Sinai-Bowen-Ruelle measures for certain Hénon maps

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0. Introduction.

We study maps $T_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T_{a,b}(x,y) = (1 - ax^2 + y, bx), \quad 0 < a < 2,\ b > 0.$$

In [**BC2**] it was proved that for a positive measure set of parameters (a,b), $T_{a,b}$ has a topologically transitive attractor $\Lambda = \Lambda_{a,b}$ on which there is some hyperbolic behavior. The aim of this paper is to study the *statistical* properties of these attractors. Using the machinery developed in [**BC2**] we prove the following

Theorem. *There is a set $\Delta \subset \mathbb{R}^2$ with* $\mathrm{Leb}(\Delta) > 0$ *such that for all $(a,b) \in \Delta$, $T = T_{a,b}$ admits a unique SBR measure λ^*. Moreover, supp $(\lambda^*) = \Lambda$ and (T, λ^*) is Bernoulli.*

A T-invariant Borel probability measure μ is called a Sinai-Bowen-Ruelle (SBR) measure if there is a positive Lyaponov exponent μ-a.e. and the conditional measures of μ on unstable manifolds are absolutely continuous with respect to the Riemannian measure induced on these manifolds. (A more precise definition is given in Section 3.4.1.) This notion is due to Sinai [**S1**], [**S2**]. See also [**LS**]. The significance of these measures is evident in the following corollary.

Corollary. *For $(a,b) \in \Delta$, $T = T_{a,b}$ has the following property: Let U be a neighborhood of Λ. Then there is a set $\tilde{U} \subset U$ with positive Lebesgue measure such that for all continuous functions $\varphi : U \to \mathbb{R}$,*

$$\frac{1}{n} \sum_{i=0}^{n-1} \varphi \circ T^i(x) \to \int \varphi \, d\lambda^*$$

for every $x \in \tilde{U}$.

This corollary follows from our theorem and general nonuniform hyperbolic theory (see [**PS**,Thm.1] or Section 4.1.2). The property expressed in this corollary is often assumed to be true in physics; it is also taken for granted in numerical experiments. In the case of Axiom A attractors, mathematical justification of this property is provided by the theory of Sinai, Bowen and Ruelle (see e.g. [**B**], [**Ru1**] and [**S2**]). It has been conjectured that many other attractors admit SBR measures, but as far as we know, the Hénon family and similar examples (see next paragraph) are the only nonuniformly hyperbolic attractors for which SBR measures have been constructed.

Mora and Viana [**MV**] proved recently that in homoclinic bifurcations of surface diffeomorphisms, very small attractors with high periods appear for a positive measure set

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of parameters. These attractors have the same qualitative estimates as those in [**BC2**]. Since our proofs rely only on these qualitative estimates, our results apply there also.

This paper is organized as follows. We assume throughout that $T = T_{a,b}$, where (a, b) is a pair of "good" parameters, meaning the ones selected in [**BC2**]. We do not concern ourselves with how these parameters are selected. In Section 1 we give a summary of the results from [**BC2**] that are relevant to this work. The primary focus of [**BC2**] is on a certain set of points called "critical points". These points live on a curve W that is the unstable manifold of a fixed point. In Section 2 we shift our attention from the critical points to the action of T on all of W. Our Main Proposition establishes a near complete correspondence between the dynamics of $T|W$ and that of certain interval maps. An SBR measure is constructed in Section 3, where Lebesgue measure on a small piece of W is pushed forward. The ergodic properties of this invariant measure are studied in Section 4.

1. Dynamics of certain Hénon maps: results from [BC2].

In this section we try to give a self-contained summary of what is known about the "good" Hénon maps. For conceptual simplicity, we will isolate for separate discussion several aspects of their dynamics. The reader should be aware that many of these properties are related, in the sense that their proofs are very delicately intertwined. Some of the ideas in [**BC2**] will be expressed here in slightly different language. For instance, we will speak of orbits and vectors as being "controlled" (see sections 1.4 and 1.5). We will also introduce the notion of "generalized tangential position" (Section 1.6).

1.1 The one-dimensional model (Section 2 of [BC2]).

The b-values considered in [**BC2**] are extremely small and the entire analysis there is modeled on that of a certain class of 1-d maps. These are maps of the form $f = f_a : [-1, 1] \circlearrowleft$, where $f_a x = 1 - ax^2$ and a is very near 2. In addition the following conditions are imposed on f:

(1) There is $c > 0$ ($c \approx \log 2$) such that $|Df^n(f0)| \geq e^{cn}$ for all $n \geq 0$.

(2) There is a small real number $\alpha > 0$ (e.g. $\alpha = 10^{-6}$) such that $|f^n 0| \geq e^{-\alpha n}$ for all $n \geq 1$.

It is proved in [**BC2**] that the set of parameter values $\mathcal{A} = \{a | f_a \text{ satisfies (1) and (2)}\}$ has positive Lebesgue measure.

To study the growth of $(f^n)'x$ for $x \in [-1, 1]$, we have three types of derivative estimates given in sections 1.1.1 – 1.1.3. First some notation: Let $\delta > 0$ be a small real number that is nevertheless $\gg 2 - a$. We assume that $\delta = e^{-\mu_0}$ for some $\mu_0 \in \mathbb{Z}^+$. For bookkeeping purposes write

$$(-\delta, \delta) = \dot{\bigcup_{|\mu| \geq \mu_0}} I_\mu,$$

where $I_\mu = (e^{-(\mu+1)}, e^{-\mu})$ for $\mu > 0$ and $I_\mu = -I_{-\mu}$ for $\mu < 0$. Each I_μ is further subdivided into μ^2 intervals $\{I_{\mu,j}\}$ of equal length.

For $a \in \mathcal{A}$, $f = f_a$ has the following properties:

1.1.1. Derivative estimates away from the critical point.

There is $c_0 > 0$ and $M_0 \in \mathbb{Z}^+$ such that

(i) if $x, \dots, f^{j-1}x \notin (-\delta, \delta)$ and $j \geq M_0$, then $|Df^j(x)| \geq e^{c_0 j}$;

(ii) if $x, fx, \dots, f^{k-1}x \notin (-\delta, \delta)$ and $f^k x \in (-\delta, \delta)$, any $k \in \mathbb{Z}^+$, then $|Df^k(x)| \geq e^{c_0 k}$.

(iii) if $x, fx, \dots, f^{k-1}x \notin (-\delta, \delta)$, then $|Df^k(x)| \geq \delta e^{c_0 k}$.

1.1.2. Derivative estimates when bound to the critical orbit.

Let $\beta = 14\alpha$. For $x \in (-\delta, \delta)$ define $p(x)$ to be the largest integer p such that

$$|f^j x - f^j 0| < e^{-\beta j} \qquad \forall j < p.$$

Then

(i) $\frac{1}{2}|\mu| \leq p(x) \leq 5|\mu| \qquad \forall x \in I_\mu$;

(ii) $|Df^p(x)| \geq e^{c' p}$ for some $c' > 0$.

The orbit of x is said to be *bound* to the critical orbit during the period $j < p$. We may assume that p is constant on each $I_{\mu,j}$.

1.1.3. Distortion of Df^n.

Let $\mathcal{P}$ be the partition of $[-1,1]$ into $[-1,-\delta] \cup [\delta, 1] \cup \bigcup_{\mu,j} I_{\mu,j}$. For $J \in \mathcal{P}$, let J^+ denote the interval consisting of J and its two adjacent intervals in $\mathcal{P}$. Then the following holds: Assume that $\omega \subset [-1,1]$ is such that for all $k < N$, $f^k\omega \subset J^+$ for some $J \in \mathcal{P}$. Then there is a constant C, independent of N such that

$$\frac{|Df^N(x)|}{|Df^N(y)|} \leq C \qquad \forall x, y \in \omega.$$

1.2. Some geometric properties of Hénon maps.

The material in this subsection is quite elementary and is valid for $T = T_{a,b}$ where (a,b) belongs to a set Δ' of the form $\{(a,b) : a_1 < a < a_2 < 2,\ 0 < b < b_0\}$.

1.2.1. The unstable manifold W.

T has a unique fixed point $\hat{z}$ in the first quadrant. This fixed point is hyperbolic. Its expanding direction has slope of order $-b/2$ and eigenvalue ≈ -2. Its contractive direction has slope ≈ 2 and eigenvalue $\approx b/2$. The global unstable manifold at $\hat{z}$ is called W. It will play the rôle of the interval $[-1,1]$ in our 1-d model.

1.2.2. Attracting sets.

To guarantee that T has a compact attracting set, we first choose $a_0 < a_1 < 2$ with a_0 sufficiently near 2. Then there exists $b_0 > 0$ sufficiently small compared to $2 - a_1$ such that for all $(a,b) \in \Delta' := [a_0, a_1] \times (0, b_0]$, W stays in a bounded region. Moreover, if $\Lambda = \Lambda_{a,b}$ is the closure of W, then there is an open neighborhood $U = U_{a,b}$ of Λ such that for all $z \in U$, dist $(T^n z, \Lambda) \to 0$ as $n \to \infty$ ([**BC2**], Thm 4; see also [**BM**]).

1.2.3. Hyperbolicity and resemblance to 1-d behavior outside of $(-\delta,\delta)\times\mathbb{R}$.

Let $\delta>0$ be at least as small as needed in our one-dimensional analysis and assume that $b_0 \ll 2-a_0 \ll \delta$. Let $s(v)$ denote the absolute value of slope of the vector v. A simple calculation shows that if $z=(x,y)\notin(-\delta,\delta)\times\mathbb{R}$ and $s(v)\le\delta$, then

$$s(DT_z v)\le\frac{b}{|x|}\le\frac{b}{\delta},$$

which we assume to be $\ll\delta$. ([**BC2**], Lemma 4.5.) This defines invariant cones outside of $(-\delta,\delta)\times\mathbb{R}$.

For $z=(x,y)\notin(-\delta,\delta)\times\mathbb{R}$ and unit vector v with $s(v)\le\delta$, we have essentially the same estimates as in 1-dimension. That is, there is $c_0>0$ and $M_0\in\mathbb{Z}^+$ such that

(i) if $z,\dots,T^{j-1}z\notin(-\delta,\delta)\times\mathbb{R}$ and $j\ge M_0$, then $|DT_z^j v|\ge e^{c_0 j}$;
(ii) if $z,Tz,\dots,T^{k-1}z\notin(-\delta,\delta)\times\mathbb{R}$ and $T^k z\in(-\delta,\delta)\times\mathbb{R}$, any $k\in\mathbb{Z}^+$, then $|DT_z^k v|\ge e^{c_0 k}$;
(iii) if $z,Tz,\dots,T^{k-1}z\notin(-\delta,\delta)\times\mathbb{R}$ then $|DT^k v|\ge\delta e^{c_0 k}$.

1.2.4. Curvature estimates.

Tangencies between stable and unstable manifolds of T are inevitable. In order to show that some of these tangencies are quadratic, we will need to control the curvature of W. In [**BC2**], a curve γ in $\mathbb{R}^2$ is called a $C^2(b)$ *curve* if $s(\gamma')\le 10b$ and its curvature $\kappa\le 10b$. The following lemma is used to verify the $C^2(b)$ property.

LEMMA (proved in Section 7.6 of [**BC2**]). *Let $t\mapsto\gamma(t)$ be a smooth curve. Write $\gamma_n=T^n\gamma$ and let κ_n denote the curvature of γ_n. Assume there is a $c>0$ such that for all t*

(i) $\kappa_0(t)\le 1$;
(ii) $|\gamma_n'(t)|\,/\,|\gamma_{n-j}'(t)|\ge e^{cj}\qquad\forall j\le n$;
(iii) $\gamma_{n-j}(t)\notin(-\delta,\delta)\times\mathbb{R}$ *and* $s(\gamma_{n-j}'(t))<\delta$ *for* $j=1,\dots,4$.

Then $\kappa_n(t)\le 10b$ for all t.

The discussion from here on applies only to $T=T_{a,b}$, where (a,b) belongs in an inductively constructed set $\Delta\subset\Delta'$. The set Δ has the property that for each sufficiently small $b>0$, $\Delta_b:=\{a:(a,b)\in\Delta\}$ is a Cantor set and has positive Lebesgue measure.

1.3. The critical set (mostly Sections 5 and 6 of [**BC2**]).

We use the following notation. For $A\in GL(2,\mathbb{R})$, if $v\mapsto|Av|/|v|$ is not constant, let $e(A)$ denote the unit vector that is contracted the most by A. Write $e_n(z)=e(DT_z^n)$ if it makes sense. For $z\in W$, let $\tau\in T_z\mathbb{R}^2$ denote a unit vector, tangent to W.

A certain subset $\mathcal{C}$ of W called the critical set is singled out in [**BC2**] to play the rôle of 0 in 1-d. This set is constructed according to the rule in 1.3.1. Each $z_0\in\mathcal{C}$ has the property that $\tau(z_0)$ is the most contracted direction, i.e. $\lim_{n\to\infty}e_n(z_0)=\tau(z_0)$. This can be thought of as the moral equivalent to $f'(0)=0$.

