and every aperiodic ergodic measure is exponentially mixing. The second type of conditions we consider is when all of the Lyapunov exponents of $\mathcal{X}$ are negative. As in finite dimensions, we show under these conditions that nearby orbits cluster together in a phenomenon known as "random sinks".

The conditions in the last paragraph give a general understanding of the structure of invariant measures; they alone do not guarantee uniqueness. (Indeed, it is not the case that for the equations in question, invariant measures are always unique;

[1]Email masmoudi@cims.nyu.edu
[2]Email lsy@cims.nyu.edu . This research is partially supported by a grant from the NSF

see Theorem 3.) For uniqueness, one needs to guarantee that there are places for distinct ergodic components to meet. To this end, we have identified some conditions expressed in terms of existence of special sequences of controls. These conditions are quite special; however, they are easily verified for the equations of interest. Assuming these conditions, the uniqueness of the invariant measure follows readily. In the case of negative Lyapunov exponents, there is, in fact, a stronger form of uniqueness or stability, namely that *all* solutions independent of initial conditions become asymptotically close to one other as time goes to infinity.

This work is inspired by a number of recent papers on the uniqueness of invariant measure for the Navier-Stokes equations ([5], [1], [3], [7], [8]), and by [4], which proves uniqueness of invariant measure for a different equation. With the exception of [7] and [8], all of the other authors worked with unbounded noise. Naturally there is overlap among these papers and with the first part of ours. More detailed references will be given as the theorems are stated.

Instead of working directly with specific PDEs, we have elected to prove our ergodic theory results for general randomly perturbed dynamical systems on infinite dimensional Hilbert spaces satisfying conditions compatible with the PDEs of interest. This allows us to make more transparent the relations between the various dynamical properties and the mechanisms responsible for them. Once our "abstract" results are in place, to apply them to specific equations, it suffices to verify that the conditions in the theorems are met. (In this regard, we are influenced by [7], which takes a similar approach.)

This paper is organized as follows. Before proceeding to a discussion of our "abstract results", we first give a sample of their applications. This is done in Section 2. Sections 3 and 4 treat the two types of conditions that lead to simpler structures for invariant measures. In each case, we begin with a general discussion and finish with proofs of concrete results for PDEs which we now state.

2 Statement of Results for PDEs

This section contains precise formulations of results on PDEs that can be deduced from our "abstract theory". The theorems below are proved in Sects. 3.3 and 4.3.

2.1 The Navier-Stokes system

The first application of our general results is to the 2-D incompressible Navier-Stokes equations in the 2−torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$. We consider the randomly forced system

$$\begin{cases} \partial_t u - \nu\Delta u + u\cdot\nabla u = -\nabla p + \sum_{k=1}^{\infty} \delta(t-k)\eta_k(x)\ , \\ \qquad\qquad \mathrm{div}(u) = 0, \quad u(t=0) = u_0 \end{cases} \tag{1}$$

where $u_0(x) \in L^2(\mathbb{T}^2)$, $\mathrm{div}(u_0) = 0$, $\int u_0 = 0$, $\nu > 0$ is the viscosity, and where the η_k's are i.i.d. random fields which can be expanded as

$$\eta_k(x) = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j(x) \ . \tag{2}$$

Here the Hilbert space in question is

$$H = \{u,\ u \in L^2(\mathbb{T}^2),\ \mathrm{div}(u) = 0,\ \text{and} \int u = 0\} \ ,$$

and $\{e_j,\ j \geq 1\}$ is the orthonormal basis consisting of the eigenfunctions of the Stokes operator $-\Delta e_j + \nabla p_j = \lambda_j e_j$, $\mathrm{div}(e_j) = 0$, with $\lambda_1 \leq \lambda_2 \leq \cdots$. We assume that $\xi_{jk},\ j,k \in \mathbb{N}$, are independent random variables where ξ_{jk} is distributed according to a law which has a positive Lipschitz density ρ_j with respect to the Lebesgue measure on $[-1,1]$. Finally, the b_j are required to satisfy $\sum_{j=1}^{\infty} b_j^2 = a^2 < \infty$ for some $a > 0$.

From (1), we define a Markov chain u_k with values in H given by $u_k = u(k+0,\cdot)$. That is to say, if S_t is the semi-group generated by the unforced Navier-Stokes equation, i.e. equation (1) without the term $\sum_{k=1}^{\infty} \delta(t-k)\eta_k(x)$, and $S = S_1$, then

$$u_{k+1} = S(u_k) + \eta_k \ .$$

Theorem 1 (Uniqueness of invariant measure and exponential mixing). *For the system above, there exists $N \geq 1$ depending only on the viscosity ν and on a such that if $b_j \neq 0$ for all $1 \leq j \leq N$, then the Markov chain u_k has a unique invariant measure μ in H. Moreover, for all $u_0 \in H$, the distribution Θ_k of u_k converges exponentially fast to μ in the sense that for every test function $f : H \to \mathbb{R}$ of class $C^{0,\sigma}$, $\sigma > 0$, there exists $C = C(f, u_0)$ such that for all $k \geq 1$,*

$$\left| \int f d\Theta_k \ - \ \int f d\mu \right| < C\tau^k$$

for some $\tau < 1$ depending only on the Hölder exponent σ.

Papers [7] and [8] together contain a proof of the uniqueness of invariant measure part of Theorem 1; these papers rely on ideas different from ours. While this manuscript was being written, we received electronic preprints [9] and [10] which together prove the results in Theorem 1 using methods similar to ours.

Theorem 1'. *The result in Theorem 1 holds if we replace L^2 by H^s, any $s \in \mathbb{N}$, and impose the restriction $\sum_{j=1}^{\infty} \lambda_j^s b_j^2 = a^2 < \infty$ on the noise.*

Remark. In the theorems above, we can also treat noises that are bounded but not compact provided that we consider the Markov chain $u_k = u(k-0,\cdot)$ or, equivalently,

$u_{k+1} = S(u_k + \eta_k)$. An example of bounded, noncompact noise satisfying the conditions of Theorems 1 and 1' is the following: Let V_N be the span of $\{e_1, e_2, ..., e_N\}$, and consider

$$\eta_k = \sum_{j=1}^{N} b_j \xi_{jk} e_j + \eta_k' \tag{3}$$

where $b_j \neq 0$ for all j, $1 \leq j \leq N$, and η_k' are i.i.d. random variables with a law supported on a bounded set in $V_N^\perp$.

Our next result gives a stronger form of uniqueness than the previous one. It guarantees, under the assumption of negative Lyapunov exponents, that independent of initial condition, *all* the solutions eventually come together and evolve as one, their time evolution depending only on the realization of the noise. Lyapunov exponents are defined in Sect. 4.1. Having only negative Lyapunov exponents means, roughly speaking, that infinitesimally the semi-group is contractive *on average* along typical orbits. More regularity is required for the next result; thus we work in H^2. Let $A(0) \subset H^2$ denote the closure of the set of points accessible under the Markov chain u_k with $u_0 = 0$.

Theorem 2 (Asymptotic uniqueness of solutions independent of initial conditions). *Consider the system defined by (1) with $H = H^2$ and where the η_k are i.i.d. with a law which has bounded support. Assume there is an invariant measure μ supported on $A(0)$ such that all of its Lyapunov exponents are strictly negative. Then*

(a) μ is the unique invariant measure the Markov chain u_k has in H;

(b) there exists $\lambda < 0$ such that for almost every sequence of η_k and every pair of initial conditions $u_0, u_0' \in H$, there exists $C = C(u_0, u_0')$ such that if $u_{k+1} = S(u_k) + \eta_k$ and $u_{k+1}' = S(u_k') + \eta_k$ for all $k \geq 0$, then

$$\|u_k - u_k'\| \leq Ce^{\lambda k} \qquad \forall\ k \geq 0\ .$$

Observe that for this result very little is required of the structure of the noise.

Remark. We will explain in Sect. 4.4 (see Remark) that for fixed positive viscosity, S is a uniform contraction near 0, and so it continues to be a contraction for sufficiently small bounded noise. Very small but *unbounded* noise is treated in [12]. As noise level increases, it is likely that there is a range where S is no longer a contraction but all of its Lyapunov exponents remain negative. Indeed, for any ergodic invariant measure μ of the Navier-Stokes system, the largest Lyapunov exponent λ_1 is either < 0, $= 0$, or > 0: $\lambda_1 > 0$ can be interpreted as "temporal chaos"; $\lambda < 0$ implies the asymptotic uniqueness of solutions as we have shown; the case $\lambda_1 = 0$ is sometimes regarded as less significant because it can often be perturbed away. Of these three possibilities, the only one that has been proved to occur is $\lambda_1 < 0$.

Remark. Theorems 1 and 2 apply to other nonlinear parabolic equations for which all solutions of the unforced equation relax to their unique stable stationary solutions.

2.2 The real Ginzburg-Landau equation

Our second application is to the following equation, which, following [4], we refer to as the real Ginzburg-Landau equation. We consider a periodic domain in one space dimension, i.e. $\mathbb{T} = (\mathbb{R}/2\pi\mathbb{Z})$, and consider the system

$$\begin{cases} \partial_t u - \nu\Delta u - u + u^3 = \sum_{k=1}^{\infty} \delta(t-k)\eta_k(x) \ , \\ \\ u(t=0) = u_0 \ . \end{cases} \tag{4}$$

Here $H = L^2(\mathbb{T})$, $\{e_j,\ j \geq 0\}$ is the orthonormal basis defined by $-\Delta e_j = \lambda_j e_j$, $\lambda_1 \leq \lambda_2 \leq \cdots$, $\nu > 0$ is a positive constant, and the η_k's are i.i.d. random fields which can be expanded as

$$\eta_k(x) = \sum_{j=0}^{\infty} b_j \xi_{jk} e_j(x) \ . \tag{5}$$

We assume the same conditions on ξ_{jk} and b_j as in the first paragraph of Sect. 2.1.

The unforced equation in (4) is somewhat more unstable than the (unforced) Navier-Stokes equation. It has at least three stationary solutions: two stable ones, namely $u = 1$ and $u = -1$, and an unstable one, namely $u = 0$. Our next result shows that the number of invariant measures vary depending on how localized the forcing is, particularly in the zeroth mode.