1.3.1. Location and rule of construction of $\mathcal{C}$.

$\mathcal{C}$ is located in $W\cap((-10b,10b)\times\mathbb{R})$. It seems likely that it does not lie on any smooth curve. To give a more precise description of where points of $\mathcal{C}$ are located, we divide

W into segments of different "generations": First, there is a unique point $z_0 \in \mathcal{C}$ on the roughly horizontal segment of W containing the fixed point $\hat{z}_0$. In [**BC2**], the segment of W between $T^2 z_0$ and Tz_0 is called G_1, the *leaf of generation 1.* Leaves of generation $g \geq 2$ are defined by

$$G_g := T^{g-1} G_1 - G_{g-1}.$$

We assume (a, b) is sufficiently near $(2, 0)$ so that $\bigcup_{g \leq 27} G_g$ consists of 2^{26} roughly horizontal segments linked by sharp turns near $x = \pm 1$, $y = 0$, and that $(\bigcup_{g \leq 27} G_g) \cap ((-\delta, \delta) \times \mathbb{R})$ consists of 2^{26} $C^2(b)$ curves (see Section 1.2.4). If $\mathcal{C}_g := \mathcal{C} \cap \bigcup_{i \leq g} G_i$, then $\mathcal{C}_{27}$ contains 2^{26} points, one on each of these $C^2(b)$ curves.

For $g > 27$, the following rule is used. Let ρ be a number with $b \ll \rho \ll e^{-72}$, and assume $\mathcal{C}_{g-1}$ is already defined. Consider a maximal piece of $C^2(b)$ curve $\gamma \subset G_g$. If γ contains a segment of length $2\rho^g$ centered at $z_0' = (x_0', y_0')$, and there is a critical point $z_0 = (x_0, y_0) \in \mathcal{C}_{g-1}$ with $x_0' = x_0$, and $|y_0' - y_0| \leq b^{g\sigma}$, $\sigma = \frac{1}{540}$, then there is a unique point $\hat{z}_0 \in \mathcal{C}_g \cap \gamma$. Moreover, $|\hat{z}_0 - z_0'| \leq |y_0' - y_0|^{\frac{1}{2}}$. These are the only points of $\mathcal{C}_g$.

1.3.2. Remarks on the construction of $\mathcal{C}$.

Roughly speaking, certain points z on W with $e_n(z) = \tau$ are picked out as approximate critical points. Once a point is designated an approximate critical point, parameters are excluded to ensure that a point in $\mathcal{C}$ is eventually constructed nearby. The construction of $\mathcal{C}$ itself involves an inductive procedure that is guaranteed to work only for the "good" parameters.

As to when to initiate the construction of a critical point, the rules in Section 1.3.1 are obviously quite arbitrary. There are two guiding principles. The first is that each "distinguishable" critical orbit of length n causes a certain measure set of parameters to be discarded, so we cannot afford to have too many critical points. On the other hand, as in 1-d, when a critical orbit returns to $(-\delta, \delta) \times \mathbb{R}$, we want it to be "bound" to a suitable critical point of earlier generation. So we must be sure there are enough of these "suitable" critical points. These issues are discussed in Sections 3 and 6 of [**BC2**].

1.3.3. Dynamical properties of $\mathcal{C}$.

The parameter selection procedure is also designed to guarantee that every $z_0 \in \mathcal{C}$ has the following properties:

(1) For all $n \geq 0$,

$$|DT^n_{z_0} \tbinom{0}{1}| \geq e^{cn} \text{ for some } c \approx \log 2,$$
$$|DT^n_{z_0} \tau| \leq (Cb)^n \text{ for some } C \text{ independent of } b;$$

(2) there is a small real number α, say $\alpha = 10^{-6}$, such that $T^n z_0$ stays a distance of $\geq e^{-\alpha n}$ from certain points of $\mathcal{C}$.

The uniform hyperbolicity expressed in (1) is analogous to condition (1) in Section 1.1. We will elaborate on (2) and other dynamical properties of $\mathcal{C}$ later on.

1.3.4. Contracting fields in neighborhoods of a critical point.

Let $z_0 \in \mathcal{C}$. When an orbit comes near z_0, the course of its interaction with z_0 in the next n iterates is determined to a great extent by the contracting field e_n and expanding field $e_n^\perp$ near z_0. The following facts are proved in Section 5 of [**BC2**]*:

(i) e_1 is defined everywhere and has slope $= 2ax + \mathcal{O}(b)$.

(ii) There is λ with $b \ll \lambda < 1$ such that for all $z_0 \in \mathcal{C}$, e_n is defined on $B_n(z_0)$:= the disk of radius $(\lambda/5)^n$ about z_0. Moreover,

$$|DT_z^n\, e_n| \le (Cb)^n, \qquad \forall z \in B_n.$$

(iii) There is a constant $C > 0$ such that for all $z_1,\ z_2 \in B_n$

$$|e_n(z_1) - e_n(z_2)| \le C|z_1 - z_2|.$$

(iv) For $(x_1, y_1), (x_2, y_2) \in B_n$ with $|y_1 - y_2| \le |x_1 - x_2|$

$$|e_n(x_1, y_1) - e_n(x_2, y_2)| = (2a + \mathcal{O}(b))|x_1 - x_2|.$$

(v) For $m < n$, $|e_n - e_m| \le \mathcal{O}(b^m)$ on $B_n(z_0)$.

Suppose z_0 lies on a $C^2(b)$ curve γ. Then it follows from (i)–(v) that there is a unique z_0^n in $\gamma \cap B_n(z_0)$ at which $e_n = \tau$, and that $|z_0^n - z_0| = \mathcal{O}(b^n)$.

1.4 Controlled orbits.

1.4.1 Definitions.

(i) Let $z_0 = (x_0, y_0)$ be a critical point lying on a piece of $C^2(b)$ curve $\gamma \subset W$, and let $\zeta_0 = (\xi_0, \eta_0)$ be an arbitrary point in $(-\delta, \delta) \times \mathbb{R}$. We say that ζ_0 is in *tangential position* with respect to z_0 if there is $\hat{z}_0 = (\hat{x}_0, \hat{y}_0) \in \gamma$ with $\hat{x}_0 = \xi_0$ and $|\hat{y}_0 - \eta_0| < |\hat{x}_0 - x_0|^4$.

(ii) Let $\zeta_0 \in W$, and let $\zeta_n = T^n\zeta_0$. We say that *the orbit of ζ_0 is controlled on the time interval* $[0, N), N \le \infty$, if the following holds. If $\zeta_0 \notin \mathcal{C}$, let $n_1 < n_2 < \cdots$ be the times when ζ_n is in $(-\delta, \delta) \times \mathbb{R}$. If $\zeta_0 \in \mathcal{C}$, take $n_1 > 0$ to be the first time ζ_n returns to $(-\delta, \delta) \times \mathbb{R}$. Then for every $n_i \in [0, N), \zeta_{n_i}$ is attached to a critical point with respect to which it is in tangential position. This critical point is denoted by $z(\zeta_{n_i})$ and is called the *binding point* of ζ_{n_i}. See 1.4.3 for a further requirement on the assignment $\zeta_{n_i} \mapsto z(\zeta_{n_i})$.

1.4.2 Control of critical orbits.

A key idea in [**BC2**] is that through parameter exclusion, it is arranged so that for every $z_0 \in \mathcal{C}$

(i) the orbit of z_0 is controlled on the time interval $[0, \infty)$;

(ii) $|z_{n_i} - z(z_{n_i})| \ge e^{-\alpha n_i} \quad \forall i \ge 1$;

(iii) $\mathrm{gen}(z(z_{n_i})) < \theta \cdot n_i$ for some small $\theta > 0$.

*The slight difference between the definition of e_n here and that in [**BC2**] is unimportant.

(ii) is the precise statement of (2) in 1.3.3. To explain the availability of binding points, consider first the following ideal situation: Suppose that z_n is a free return, and that $\bigcup_{g<\theta n} G_g$ contains $2^{[\theta n]-1}$ $C^2(b)$ curves stretched across $(-\delta, \delta) \times \mathbb{R}$, each containing a critical point. In this case one needs only to consider the curve nearest to z_n. Most locations of z_n are in tangential position relative to the critical point on this curve; bad locations of z_n correspond to deleted parameters. In general, some of the $2^{[\theta n]-1}$ pieces of W are missing or partially missing. What is true is that at free returns, z_n is always surrounded by a "fairly regular" collection of $C^2(b)$ segments $\{\gamma_j\}$ of W. "Fairly regular" here means that $\text{gen}(\gamma_j) \sim 3^j$, $\text{length}(\gamma_j) \sim \rho^{3^j}$ and $\text{dist}(z_n, \gamma_j) \sim b^{3^j}$ $(3^j < \theta n)$. It is then shown that such a family contains enough critical points for our purposes. (See sections 6 and 7 of [**BC2**].)

1.4.3 Bound periods.

Let $z_0 \in \mathcal{C}$, and let ζ_0 be in tangential position with respect to z_0. As a first approximation we define the bound period $\tilde{p}(\zeta_0, z_0)$ to be the largest k such that

$$|T^j\zeta_0 - T^j z_0| < e^{-\beta j} \qquad \forall j < k.$$

(Recall that $\beta = 14\alpha$.)

Now consider ζ_0 whose orbit is controlled on $[0, \infty)$, and let $\tilde{p}_i = \tilde{p}(\zeta_{n_i}, z(\zeta_{n_i}))$. Note that it is entirely possible for ζ_n with $n_i < n < n_i + \tilde{p}_i$ to return to $(-\delta, \delta) \times \mathbb{R}$, so that bound periods can be initiated in the middle of bound periods. It is shown in Section 6.2 of [**BC2**] that by modifying slightly the definition of $\tilde{p}$, bound periods can be made "nested". More precisely, one can choose $\{p_i\}_{i=1}^{\infty}$ in such a way that for all i,

(i) $p_i \le \tilde{p}_i$ and $|T^{p_i}\zeta_{n_i} - T^{p_i} z(\zeta_{n_i})| \ge e^{-\beta^* p_i}$ for some $\beta^* \approx \beta$,
(ii) if n_j is such that $n_i < n_j < n_i + p_i$, then $n_j + p_j \le n_i + p_i$.

The bound period between ζ_{n_i} and $z(\zeta_{n_i})$ is then defined to be p_i. It is further required that if the bound relation between z_{n_i} and $z(z_{n_i})$ is still in effect at time $n_j > n_i$, then we must have $z(z_{n_j}) = z(T^{n_j - n_i} z(z_{n_i}))$.

1.4.4 Bound and free states.

Conceptually, it is convenient to divide a controlled orbit into "free" and "bound" states. Let ζ_0 be as above.

- For $n \le n_1$, ζ_n is *free*.
- For $n_1 < n < n_1 + p_1$, ζ_n is said to be in *bound* state. (During this period ζ_n may make multiple returns to $(-\delta, \delta) \times \mathbb{R}$, at which times new bindings are formed. But by our definition of p_i, these "secondary bound periods" expire at or before $n_1 + p_1$.)
- Let n_i be the first return to $(-\delta, \delta) \times \mathbb{R}$ after $n_1 + p_1$. Then ζ_n is free for $n_1 + p_1 \le n \le n_i$, and ζ_{n_i} is called a *free return*.
- ζ_n is in bound state again during the period $n_i < n < n_i + p_i$, and so on.

1.5 Keeping track of derivatives.

Throughout this subsection we assume that the orbit ζ_0 is controlled on $[0, \infty)$, with return times $n_1 < n_2 < \dots$ and binding points $\zeta_{n_i} \leftrightarrow z(\zeta_{n_i})$ as defined in 1.4.1.

1.5.1 The fold period.

For each n_i, the fold period of ζ_{n_i} with respect to $z(\zeta_{n_i})$ is defined to be a number l_i with $2m \le l_i \le 3m$, where

$$(5b)^m \le |z(\zeta_{n_i}) - \zeta_{n_i}| \le (5b)^{m-1}.$$

For convenience we will choose l_i such that $n_i + l_i \ne n_j, n_j + 1$ for any j. (The precise definition is given in [**BC2**], Section 6.3 and does not particularly concern us.) We mention three facts about fold periods:

(i) The fold period initiated at a return to $(-\delta, \delta) \times \mathbb{R}$ is very short compared to the bound period initiated at that time. In fact, $l/p \le$ const $/\log(1/b)$, which tends to 0 as $b \to 0$.

(ii) It follows from (i) above and Section 1.4.2 that every $z_0 \in \mathcal{C}$ has the following property: for every $m \in \mathbb{Z}^+$, there are integers m_1 and m_2, with $m_1 \le m \le m_2$ and $m_2 - m_1 \le$ const. $m/\log(1/b)$, such that z_{m_1} and z_{m_2} are outside all fold periods. (See Lemma 6.5 in [**BC2**].)

(iii) If $n_i < n_j \le n_i + l_i$ then $n_j + l_j \le n_i + l_i$, i.e. a fold period initiated within another fold period does not extend beyond that fold period.

The role of the fold period will become clear in 1.5.4.

1.5.2 The splitting algorithm (section 7.1 of [**BC2**]).

Let $v \in T_{\zeta_0}\mathbb{R}^2$ be a tangent vector. The following algorithm is devised in [**BC2**] to keep track of $w_n(\zeta_0, v) := DT^n_{\zeta_0} v$. For each n, we decompose $w_n(\zeta_0, v)$ into

$$w_n(\zeta_0, v) = w_n^*(\zeta_0, v) + E_n(\zeta_0, v),$$

where w_n^* and E_n are defined according to the following rules. (We will omit all references to (ζ_0, v) from now on.)

- For $n < n_1$, let $w_n^* = w_n$.
- At time n_1, let $e = e_{l_1}$ be the contractive vector field around $z(\zeta_{n_1})$. Write

  $$w_{n_1} = A_{n_1} e + B_{n_1} \binom{0}{1}$$

  and let

  $$w_{n_1}^* = B_{n_1} \binom{0}{1}, \quad E_{n_1} = A_{n_1} e.$$

- For $n > n_1$, if $n \ne$ any n_i, define

  $$E_n = \sum_{\substack{n_k < n \\ \text{s.t. } n_k + l_k > n}} DT^{n-n_k}_{\zeta_{n_k}} (A_{n_k} e_{l_k}).$$

  This of course defines w_n^* as well.
- If $n = n_i$ for some $i > 1$, we let $e = e_{l_i}$, write

  $$DT_{\zeta_{n-1}} w_{n-1}^* = A_n e + B_n \binom{0}{1}$$

  and let

  $$w_n^* = B_n \binom{0}{1}.$$

This algorithm has no geometric meaning for arbitrary (ζ_0, v). See 1.5.4 for a discussion of the situation for which it is designed.

1.5.3. Controlled derivatives along controlled orbits.

Definition. Let $\zeta_0 = W$ and let $v \in T_{\zeta_0}\mathbb{R}^2$. We say that the pair (ζ_0, v) is *controlled* on the time interval $[0, N)$ if the orbit of ζ_0 is controlled on $[0, N)$ and whenever $\zeta_n \in (-\delta, \delta) \times \mathbb{R}$, the splitting algorithm in Section 1.5 gives

$$(*) \qquad 3|\zeta_n - z(\zeta_n)| \le |\overline{B}_n| \le 5|\zeta_n - z(\zeta_n)|, \qquad M \le n \le N,$$

where

$$\overline{B}_n = \frac{B_n}{|DT_{\zeta_{n-1}} w^*_{n-1}|}.$$

If $(*)$ holds, we say that the vector $v_n = DT^n_{\zeta_0} v$ *splits correctly.*

One of the most important properties of T proved in [**BC2**] is that

for every $z_0 \in \mathcal{C}$, $(z_0, \binom{0}{1})$ *is controlled during the time interval* $[0, \infty)$.

It is also proved that if ζ_0 is bound to $z_0 \in \mathcal{C}$ then $(\zeta_0, \binom{0}{1})$ is also controlled during the bound period $[0, p)$.

1.5.4 1-dimensional behavior in 2-dimensions.

We now give some idea of how the splitting algorithm is used to study $D^n_{\zeta_0} v$ where (ζ_0, v) is controlled on $[0, \infty)$. For definiteness consider $\zeta_0 \in \mathcal{C}$ and $v = \binom{0}{1}$. We claim that $\{w^*_n\}_{n=0}^{\infty}$ is essentially 1-d in character, and has properties similar to $(f^n)'(f0)$ for $f(x) = 1 - ax^2$.

As before, let $n_1 < n_2 < \ldots$ be the return times of ζ_0 to $(-\delta, \delta) \times \mathbb{R}$. First $w^*_1 = \binom{1}{0}$. For $0 < n < n_1$, $\{|w^*_n|\}$ behaves qualitatively like $\{|(f^n)'(f0)|\}$; see 1.2.3. Let us consider $n = n_1$, and let $w_n = A_n e + B_n\binom{0}{1}$ be the decomposition given by the splitting algorithm.

Ignoring the A-term for now, and using the fact that $(\zeta_0, \binom{0}{1})$ is controlled, we see that

$$\begin{aligned} |w^*_{n+1}| = |w^*_n| &= |\overline{B}_n| \cdot |DT_{\zeta_{n-1}} w^*_{n-1}| \\ &\sim 2a\,|\zeta_n - z(\zeta_n)| \cdot |DT^{n-1}_{\zeta_1} w^*_1|, \end{aligned}$$

which has an obvious resemblance to $|(f^n)'(f0)|$. Moreover, since $w^*_{n+1} = B_n\binom{1}{0}$, the action of DT^j on w^*_{n+1} for $j < n_2 - n_1$ is again essentially 1-d. The vector $DT^{n_2-n_1} w^*_{n_1}$ is split at time n_2, and the new B-term is treated similarly.

The general philosophy is that the A-terms can, in some sense, be neglected. Let us consider a simple situation, where the fold period l at time $n = n_1$ expires before the next return to $(-\delta, \delta) \times \mathbb{R}$. We claim that $DT^l_{\zeta_n} A_n e$ is extremely short compared to $DT^l_{\zeta_n} B_n\binom{0}{1}$, so that the effect of adding this term at time $n + l$ is negligible. This is because

$$|DT^l_{\zeta_n} A_n e| \le |DT_{\zeta_{n-1}} w^*_{n-1}| \cdot (5b)^l,$$

whereas

$$|DT^l_{\zeta_n} B_n \tbinom{0}{1}| \geq |DT_{\zeta_{n-1}} w^*_{n-1}| \cdot |\zeta_n - z(\zeta_n)| \cdot |DT^l_{\zeta_n} \tbinom{0}{1}|,$$

and we know that $|\zeta_n - z(\zeta_n)| \geq (5b)^{l/2}$ by the definition of l, and also that $|DT^{l-1}_{\zeta_{n+1}} \tbinom{1}{0}| \geq \delta e^{c(l-1)}$ by 1.2.3(iii).

In general, the computation is more complicated, but if (ζ_0, v) is controlled one has $|\overline{B}_j| \geq |\zeta_j - z(\zeta_j)|$ at all returns, and the rejoining of the A-terms to w^*_n at the end of fold periods has negligible effects. (See Section 7.3 of [**BC2**].)

We have made a point of comparing $w^*_n(z_0, \tbinom{0}{1})$, $z_0 \in \mathcal{C}$, to $(f^n)'(f0)$. This parallel is not complete unless we could guarantee, through parameter exclusion, that for all $z_0 \in \mathcal{C}$, $|w^*_n(z_0, \tbinom{0}{1})| \geq e^{cn}$ for some $c > 0$. This is not exactly what we stated in 1.3.3, but it is in fact true.

1.6. Dynamics near the "turns".

1.6.1. Bound state estimates.

Assume that ζ_0 is in tangential position with respect to $z_0 \in \mathcal{C}$, and let p and l be its bound and fold periods. We now record some estimates from [**BC2**] pertaining to the bound states $\{\zeta_j\}_{j=0}^p$. These estimates rely on the fact that critical orbits have the properties in sections 1.3.1 and 1.5.3.

(i) $|T^j\zeta_0 - T^j z_0| \sim |\zeta_0 - z_0|^2 |DT^j_{z_0} \tbinom{0}{1}|$ for $l \leq j \leq p$.
(For a precise statement, see Section 7.5, in particular Lemma 7.4, of [**BC2**].)

(ii) If $|\zeta_0 - z_0| \approx e^{-\mu}$, $\mu > 0$, then $\frac{1}{2}\mu \leq p \leq 5\mu$.
(This follows essentially from (i) and Section 1.3.1.)

(iii) $\left|DT^p_{\zeta_0} \tbinom{0}{1}\right| \cdot |\zeta_0 - z_0| \geq e^{c'p}$ for some $c' > 0$.
(See Lemma 7.5 of [**BC2**].)

Two points are to be noted here. The first is the striking resemblance between these estimates and those in 1-d. Assertion (i), for instance, says that ζ_0 experiences a quadratic contraction from its interaction with z_0. The second observation we wish to make is that unlike the situation in 1-d, not all the points near z_0 have this "quadratic" behavior. For instance, ζ_0 may lie on a piece of stable manifold through z_0, and be attracted to z_0 forever. Or, if ζ_0 is directly above z_0, then we will have $|T^j\zeta_0 - T^j z_0| \approx |\zeta_0 - z_0| \cdot |DT^j_{z_0} \tbinom{0}{1}|$ instead of (i) for the first few iterates.

1.6.2 Generalized tangential positions.

We wish to say more precisely for which region the estimates in 1.6.1 hold. For $z \in W$, we say that (x', y') is the natural coordinate system at z if $(0,0)$ is at z, the x'-axis lines up with $\tau(z)$, and the y'-axis with $\tau(z)^\perp$. The following definition is *not* contained in [**BC2**] but will be useful for us in the next section.

Definition. Let $c > 0$ be a small number $\ll 2a$, say $c = \frac{1}{100}$, and let $z_0 \in \mathcal{C}$. A point ζ_0 near z_0 is said to be in *generalized tangential position* with respect to z_0 if in the natural coordinate system at $z_0, \zeta_0 = (\xi', \eta')$ with $|\eta'| \leq c\xi'^2$.

Clearly, if ζ_0 is in tangential position wrt z_0, then it is in generalized tangential position, because the segment of W containing z_0 is a $C^2(b)$ curve tangent to the x'-axis at $(0,0)$.

While it is not explicitly stated this way, the proofs in [**BC2**] show in fact that the bound estimates in 1.6.1 hold for all ζ_0 in generalized tangential position wrt $z_0 \in \mathcal{C}$. Let us recall briefly why this is so:

Let p be the bound period between ζ_0 and z_0. From 1.3.4 we know that there is a contractive direction field defined on a small ball about z_0. The integral curves of e_p are roughly parabolas of the form $y' = \text{const} + a(x' - x'_p)^2$, and have a unique tangency with the x'-axis at $z_0^{(p)} := (x'_p, 0)$. Since $|z_0^{(p)} - z_0| = O(b^p) \ll |z_0 - \zeta_0|$, for our purposes we may as well confuse $z_0^{(p)}$ with z_0. With this simplification we can think of z_0 and ζ_0 as lying on the graph of a function $\varphi : x'\text{-axis} \to y'\text{-axis}$ with $|\varphi'(x')| \le 2c|x'|$. The estimates in sections 7.4 and 7.5 of [**BC2**] apply to points on such a curve. What really matters is that all the tangent vectors to graph(φ) split correctly wrt e_p, which a priori is defined only in a small ball about z_0 but is shown in a local induction in [**BC2**] to be defined on the entire segment of graph(φ) between z_0 and ζ_0. These estimates imply those in 1.6.1.

1.7 Distortion estimates during bound periods.

In the course of proving some of the estimates in the last few subsections, the following estimates are used.

(i) For all $z_0 \in \mathcal{C}$, if ζ_0 is bound to z_0 with bound period p then

$$\frac{1}{2} \le \frac{|w_j^*(\zeta_0, \binom{0}{1})|}{|w_j^*(z_0, \binom{0}{1})|} \le 2 \qquad \forall j < p.$$

This is Assertion 4(a) in [**BC2**]. It is proved in Lemmas 7.8, 7.9 and on p. 151.

(ii) There is a constant $C_0 > 0$ such that $\forall z_0 \in \mathcal{C}$, if ζ_0 and ζ'_0 are bound to z_0 during $[0, \nu]$, then

a. $$\frac{|w_\nu^*(\zeta_0, \binom{0}{1})|}{|w_\nu^*(\zeta'_0, \binom{0}{1})|} \le \exp\left\{ C_0 \sum_{j=0}^{\nu-1} \frac{\Delta_j(\zeta_0, \zeta'_0)}{d_{\mathcal{C}}(z_j)} \right\},$$

where $\Delta_j(\zeta_0, \zeta'_0) = \max\limits_{0 \le i \le j} |\zeta_i - \zeta'_i|$ and

$$d_{\mathcal{C}}(z_j) := \begin{cases} |x_j| & \text{if} \quad z_j = (x_j, y_j) \quad \text{and} \quad |x_j| \ge \delta \\ |z_j - z(z_j)| & \text{if} \quad |x_j| \le \delta \quad \text{and} \quad z(z_j) \quad \text{is the binding point.} \end{cases}$$

b. $$\sphericalangle \left(w_\nu^* \left(\zeta_0, \binom{0}{1} \right), w_\nu^* \left(\zeta'_0, \binom{0}{1} \right) \right) < 2b^{\frac{1}{2}} \Delta_\nu.$$

These estimates are proved the same time (i) is proved.

(iii) There is a constant $C'_0 > 0$ such that if z_0, ζ_0 and ζ'_0 are as in (ii) and in addition ζ_0 and ζ'_0 lie on a curve all the tangent vectors of which split correctly with respect to z_0 (compare 1.5.3) then

$$\sum_{j=0}^{p} \frac{\Delta_j(\zeta_0, \zeta'_0)}{d_{\mathcal{C}}(z_j)} \le C'_0 \frac{|\zeta_0 - \zeta'_0|}{|\zeta_0 - z_0|}.$$

This is proved on p.p. 151 and 163 of [**BC2**].

The constants C_0 and C_0' above remain uniformly bounded as $\delta \to 0$.

2. Dynamics on the unstable manifold W.

2.1. Statements of results.

Let $T = T_{a,b}$ be as in Section 1. The purpose of this section is to use the dynamics of points in $\mathcal{C}$ to help us understand the dynamics of all points on W. We will prove that the derivative estimates for $DT^n_\zeta \tau$, $\zeta \in W$, are completely parallel to those for $(f^n)'x$ where $f : [-1, 1] \circlearrowleft$ satisfies the conditions in Section 1.1.