Theorem 3 (Number of ergodic measures). *Consider the Markov chain u_k defined by the system in (4).*

(a) There exists $\alpha > 0$ such that if $\sum_{j=0}^{\infty} |b_j|^2 = a^2 \leq \alpha^2$, then there are at least two different invariant measures.

(b) There exists N depending only on ν and on a such that if $b_j \neq 0$ for all $0 \leq j \leq N$, then the number of ergodic invariant measures is finite.

(c) If $b_j \neq 0$ for all $0 \leq j \leq N$ and $b_0 > 1$, then the invariant measure is unique, and for every initial condition $u_0 \in H$, the distribution of u_k converges to it exponentially fast in the sense of Theorem 1.

In contrast to part (a), we observe that to obtain uniqueness of the invariant measure, we may take b_j, $1 \leq j \leq N$, to be arbitrarily small as long as they are > 0, and the forcing in the zeroth mode, i.e. $b_0\xi_{0k}$, can be arbitrarily weak as long as its law has a tail which extends beyond $[-1, 1]$. As will be explained in Sect. 3.4, the condition $b_0 > 1$ above can, in fact, be replaced by $b_0 > \kappa$ for a smaller κ.

Theorem 3 complements [4], which drives high rather than low modes, and proves uniqueness for unbounded noise using techniques very different from ours.

3 Invariant Measures and their Ergodic Properties

3.1 Formulation of abstract results

Setting and notation. Let $S : H \to H$ be a transformation of a separable Hilbert space H, and let ν be a probability measure on H. We consider the Markov chain $\mathcal{X} = \{u_n, n = 0, 1, 2, \cdots\}$ on H defined by either

$$\textbf{(I)}\ \ u_{n+1} = S(u_n) + \eta_n \qquad \text{or} \qquad \textbf{(II)}\ \ u_{n+1} = S(u_n + \eta_n)$$

where $\eta_0, \eta_1, \cdots$ are $i.i.d.$ with law ν. The following notation is used throughout this paper: $B_H(R)$ or simply $B(R)$ denotes the ball of radius R in H, i.e. $B(R) = \{u \in H, \|u\| \le R\}$; K denotes the support of ν; and given an initial distribution Θ_0 of u_0, the distribution of u_n under $\mathcal{X}$ is denoted by Θ_n. If $T : H \to H$ is a mapping and μ is a measure on H, then $T_*\mu$ is the measure defined by $(T_*\mu)(E) = \mu(T^{-1}(E))$.

Standing Hypotheses

(P1) **(a)** $S(B(R))$ is compact $\forall R > 0$;

(b) $\forall R > 0,\ \exists M_R > 0$ such that $\forall u, v \in B(R),\ \|Su - Sv\| \le M_R \|u - v\|$.

(P2) $\forall a > 0, \exists R_0 = R_0(a)$ such that if $K \subset B(a)$, then $\forall R > 0$, $\exists N_0 = N_0(R) \in \mathbb{Z}^+$ such that for $u_0 \in B(R)$, $u_n \in B(R_0)$ $\forall n \ge N_0$.

(P3) $\exists \gamma < 1$ such that given $R > 0$, there is a finite dimensional subspace $V \subset H$ such that if P_V and $P_{V^\perp}$ denote orthogonal projections from H onto V and $V^\perp$ respectively, then $\forall u, v \in B(R),\ \|P_{V^\perp} S(u) - P_{V^\perp} S(v)\| \le \gamma \|u - v\|$.

(P4) **(a)** K is compact if $\mathcal{X}$ is defined by (I), bounded if $\mathcal{X}$ is defined by (II).

(b) Let V be given by (P3) with $R = R_0$. Then $\nu = (P_V)_*\nu \times (P_{V^\perp})_*\nu$ where $(P_V)_*\nu$ has a density ρ with respect to the Lebesgue measure on V, $\Omega := \overline{\{\rho > 0\}}$ has piecewise smooth boundary and $\rho|_\Omega$ is Lipschitz.

We remark that (P1)–(P3) are selected to reflect the properties of general (nonlinear) parabolic PDEs.

Definition 3.1 *A probability measure μ on H is called an* **invariant measure** *for $\mathcal{X}$ if $\Theta_0 = \mu$ implies $\Theta_n = \mu$ for all $n > 0$.*

Lemma 3.1 *Assume* **(P1), (P2)** *and* **(P4)(a)**. *Then*

(i) $\mathcal{X}$ has an invariant measure;

(ii) there exists a compact set $A \subset B(R_0)$ on which all invariant measures of $\mathcal{X}$ are supported.

Proof. Let $A_0 = B(R_0)$. For $n > 0$, let $A_n = S(A_{n-1}) + K$ in the case of (I) and $A_n = S(A_{n-1}+K)$ in the case of (II). Then each A_n is compact, and by (P2), $A_n \subset A_0$ for all $n \geq$ some N_0. Let

$$A \;=\; \cup_{i=0}^{N_0-1} \left(\cap_{k=0}^{\infty} A_{kN_0+i}\right) .$$

Then A is compact, contained in $B(R_0)$, and satisfies $S(A) + K = A$. To construct an invariant measure for $\mathcal{X}$, pick an arbitrary $u_0 \in A$, and let $\Theta_0 = \delta_{u_0}$, the Dirac measure at u_0. Then any accumulation point of the sequence $\{\frac{1}{n}\sum_{i<n}\Theta_i\}_{n=1,2,\cdots}$ is an invariant measure for $\mathcal{X}$. That all invariant measures are supported on A follows from the fact that for every $u_0 \in H$ and any sequence of kicks $\{\eta_k\}$, $\text{dist}(u_n, A) \to 0$ as $n \to \infty$. ■

Definition 3.2 *Let μ be an invariant measure for $\mathcal{X}$, and let $\Theta_0 = \delta_{u_0}$ for the u_0 specified.*

(1) We say $(\mathcal{X}, \mu)$ is **ergodic** *if for μ-a.e. u_0, $\frac{1}{n}\sum_{i=0}^{n-1}\Theta_i \;\to \mu$ weakly as $n \to \infty$.*

(2) We say $(\mathcal{X}, \mu)$ is **mixing** *if for μ-a.e. u_0, $\Theta_n \;\to \mu$ weakly as $n \to \infty$.*

(3) We say $(\mathcal{X}, \mu)$ is **exponentially mixing** *for Hölder continuous observables if for each $\sigma > 0$, there exists $\tau = \tau(\sigma) < 1$ such that the following holds for every $f : H \to \mathbb{R}$ of class $C^{0,\sigma}$: for μ-a.e. u_0, there exists $C = C(f, u_0)$ such that*

$$\left| \int f d\Theta_n \;-\; \int f d\mu \right| < C\tau^n \quad \text{for all } n \geq 1.$$

Let $\mathcal{X}^n$ denote the n-step Markov chain associated with $\mathcal{X}$.

Theorem A (Structure of invariant measures). *Assume* **(P1)–(P4)**. *Then*

(1) $\mathcal{X}$ has at most a finite number of ergodic invariant measures.

(2) If $(\mathcal{X}^n, \mu)$ is ergodic for all $n \geq 1$, then $(\mathcal{X}, \mu)$ is exponentially mixing for Hölder continuous observables.

The reasons behind these results are that under (P1)–(P4), $\mathcal{X}$ resembles a Markov chain on $\mathbb{R}^N$ whose transition probabilities have densities. One expects, therefore, the same type of decomposition into ergodic and mixing components.

We now give a condition that guarantees the uniqueness of invariant measures and other convergence properties. This condition is expressed in terms of the existence of special sequences of controls; it is quite strong, but is easily verified for the PDEs under consideration.

(C) Given $\varepsilon_0 > 0$ and $R > 0$, there is a finite sequence of controls $\hat{\eta}_0, \cdots \hat{\eta}_n$ such that for all $u_0,\ u_0' \in B(R)$, if $u_{k+1} = S(u_k) + \hat{\eta}_k$ and $u_{k+1}' = S(u_k') + \hat{\eta}_k$ for $k < n$, then $\|u_n - u_n'\| < \varepsilon_0$.

Theorem B (Sufficient condition for uniqueness and mixing). *Assume* **(P1)**–**(P4)** and **(C)**. *Then*

(1) $\mathcal{X}$ *has a unique invariant measure* μ*, and* $(\mathcal{X}, \mu)$ *is exponentially mixing;*

(2) $\exists \tau = \tau(\sigma) < 1$ *such that* $\forall f \in C^{0,\sigma}$ *and for every* $u_0 \in H$*, there exists* C *s.t.*

$$\left| \int f d\Theta_n \ - \ \int f d\mu \right| < C\tau^n \quad \text{for all } n \geq 1.$$

Recalling that the invariant measure μ is supported on a (relatively small) compact subset of H, we remark that the assertion in (2) above is considerably stronger than the usual notion of exponential mixing: it tells us about initial conditions far away from the support of μ. This property is reminiscent of the idea of **Sinai-Ruelle-Bowen measures** for attractors in finite dimensional dynamical systems.

3.2 Proofs of abstract results (Theorems A and B)

We will prove Theorems A and B for the case where $\mathcal{X}$ is defined by (I); the proofs for (II) are very similar. Also, to avoid the obstruction of main ideas by technical details, we will assume $(P_V)_*\nu$ is the normalized Lebesgue measure on $\Omega := \{u \in V, \|u\| \leq r\}$ for some $r > 0$; the general case is messier but conceptually not different.

Let $M = M_{R_0}$ where R_0 is given by (P2) and M_{R_0} is as defined in (P1). The following notation is used heavily: Given u_0 and $\eta = (\eta_0, \eta_1, \eta_2, \cdots) \in K^{\mathbb{N}}$, we define $u_i(\eta)$ inductively by letting $u_0(\eta) = u_0$ and $u_i(\eta) = S(u_{i-1}(\eta)) + \eta_{i-1}$ for $i > 0$. Notation such as $u_i(\eta_0, \cdots, \eta_{n-1})$ for a finite sequence $(\eta_0, \cdots, \eta_{n-1})$ with $i \leq n$ has the obvious meaning, as does $u_i'(\eta)$ for given u_0'.