As usual, $\{\zeta_n\}_{n=-\infty}^{\infty}$ denotes the orbit of ζ_0. Before we begin we need to modify or extend slightly a couple of our definitions from Section 1. First we relax our definition of "control" for the orbit of ζ_0 to requiring only that whenever $\zeta_n \in (-\delta, \delta) \times \mathbb{R}$, there is a critical point $z(\zeta_n)$ wrt which ζ_n is in *generalized* tangential position. (See 1.6.2.) Second, we say that the pair (ζ_0, v) is controlled on the time interval $[j, N)$, $-\infty < j < N$, if $(\zeta_j, DT^j_{\zeta_0} v)$ is controlled on $[0, N-j)$, and that (ζ_0, v) is controlled on $(-\infty, N)$ if it is controlled on $[j, N)$ for all $j < N$. Note that for all $\zeta_0 \in W, \zeta_j$ tends to the unique hyperbolic fixed point $\hat{z}$ as $j \to -\infty$.

2.1.1 Main Proposition and Main Corollary.

PROPOSITION 1. (Main Proposition). *For all $\zeta_0 \in W$, if $\zeta_i \notin \mathcal{C}$ for all $i < N$, then the pair $(\zeta_0, \tau(\zeta_0))$ is controlled on the time interval $(-\infty, N)$.*

Proposition 1 enables us to describe each $\zeta_i, i \le N$, as being in a "free state" or "bound state" as discussed in 1.4.4. (Technically one needs to specify a starting point for this to make sense, but since $\zeta_i \to \hat{z}$ as $i \to -\infty$, we can think of our trajectory as starting from ζ_j for some negatively very large j.

COROLLARY 1. (Main Corollary). *Let $\zeta_0 \in W$, and assume that $\zeta_i \notin \mathcal{C}$ for all i. Then ζ_i is in a "free state" infinitely often, and the following holds for $\tau_i := DT^i_{\zeta_0} \tau$:*

I. Expansion outside of $(-\delta, \delta) \times \mathbb{R}$.

There is $c_0 > 0$ and $M_0 \in \mathbb{Z}^+$ such that

(i) *if ζ_i is free and $\zeta_i, \ldots, \zeta_{i+M_0} \notin (-\delta, \delta) \times \mathbb{R}$, then*

$$\frac{|\tau_{i+M_0}|}{|\tau_i|} \ge e^{cM_0};$$

(ii) *if $\zeta_i \notin (-\delta, \delta) \times \mathbb{R}$ is free, and $k > i$ is the first time $\zeta_k \in (-\delta, \delta) \times \mathbb{R}$ then*

$$\frac{|\tau_k|}{|\tau_i|} \ge e^{c_0(k-i)}.$$

II. Bound period estimates.

There is $c_1 \approx \log 2$ such that the following holds: if $\zeta_i \in (-\delta, \delta) \times \mathbb{R}$ is free and becomes bound at this time to $z(\zeta_i) \in \mathcal{C}$ with bound period p, then

(i) *if $e^{-\mu-1} \le |\zeta_i - z(\zeta_i)| \le e^{-\mu}$, then $\frac{1}{2}\mu \le p \le 5\mu$;*

(ii) $|\tau_{i+j}|/|\tau_i| \ge 3\,|\zeta_i - z(\zeta_i)| \cdot \left|DT^j_{z(\zeta_i)}\binom{0}{1}\right| \ge |\zeta_i - z(\zeta_i)|\, e^{c_1 j}, \qquad 0 < j < p;$

(iii) $|\tau_{i+p}|/|\tau_i| \ge e^{\frac{1}{3}c_1 p}.$

2.1.2. Global distortion estimates.

We will need the following distortion estimate for $DT^n\tau$. Let $\mathcal{P} = \{I_{\mu j}\}$ be the partition of $(-\delta, \delta)$ as described in Section 1.1, i.e.

$$(-\delta,\delta) = \dot{\bigcup_{|\mu|\geq\mu_0}} I_\mu,$$

where $I_\mu = (e^{-(\mu+1)}, e^{-\mu})$ for $\mu > 0$, $I_\mu = -I_{-\mu}$ for $\mu < 0$, and each I_μ is further subdivided into μ^2 intervals $\{I_{\mu j}\}$ of equal length.

For x_0 with $|x_0| \ll \delta$, we let $\mathcal{P}_{[x_0]}$ denote a copy of $\mathcal{P}$ with 0 "moved" to x_0. More precisely, let $h : (-\delta,\delta) \to (-\delta,\delta)$ be the piecewise linear homeomorphism taking the points $-\delta, -e^{-(\mu_0+1)}, x_0, e^{-(\mu_0+1)}, \delta$ to $-\delta, -e^{-(\mu_0+1)}$, 0, $e^{-(\mu_0+1)}, \delta$ respectively and let $\mathcal{P}_{[x_0]} = h^{-1}\mathcal{P}$.

Furthermore, if γ is a roughly horizontal curve and $z_0 = (x_0, y_0)$ is such that $|x_0| \ll \delta$, then $\mathcal{P}_{[z_0]} = \mathcal{P}_{[x_0]}$ is the obvious partition on $\gamma \cap ((-\delta,\delta)\times\mathbb{R})$. Once γ and z_0 are specified, we will speak about $I_{\mu j}$ as though it was a subsegment of γ. Also, for $J = I_{\mu j}$, let J^+ denote the union of J with its adjacent intervals in $\mathcal{P}_{[\cdot]}$.

We will use the following notation: if γ_0 is a curve segment then $\gamma_j = T^j\gamma_0$; ζ_j denotes $T^j\zeta_0$ and $\tau_j(\zeta_0) = DT^j_{\zeta_0}\tau$ etc.

Proposition 2. *Let $\gamma_0 \subset W \cap ((-\delta,\delta)\times\mathbb{R})$ be a curve segment, and let $0 = t_0 < t_1 < \cdots < t_q = N$ be its free return times. More precisely, for all $k < q$,*

1. *all points in γ_{t_k} have a common binding point, which we denote by $z^{(k)}$, and $\gamma_{t_k} \subset J^+$ for some $J \in \mathcal{P}_{z^{(k)}}$;*
2. *if p_k is the bound period between γ_{t_k} and $z^{(k)}$, then t_{k+1} is the smallest $j \geq t_k + p_k$ such that $\gamma_j \cap ((-\delta,\delta)\times\mathbb{R}) \neq \emptyset$.*

Then for all $\zeta_0,\ \zeta_0' \in \gamma_0$,

$$\frac{|\tau_N(\zeta_0)|}{|\tau_N(\zeta_0')|} \leq C_1$$

for some C_1 independent of γ_0 or N.

2.1.3 Remarks on the critical set.

Recall that in [**BC2**], the critical set $\mathcal{C}$ is constructed according to some seemingly arbitrary rules. We wish to point out here that $\mathcal{C}$ in fact admits certain intrinsic characterizations. For instance, Corollary 1 gives the following dynamical characterization of $\mathcal{C}$: Let $z_0 \in W$. Then

$$z_0 \quad \text{lies on a critical orbit} \iff \limsup_{n\to\infty}|DT^n_{z_0}\tau| < \infty \iff \lim_{n\to\infty}|DT^n_{z_0}\tau| = 0.$$

In fact, $z_0 \in \mathcal{C}$ iff

$$|DT^j_{z_0}\tau| \leq (5b)^j \quad \forall j > 0 \quad \text{and} \quad |DT_{z_{-1}}\tau| > 1.$$

We mention also that $\mathcal{C}$ admits geometric characterizations. For instance, it is straightforward to verify using the curvature computations in Section 7.6 of [**BC2**] that $z_0 \in \mathcal{C}$ iff

$$\kappa(z_0) \ll 1 \quad \text{and} \quad \kappa(z_n) > b^{-n} \quad \forall n > 1.$$

2.2 Some lemmas.

The lemmas in Section 2.2 are not explicitly stated or proved in [**BC2**], but their proofs resemble those in [**BC2**]. We will repeat the shorter arguments and refer the reader to [**BC2**] for the longer ones.

We will write $w_j(z) := w_j(z, \binom{0}{1})$ and $\tau_j(z) := w_j(z, \tau)$, and let $w_j^*(z)$ and $\tau_j^*(z)$ have the obvious meanings. When we say that a segment $\gamma \subset W$ is "free", it will be assumed implicitly that $\forall z \in \gamma$, (z, τ) is controlled during the time interval $(-\infty, 0)$. The absolute value of the slope of a vector is denoted $s(v)$.

2.2.1 Slope, curvature and derivative estimates.

The purpose of this subsection is to prove the following three lemmas:

LEMMA 1. *Let γ be a free segment of W. Then*

(i) $\forall z \in \gamma$, $s(\tau(z)) < 2b/\delta$;
(ii) $\forall z \in \gamma \cap ((-\delta, \delta) \times \mathbb{R})$, $s(\tau(z)) < 10b$.

LEMMA 2. *Let γ be a free segment of W in $(-\delta, \delta) \times \mathbb{R}$. Then $\kappa(\gamma) < 10b$.*

It follows immediately from these two lemmas that free segments of W in $(-\delta, \delta) \times \mathbb{R}$ are $C^2(b)$ curves. The next lemma is needed in Section 3.

LEMMA 3. *There exists $c_1 > 0$ such that if $z \in W \cap ((-\delta, \delta) \times \mathbb{R})$ is in a free state, then*

$$|DT_z^{-j}\tau| \le e^{-c_1 j} \qquad \forall j \ge 0.$$

We begin with the following sublemma:

SUBLEMMA 1. *Let z_0 be a point in G_1 near the fixed point $\hat{z}$, and assume that (z_0, τ) is controlled on $[0, j)$. If $z_j \notin (-\delta, \delta) \times \mathbb{R}$ then, $s(\tau_j^*(z_0)) < 2b/\delta$.*

PROOF: Proceed by induction exactly as is done in the proof of LI(ν')(a) in Section 7.3 of [**BC2**]. $\square$

PROOF OF LEMMA 1: (i) follows immediately from Sublemma 1. To see (ii), note that $T^{-1}z$ is outside of all fold periods. Apply Sublemma 1 and Section 1.2.3. $\square$

SUBLEMMA 2. *Let ζ_0 be in generalized tangential position with respect to $z_0 \in \mathcal{C}$. Let p be the bound period between ζ_0 and z_0, and let n be the first free return of ζ_0 to $(-\delta, \delta) \times \mathbb{R}$. Then there is $c > 0$ such that*

(i) $|w_j(\zeta_0)| \ge e^{cj} \qquad \forall j < p;$
(ii) $\dfrac{|w_n(\zeta_0)|}{|w_j(\zeta_0)|} \ge e^{c(n-j)} \qquad \forall j < n.$

PROOF: (i) If z_j is outside of all fold periods, then

$$|w_j(\zeta_0)| = |w_j^*(\zeta_0)| \gtrsim |w_j^*(z_0)| \geq e^{c^* j}.$$

(See 1.7.1. for "$\gtrsim$" and 1.5.4 for the inequality.) If not, choose $k > j$ such that z_k is outside of all fold periods and $k - j < (C/\log(1/b))\, j$. (See 1.5.1.) We then have

$$|w_j(\zeta_0)| \geq 5^{-(C/\log(1/b))j} e^{c^* k} \geq e^{cj}.$$

(ii) If $j \geq p$ the estimate follows from 1.2.3(ii) so we only need to consider $j < p$

Case 1. z_j is outside of all fold periods. We have

$$\frac{|w_n(\zeta_0)|}{|w_j(\zeta_0)|} \gtrsim \frac{|w_n(\zeta_0)|}{|w_p(\zeta_0)|} \cdot \frac{|w_p^*(z_0)|}{|w_j^*(z_0)|}.$$

To see that the second factor $\geq e^{c(p-j)}$, let $n_1 < n_2 < \cdots < n_k$ be the times between j and p when z_j returns freely to $(-\delta, \delta) \times \mathbb{R}$. Write $\overline{w}_i(z_0) = DT_{z_{i-1}} w_{i-1}^*(z_0)$. We know from 1.2.3, 1.5.4 and 1.6.1 (iii) that every factor in

$$\frac{|w_p^*(z_0)|}{|\overline{w}_{n_k}(z_0)|} \cdot \frac{|\overline{w}_{n_k}(z_0)|}{|\overline{w}_{n_{k-1}}(z_0)|} \cdot \ldots \cdot \frac{|\overline{w}_{n_1}(z_0)|}{|w_j^*(z_0)|}$$

is exponential.

Case 2. z_j is in a fold period. Let k be the last time it was outside all fold periods. Because of the rule for construction of the fold periods (see 1.5.1(iii)) ζ_j must still be in fold relation to $z(\zeta_k)$. Let p' and l' be the length of the bound period and fold period resp., initiated at time k. Then $p' < n - k$ since z_n is free and

$$\begin{aligned} j - k \leq l' \leq \frac{Cp'}{\log(1/b)} &\leq \frac{C(n-k)}{\log(1/b)} \\ &\leq \frac{C(n-j)}{\log(1/b)} + \frac{C(j-k)}{\log(1/b)}. \end{aligned}$$

We move the second term to the left and conclude that

$$j - k \leq \frac{C'(n-j)}{\log(1/b)}.$$

The same argument as before finishes the proof. □

PROOF OF LEMMA 3: Let $z \in \gamma$. If $T^{-i}z \notin (-\delta, \delta) \times \mathbb{R}\ \forall i > 0$ our exponential estimate follows directly from (1.2.3)(ii). Otherwise suppose $\zeta_0 = T^{-n}z$ is the previous free return. It is enough to verify

$$\frac{|\tau_n(\zeta_0)|}{|\tau_j(\zeta_0)|} \geq e^{c_1(n-j)} \qquad 0 \leq j \leq n.$$

(If there is more than one free return between $T^{-j}z$ and z, repeat the argument and use the chain rule.)