Lemma 3.2 (Matching Lemma) *Let* $\delta = r(2M)^{-1}$*. There is a set* $\Gamma \subset K^{\mathbb{N}}$ *with* $\nu^{\mathbb{N}}(\Gamma) > 0$ *such that* $\forall u_0, u_0' \in B(R_0)$ *with* $\|u_0 - u_0'\| < \delta$*, there is a measure-preserving map* $\Phi : \Gamma \to K^{\mathbb{N}}$ *with the property that* $\forall \eta \in \Gamma$,

$$\|u_n(\eta) - u_n'(\Phi(\eta))\| \leq \|u_0 - u_0'\|\gamma^n \qquad \forall n \geq 0 \ .$$

By virtue of (P4)(b), $K = \Omega \times E$ where $\Omega \subset V$ is as above and $E \subset V^\perp$. We write $\eta_0 = (\eta_0^1, \eta_0^2)$ with $\eta_0^1 \in \Omega$, $\eta_0^2 \in E$. Since all of our operations take place in V, it is convenient to introduce the notation $K_\varepsilon := \{u \in \Omega, \|u\| \leq \varepsilon\} \times E$, so that in particular $K_r = K$.

Proof. Suppose $\|u_0 - u_0'\| < \delta$. We define $\Phi^{(1)} : K_{\frac{r}{2}} = K_{r-M\delta} \to H$ by

$$({\eta_0'}^1, {\eta_0'}^2) \ = \ \Phi^{(1)}(\eta_0) \ := \ (\eta_0^1 + P_V S(u_0) - P_V S(u_0'), \ \eta_0^2) \ .$$

Observe that

(i) $\|{\eta_0'}^1\| < (r - M\delta) + M \cdot \|u_0 - u_0'\| < r$, so that $\Phi^{(1)}(K_{\frac{r}{2}}) \subset K$;
(ii) $\Phi^{(1)}$ preserves ν-measure; and
(iii) for $\eta_0 \in K_{\frac{r}{2}}$, if $u_1 = u_1(\eta_0)$ and $u_1' = u_1'(\Phi^{(1)}(\eta_0))$, then

$$P_V u_1 = P_V u_1' \qquad \text{and} \qquad \|P_{V^\perp} u_1 - P_{V^\perp} u_1'\| < \gamma \|u_0 - u_0'\|.$$

We may, therefore, repeat the argument above with (u_1, u_1') in the place of (u_0, u_0'), defining for each $u_1 = u_1(\eta_0)$, $\eta_0 \in K_{\frac{r}{2}}$, a map from $K_{r-M\delta\gamma} = K_{r(1-\frac{1}{2}\gamma)}$ to K. Put together, this defines an injective map $\Phi^{(2)} : K_{\frac{r}{2}} \times K_{r(1-\frac{1}{2}\gamma)} \to K^2$ which carries ν^2-measure to ν^2-measure such that for each $(\eta_0, \eta_1) \in K_{\frac{r}{2}} \times K_{r(1-\frac{1}{2}\gamma)}$, if $u_2 = u_2(\eta_0, \eta_1)$ and $u_2' = u_2'(\Phi^{(2)}(\eta_0, \eta_1))$, then $P_V u_2 = P_V u_2'$ and $\|P_{V^\perp} u_2 - P_{V^\perp} u_2'\| < \gamma^2 \|u_0 - u_0'\|$.

Continued *ad infinitum*, this process defines a map

$$\Phi \ : \ \Gamma := K_{\frac{r}{2}} \times K_{r(1-\frac{1}{2}\gamma)} \times K_{r(1-\frac{1}{2}\gamma^2)} \times \cdots \ \to \ K^{\mathbb{N}}$$

with the desired properties. Clearly, $\nu(\Gamma) = \Pi_{i \geq 0}(1 - \frac{1}{2}\gamma^i)^D > 0$ where $D = \dim V$. ■

Proof of Theorem A(1). Recall that if μ is an ergodic invariant measure for $\mathcal{X}$, then by the Birkhoff Ergodic Theorem, $\frac{1}{n}\sum_0^{n-1} \delta_{u_i(\eta)} \to \mu$ for μ-a.e. u_0 and $\nu^{\mathbb{N}}$-a.e. $\eta = (\eta_0, \eta_1, \cdots)$. This together with Lemma 3.2 implies that if μ, and μ' are ergodic measures and there exist $u_0 \in \mathrm{supp}(\mu)$ and $u_0' \in \mathrm{supp}(\mu')$ with $||u_0 - u_0'\| < \delta$, then $\mu = \mu'$. Since all invariant measures of $\mathcal{X}$ are supported on the compact set A (Lemma 3.1), it follows that there cannot be more than a finite number of them. ■

Proof of the uniqueness of invariant measure part of Theorem B. From the last paragraph, we know that all the ergodic components of μ are pairwise $\geq \delta$ apart in distance. Thus condition (C) with $\varepsilon_0 = \delta$ and $R = R_0$ guarantees that there is at most one ergodic component. ■

We remark that the uniqueness of invariant measure results in Theorem 1 and Theorem 3(c) follow immediately from the preceding discussion once the abstract hypotheses (P1)–(P4) and (C) are checked for these equations.

The next lemma is used only to prove the general result in Theorem A(2); it is not needed for the applications in Theorems 1–3. (Both the Navier-Stokes and Ginzberg-Landau equations satisfy much stronger conditions, making this argument unnecessary.) Let $B(u, \varepsilon)$ denote the ball of radius ε centered at u, and let $P^n(\cdot|u)$ denote the n-step transition probability given u. In the language introduced earlier, if $\Theta_0 = \delta_u$, then $P^n(\cdot|u) = \Theta_n(\cdot)$.

Lemma 3.3 *Let μ be an invariant measure with the property that $(\mathcal{X}^n, \mu)$ is ergodic for all $n \geq 1$. We fix $B = B(\tilde{u}, \tilde{\varepsilon})$ where $\tilde{u} \in$ supp μ and $\tilde{\varepsilon} > 0$. Then there exist $N_0 \in \mathbb{Z}^+$ and $\alpha_0 > 0$ such that $P^{N_0}(B|u) \geq \alpha_0$ for every $u \in$ supp μ.*

Proof. Pick arbitrary $u_0 \in \text{supp}\mu$. Until nearly the end of the proof, the discussion pertains to this one point. Consider the "restricted distribution" $\hat{\Theta}_n$ defined by

$$\hat{\Theta}_n(G) = \nu^n\{(\eta_0, \cdots, \eta_{n-1}) : \eta_i \in K_{r-M\delta\gamma^i} \ \forall i < n \text{ and } u_n(\eta_0, \cdots, \eta_{n-1}) \in G\}$$

where δ is as in Lemma 3.2, and let W_n denote the support of $\hat{\Theta}_n$.

Claim 1. $d(u_0, \cup_{n>0} W_n) = 0$.

Proof. By compactness, a subsequence of $\frac{1}{n}\sum_{i=0}^{n-1}(\hat{\Theta}_i(A))^{-1}\hat{\Theta}_i$ converges weakly to a probability measure $\tilde{\mu}$ on A (where A is as in Lemma 3.1). Since the restrictions on η_i become milder and milder as $i \to \infty$, $\tilde{\mu}$ is an invariant measure for $\mathcal{X}$. By construction, all the $\hat{\Theta}_i$ are supported on supp μ, so we must have $\tilde{\mu} = \mu$, for we know from Theorem A(1) that all the other ergodic invariant measures have their supports bounded away from supp μ.

Let $N = N(u_0)$ be such that $d(u_0, W_N) < \varepsilon$ where $\varepsilon < \delta$ is a small positive number to be determined.

Claim 2. For all $k \geq 0$ and $u \in W_{kN}, \exists u' \in W_{(k+1)N}$ such that $\|u - u'\| < \gamma^{kN}\varepsilon$.

Proof. The claim is true for $k = 0$ by choice of N. We prove it for $k = 1$: Let $u_0' \in W_N$ be such that $\|u_0 - u_0'\| < \varepsilon$, and fix an arbitrary $u \in W_N$. By definition, there exist $\eta_i \in K_{r-M\delta\gamma^i}$ such that $u = u_N(\eta_0, \cdots, \eta_{N-1})$. We wish to use the proximity of u_0' to u_0 and the Matching Lemma to produce $(\eta_0', \cdots, \eta_{N-1}')$ with the property that $u_N'(\eta_0', \cdots, \eta_{N-1}') \in W_{2N}$ and $\|u_N - u_N'\| < \varepsilon\gamma^N$. To obtain the first property, it is necessary to have $\eta_i' \in K_{r-M\delta\gamma^{i+N}}$ for all $i < N$. We proceed as follows: since $\|u_0 - u_0'\| < \varepsilon$ and $\eta_0 \in K_{r-M\delta}$, $\exists \eta_0' \in K_{r-M\delta+M\varepsilon}$ such that $\|u_1(\eta_0) - u_1'(\eta_0')\| < \varepsilon\gamma$; similarly $\exists \eta_1' \in K_{r-M\delta\gamma+M\varepsilon\gamma}$ such that $\|u_2(\eta_0, \eta_1) - u_1'(\eta_0', \eta_1')\| < \varepsilon\gamma^2$, and so on. (See the proof of Lemma 3.2.) Thus $\eta_i' \in K_{r-M\gamma^i(\delta-\varepsilon)}$, and assuming ε is sufficiently small that $\delta\gamma^N < (\delta - \varepsilon)$, we have $\eta_i' \in K_{r-M\delta\gamma^{i+N}}$. To prove the assertion for $k = 2$, we pick an arbitrary $u \in W_{2N}$, which, by definition, is equal to v_N from some $v_0 \in W_N$. Since we have shown that there exists $v_0' \in W_{2N}$ with $\|v_0 - v_0'\| < \gamma^N\varepsilon$, it suffices to repeat the argument above to obtain $v_N' \in W_{3N}$ with $\|v_N - v_N'\| < \gamma^{2N}\varepsilon$.

Claim 3. There exists $k_1 = k_1(u_0)$ s.t. for $k \geq k_1$, $P^{kN}(B|u) \geq \hat{\Theta}_{kN}(B(\tilde{u}, \frac{\tilde{\varepsilon}}{2})) > 0$ for all $u \in H$ with $\|u - u_0\| < \delta$.