From Sublemma 2 (i) we know that $e_n(\zeta_0)$ is well-defined. We write

$$\tau(\zeta_0) = Ae_n(\zeta_0) + B\binom{0}{1}$$

and let l be the fold period between ζ_0 and $z_0 = z(\zeta_0)$.

Case 1. $j \geq l$. Since τ splits correctly at ζ_0, we have

$$|DT^k_{\zeta_0} B\tbinom{0}{1}| \geq |B|\, e^{ck} \gg |DT^k_{\zeta_0} Ae_n|$$

$\forall k \geq l$, and so

$$\frac{|\tau_n(\zeta_0)|}{|\tau_j(\zeta_0)|} \gtrsim \frac{|B|\,|w_n(\zeta_0)|}{|B|\,|w_j(\zeta_0)|} \geq e^{c(n-j)}$$

by Sublemma 2 (ii).

Case 2. $j < l$. At most we have $|\tau_j(\zeta_0)| \leq 5^l$. Since $l \leq (C/\log(1/b))\,n$, we are still guaranteed that

$$\frac{|\tau_n(\zeta_0)|}{|\tau_j(\zeta_0)|} \geq \frac{e^{cn}}{5^{(C/\log(1/b))n}} \geq e^{c_1(n-j)}.$$

This completes the proof. □

Proof of Lemma 2: Choose n so that $\zeta_0 = T^{-n}z$ is on the first generation G_1 of W close to the fixed point. Then $\kappa(\zeta_0) \leq 1$ and the curvature estimate follows immediately from Lemma 3 and 1.2.4. □

2.2.2 Abundance of $C^2(b)$ segments of W near free returns.

Lemma 4. *Let $\zeta_0 \in W \cap ((-\delta,\delta)\times\mathbb{R})$ be free. Then for every $m \in \mathbb{Z}^+$ with $3m < gen(\zeta_0)$, $\exists m'$ with $m < m' \leq 3m$ and two $C^2(b)$ curves γ and γ' in W with the following properties:*

(i) *ζ_0 is sandwiched between γ and γ' extending $\geq 3\rho^{m'}$ to each side of ζ_0;*
(ii) *$dist(\zeta_0,\gamma)$, $dist(\zeta_0,\gamma') \leq (Cb)^{m'}$, $C = 5e^{72}$;*
(iii) *if η and η' are the two points in γ and γ' resp. with the same x-coordinate as ζ_0, then $|\tau(\eta)-\tau(\zeta_0)|$, $|\tau(\eta')-\tau(\zeta_0)| \leq (Cb)^{m'/6}$;*
(iv) *$gen(\gamma) = m'+1$, $gen(\gamma') \leq m'$.*

The proof of Lemma 4 is virtually identical to that of the corresponding result in [**BC2**] for z_n (instead of ζ_0), where $z_0 \in \mathcal{C}$ and z_n is a free return to $(-\delta,\delta)\times\mathbb{R}$. (See Section 1.4.2.) We sketch here the main steps of the proof, referring to [**BC2**] for more details:

Outline of Proof of Lemma 4:

Step 1. Show that $\exists m' \in [m, 3m]$ such that $\zeta_{-m'}$ is in a "favorable" position relative to ζ_0. This means that $\zeta_{-m'}$ is outside of all folding periods, and that $\forall k < m'$, $|\zeta_{-m'+k} - z(\zeta_{-m'+k})| \geq e^{-36k}$ whenever $\zeta_{-m'+k} \in (-\delta,\delta)\times\mathbb{R}$. (See Lemma 6.6 in [**BC2**] for a proof.)

Step 2. Show that $|DT^k_{\zeta_{-m'}}\tau| \geq e^{-36k}$ $\forall k < m'$. Idea of proof: Let $t_1 < \cdots < t_j$ be the free return times of $\zeta_{-m'}$. First note that $s(\tau(\zeta_{-m'})) < 2b/\delta$ (Sublemma 1), so that

$|DT^{t_1}_{\zeta_{-m'}}\tau| \geq e^{ct_1}$. Between t_i and t_{i+1}, use the fact that τ splits correctly, and apply Sublemma 2(i) and 1.6.1(iii). (This is also proved in Lemma 7.2 of [**BC2**].)

Step 3. Show that there is a contractive vector field $e = e_{m'}$ defined on a strip containing $\zeta_{-m'}$ such that

(i) $|DT^j e| \leq (Cb)^j \quad \forall j < m,\ C = 5e^{72}$;

(ii) there is an integral curve joining $\zeta_{-m'}$ to two points η_0^1 and η_0^2 on G_1 and G_2 respectively;

(iii) About η_0^r, $r = 1, 2$, there is a segment $\gamma^r \subset W$ extending $\geq 3\rho^{m'}$ to each side of η_0^r such that each point on γ^1 (resp γ^2) is joined to a point on G_2 (resp G_1) by an integral curve of e.

This is proved in Section 5.3 of [**BC2**].

Step 4. Show $T^{m'}\gamma^1$ and $T^{m'}\gamma^2$ contain $C^2(b)$ curves with the properties as claimed. Idea of proof: Again consider the free return times $t_1 < \cdots < t_j$ of $\zeta_{-m'}$. Assuming that γ^r is not so near the "tips", we verify using the criteria in 1.2.4 that $T^{t_1}\gamma^r$ $(r = 1, 2)$ is a $C^2(b)$ curve. Assume now that $T^{t_i}\gamma^r$ contains a $C^2(b)$ curve $\gamma^{r,i}$, trimmed so that it extends by $3\rho^{m'}$ to each side of $\eta^r_{t_i}$. We need to know that τ splits correctly at every point on $\gamma^{r,i}$. Assuming that, we may view $\gamma^{r,i}$ as being tied to $\zeta_{-m'+t_i}$ for the next $t_{i+1} - t_i$ iterates, and the same argument as in Lemma 2 will tell us that $\gamma^{r,i+1}$ is again $C^2(b)$. To see that τ splits correctly on $\gamma^{r,i}$, use the following facts: (i) τ splits correctly at $\zeta_{-m'+t_i}$; (ii) $|\tau(\eta^r_{t_i}) - \tau(\zeta_{-m'+t_i})| < (Cb)^{t_i/6}$ (see Lemma 7.3 in [**BC2**]), and $(Cb)^{t_i/6} \ll e^{-36t_i} < |\zeta_{-m'+t_i} - z(\zeta_{-m'+t_i})|$; (iii) $\gamma^{r,i}$ is $C^2(b)$ and has length $\ll |\zeta_{-m'+t_i} - z(\zeta_{-m'+t_i})|$. □

Let ζ_0 be as in Lemma 4. Then for every j with $3^{j+1} < \text{gen}(\zeta_0)$, there is m_j with $3^j < m_j \leq 3^{j+1}$ and $C^2(b)$ curves γ_j and γ_j' with the properties in Lemma 4. In the future we will refer to $\{\gamma_j\}$ and $\{\gamma_j'\}$ as "stacks captured by ζ_0".

2.3 Geometry of the critical set.

The purpose of this subsection is to establish some regularity in the structure of the fractal set $\mathcal{C}$.

2.3.1 A basic lemma.

LEMMA 5. *Let $z_0 \in \mathcal{C}$ be located in a $C^2(b)$ curve $\gamma \subset W$. Assume that γ extends to $\geq 2d$ to each side of z_0 and let $\zeta_0 \in \gamma$ be such that $|\zeta_0 - z_0| = d$. Then there are no critical points in the disk $B_{d^2}(\zeta_0) := \{z : |z - \zeta_0| \leq d^2\}$.*

PROOF: We will assume that there is a critical point $\tilde{z}_0 \in B_{d^2}(\zeta_0)$ and try to obtain a contradiction.

Let us first assume that $\tilde{z}_0$ lies on a $C^2(b)$ curve $\tilde{\gamma} \subset W$ that extends $\geq d$ to each side of $\tilde{z}_0$. Using the notation in Section 1.3, we choose m with $b^m \ll d^2$ and $2d < (\lambda/5)^m$, and estimate $|\tau(\zeta_0) - e_m(\zeta_0)|$ in 2 different ways:

First note that e_m is defined on all of $B_{2d}(z_0)$, and that for purpose of comparing $e_m(z_0)$ with $e_m(\zeta_0)$, we may assume $e_m(z_0) = \tau(z_0)$ (see 1.3.4). We therefore have

$$|\tau(\zeta_0) - e_m(\zeta_0)| = (2a + \mathcal{O}(b))d.$$

Next, we let $\tilde{\zeta}_0$ be the point in $\tilde{\gamma}$ with the same x-coordinate as ζ_0 and write

$$|e_m(\zeta_0) - \tau(\zeta_0)| \leq |e_m(\zeta_0) - e_m(\tilde{z}_0)| + |e_m(\tilde{z}_0) - \tau(\tilde{\zeta}_0)| + |\tau(\tilde{\zeta}_0) - \tau(\zeta_0)|.$$

The first two terms are $\leq 5d^2$ (again see 1.3.4). As for the third term, since γ and $\tilde{\gamma}$ are nonintersecting $C^2(b)$ curves of length $\geq d$ and $|\zeta_0 - \tilde{\zeta}_0| \leq d^2$, an easy computation gives $|\tau(\zeta_0) - \tau(\tilde{\zeta}_0)| \leq 2d$. These three terms add up to $2d + 10d^2$, which is $< (2a + \mathcal{O}(b)) \cdot d$, and we obtain a contradiction.

Now the $C^2(b)$ curve containing $\tilde{z}_0$ may not be long enough. In that case we use the rules of construction of $\mathcal{C}$ (see 1.3.1) to successively obtain critical points $\tilde{z}_0^{(1)}$, $\tilde{z}_0^{(2)}$, ..., $\tilde{z}_0^{(k)}$ of lower and lower generation. Let G be such that $\rho^{2G} \leq d \leq \rho^G$, and let i be the smallest integer with $\mathrm{gen}(\tilde{z}_0^{(i)}) < G$. Then

$$|\tilde{z}_0^{(i)} - \tilde{z}_0| \leq \sum_{j=\mathrm{gen}(\tilde{z}_0^{(i-1)})}^{\infty} (b^{\sigma j})^{\frac{1}{2}} \leq 2b^{\sigma G/2},$$

which we may assume is $< d^2$ and $\tilde{z}_0^{(i)}$ lies on a $C^2(b)$ curve extending $\geq 3\rho^G > d$ to each side on $\tilde{z}_0^{(i)}$. The previous argument can now be repeated with $\tilde{z}_0^{(i)}$ in place of $\tilde{z}_0$. $\square$

2.3.2 View of $\mathcal{C}$ from a point in the free state.

LEMMA 6. *Let $\zeta \in G_n \cap (-\delta, \delta) \times \mathbb{R}$ be free, and let z_0 and z_0' be two critical points wrt which ζ is in tangential position. Assume that $|\zeta - z_0|, |\zeta - z_0'| \geq b^{\frac{n}{100}}$. Then the x-coordinate of ζ cannot lie between those of z_0 and z_0'.*

PROOF: We assume this scenario occurs. Let $d = |\zeta - z_0|, d' = |\zeta - z_0'|$, and suppose that $d \leq d'$. Let η be the point on $W^u_{\mathrm{loc}}(z_0')$ with the same x-coordinate as z_0, and let $D = |\eta - z_0|$. We will show that $D \ll d^2 + d'^2 < (d + d')^2$, which will contradict Lemma 5.

The difficulty in comparing $W^u_{\mathrm{loc}}(z_0)$ and $W^u_{\mathrm{loc}}(z_0')$ directly is that the two $C^2(b)$ curves may be too far apart and one of them may not be long enough. (We had a similar problem in the proof of Lemma 5.) So again we rely on curves captured by ζ.

First we use ζ to capture a segment W_0 of W of length $> d$ and with $\mathrm{dist}(\zeta, W_0) < d^4$. (This is clearly possible, even with $d \approx b^{\frac{n}{100}}$.) We further require that W_0 be on the opposite side of ζ as $W^u_{\mathrm{loc}}(z_0)$. (See Lemma 4.) Let ζ_{z_0} and ζ_{W_0} be the points on $W^u_{\mathrm{loc}}(z_0)$ and W_0 respectively with the same x-coordinate as ζ. Then $|\zeta_{z_0} - \zeta_{W_0}| < 2d^4$; and $|\tau(\zeta_{z_0}) - \tau(\zeta_{W_0})| < 2d^2$ because $W^u_{\mathrm{loc}}(z_0)$ and W_0 are nonintersecting $C^2(b)$ curves that extend $> d$ to each side of ζ. Putting this together we have

$$D_0 := \mathrm{dist}(z_0, W_0) < 2d^4 + 2d^2 \cdot d + 20b \cdot d^2 \ll d^2.$$

Similarly we let W_0' be a segment captured by ζ on the opposite side of $W^u_{\mathrm{loc}}(z_0')$. We require that W_0' has length $> d'$ and $\mathrm{dist}(\zeta, W_0') < d'^4$, so that

$$D_0' := \mathrm{dist}(\eta, W_0') < 2d'^4 + 2d'^2 \cdot d + 20b \cdot d^2 \ll d'^2.$$

Now D is easily estimated as follows: If $W^u_{\text{loc}}(z_0)$ and $W^u_{\text{loc}}(z_0')$ are on the same side of ζ, then $D \le \max(D_0, D_0')$. If $W^u_{\text{loc}}(z_0)$ and $W^u_{\text{loc}}(z_0')$ are on opposite sides of ζ, then $D \le D_0 + D_0'$. $\square$

Consider a point $\zeta \in (-\delta,\delta)\times\mathbb{R}$ in the free state. Let $\{\gamma_j\}_{j=1}^k$ be a stack captured by ζ as discussed in 2.2.2. We may assume that there is a critical point on γ_1. Let $j(\zeta)$ be such that $\gamma_{j(\zeta)}$ is the γ_j of highest generation that contains a critical point, and call this critical point $\hat{z}(\zeta)$. We now fix some terminology to describe the location of $\mathcal{C}$ relative to ζ: We say that $\mathcal{C}$ is *"in the middle"* if there exists a stack $\{\gamma_j\}_{j=1}^k$ captured by ζ with the property that $j(\zeta) = k$ and $|\zeta - \hat{z}(\zeta)| < b^{\frac{1}{100}\text{gen}(\zeta)}$. If this is not the case, then we say that $\mathcal{C}$ is *"on the left"* (or *"on the right"*) if for every stack captured by $\zeta, \hat{z}(\zeta)$ lies on the left half (resp. right half) of $\gamma_{j(\zeta)}$.

We remark that if $\mathcal{C}$ is not "in the middle", then it has to be either "on the left" or "on the right" of ζ, because ζ is always in tangential position wrt $\hat{z}(\zeta)$, and Lemma 6 applies.

2.4 Proof of Main Proposition.

2.4.1 General strategy.

Let G_n be the leaf of generation n (see 1.3.1 for definition). We assume it has been proved that for all $\zeta_0 \in G_n$ the pair (ζ_0, τ) is controlled on the time interval $(-\infty, 0)$, and will show that (ζ_0,τ) is controlled on $(-\infty, 0]$. Clearly, we need only to consider $\zeta_0 \in (-\delta,\delta)\times\mathbb{R}$ and may assume that $\zeta_0 \notin T^i\mathcal{C}$ for any $i \ge 0$.

If ζ_0 is in a bound state, let ζ_{-i} be the last time it was free and let $\tilde{z}_0 = z(\zeta_{-i})$. Then ζ_0 is in tangential position wrt $z(\tilde{z}_i)$. Moreover, since $(\zeta_{-i}, \binom{0}{1})$ is controlled on the time interval $[0,i]$ (see Section 1.5.3) and $\tau(\zeta_{-i})$ splits correctly into $A_{-i}e + B_{-i}\binom{0}{1}$ by our induction hypothesis, the rejoining of the A_{-i}-term (if it has already taken place) has negligible effect. This proves that $\tau(\zeta_0)$ again splits correctly and we are done.

The case where ζ_0 is free is handled in the following lemma:

LEMMA 7. *Let γ be a maximal free segment of G_n. If $\gamma \cap ((-\delta,\delta)\times\mathbb{R}) \ne \emptyset$, then there is a critical point z_0 with respect to which every $\zeta \in \gamma\cap((-\delta,\delta)\times\mathbb{R})$ is in generalized tangential position. Moreover, if $\tau(\zeta) = A(\zeta)e + B(\zeta)\binom{0}{1}$ is the splitting in 1.5.2 with respect to our binding point $z_0 := z(\gamma)$, then*

$$3|\zeta - z_0| \le |B(\zeta)| \le 5|\zeta - z_0|.$$

Let γ_- and γ_+ denote the left and right endpoints of γ respectively. (This makes sense since free segments are roughly horizontal.) If $\gamma_+ \in (-\delta,\delta)\times\mathbb{R}$, then it is attached to some $\tilde{z}_+ \in \mathcal{C}$ because it is also in a bound state. Similarly, let $\tilde{z}_- = z(\gamma_-)$ if $\gamma_- \in (-\delta,\delta)\times\mathbb{R}$. The following are all the possible geometric configurations:

Case 1. γ is stretched across $(-\delta,\delta)\times\mathbb{R}$, i.e. neither γ_+ nor γ_- is in $(-\delta,\delta)\times\mathbb{R}$. We will show that z_0 lies on γ.

Case 2. γ_+ is in $(-\delta,\delta)\times\mathbb{R}$ and $\tilde{z}_+$ lies to the left of γ_+; $\gamma_- \notin (-\delta,\delta)\times\mathbb{R}$. As in Case 1, we will show that $z_0 \in \gamma$.

Case 3. Both γ_- and γ_+ are in $(-\delta,\delta)\times\mathbb{R}$; $\tilde{z}_-$ lies to the right of γ_- and $\tilde{z}_+$ to the left of γ_+. We will also show that $z_0 \in \gamma$.

Case 4. $\gamma_+ \in (-\delta,\delta)\times\mathbb{R}$ and $\tilde z_+$ is to the right of γ_+. In this case we will show that $\tilde z_+$ is a viable candidate for z_0.

2.4.2 Proof of Lemma 7.

We continue to use the terminology introduced in Section 2.3.2. For $n \in \mathbb{Z}^+$, let $\Delta_n > 0$ be sufficiently small that the following holds: Let γ be a free segment in $G_n \cap (-\delta,\delta)\times\mathbb{R}$, and let $\zeta = (\xi,\eta)$ and $\zeta' = (\xi',\eta')$ be two points on γ with $\xi - \Delta_n \le \xi' \le \xi$. If $\mathcal{C}$ is "on the left" of ζ, then it is either "on the left" or "in the middle" for ζ'. We assume also the analogous statement if $\mathcal{C}$ is "on the right" of ζ'. The existence of Δ_n follows from the proof of Lemma 6.

We now deal with the four cases discussed in Section 2.4.1.

Case 1. Since $\gamma \cap (-\delta,\delta)\times\mathbb{R}$ is a $C^2(b)$ curve (Corollary to Lemmas 1 and 2), once we produce a critical point $z_0 \in \gamma$, the rest of the assertion will follow. To produce z_0 we start with $\zeta^{(0)}$ on γ with x-coordinate $= \frac{1}{2}\delta$. Then $\hat z(\zeta^{(0)})$ must be on the left. We move left along γ by steps of Δ_n to obtain successively $\zeta^{(1)},\zeta^{(2)},\cdots$. If for some $k, \mathcal{C}$ is "in the middle", then we are done, because some point on γ "sees" $\hat z(\zeta^{(k)})$ (see Sublemma 3 below) and the rules of construction of $\mathcal{C}$ says that a critical point must have been constructed on γ. Clearly we cannot move left indefinitely and continue to have $\mathcal{C}$ "on our left."

SUBLEMMA 3. *Let $\zeta = \zeta^{(k)}$, where $\zeta^{(k)}$ and γ are as above. Then vert dist$(\hat z(\zeta),\gamma) < b^{\frac{1}{540}n}$.*

PROOF: Let η be the point on γ with the same x-coordinate as $\hat z(\zeta)$, and let $\hat\zeta$ be the point on $\gamma_{j(\zeta)}$ with the same x-coordinate as ζ. Then

$$\begin{aligned}|\hat z(\zeta) - \eta| &\le |\zeta - \hat\zeta| + 20b\,|\hat z(\zeta) - \hat\zeta| \\ &< (Cb)^{\frac{n}{9}} + 20b\cdot b^{\frac{n}{100}} \ll b^{\frac{n}{540}}.\end{aligned}$$

□

Case 2. Let $d_+ = |\gamma_+ - \tilde z_+|$. We need the following simple estimate:

SUBLEMMA 4. *Let γ be as in Case 2, and let ζ be a point on γ with $|\zeta - \gamma_+| < d_+$. Then*

$$\operatorname{dist}(\zeta, W^u_{\text{loc}}(\tilde z_+)) \ll d_+^2.$$

PROOF: Use γ_+ to capture an unstable leaf γ_j on the opposite side of $W^u_{\text{loc}}(\tilde z_+)$ with length$(\gamma_j) \approx d_+$. Let η_j and $\hat\eta$ be the points on γ_j and $W^u_{\text{loc}}(\tilde z_+)$ resp with the same x-coordinate as γ_+. Then $|\hat\eta - \gamma_+| < d_+^4$ because γ_+ is in tangential position wrt $\tilde z_+$, and $|\eta_j - \gamma_+| < (Cb)^{m_j} \ll d_+^4$. Also, γ_j and $W^u_{\text{loc}}(\tilde z_+)$ are long enough to guarantee that $|\tau(\eta_j) - \tau(\hat\eta)| < d_+^2$. So

$$\operatorname{dist}(\zeta, W^u_{\text{loc}}(\tilde z_+)) < 2d_+^4 + 2d_+^2\cdot d_+ + (20b)d_+^2 \ll d_+^2.$$

□

Let $R_+ = \{z \in \gamma : |z - \gamma_+| < \frac{1}{2}d_+\}$. We note that if $\zeta \in R_+$ then $\mathcal{C}$ cannot be "in the middle". This is because $d_+ > e^{-\alpha n}$, which is $\gg b^{\frac{n}{100}}$, and if $|\hat z(\zeta) - \zeta| < b^{\frac{n}{100}}$ for any stack, then we will have

$$\text{horiz dist}(\hat z(\zeta), \tilde z_+) > \tfrac{1}{3}d_+,$$

and

$$\mathrm{dist}(\hat{z}(\zeta), W^u_{\mathrm{loc}}(\tilde{z}_+)) \ll d_+^2,$$

which contradicts the geometry of the critical set.

We start with $\zeta^{(0)} = \gamma_+$. First we claim that $\mathcal{C}$ is on the left: "middle" has been ruled out in the last paragraph, and "right" is not compatible with the position of $\tilde{z}_+$ (Lemma 6). We move left by steps of Δ_n as before until we reach $\zeta^{(k)}$ with $\mathcal{C}$ in the middle. Now $\zeta^{(k)} \notin R_+$, which guarantees that γ extends $> \frac{1}{2}e^{-\alpha n} \gg 2\rho^n$ to each side of $\zeta^{(k)}$, and so a critical point z_0 on γ is assured. The correct splitting of τ wrt z_0 is automatic as before.

Case 3. We begin as in Case 2, taking $\zeta^{(0)} = \gamma_+$ and moving left in small steps. We must reach some $\zeta^{(k)}$ with $\hat{z}(\zeta^{(k)})$ in the middle before arriving at γ_-, otherwise $\hat{z}(\gamma_-)$ would be on the left, contradicting the fact that $\tilde{z}_-$ is to the right of γ_-. Now $\zeta^{(k)} \notin R_+ \cup R_-$, where R_- has the obvious definition. So again a critical point on γ is guaranteed.

Case 4. Let $d = |\gamma_+ - \tilde{z}_+|$, and let η be the point on $W^u_{\mathrm{loc}}(\tilde{z}_+)$ with the same x-coordinate as γ_+. Then $|\eta - \gamma_+| < 2d^4$ and $|\tau(\eta) - \tau(\gamma_+)| < d^2$ (an exercise: c.f. the proof of Sublemma 4). We need to show that every point in γ is in a generalized tangential position with respect to $\tilde{z}_+$. Let (x', y') be the natural coordinate system at $\tilde{z}_+$ (see 1.6.2) and let φ be the function whose graph is the curve γ. Then

$$|\varphi(-d)| < 2d^4 + (10b)d^2 \ll \tfrac{1}{100}d^2,$$
$$|\varphi'(-d)| < d^2 + (10b)d \ll \tfrac{2}{100}d,$$

and $|\varphi''| \le 10b$. This proves that $|\varphi(x')| < \frac{1}{100}x'^2$ for all the relevant x'.

We also need to know that τ splits "correctly" at every point on γ. This is true because $\tau(\gamma_+)$ splits correctly (it is in a bound state) and γ is a $C^2(b)$ curve.

This completes the proof of Lemma 7. $\square$

We remark that an alternate proof of Lemma 7 is to try to carry out the "capture" argument in Lemma 4 simultaneously for all points on γ.

2.4 Proof of Corollary 1.

Part I of Corollary 1 follows from Lemma 1 and Section 1.2. Part II follows from the correct splitting guaranteed in Proposition 1, the properties of $\mathcal{C}$ as discussed in 1.3.3, the distortion estimates in 1.7.1, and the bound period estimates in 1.6.

2.5 Proof of Proposition 2.

A proof of Proposition 2 is given in sections 2 and 8 of [**BC2**]; see Lemma 8.9 in particular. Because the arguments there are a bit sketchy, and this estimate is of central importance in the construction of SBR measures, we are going to fill in some details. We will assume the distortion estimate for $DT^j_{(\cdot)}\binom{0}{1}$ during bound periods (see 1.7.1), and try to explain how these estimates can be used to control the distortion of τ_j over arbitrarily long periods of time. The fact that (ζ, τ) is controlled for all ζ in question is used implicitly throughout.