Proof. Let $\mathcal{N}(W, \varepsilon)$ denote the ε-neighborhood of $W \subset H$. If follows from Claim 2 that if $\mathcal{N}_{kN} := \mathcal{N}(W_{kN}, \ 2\varepsilon\sum_{i=0}^{k}\gamma^{iN})$, then $\mathcal{N}_{kN} \subset \mathcal{N}_{(k+1)N}$ for all k. Moreover, the ergodicity of $(\mathcal{X}^N, \mu)$ together with an observation similar to that in Claim 1 shows that the closure of $\cup_k \mathcal{N}_{kN}$ contains suppμ. Thus $\mathcal{N}_{kN} \cap B(\tilde{u}, \frac{\tilde{\varepsilon}}{4}) \neq \emptyset$ for large enough k. If $2\varepsilon\sum_{i=1}^{\infty}\gamma^{iN} < \frac{\tilde{\varepsilon}}{4}$, then $\hat{\Theta}_{kN}(B(\tilde{u}, \frac{\tilde{\varepsilon}}{2})) > 0$. Now for u with $\|u - u_0\| < \delta$, the entire restricted distribution $\hat{\Theta}_n$ starting from u_0 can be coupled to a part of the (unrestricted) distribution starting from u. Thus for sufficiently large n, $P^n(B|u) \geq \hat{\Theta}_n(B(\tilde{u}, \frac{\tilde{\varepsilon}}{2}))$.

To finish, we cover supp μ with a finite number of δ-balls centered at $u_0^{(1)}, \cdots, u_0^{(n)}$, and choose $N_0 = \hat{k}_1 \hat{N}$ where $\hat{k}_1 = max_i\ k_1(u_0^{(i)})$ and $\hat{N} = \Pi_i\ N(u_0^{(i)})$. The lemma is proved with $\alpha_0 = \min_i \hat{\Theta}_{N_0}(B(\tilde{u}, \frac{\tilde{\varepsilon}}{2}))$ where $\hat{\Theta}_{N_0}$ is the restricted distribution starting from $u_0^{(i)}$. ■

From Lemma 3.2, we see that associated with each pair of points (u_0, u_0') with $\|u_0 - u_0'\| < \delta$, there is a cascade of matchings between u_n and u_n', leading to the definition of a measure-preserving map

$$\Phi\ :\ \Gamma := K_{\frac{r}{2}} \times K_{r(1-\frac{1}{2}\gamma)} \times K_{r(1-\frac{1}{2}\gamma^2)} \times \cdots\ \to\ K^{\mathbb{N}}$$

with the property that for $\eta \in \Gamma$,

$$\|u_i(\eta) - u_i'(\Phi(\eta))\| \leq \gamma^i \|u_0 - u_0'\| \quad \text{for all}\ \ i \leq n.$$

The main goal in the next proof is, in a sense, to extend Φ to all of $K^{\mathbb{N}}$ by attempting repeatedly to match the orbits that have not yet been matched.

Proof of Theorem A(2). We consider for simplicity the case $N_0 = 1$. Let $u_0, u_0' \in$ supp μ, and let Θ_n and Θ_n' denote the distributions of u_n and u_n' respectively. We seek to define a measure-preserving map $\Phi : K^{\mathbb{N}} \to K^{\mathbb{N}}$ and to estimate the difference between Θ_n and Θ_n' by

$$I_n\ :=\ \left| \int f d\Theta_n - \int f d\Theta_n' \right|\ \leq\ \int |f(u_n(\eta)) - f(u_n'(\Phi(\eta)))|\ d\nu^{\mathbb{N}}(\eta)\ .$$

Let B be a ball of diameter δ centered at some point in suppμ. By Lemma 3.3, $P(B|u_0) \geq \alpha_0$, and $P(B|u_0') \geq \alpha_0$. Matching $u_1 \in B$ to $u_1' \in B$, we define a measure-preserving map $\Phi^{(1)} : \tilde{\Gamma}_1 \to K$ for some $\tilde{\Gamma}_1 \subset K$ with $|\tilde{\Gamma}_1| = \alpha_0$. This extends, by the Matching Lemma, to a measure-preserving map $\Phi : \Gamma_1 = \tilde{\Gamma}_1 \times \Gamma \to K^{\mathbb{N}}$. The map $\Phi|_{\Gamma_1}$ represents the cascade of future couplings initiated by $\Phi^{(1)}$.

Suppose now that Φ has been defined on $\cup_{k \leq n} \Gamma_k$ where Γ_k is the set of η matched at step k. More precisely, $\Gamma_1, \Gamma_2, \cdots, \Gamma_n$ are disjoint subsets of $K^{\mathbb{N}}$, and each Γ_k is of the form $\Gamma_k = \tilde{\Gamma}_k \times \Gamma$ for some $\tilde{\Gamma}_k \subset K^k$; the matching of u_k and u_k' in B that takes place at step k defines a map $\Phi^{(k)} : \tilde{\Gamma}_k \to K^k$, while the cascade of future matchings initiated by $\Phi^{(k)}$ results in the definition of $\Phi : \tilde{\Gamma}_k \times \Gamma \to K^{\mathbb{N}}$. We now explain how to define Γ_{n+1}. Let $\tilde{G}_n = K^n \setminus \cup_{k \leq n} \Gamma_k^{(n)}$ where $\Gamma_k^{(n)} = \tilde{\Gamma}_k \times \Gamma^{(n-k-1)}$ is the first n-factors in Γ_k. Consider the restricted distribution $\tilde{\Theta}_{n+1}$ defined by $(\eta_0, \cdots, \eta_{n-1}) \in \tilde{G}_n$; the corresponding distribution $\tilde{\Theta}_{n+1}'$ is defined similarly. By Lemma 3.3, an α_0-fraction of these two distributions can be matched, defining an immediate matching

$\Phi^{(n+1)} : \tilde\Gamma_{n+1} \to K^{n+1}$ with $\tilde\Gamma_{n+1} \subset \tilde G_n \times K$ and $|\tilde\Gamma_{n+1}| = \alpha_0 |\tilde G_n|$. Future couplings that result from $\Phi^{(n+1)}$ define $\Phi : \Gamma_{n+1} \to K^{\mathbb N}$ with $\Gamma_{n+1} = \tilde\Gamma_{n+1} \times \Gamma$.

We claim that $\nu^n(\tilde G_n)$ decreases exponentially. This requires a little argument, for even though at each step a fraction of α_0 of what is left is matched, our matchings are "leaky", meaning not every orbit defined by a sequence in $\Gamma_k^{(n)}$ can be matched to something reasonable at the $(n+1)$st step. To estimate $\nu^n(\tilde G_n)$, we write $K^{\mathbb N} \setminus \cup_{k\le n} \Gamma_k$ as the disjoint union $G_n \cup H_n$ where $G_n = \tilde G_n \times K^{\mathbb N}$. The dynamics of $(G_n, H_n) \to (G_{n+1}, H_{n+1})$ are as follows: An α_0-fraction of G_n leaves G_n at the next step; of this part, a fraction of $\Pi_{i\ge 0}(1-\frac12\gamma^i)^D$ (recall that D is the dimension of V) goes into Γ_{n+1} (see Lemma 3.2) while the rest goes into H_{n+1}. At the same time, a fraction of H_n returns to G_{n+1}. We claim that this fraction is bounded away from zero for all n. To see this, consider one Γ_k at a time, and observe (from the definition of $\Gamma_k^{(n)}$) that $|(\Gamma_k^{(n)} \times K) \setminus \Gamma_k^{(n+1)}| \sim \text{const } |\tilde\Gamma_k| \gamma^{n-k}$.

Combinatorial Lemma *Let $a_0, b_0 > 0$, and suppose that a_n and b_n satisfy recursively*

$$a_{n+1} \ge (1-\alpha_0)a_n + \alpha_1 b_n \qquad \text{and} \qquad b_{n+1} \le (1-\alpha_1)b_n + \alpha_0 a_n$$

for some $0 < \alpha_0, \alpha_1 < 1$. Then there exits $c > 0$ such that $\frac{a_n}{b_n} > c$ for all n.

The proof of this purely combinatorial lemma is left as an exercise. We deduce from it that $\inf_n |G_n|/|H_n| > 0$, which implies $\nu^n(\tilde G_n) \le C\beta^n$ for some $C > 0$ and $\beta < 1$. This in turn implies that $|\Gamma_{n+1}| \le C\beta^n$.

Proceeding to the final count, we let $f : \text{supp } \mu \to \mathbb R$ be such that $|f| < C_1$ and $|f(u) - f(v)| < C_1 \|u - v\|^\sigma$. Then

$$\begin{aligned} I_n \ \le\ & \int_{\tilde G_n} |f(u_n(\eta_0, \cdots, \eta_{n-1}))| d\nu^n \ + \int_{K^n - \Phi^{(n)}(\cup_{k\le n}\Gamma_k^{(n)})} |f(u_n'(\eta_0, \cdots, \eta_{n-1}))| d\nu^n \\ & + \sum_{k\le n} \int_{\Gamma_k^{(n)}} |f(u_(\eta_0, \cdots, \eta_{n-1})) - f(u_n'(\Phi^{(n)}(\eta_0, \cdots, \eta_{n-1})))| d\nu^n \qquad (6) \\ \le\ & 2C_1 \cdot C\beta^n \ + \sum_{k\le n} C\beta^{k-1} \cdot C_1(\delta\gamma^{n-k})^\sigma \\ \le\ & \text{const } n \cdot [\max(\beta, \gamma^\sigma)]^n \ \le\ \text{const} \cdot \tau^n \ . \end{aligned}$$

Since these estimates are uniform for all pairs u_0, u_0', we obtain by integrating over u_0' that

$$\left| \int f d\Theta_n \ - \ \int f d\mu \right| \le \ \text{const} \cdot \tau^n \ .$$

■

Proof of Theorem B. We will prove, in the next paragraph, that assertion (2) in Theorem B holds for any invariant measure μ of $\mathcal{X}$. From this (1) follows immediately: since $(\mathcal{X},\mu)$ is exponentially mixing, it is ergodic; and since μ is chosen arbitrarily, it must be the unique invariant measure.

To prove the claim above, we pick arbitrary $u_0 \in H$, $u_0' \in A$, and compare their distributions Θ_n and Θ_n' as we did in the proof of Theorem A(2). First, by waiting a suitable period, we may assume that Θ_n is supported in $B(R_0)$ (where R_0 is as in (P2)). By condition (C) with $\varepsilon_0 = \delta$ where δ is as in Lemma 3.2, there is a set of controls of length N_0 and having ν^{N_0}-measure α_0 for some $\alpha_0 > 0$ that steer the entire ball $B(R_0)$ into a set of diameter $< \delta$. The estimate for $|\int f d\Theta_n - \int f d\hat{\Theta}_n|$ now proceeds as in Theorem A(2), with the use of these special controls taking the place of Lemma 3.3 to guarantee that an α_0-fraction of what is left is matched every N_0 steps. Averaging u_0' with respect to μ, we obtain the desired result. ■

3.3 Applications to PDEs: Proofs of Theorems 1 and 3

In this subsection, we prove the theorems related to PDEs stated in Sect. 2.1.

Proof of Theorem 1. We will prove that the abstract hypotheses (P1)–(P4) and (C) hold for the incompressible Navier-Stokes equation in L^2 for the type of noise specified. Let $S(u_0) = u(t=1)$ where u is the solution of the Navier-Stokes equation with initial data u_0, and let $u_k = S(u_{k-1}) + \eta_k$. Most of the computations below are classically known (see for instance [2], [14]); we include them for completeness.

We start by recalling a few properties of the Navier-Stokes equation in the 2-D torus. First, the following energy estimate holds for all $t > 0$:

$$\frac{1}{2}||u(t)||_{L^2}^2 + \nu \int_0^t ||\nabla u||_{L^2}^2 = \frac{1}{2}||u_0||_{L^2}^2. \tag{7}$$

Since $\int u = 0$, we have the Poincare inequality

$$||\nabla u||_{L^2} \geq ||u||_{L^2}. \tag{8}$$

From (7) and (8), it follows that

$$||S(u)||_{L^2} \leq e^{-\nu}||u||_{L^2} \ ; \tag{9}$$

thus (P2) is satisfied by taking $R_0(a) > \frac{1}{1-e^{-\nu}}a$. On the other hand, for any two solutions u and v with initial conditions u_0 and v_0, we have

$$\begin{aligned} \frac{1}{2}\partial_t ||u-v||_{L^2}^2 + \nu||\nabla(u-v)||_{L^2}^2 &\leq \left|\int (u-v).\nabla v(u-v)\right| \\ &\leq C||\nabla v||_{L^2}||u-v||_{L^2}\ ||u-v||_{H^1} \\ &\leq \frac{\nu}{2}||u-v||_{H^1}^2 + \frac{C}{\nu}||\nabla v||_{L^2}^2||u-v||_{L^2}^2\ . \end{aligned} \tag{10}$$

(Hölder and Sobolev inequalities are used to get the second line, and the Cauchy-Schwartz inequality is used to get the third.) Then, applying a Gronwall lemma, we get

$$||S(u_0) - S(v_0)||^2_{L^2} + \nu \int_0^1 ||(u-v)(s)||^2_{H^1} ds \leq C_R ||u_0 - v_0||^2_{L^2}. \tag{11}$$

Here and below, C_R denotes a generic constant depending only on R, an upper bound on the L^2 norm of u_0, and on the viscosity ν. (P1)(b) follows from (11).

To prove that (P3) holds, we use (11), (7) and a Chebychev inequality to deduce the existence of a time s, $0 < s < 1$, such that $\nu||(u-v)(s)||^2_{H^1} \leq 4C_R||u_0 - v_0||^2_{L^2}$, $\nu||u(s)||^2_{H^1} < 2R^2$ and $\nu||v(s)||^2_{H^1} < 2R^2$. Combining these estimates with energy estimates in H^1 for $t > s$, namely,

$$\frac{1}{2}||\nabla u(t)||^2_{L^2} + \nu \int_s^t ||\Delta u||^2_{L^2} = \frac{1}{2}||\nabla u(s)||^2_{L^2} \ , \tag{12}$$

$$\frac{1}{2}||\nabla v(t)||^2_{L^2} + \nu \int_s^t ||\Delta v||^2_{L^2} = \frac{1}{2}||\nabla v(s)||^2_{L^2} \ , \tag{13}$$

$$\begin{aligned} \frac{1}{2}\partial_t ||u-v||^2_{H^1} + \nu||u-v||^2_{H^2} &\leq ||u-v||_{H^2} \ ||u-v||_{H^1}\big(||u||_{H^2} + ||v||_{H^2}\big) \qquad (14) \\ &\leq \frac{\nu}{4}||u-v||^2_{H^2} + \frac{1}{\nu}\big(||u||^2_{H^2} + ||v||^2_{H^2}\big)||u-v||^2_{H^1}, \end{aligned}$$

integrating (14) between s and 1 and using again a Gronwall lemma, we deduce easily that

$$||S(u_0) - S(v_0)||_{H^1} \leq C_R ||u_0 - v_0||_{L^2}. \tag{15}$$

For any $\gamma > 0$ and $R > 0$, we may take N large enough that if $V_N :=$span$\{e_1, e_2, ..., e_N\}$, then $C_R \ ||u||_{L^2} \leq \gamma ||u||_{H^1} \ \forall u \in V_N^\perp$. This together with (15) proves (P3).

Finally, property (C) is satisfied by taking $\eta_i = 0$ for $1 \leq i \leq n_0$ where n_0 is large enough that $Re^{-\nu n_0} \leq \varepsilon_0$ (see (9)). The product structure of the noise ν [3] in property (P4)(b) holds because ξ_{jk} in (2) are independent; the assumption on $P_{V*}\nu$ holds because $b_j \neq 0$ for $1 \leq j \leq N$ where N is as in (P3) and the law for ξ_{jk} has density ρ_j. ■

Proof of Theorem 1'. We now prove (P1)–(P4) and (C) in H^s.

To prove (P1)(b), we use the energy estimates

$$\begin{aligned} \frac{1}{2}\partial_t ||u||^2_{H^s} + \nu||u||^2_{H^{s+1}} &\leq C||u||_{H^s}||u||_{H^{s+1}}||u||_{H^1} \\ &\leq \frac{\nu}{2}||u||^2_{H^{s+1}} + \frac{C}{\nu}||u||^2_{H^1}||u||^2_{H^s} \ , \qquad (16) \end{aligned}$$

[3] We hope our dual use of the symbol ν as viscosity and as noise does not lead to confusion.

$$\frac{1}{2}\partial_t||u-v||^2_{H^s} + \nu||u-v||^2_{H^{s+1}} \leq C||u-v||_{H^s}||u-v||_{H^{s+1}}(||u||_{H^{s+1}} + ||v||_{H^{s+1}}) \tag{17}$$
$$\leq \frac{\nu}{2}||u-v||^2_{H^{s+1}} + \frac{C}{\nu}(||u||^2_{H^{s+1}} + ||v||^2_{H^{s+1}})||u-v||^2_{H^s} \ ,$$

and Gronwall's lemma between times 0 and 1.

To prove (P3), we proceed as in the case of L^2, showing the existence of a time τ, $0 < \tau < 1$, such that $||(u-v)(\tau)||_{H^{s+1}} \leq 4C_R||u_0-v_0||_{H^s}$ and $||u(\tau)||_{H^{s+1}}, ||v(\tau)||_{H^{s+1}} \leq 4C_R$ where $||u_0||_{H^s}, ||v_0||_{H^s} < R$. Then using (16) and (17) with s replaced by $s+1$ and integrating between τ and 1, we deduce that

$$||S(u_0) - S(v_0)||_{H^{s+1}} \leq C_R||u_0 - v_0||_{H^s}, \tag{18}$$

from which we obtain (P3).

To prove (P2), we make use of the regularizing effect of the Navier-Stokes equation in 2-D

$$||S(u_0)||_{H^s} \leq C_s(||u||_{L^2}) \tag{19}$$

where C_s is a function depending only on s (see [14]). Since $B_{H^s}(a) \subset B_{L^2}(a)$, we know from (P3) for L^2 that if $u_0 \in B_{L^2}(R)$, we have $u_n \in B_{L^2}(R_0)$ $\forall n \geq$ some N_0. Taking $R_s = C_s(R_0) + a$, we get that $u_n \in B_{H^s}(R_s)$ $\forall n \geq N_0$. To prove (C), we argue as in L^2, taking $\eta_i = 0$, $1 \leq i \leq n_0$, for large enough n_0 and appealing to the fact that $C_s(r) \to 0$ as $r \to 0$. ■

We remark that (P2) and (C) above can be proved directly without going through L^2. Next we move on to the real Ginzburg-Landau equation.

Proof of Theorem 3. For simplicity, we take $\nu = 1$.