We write

$$\log \frac{|\tau_N(\zeta_0)|}{|\tau_N(\zeta_0')|} = \sum_{k<q} S_k' + \sum_{k<q} S_k'',$$

where

$$S'_k = \log \frac{|\tau_{p_k}(\zeta_{t_k})|}{\left|\tau_{p_k}(\zeta_{t'_k})\right|} \qquad \text{and} \qquad S''_k = \log \frac{\left|\tau_{t_{k+1}-p_k}(\zeta_{t_k+p_k})\right|}{\left|\tau_{t_{k+1}-p_k}(\zeta'_{t_k+p_k})\right|}.$$

First we show that $\sum S''_k <$ some C''_1. Consider j with $t_k + p_k \leq j < t_{k+1}$. Since γ_j is free and free segments have uniformly bounded slopes and curvatures, it follows that

$$\left| DT_{\zeta_j}\tau - DT_{\zeta'_j}\tau \right| \leq \text{const}\, |\gamma_j|.$$

So

$$S''_k \lesssim \sum_{j=t_k+p_k}^{t_{k+1}-1} \frac{\left| DT_{\zeta_j}\tau - DT_{\zeta'_j}\tau \right|}{\left| DT_{\zeta'_j}\tau \right|} \leq \frac{1}{\delta} \cdot \text{const} \sum_j |\gamma_j|,$$

which is $<$ const $|\gamma_{t_{k+1}}|$ because Corollary 1 (part I) tells us that

$$\left|\gamma_{t_{k+1}}\right| \geq e^{c_0(t_{k+1}-j)}\, |\gamma_j|\,.$$

Using Corollary 1 again (both parts I and II) we conclude that

$$\left|\gamma_{t_{k+1}}\right| \geq |\gamma_{t_k+p_k}| \geq 2\, |\gamma_{t_k}|\,.$$

Hence

$$\sum S''_k < \text{const}\, |\gamma_N| < \text{ some } C''_1.$$

To estimate S'_k we first prove the following

Sublemma 5. *Let $\eta_0,\ \eta'_0 \in W$ lie in the same $I_{\mu j}$ with respect to some $\tilde{z}_0 \in \mathcal{C}$, and let p be their common bound period. Then there is $C > 0$ not depending on $\eta_0,\ \eta'_0$ or z_0 such that*

$$\log \frac{|\tau_p(\eta_0)|}{|\tau_p(\eta'_0)|} \leq C \frac{|\eta_0 - \eta'_0|}{e^{-\mu}}.$$

Throughout the proof we will use c to denote a generic constant that is positive and very small.

Proof of sublemma 5: Split

$$\tau(\eta_0) = Ae + B\tbinom{0}{1}$$
$$\tau(\eta'_0) = A'e' + B'\tbinom{0}{1},$$

where e and e' are the contractive (unit) vectors for the period $[0, p]$. We will use the notation

$$e_j := DT^j_{\eta_0} e, \qquad e'_j := DT^j_{\eta'_0} e',$$
$$w_j := DT^j_{\eta_0}\tbinom{0}{1}, \qquad w'_j := DT^j_{\eta'_0}\tbinom{0}{1}.$$

Let R_θ be rotation by θ, and let θ be chosen such that R_θ carries w_p to a positive multiple of w'_p. We will write

$$\frac{|\tau_p(\eta_0)|}{|\tau_p(\eta'_0)|} \leq (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}) + (\mathrm{V}),$$

where (I)–(V) are defined and estimated as follows:

$$(\mathrm{I}) = \frac{\left|R_\theta B w_p + \frac{|Bw_p|}{|B'w'_p|} A' e'_p\right|}{\left|B'w'_p + A'e'_p\right|} = \frac{|Bw_p|}{|B'w'_p|},$$

where

$$\frac{|B|}{|B'|} \leq 1 + 5\frac{|\eta_0 - \eta'_0|}{e^{-\mu}}$$

because of the way τ and e change with z, and

$$\frac{|w_p|}{|w'_p|} \leq 1 + 2C_0C'_0\frac{|\eta_0 - \eta'_0|}{e^{-\mu}}$$

by 1.7.1(ii), (iii). (Note that the constants C_0 and C'_0 are purely numerical so

$$C_0C'_0\frac{|\eta_0 - \eta'_0|}{e^{-\mu}}$$

is small if μ_0 is chosen sufficiently large.)

$$(\mathrm{II}) = \frac{1}{|\tau(\eta'_0)|} \cdot \left|\frac{|Bw_p|}{|B'w'_p|}A'e_p - \frac{|Bw_p|}{|B'w'_p|}A'e'_p\right| \leq |e_p - e'_p|,$$

which is $< c|\eta_0 - \eta'_0|$ by Lemma 5.5 of [**BC2**].

$$(\mathrm{III}) = \frac{1}{|\tau_p(\eta'_0)|} \cdot \left|A'e_p - \frac{|Bw_p|}{|B'w'_p|}A'e_p\right| \leq \left|1 - \frac{|Bw_p|}{B'w'_p|}\right|,$$

which is $<$ const $|\eta_0 - \eta'_0|e^\mu$ (same as (I)).

$$(\mathrm{IV}) = \frac{1}{|\tau(\eta'_0)|} \cdot |R_\theta A'e_p - A'e_p| \leq |\theta|,$$

which is $< c\Delta_p \leq cC'_0|\eta_0 - \eta'_0|e^\mu$ by 1.7.1 (ii) and (iii).

$$(\mathrm{V}) = \frac{1}{|\tau_p(\eta'_0)|} \cdot |R_\theta Ae_p - R_\theta A'e_p| \leq c|A' - A| \leq c|\eta_0 - \eta'_0|.$$

Together these estimates prove the sublemma. $\square$

Applying this sublemma to each S'_k, we obtain

$$\sum_{k=0}^{q-1} S'_k \leq C \sum_{k=0}^{q-1} \frac{|\gamma_{t_k}|}{e^{-\mu_k}},$$

where $\gamma_{t_k} \subset I_{\mu_k j_k}$. To estimate this sum, we let $m(\mu) = \max\{t_k : \mu_k = \mu\}$ for each μ, and use the fact that $|\gamma_{t_{k+1}}| \geq 2|\gamma_{t_k}|$ to conclude that

$$\sum_{k<q} \leq \frac{|\gamma_{t_k}|}{e^{-\mu_k}} \leq \text{ const } \sum_{\mu} \frac{|\gamma_{m(\mu)}|}{e^{-\mu}} \leq \text{ const } \sum_{\mu} \frac{1}{\mu^2}.$$

This completes the proof of Proposition 2. □

3. Construction of SBR-measures.

We continue to assume that $T = T_{a,b}$, where (a, b) is one of the "good" parameters. Our strategy is as follows. Put Lebesgue measure m on a piece of W. Transport m forward by T, and take the ergodic averages of these measures. We will show that any limit point of these ergodic averages contains at least one component that has absolutely continuous conditional measures on unstable manifolds. This construction is standard for Axiom A attractors. The piecewise uniformly hyperbolic case is dealt with in e.g. [**Y2**], which contains a simple version of what is done here.

3.1. Bookkeeping on the unstable manifold.

We showed in Lemma 7, Section 2, that every maximal free segment γ in $W \cap ((-\delta, \delta) \times \mathbb{R})$ is assigned to a critical point with respect to which it is in generalized tangential position. This critical point will be denoted by $z(\gamma)$. On γ, it is natural to consider the partition $\mathcal{P}_{[z(\gamma)]}$, where $\mathcal{P}_{[\cdot]}$ is the partition defined in Section 2.1.4. In the construction below we will often speak of $I_{\mu j}$ on γ without explicit mention of $\mathcal{P}_{[z(\gamma)]}$.

We select a piece of W on which to begin our construction. Let, for instance, z_0 be the critical point on G_1, and let $\Delta \subset G_1 \cap ((-\delta, \delta) \times \mathbb{R})$ correspond to $(e^{-(\mu_0+1)}, e^{-\mu_0})$. Let $\mathcal{P}_0 = \mathcal{P}_{[z_0]}|\Delta$. We will describe in the next few paragraphs a sequence of partitions $\mathcal{P}_0 \prec \mathcal{P}_1 \prec \mathcal{P}_2 \ldots$, such that each $\mathcal{P}_n$ divides Δ into a countable number of intervals — or curve segments rather. Points in $\mathcal{P}_n(z)$ can be regarded as having trajectories "indistinguishable" from that of z up to time n.

We consider one element ω of $\mathcal{P}_0$ at a time. Regarding ω as bound to $z(\omega) = z_0$, we let $j_1 > 0$ be the first time when $T^j\omega$ is free and intersects $(-\delta, \delta)$. (We may assume that all points in $T^j\omega$ become free simultaneously.) If $T^{j_1}\omega$ contains some $I_{\mu j}$ then we let $k_1 = j_1$ and go to the next paragraph. If not, then we consider $T^{j_1}\omega$ as bound to $z(T^{j_1}\omega)$ and wait for it to return again in a free state, say at time j_2. From Corollary 1 in Section 2 it is clear that $\left|T^{j_2}\omega\right| \geq 2\left|T^{j_1}\omega\right|$, so repeating this process for at most a finite number of times, there will be a free return at time j_l, when $T^{j_l}\omega \supseteq$ some $I_{\mu j}$. Set $k_1 = j_l$.

We now define $\mathcal{P}_n|\omega$ for $n \leq k_1$. Let $\mathcal{P}_n = \mathcal{P}_0$ for $n < k_1$. For $n = k_1$, we first let $\mathcal{P}'_{k_1}|\omega = T^{-k_1}(\mathcal{P}_{[z(T^{k_1}\omega)]} \cup \{(-1, -\delta), (\delta, 1)\})$, and obtain $\mathcal{P}_{k_1}$ from $\mathcal{P}'_{k_1}$ by adjoining each of the two end intervals on $\mathcal{P}'_{k_1}|\omega$ to its neighbor unless the T^{k_1}-image of this end interval lies outside of $(-\delta, \delta)$ and has length $\geq |I_{\mu_0 j}|$.

We then repeat the argument in the last two paragraphs for each element ω' of $\mathcal{P}_{k_1}$. That is, if k_2 is the first time after k_1 when part of $T^{k_2}\omega'$ returns freely to $(-\delta, \delta)$ and $T^{k_2}\omega' \supset$ some $I_{\mu j}$ then we cut up ω' again at this time according to the locations of $T^{k_2}z$.

Next we introduce a sequence of stopping times $t_0 < t_1 < \ldots$ on Δ. Let Δ^+ and Δ^- be the rightmost and leftmost intervals in the partition $\mathcal{P}$ of $(-\delta, \delta)$. (We may assume that Δ^+ and Δ^- are fixed intervals not depending on the location of critical points.) Let $t_0 \equiv 0$. For $z \in \Delta$, we define $t_1(z)$ to be the smallest $k > 0$ such that $T^k(\mathcal{P}_{k-1}(z))$ contains either Δ^+ or Δ^-, $t_2(z)$ to be the smallest $k > t_1(z)$ when $T^k(\mathcal{P}_{k-1}(z))$ contains either Δ^+ or Δ^-, and so on. Note that $t_n(z)$ could take on the value ∞, since it is possible for a point to keep returning to the shorter intervals which get cut before they get a chance to grow long. We will prove in Section 3.3, however, that this is an extremely improbable event.

Our construction on $\mathcal{P}_n$ is virtually identical to that of a similar partition in Section 1 of [**BC2**] — except of course that our construction takes place on W whereas the one in [**BC2**] is carried out in parameter space. Our t_n's correspond essentially (though not exactly) to the "escape times" in [**BC2**].

3.2. Derivative estimates.

Let m denote Lebesgue measure on W, i.e. if $\gamma \subset W$ is a curve segment, then $m(\gamma)$ is equal to the arc length of γ. Let $T_*^j(m|\Delta)$ denote the measure with $T_*^j(m|\Delta)(E) = m(T^{-j}E \cap \Delta)$. Clearly the density of $T_*^j(m|\Delta)$ on $T^j\Delta$ is given by

$$\frac{dT_*^j(m|\Delta)}{dm}(z) = \left|DT_z^{-j}\tau\right|.$$

Our first task is to study how $T_*^j(m \mid \Delta)$ is distributed along W for $j = 1, 2, \ldots$. For this we use the derivative estimates in Corollary 1. The distortion estimate in Proposition 2 is crucial for controlling local fluctuations in densities along certain segments of W. This will be important in our construction. Lemma 3 will also be used.

3.3. Frequency of returns.

The aim of this section is to show that a positive measure set of points in Δ return with positive frequency to $\Delta^+ \cup \Delta^-$.

3.3.1. First escape time estimates.

Consider first the 1-d map $f : [-1, 1] \circlearrowleft$ satisfying the conditions in Section 1.1. Let γ be an interval in $[-1, 1]$. We assume that γ is either $\approx I_{rj}$ for some r, j, or $\gamma \cap (-\delta, \delta) = \emptyset$. (The notation $\gamma \approx I_{rj}$ means that $I_{rj} \subset \gamma \subset I_{rj}^+$, where I_{rj}^+ is defined to be the union of I_{rj} and its two adjacent intervals.) We define the *first escape time function* $t|_\gamma$ exactly as t_1 is defined in Section 3.1 — except that we start from γ. This definition is related to our earlier definition of stopping times $t_1 < t_2 < \ldots$ on Δ (had we defined them for our interval map) by

$$t_{i+1}(x) - t_i(x) = \left(t|_{f^{t_i}(\mathcal{P}_{t_i}(x))}\right)(x).$$

Note that if $\gamma = f^{t_i}(\mathcal{P}_{t_i}(x))$ for some x and does not intersect $(-\delta, \delta)$, then it has δ or $-\delta$ as one of its end points and has length $\geq |\Delta^+|$. We claim that for such an interval γ,

$t|_\gamma$ is constant on γ and is $\leq M$ for some M independent of γ. To see this let k_1 be the first time $f^{k_1}\gamma \supset$ some $I_{\mu j}$. Using bound estimates similar to those for $\Delta^\pm$ it is easy to verify that $k_1 \leq M = C\log(1/\delta)$ and that $f^j\gamma \geq C/\log(1/\delta) \gg 2\delta$. Therefore $f^{k_1}\gamma$ must contain either Δ^+ or Δ^- and $t|_\gamma = k_1$.