(a) We need to prove that there exist two disjoint stable sets A_1 and A_{-1}, stable in the sense that $\forall u \in A_{\pm 1},\ S(u) + \eta \in A_{\pm 1}$ $\forall \eta \in K$. Let

$$A_1 = \{u \in H,\ ||u-1||_{L^2} \leq \beta\} \tag{20}$$

where β is a constant to be determined. We recall for each $\phi \in \mathbb{R}$ the energy estimate

$$\frac{1}{2}\partial_t||u-\phi||^2_{L^2} + ||\nabla(u-\phi)||^2_{L^2} + \int_{\mathbb{T}} u(u-1)(u+1)(u-\phi)\ dx = 0\ . \tag{21}$$

Substituting $\phi = 1$ in (21), we get

$$\frac{1}{2}\partial_t||u-1||^2_{L^2} + ||\nabla(u-1)||^2_{L^2} \leq -\int_{\mathbb{T}} u(u+1)(u-1)^2\ dx. \tag{22}$$

Now for any ϕ with $0 < \phi < 1$, we have

$$u(u+1)(u-1)^2 \geq \phi(\phi+1)(u-1)^2 \qquad \text{if} \quad u \geq \phi \quad \text{or} \quad u \leq -1-\phi, \tag{23}$$
$$u(u+1)(u-1)^2 \geq -1 \qquad \forall u.$$

Hence

$$\int_{\mathbb{T}} u(u+1)(u-1)^2\ dx \geq \int_{\mathbb{T}} (1_{\{u\geq\phi\}} + 1_{\{u\leq -1-\phi\}})\phi(\phi+1)(u-1)^2 - \text{meas}\{u \leq \phi\}.$$

Since the first term on the right side is

$$\geq\ \phi(\phi+1)||u-1||_{L^2}^2\ -\ \int_{\mathbb{T}} 1_{\{-1-\phi<u<\phi\}}|\phi(\phi+1)(u-1)^2|,$$

we see for $\phi \leq 1/4$ that $\phi(\phi+1)(\phi+2)^2 \leq 3$, so that

$$\int_{\mathbb{T}} u(u+1)(u-1)^2\ dx\ \geq\ \phi(\phi+1)||u-1||_{L^2}^2\ -\ 4\ \text{meas}\{u \leq \phi\}\ . \tag{24}$$

Assuming $\beta < 1/4$ so that $A_1 \cap \{u > 3/4\} \neq \emptyset$, the Poincare inequality yields for $\psi < 3/4$ that

$$\begin{aligned} ||(u-\psi)1_{\{u<\psi\}}||_{L^2} &\leq\ C\ \text{meas}\ \{u \leq \psi\}||\nabla(u1_{\{u<\psi\}})||_{L^2} \\ &\leq\ C\ \text{meas}\ \{u \leq \psi\}||\nabla u||_{L^2}, \end{aligned} \tag{25}$$

the factor meas$\{u \leq \psi\}$ coming from the scale invariance. On the other hand, for $\phi < \psi$, we have

$$\text{meas}\ \{u \leq \phi\} \leq \frac{1}{(\phi-\psi)^2}||(u-\psi)1_{\{u<\psi\}}||_{L^2}^2\ . \tag{26}$$

For $u \in A_1$, we also have

$$\text{meas}\ \{u \leq \psi\} \leq \frac{\beta^2}{(1-\psi)^2}. \tag{27}$$

Putting together (25), (27) and (26), and choosing for instance $\phi = 1/4$ and $\psi = 1/2$, we have

$$||\nabla u||_{L^2}^2\ \geq\ \frac{1}{\beta^4}||(u-\psi)1_{\{u<\psi\}}||_{L^2}^2(1-\psi)^4 \tag{28}$$

$$\geq\ \frac{1}{\beta^4}\text{meas}\ \{u \leq \phi\}(\phi-\psi)^2(1-\psi)^4\ . \tag{29}$$

Taking β so that $\beta^4 \leq \frac{1}{8}(\phi-\psi)^2(1-\psi)^4$ (e.g. $\beta \leq 1/8$), (22) and (24) yield

$$\frac{1}{2}\partial_t||u-1||_{L^2}^2 + \frac{1}{2}||\nabla u||_{L^2}^2 \leq -\phi(\phi+1)||u-1||_{L^2}^2 \tag{30}$$

as long as $u \in A_1$. Hence

$$||S(u)-1||_{L^2} \leq e^{-\phi(\phi+1)}||u-1||_{L^2}\ . \tag{31}$$

Finally, taking a small enough, namely

$$a \leq \beta(1 - e^{-\phi(\phi+1)}), \tag{32}$$

we see that A_1 is stable under the Markov chain. Applying Lemma 3.1 ((P1) and (P4)(a) are easily satisfied and (P2) is replaced by the stability of A_1), we deduce that there is at least one invariant measure supported in A_1. A symmetric argument produces an invariant measure in A_{-1}. Clearly these two measures are distinct.

(b) We need to verify (P3) and (P4)(b); the arguments are similar to those in the proof of Theorem 1. The assertion then follows from Theorem A(1).

(c) We explain how to verify condition (C). First, we use the regularizing effect of the Laplacian to deduce that for u_0 with $\|u_0\|_{L^2} < R$, $||u_1||_{L^\infty} \leq ||u_1||_{H^1} \leq C_R$. Then, using the maximum principle for parabolic equations, we get

$$\partial_t g \leq g - g^3 \quad \text{where} \quad g(t) = \max_{x \in \mathbb{T}} |u| \ . \tag{33}$$

Choosing n_0 large enough that $-b_0 < g(n_0) < b_0$ and taking $\eta_0 = \eta_1 = \cdots = \eta_{n_0} = 0$, we obtain $\|u_{n_0+1}\|_\infty < b_0$. Let $\eta_{n_0+1} = \eta_{n_0+2} = b_0 e_0$. Then $u_{n_0+1} = S(u_{n_0}) + \eta_{n_0+1} > 0$, and so $S(u_{n_0+1}) > 0$. Thus $1 < u_{n_0+2} < C = 3b_0$. Taking $\eta_{n_0+3} = \eta_{n_0+4} = \cdots \eta_{n_0+n_1} = 0$ for large enough n_1, we can arrange to have $||u_{n_0+n_1} - 1||_{L^2}$ as small as we wish. Notice that in the argument above, we took $b_0 > 1$ to make sure that after arranging for $\|u_{n_0+1}\|_\infty$ to be ≈ 1, we obtain $u_{n_0+2} > 1$ after two kicks in a suitable direction. It is clear that with more kicks the condition $b_0 > 1$ can be relaxed. ■

4 Dynamics with Negative Lyapunov Exponents

4.1 Formulation of abstract results

We consider a semi-group S_t on H and a Markov chain $\mathcal{X}$ defined by (I) or (II) in the beginning of Sect. 3.1. In order for Lyapunov exponents to make sense, we need to impose differentiability assumptions.

(P1') **(a)** $S(B(R))$ is compact $\forall R > 0$;

(b) S is C^{1+Lip}, meaning for every $u \in H$, there exists a bounded linear operator $L_u : H \to H$ with the property for all $h \in H$,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \{S(u + \varepsilon h) - S(u) - L_u(\varepsilon h)\} = 0 \tag{34}$$

and $\forall R > 0, \exists M_R$ such that $\forall u, v \in B(R)$, $\|L_u - L_v\| \leq M_R \|u - v\|$.

Since Lemma 3.1 clearly holds with (P1) replaced by (P1'), we let A be as in Section 3.

Proposition 4.1 *Assume (P1'), (P2) and (P4)(a), and let μ be an invariant measure for $\mathcal{X}$. Then there is a measurable function λ_1 on H with $-\infty \le \lambda_1 < \infty$ such that for μ-a.e. u_0 and $\nu^{\mathbb{N}}$-a.e. $\eta = (\eta_0, \eta_1, \eta_2, \cdots)$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \|L_{u_{n-1}} \circ \cdots \circ L_{u_1} \circ L_{u_0}\| = \lambda_1(u_0) .$$

Moreover, λ_1 is constant μ-a.e. if $(\mathcal{X}, \mu)$ is ergodic.

This proposition follows from a direct application of the Subadditive Ergodic Theorem [6] together with the boundedness of $\|L_u\|$ on A (see also Lemma 4.1 below). We will refer to the function or, in the ergodic case, number λ_1 as the **top Lyapunov exponent** of $(\mathcal{X}, \mu)$. This section is concerned with the dynamics of $\mathcal{X}$ when $\lambda_1 < 0$.

We begin by stating a result, namely Theorem C, which gives a general description of the dynamics when $\lambda_1 < 0$. This result, however, is not needed for our application to PDEs. The proof of Theorem 2 uses only Theorem D, which is independent of Theorem C.

Let μ be an invariant measure of $\mathcal{X}$. Theorem C concerns the conditional measures of μ *given the past*. That is to say, we view $\mathcal{X}$ as starting from time $-\infty$, i.e. consider $\cdots, u_{-2}, u_{-1}, u_0, u_1, u_2, \cdots$ defined by $u_{n+1} = Su_n + \eta_n$ $\forall n \in \mathbb{Z}$ where $\cdots, \eta_{-2}, \eta_{-1}, \eta_0, \eta_1, \eta_2, \cdots$ are ν-i.i.d. Then for $\nu^{\mathbb{Z}}$-a.e. $\eta = (\cdots, \eta_{-1}, \eta_0, \eta_1, \cdots)$, the conditional probability of μ given $\eta^- := (\cdots, \eta_{-2}, \eta_{-1})$ is well defined. We denote it by μ_η.

Theorem C (Random sinks). *Assume* **(P1'), (P2)** *and* **(P4)(a)**, *and let μ be an ergodic invariant measure with $\lambda_1 < 0$. Then there exists $k_0 \in \mathbb{Z}^+$ such that for $\nu^{\mathbb{Z}}$-a.e. $\eta \in K^{\mathbb{Z}}$, μ_η is supported on exactly k_0 points of equal mass.*

This result is well known for stochastic flows in finite dimensions (see [11]). In the next theorem we impose a condition slightly stronger than **(C)** in Sect. 3.1 to obtain the type of uniqueness result needed for Theorem 2.

(C') There exists $\hat{u}_0 \in H$ such that for all $\varepsilon_0 > 0$ and $R > 0$, there is a finite sequence of controls $\hat{\eta}_0, \cdots \hat{\eta}_n$ such that for all $u_0 \in B(R)$, if $u_{k+1} = Su_k + \hat{\eta}_k$ and $\hat{u}_{k+1} = S\hat{u}_k + \hat{\eta}_k$ for all $k < n$, then $\|u_n - \hat{u}_n\| < \varepsilon_0$.

For $u \in H$, we define the accessibility set $A(u)$ as follows: let $A_0(u) = \{u\}$, $A_n(u) = S(A_{n-1}(u)) + K$ for $n > 0$, and $A(u) = \overline{\cup_{n\ge0} A_n(u)}$.

Theorem D (Asymptotic uniqueness of solutions independent of initial condition). *Assume* **(P1'), (P2), (P4)(a)** *and* **(C')**. *Suppose there is an ergodic invariant measure μ supported on $A(\hat{u}_0)$ for which $\lambda_1 < 0$. Then μ is the only invariant measure $\mathcal{X}$ has, and the following holds for $\nu^{\mathbb{N}}$-a.e. $\eta = (\eta_0, \eta_1, \cdots)$:*

$$\forall u_0, u_0' \in H, \qquad \|u_n(\eta) - u_n'(\eta)\| \le Ce^{\lambda n} \qquad \forall n > 0$$

where λ is any number $> \lambda_1$ and $C = C(u_0, u_0', \lambda)$.

Roughly speaking, Theorem D allows us to conclude that *all* the orbits are eventually "the same" once we know that the linearized flows along *some* orbits are contractive. This passage from a local to a global phenomenon is made possible by condition (C'), which in the abstract is quite special but is satisfied by a number of standard parabolic PDEs.

4.2 Proofs of abstract results (Theorems C and D)

Let A be the compact set in Lemma 3.1, and let K denote the support of ν as before. We consider the dynamical system $F : K^{\mathbb{N}} \times A \to K^{\mathbb{N}} \times A$ defined by

$$F(\eta, u) \;=\; (\sigma\eta, S(u) + \eta_0)$$

where $\eta = (\eta_0, \eta_1, \eta_2, \cdots)$ and σ is the shift operator, i.e. $\sigma(\eta_0, \eta_1, \eta_2, \cdots) = (\eta_1, \eta_2, \cdots)$. The following is straightforward.

Lemma 4.1 *Let μ be an invariant measure of $\mathcal{X}$ in the sense of Definition 3.1. Then F preserves $\nu^{\mathbb{N}} \times \mu$, and $(F, \nu^{\mathbb{N}} \times \mu)$ is ergodic if and only if $(\mathcal{X}, \mu)$ is ergodic in the sense of Definition 3.2.*

Our next lemma relates the top Lyapunov exponent of a system, which describes the average infinitesimal behavior along its typical orbits, to the local behavior in neighborhoods of these orbits. A version applicable to our setting is contained in [13]. Let $B(u, \alpha) = \{v \in H, \|v - u\| < \alpha\}$.

Proposition 4.2 [13] *Let μ be an invariant measure, and assume that $\lambda_1 < 0$ μ-a.e. Then given $\varepsilon > 0$, there exist measurable functions $\alpha, \gamma : K^{\mathbb{N}} \times A \to (0, \infty)$ and a measurable set $\Lambda \subset K^{\mathbb{N}} \times A$ with $(\nu^{\mathbb{N}} \times \mu)(\Lambda) = 1$ such that for all $(\eta, u_0) \in \Lambda$ and $v_0 \in B(u_0, \alpha(\eta, u_0))$,*

$$\|v_n(\eta) - u_n(\eta)\| \;<\; \gamma(\eta, u_0)\; e^{(\lambda_1 + \varepsilon)n} \qquad \forall n \geq 0\;.$$

We first prove Theorem D, from which Theorem 2 is derived.

Proof of Theorem D. From (P2), it follows that we need only to consider initial conditions in $B(R_0)$. Fix $\varepsilon > 0$ and let α and Λ be as in Proposition 4.2 for the dynamical system $(F, \nu^{\mathbb{N}} \times \mu)$. We make the following choices:

(1) Let $\alpha_0 > 0$ be a number small enough that $(\nu^{\mathbb{N}} \times \mu)\{\alpha > 2\alpha_0\} > \frac{99}{100}$. Covering the compact set $A(\hat{u}_0)$ with a finite number of $\frac{1}{2}\alpha_0$-balls, we see that there exists $\tilde{u}_0 \in A(\hat{u}_0)$ such that

$$\Gamma_1 := \{\eta \in K^{\mathbb{N}} : B(\tilde{u}_0, \alpha_0) \subset B(u, \alpha(\eta, u)) \text{ for some } u \text{ with } (\eta, u) \in \Lambda\}$$

has positive $\nu^{\mathbb{N}}$-measure.

(2) Since $\tilde{u}_0 \in A(\hat{u}_0)$, there is a sequence of controls $(\tilde{\eta}_0, \cdots, \tilde{\eta}_{k-1})$ that puts $\hat{u}_0$ in $B(\tilde{u}_0, \frac{1}{2}\alpha_0)$. Choose $\delta > 0$ and $\Gamma_2 \subset K^k$ with $\nu^k(\Gamma_2) > 0$ such that if $u_0 \in B(\hat{u}_0, \delta)$ and $(\eta_0, \cdots, \eta_{k-1}) \in \Gamma_2$, then $u_k(\eta_0, \cdots, \eta_{k-1}) \in B(\tilde{u}_0, \alpha_0)$.

(3) Condition (C') guarantees that there exists a sequence of controls $(\hat{\eta}_0, \cdots, \hat{\eta}_{j-1})$ that puts the entire ball $B(R_0)$ inside $B(\hat{u}_0, \frac{1}{2}\delta)$. Choose $\Gamma_3 \subset K^j$ with $\nu^j(\Gamma_3) > 0$ such that every sequence $(\eta_0, \cdots, \eta_{j-1}) \in \Gamma_3$ puts $B(R_0)$ inside $B(\hat{u}_0, \delta)$.

Let $\Gamma \subset K^{\mathbb{N}}$ be the set defined by

$$\{(\eta_0, \cdots, \eta_{j-1}) \in \Gamma_3;\ (\eta_j, \cdots, \eta_{j+k-1}) \in \Gamma_2;\ (\eta_{j+k}, \eta_{j+k+1}, \cdots) \in \Gamma_1\}\ .$$

Clearly, $\nu^{\mathbb{N}}(\Gamma) > 0$. The following holds for $\nu^{\mathbb{N}}$-a.e. η : Fix η, and let B_n denote the nth image of $B(R_0)$ for this sequence of kicks. By the ergodicity of $(\sigma, \nu^{\mathbb{N}})$, there exists N such that $\sigma^N\eta \in \Gamma$. Choosing $N \geq N_0(R_0)$, we have, by (P2), that $B_N \subset B(R_0)$. The choice in (3) then guarantees that $B_{N+j} \subset B(\hat{u}_0, \delta)$, and the choice in (2) guarantees that $B_{N+j+k} \subset B(\tilde{u}_0, \alpha_0)$. By (1), $B_{N+j+k} \subset B(u, \alpha(u, \sigma^{N+j+k}\eta))$ for some u with $(\sigma^{N+j+k}\eta, u) \in \Lambda$. Proposition 4.2 then says that when subjected to the sequence of kicks defined by $\sigma^{N+j+k}\eta$, all orbits with initial conditions in B_{N+j+k} converge exponentially to each other as $n \to \infty$. Hence this property holds for all orbits starting from $B(R_0)$ when subjected to η. Theorem D is proved. ■

Proceeding to Theorem C, the measures μ_η defined in Sect. 4.1 are called the *sample* or *empirical measures* of μ. They have the interpretation of describing what one sees at time 0 given that the system has experienced the sequence of kicks $\eta^- = (\cdots, \eta_{-2}, \eta_{-1})$. The characterization of μ_η in the next lemma is useful. We introduce the following notation: Let $S_{\eta_0} : H \to H$ be the map defined by $S_{\eta_0}(u) = Su + \eta_0$; for a measure μ on H, $S_{\eta_0 *}\mu$ is the measure defined by $(S_{\eta_0 *}\mu)(E) = \mu(S_{\eta_0}^{-1}E)$.

Lemma 4.2 *Let μ be an invariant measure for $\mathcal{X}$. Then for $\nu^{\mathbb{Z}}$-a.e. $\eta = (\cdots, \eta_{-2}, \eta_{-1}, \eta_0, ...)$, $(S_{\eta_{-1}} S_{\eta_{-2}} \cdots S_{\eta_{-n}})_*\mu$ converges weakly to μ_η.*

Proof: Fix a continuous function $\varphi : A \to \mathbb{R}$, and define $\varphi^{(n)} : K^{\mathbb{Z}} \to \mathbb{R}$ by

$$\begin{aligned}\varphi^{(n)}(\eta) &= \int \varphi\ d((S_{\eta_{-1}} S_{\eta_{-2}} \cdots S_{\eta_{-n}})_*\mu) \\ &= \int \varphi(S_{\eta_{-1}} S_{\eta_{-2}} \cdots S_{\eta_{-n}}(u)) d\mu(u). \end{aligned} \tag{35}$$

Then $\varphi^{(n)}$ is $\mathcal{B}_{-1}^{-n}$-measurable where $\mathcal{B}_{-1}^{-n}$ is the σ-algebra on $K^{\mathbb{Z}}$ generated by coordinates $\eta_{-1}, \cdots, \eta_{-n}$. Since $\int S_{\eta_{-n} *}\mu\ d\nu(\eta_{-n}) = \mu$, we have $E(\varphi^{(n)} | \mathcal{B}_{-1}^{-n+1}) = \varphi^{(n-1)}$. The martingale convergence theorem then tells us that $\varphi^{(n)}$ convergence $\nu^{\mathbb{Z}}$-a.e. to a function measurable on $\mathcal{B}_{-1}^{-\infty}$. It suffices to carry out the argument above for a countable dense set of continuous functions φ. ■

Lemma 4.3 *Given $\delta > 0$, $\exists N = N(\delta) \in \mathbb{Z}^+$ such that for $\nu^{\mathbb{Z}}$-a.e. η, there is a set E_η consisting of $\leq N$ points such that $\mu_\eta(E_\eta) > (1-\delta)$.*

Proof: Let α and γ be the functions in Proposition 4.2 for the dynamical system $(F, \nu^{\mathbb{N}} \times \mu)$. Given $\delta > 0$, we let $\alpha_0, \gamma_0 > 0$ be constants with the property that if

$$G \;=\; \{(\eta, u) : \alpha(\eta, u) \geq \alpha_0, \gamma(\eta, u) \leq \gamma_0\}$$

and

$$\Gamma \;=\; \{\eta \in K^{\mathbb{N}} : \mu\{u : (\eta, u) \in G\} > 1 - \delta\}\ ,$$

then $\nu^{\mathbb{N}}(\Gamma) > 1 - \delta$. Consider $\eta \in K^{\mathbb{Z}}$ such that

(i) $\mu_\eta = \lim (S_{\eta_{-1}} S_{\eta_{-2}} \cdots S_{\eta_{-n}})_* \mu$ and

(ii) $(\eta_{-n}, \eta_{-n+1}, \cdots) \in \Gamma$ for infinitely many $n > 0$.

By Lemma 4.2 and the ergodicity of $(\sigma, \nu^{\mathbb{Z}})$, we deduce that the set of η satisfying (i) and (ii) has full measure. We will show that the property in the statement of the lemma holds for these η.

Fix a cover $\{B_1, \cdots, B_N\}$ of A by $\frac{\alpha_0}{2}$-balls, and let η be as above. We consider n arbitrarily large with $(\eta_{-n}, \eta_{-n+1}, \cdots) \in \Gamma$. For each i, $1 \leq i \leq N$, such that $B_i \cap \{u \in H : ((\eta_{-n}, \eta_{-n+1}, \cdots), u) \in G\} \neq \emptyset$, pick an arbitrary point $u^{(i)}$ in this set. Our choices of G and Γ ensure that $\mu(\cup_i B(u^{(i)}, \alpha_0)) > 1 - \delta$, and that the diameter of $(S_{\eta_{-1}} S_{\eta_{-2}} \cdots S_{\eta_{-n}}) B(u^{(i)}, \alpha_0)$ is $\leq\ \gamma_0 \alpha_0 e^{(\lambda+\varepsilon)n}$. We have thus shown that a set of μ_η-measure $> 1 - \delta$ is contained in $\leq N$ balls each with diameter $\leq \gamma_0 \alpha_0 e^{(\lambda+\varepsilon)n}$. The result follows by letting $n \to \infty$. ■

To prove Theorem C, we need to work with a version of $(F, \nu^{\mathbb{N}} \times \mu)$ that has a past. Let $\tilde{F} : K^{\mathbb{Z}} \times A \to K^{\mathbb{Z}} \times A$ be such that $\tilde{F} : (\eta, u) \mapsto (\sigma\eta, S_{\eta_0} u)$, and let $\nu^{\mathbb{Z}} * \mu$ be the measure which projects onto $\nu^{\mathbb{Z}}$ in the first factor and has conditional probabilities μ_η on η-fibers. That $\nu^{\mathbb{Z}} * \mu$ is $\tilde{F}$-invariant follows immediately from Lemma 4.2. It is also easy to see that $(\tilde{F}, \nu^{\mathbb{Z}} * \mu)$ is ergodic if and only if $(F, \nu^{\mathbb{N}} \times \mu)$ is.

Proof of Theorem C. It follows from Lemma 4.3 that for $\nu^{\mathbb{Z}}$-a.e. η, μ_η is atomic, with possibly a countable number of atoms. We now argue that there exists $k_0 \in \mathbb{Z}^+$ such that for a.e. η, μ_η has exactly k_0 atoms of equal mass.

Let

$$h(\eta) \;=\; \sup_{u \in H}\ \ \mu_\eta\{u\}\ .$$

To see that h is a measurable function on $K^{\mathbb{Z}}$, let $\mathcal{P}^{(n)}$, $n = 1, 2, \cdots$, be an increasing sequence of finite measurable partitions of A such that $\text{diam}\mathcal{P}^{(n)} \to 0$ as $n \to \infty$. Then for each $P \in \mathcal{P}^{(n)}$, $\eta \mapsto \mu_\eta(P)$ is a measurable function, as are $h_n := \max_{P \in \mathcal{P}^{(n)}} \mu_\eta(P)$ and $h := \lim_n h_n$. Observe that $h(\sigma\eta) \geq h(\eta)$, with $>$ being possible in principle since S_{η_0} is not necessarily one-to-one. However, the measurability of h together with the ergodicity of $(\sigma, \nu^{\mathbb{Z}})$ implies that h is constant a.e. Let us call this value h_0. From the last lemma we know that $h_0 > 0$.

To finish, we let $X = \{(\eta, u) \in K^{\mathbb{Z}} \times A : \mu_\eta\{u\} = h_0\}$. Then X is a measurable set, $(\nu^{\mathbb{Z}} * \mu)(X) > 0$ and $\tilde{F}^{-1}X \supset X$. This together with the ergodicity of $(\tilde{F}, \nu^{\mathbb{Z}} * \mu)$ implies that $(\nu^{\mathbb{Z}} * \mu)(X) = 1$, which is what we want. ∎

4.3 Application to PDEs: Proof of Theorem 2

Let S_t be the semi-group generated by the (unforced) Navier-Stokes system, and let $S = S_1$.

Lemma 4.4 *S is C^{1+Lip} in $H^2(\mathbb{R}^2)$.*

Proof. It is easy to see that L_u is defined by $L_u w = \psi(1)$ where ψ is the solution of the linear problem

$$\begin{cases} \partial_t \psi + U.\nabla\psi + \psi.\nabla U - \nu\Delta\psi = -\nabla p, \\ \psi(t=0) = w \ , \qquad \mathrm{div}\psi = 0 \end{cases} \tag{36}$$

where U denotes the solution of the Navier-Stokes system with initial data u. That L_u is linear, continuous and goes from H^2 to H^2 is obvious. To prove that (34) holds, let U and V be the solutions of the Navier-Stokes system with initial data u and $u + \epsilon w$ respectively. Then $y = V - U - \epsilon\psi$ satisfies

$$\begin{cases} \partial_t y + (U + \epsilon\psi).\nabla y + y.\nabla V + \epsilon^2\psi.\nabla\psi - \Delta y = -\nabla p \\ y(t=0) = 0 \ , \qquad \mathrm{div}(y) = 0. \end{cases} \tag{37}$$

By a simple computation, we get that $||y(t=1)||_{H^2} \leq C(1 + ||w||^2_{H^2})\epsilon^2$, where here and below C denotes a constant depending only on the H^2 norm of u.

To prove that L_u is Lipschitz, i.e.,

$$||(L_u - L_v)w||_{H^2} \leq C||u - v||_{H^2}||w||_{H^2}, \tag{38}$$

we define $L_v w = \phi(1)$ where ϕ solves an equation analogous to (36) with V in the place of U, V being the solution with initial condition v. The desired estimate $||(\psi - \phi)(t = 1)||_{H^2}$ is obtained by subtracting this equation from (36). ∎

Remark. We observe here that the top Lyapunov exponent is negative if the noise is sufficiently small. We will show, in fact, that given any positive viscosity ν, if a (see Sect. 2.1 for definition) is small enough, then $S : H^2 \to H^2$ is a contraction on the ball of radius $\frac{\nu}{2C}$.

Rewriting equations (16) and (17) with $s = 2$, we have

$$\partial_t||u||^2_{H^2} + \nu||u||^2_{H^3} \leq \frac{C^2}{\nu}||u||^4_{H^s} \ , \tag{39}$$

$$\partial_t ||u-v||^2_{H^2} + \nu ||u-v||^2_{H^3} \le \frac{C^2}{\nu}(||u||^2_{H^3} + ||v||^2_{H^3})||u-v||^2_{H^2} \ . \tag{40}$$

For u_0 with $||u_0||_{H^2} \le \frac{\nu}{2C}$ and a noise with $a \le \frac{\nu}{2C}(1-e^{-\nu/4})$, it follows from (39) and a Gronwall lemma that

$$||S(u_0)||^2_{H^2} \le \left(\frac{\nu}{2C}\right)^2 e^{-\nu/2} \ ,$$

from which we obtain $||u_1||_{H^2} \le \frac{\nu}{2C}$. Moreover, from (39), we have that

$$\nu \int_0^1 ||u||^2_{H^3} \le \frac{\nu^3}{16C^2},$$

so that if v is another solution of the Navier-Stokes system with $||v_0||_{H^2} \le \frac{\nu}{2C}$, then (40) gives

$$||u-v||^2_{H^2} \le ||u_0 - v_0||^2_{H^2} e^{-\nu+\frac{\nu}{8}} \ .$$

Proof of Theorem 3. It suffices to check the hypotheses of Theorem D: (P1') is proved in Lemma 4.4, and we explained in the proof of Theorem 1' why (C') holds with $\hat{u}_0 = 0$. ■

References

[1] J. Bricmont, A. Kupiainen and R. Lefevere, Exponential mixing for the $2D$ Navier-Stokes dynamics, 2000 preprint.

[2] P. Constantin and C. Foias, Navier-Stokes equations, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1988. x+190 pp.

[3] W. E, J. Mattingly and Ya. G. Sinai, Gibbsian Dynamics and ergodicity for the stochastically forced $2D$ Navier-Stokes equation, 2000 preprint.

[4] J.-P. Eckmann and M. Hairer, Uniqueness of the invariant measure for a stochastic PDE driven by degenerate noise, 2000 preprint.

[5] F. Flandoli and B. Maslowski, Ergodicity of the $2D$ Navier-Stokes equations, NoDEA 1 (1994), 403-426.

[6] J. F. C. Kingman, Subadditive processes, Ecole d'été des Probabilités de Saint-Flour, Lecture Notes in Math., 539, Springer (1976)

[7] S. Kuksin and A. Shirikyan, Stochastic dissipative PDEs and Gibbs measures, Commun. Math. Phys. 213 (2000), no. 2, 291–330.

[8] S. Kuksin and A. Shirikyan, On dissipative systems perturbed by bounded random kick-forces, 2000 preprint.

[9] S. Kuksin and A. Shirikyan, A coupling approach to randomly forced nonlinear PDE's, 2001 preprint.

[10] S. Kuksin, A. L. Piatnitski and A. Shirikyan, A coupling approach to randomly forced nonlinear PDEs. II, 2001 preprint

[11] Y. Le Jan, Equilibre statistique pour les produits de diffeomorphismes aleatoires independants, Ann. Inst. Henri Poincare (Probabilites et Statistiques), 23, (1987), 111-120.

[12] J. Mattingly, Ergodicity of $2D$ Navier-Stokes Equations with random forcing and large viscosity, Commun. Math. Phys., 206 (1999), 273-288.

[13] D. Ruelle, Characteristic exponents and invariant manifolds in Hilbert space, Ann. Math., 115, (1982), 243-290.

[14] R. Temam, Navier-Stokes equations and nonlinear functional analysis. Second edition. CBMS-NSF Regional Conference Series in Applied Mathematics, 66. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1995. xiv+141 pp.

[15] R. Temam, Infinite-dimensional dynamical systems in mechanics and physics. Second edition. Applied Mathematical Sciences, 68. Springer-Verlag, New York, 1997. xxii+648 pp.