When $\gamma \approx I_{rj}$, the following large deviation estimate for $t|_\gamma$ is proved in Section 2.2 of [**BC2**].

Lemma 8. ([**BC2**]). *For $\gamma \approx I_{rj}$ and $n \geq 6r$*

$$m\{x \in \gamma : t|_\gamma(x) \geq n\} \leq e^{-n/20}\, m(\gamma).$$

Because of the close correspondence between the derivative estimates in 1 and 2 dimensions (see Corollary 1), these first escape time estimates apply without change to free segments $\gamma \subset T^k\Delta$. We now derive from these estimates the main lemma of Section 3.3.

3.3.2. A lemma.

Lemma 9. *There is a constant C^* such that for all $i \geq 0$,*

$$\int_\Delta (t_{i+1} - t_i)\, dm \leq C^*.$$

Proof: For $i \geq 1$, let $\widehat{\mathcal{P}}_i$ be the partition of Δ into "distinguishable orbits" up to time $t_i - 1$, i.e. $\widehat{\mathcal{P}}_i$ refines the partition of Δ by values of t_i and

$$\widehat{\mathcal{P}}_i|_{\{t_i = k\}} = \mathcal{P}_{k-1}|_{\{t_i=k\}}.$$

This means in particular that for each $\omega \in \widehat{\mathcal{P}}_i$, $T^{t_i}\omega \supset \Delta^+$ or Δ^-. Also, since $m\{t_i = \infty\} = 0$, $\widehat{\mathcal{P}}_i$ is a genuine partition of Δ up to a set of measure 0. It suffices to give a uniform upper bound for

$$\frac{1}{m(\omega)} \int_\omega (t_{i+1} - t_i)\, dm, \qquad \omega \in \widehat{\mathcal{P}}_i.$$

We fix $\omega \in \widehat{\mathcal{P}}_i$ and let $\gamma = T^{t_i}\omega$ Then

$$\int_\gamma (t_{i+1} - t_i) \circ T^{-t_i}\, dm \leq \sum_{r,j} \int_{I_{r,j}} t|_{I_{rj}} + \int_{\gamma_-} t|_{\gamma_-} + \int_{\gamma_+} t|_{\gamma_+},$$

where γ_- and γ_+ are those parts of γ in $(-1, -\delta) \times \mathbb{R}$ and $(\delta, 1) \times \mathbb{R}$ resp. From Lemma 8, we see that

$$\begin{aligned}\sum_{r,j} \int_{I_{r,j}} t|_{I_{r,j}}\, dm &\leq \sum_{r,j} \sum_{n=0}^{\infty} m\left\{z \in I_{r,j} : t|_{I_{r,j}}(z) \geq n\right\} \\ &\leq \sum_{n=0}^{\infty} \left\{ m\left(\bigcup_{|r| \geq \frac{n}{6}} I_{r,j}\right) + e^{-\frac{n}{20}} m\left(\bigcup_{|r| \leq \frac{n}{6}} I_{r,j}\right)\right\},\end{aligned}$$

so that

$$\frac{1}{m(\gamma)}\int_\gamma (t_{i+1}-t_i)\circ T^{-t_i}\,dm \le \frac{1}{|\Delta^+|}\left\{\sum_{n=0}^{\infty}\left(2e^{-\frac{n}{6}}+e^{-\frac{n}{20}}(2\delta)\right)+2M\right\}$$
$$\le \text{ some } C.$$

Our desired estimate then is given by

$$\frac{1}{m(\omega)}\int_\omega (t_{i+1}-t_i)\,dm = \frac{1}{m(\omega)}\int_\gamma (t_{i+1}-t_i)\circ T^{-t_i}\,d\left(T^{t_i}(m|\omega)\right)$$
$$\le C_1 C,$$

where C_1 is the distortion constant in Proposition 2. The case $i=0$ is simple. □

3.3.3. A lower bound on the frequency of returns to $\Delta^\pm$.
Let $\mathcal{P}_{n,\Delta^\pm} = \bigcup\{\omega \in \mathcal{P}_n : T^n\omega = \Delta^+ \text{ or } \Delta^-\}$

LEMMA 10. *There exists a constant $\alpha^* > 0$ such that*

$$\varliminf_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N} m\,\mathcal{P}_{n,\Delta^\pm} \ge \alpha^* > 0.$$

PROOF: From Lemma 9 we know that

$$\int_\Delta t_n\,dm \le C^* n$$

and so

$$m\{z\in\Delta : t_n(z) \le 2n\,C^*\} \ge \tfrac{1}{2}\,m(\Delta).$$

This means that

$$\sum_{k=1}^{2nC^*} m\{z\in\Delta : k = t_i(z) \text{ some } i\} \ge n\cdot\tfrac{1}{2}\,m(\Delta).$$

Now for each $\omega\in\mathcal{P}_{k-1}$ with $k = t_i(\omega)$, there is $\omega'\in\mathcal{P}_k$, $\omega'\subset\omega$, such that $T^k\omega\approx\Delta^+$ or Δ^-. Moreover

$$m(\omega') \ge m(\omega)\cdot\frac{|\Delta^+|}{2}\cdot\frac{1}{C_1}.$$

These estimates together give the desired result. □

3.4. SBR measures as limits of Lebesgue measure on W.

3.4.1 Definition of SBR measures. In this subsection we define precisely what we mean by SBR-measures. Let $F : M \circlearrowleft$ be an arbitrary C^2 diffeomorphism of a finite dimensional manifold and let μ be an F-invariant Borel probability measure on M with compact support. We will assume throughout that at μ-a.e. point, there is a strictly positive Lyaponov exponent. Under these conditions, the unstable manifold theorem of Pesin [**P1**] or Ruelle [**Ru2**] tells us that passing through μ-a.e. x there is an *unstable manifold* which we denote by $W^u(x)$.

A measurable partition $\mathcal{Q}$ of M is said to be *subordinate to* W (with respect to the measure μ) if at μ-a.e. x, $\mathcal{Q}(x)$ is contained in $W^u(x)$ and contains an open neighborhood of x in $W^u(x)$. On each $\mathcal{Q}(x)$, there are two measures that are of interest to us. One is the restriction to $\mathcal{Q}(x)$ of the Riemann measure induced on $W^u(x)$; let us call this $m_x^{\mathcal{Q}}$. The other is $\mu_x^{\mathcal{Q}}$, where $\{\mu_x^{\mathcal{Q}}\}$ is a canonical family of conditional measures of μ with respect to the partition $\mathcal{Q}$. (For a reference see e.g. Rohlin [**Ro**].)

Definition. Let $F : (M, \mu) \circlearrowleft$ be as above. We say that μ has *absolutely continuous conditional measures on unstable manifolds* if for everly measurable partition $\mathcal{Q}$ subordinate to W^u, $\mu_x^{\mathcal{Q}}$ is absolutely continuous with respect to $m_x^{\mathcal{Q}}$ for μ-a.e. x.

For ease of reference, we will in this paper refer to invariant probability measures with absolutely continuous conditional measures on unstable manifolds as *Sinai-Bowen-Ruelle measures* or simply *SBR-measures**.

3.4.2. Pushing forward Lebesgue measure on W.

Let

$$m_0 = m|_\Delta$$

and

$$m_n = \frac{1}{n} \sum_{k=0}^{n-1} T_*^k \, m_0.$$

We define $\hat{m}_n^+$ to be the restriction to $\Delta^+ \times \mathbb{R}$ of

$$\frac{1}{n} \sum_{\substack{\omega \in \widehat{\mathcal{P}}_i \\ t_i(\omega) < n}} T_*^{t_i} \, (m_0|\omega),$$

and let $\hat{m}_n^-$ be defined similarly. From lemma 10 we know that for either $\{\hat{m}_n^+\}$ or $\{\hat{m}_n^-\}$ — let us say $\{\hat{m}_n^+\}$ — there is a sequence $N_1 < N_2 < \cdots$ such that

$$\hat{m}_{N_i}^+(\mathbb{R}^2) \geq \frac{\alpha^*}{3} \quad \text{for all} \quad i.$$

*SBR measures are sometimes defined differently. All the definitions are equivalent for Axiom A attractors, but not all of them have been shown to be equivalent in the nonuniformly hyperbolic setting.

Passing to a subsequence if necessary, we may assume that

$$\hat{m}^+_{N_i} \overset{\text{weakly}}{\longrightarrow} \text{some } \hat{\lambda}$$

and

$$m_{N_i} \overset{\text{weakly}}{\longrightarrow} \text{some } \lambda.$$

The following are immediate:

(1) λ is a T-invariant Borel measure, whose support is contained in the attractor $\Lambda = \overline{W}$;

(2) the total mass of $\hat{\lambda}$ is $\geq \alpha^*/3$;

(3) λ is the sum of $\hat{\lambda}$ and another Borel measure.

Our plan is to use the geometric properties of $\hat{\lambda}$ to show that λ has at least one component with absolutely continuous conditional measures on unstable manifolds.

3.4.3. Geometric properties of $\hat{\lambda}$.

Let c_1 be the constant in Lemma 3, and let

$$\Gamma = \{z \in \Lambda \cap (\Delta^+ \times \mathbb{R}) : \exists\, v \neq 0 \in T_z\mathbb{R}^2 \text{ with } \left|DT_z^{-j} v\right| \leq e^{-c_1 j} \qquad \forall j \geq 0\}.$$

Since $\left|\det DT^{-1}\right| = b^{-1}$ and the contraction above is uniform, it follows easily that Γ is compact, and that for $z \in \Gamma$, if v_z is the direction contracted by DT^{-j} in the definition of Γ, then $z \mapsto v_z$ is continuous. Moreover, by Lemma 3 we know that supp $\hat{m}^+_n \subset \Gamma$ for all n and that $\tau(z) = v_z$.

Sublemma 6. *Let z be an accumulation point of $\bigcup_n \operatorname{supp}(\hat{m}^+_n)$. Then there is a C^1-curve $\gamma(z)$ passing through z such that*

(1) $\gamma(z) = graph(\varphi)$ for some $\varphi : \Delta^+ \to \mathbb{R}$;

(2) $\gamma(z) \subset \Gamma$ and its tangent vector at z' is $v_{z'}$.

Proof: Let $z_i \in \bigcup_n \operatorname{supp}(\hat{m}^+_n)$ be such that $z_i \to z$. For each z_i, let $\varphi_i : \Delta^+ \to \mathbb{R}$ be the function whose graph is the component of $W \cap (\Delta^+ \times \mathbb{R})$ containing z_i. By the $C^2(b)$-property of free segments the sequence of second derivatives $\{\varphi_i''\}$ is uniformly bounded. Hence a subsequence $\{\varphi_{i_k}'\}$ of $\{\varphi_i'\}$ converges uniformly. It follows immediately that φ_{i_k} converges in the C^1 sense to some φ with the properties in Sublemma 6. □

¿From Oseledec's theorem we know that Lyapunov exponents are well defined $\hat{\lambda}$-a.e. Sublemma 6 tells us that one of the exponents is positive. We also know from general theory (see [**P1**] or [**Ru2**]) that for a.e. point, its local unstable manifold is the unique curve passing through that point that contracts exponentially in backward time. Thus for a typical z, $\gamma(z)$ in Sublemma 6 must be the component of $W^u(z) \cap (\Delta^+ \times \mathbb{R})$ containing z. Let us call it $W^u_{\Delta^+}(z)$.

Let $X \subset \Delta^+ \times \mathbb{R}$ be a measurable set with the following properties: (i) $\hat{\lambda}(\mathbb{R}^2 - X) = 0$, and (ii) X is the disjoint union of $W^u_{\Delta^+}$-curves. Let $\mathcal{Q}$ be the partition of X into $W^u_{\Delta^+}$-leaves, and let $\{\hat{\lambda}^{\mathcal{Q}}_z\}$ be a canonical family of conditional measures of $\hat{\lambda}$. (See Section 3.4.1 for notations.)

SUBLEMMA 7. *The measures $\hat{\lambda}_z^{\mathcal{Q}}$ and $m_z^{\mathcal{Q}}$ are equivalent for $\hat{\lambda}$-a.e. z.*

PROOF: We claim that there is $C > 0$ such that for all intervals $J \subset \Delta^+$, one has

$$\frac{1}{C}|J| \leq \hat{\lambda}_z^{\mathcal{Q}}(J \times \mathbb{R}) \leq C\,|J|$$

for a.e. z. To see this let $\mathcal{Q}_n$ be a sequence of finite partitions of X such that $\forall z \in X$, $\mathcal{Q}_n(z) \supset W^u_{\Delta^+}(z)$ for all n, and $\bigcap_n \mathcal{Q}_n(z) = W^u_{\Delta^+}(z)$. If some $W^u_{\Delta^+}$-curve γ is contained in the support of $\hat{m}_n^+$, let ρ_n denote the density of $\hat{m}_n^+|\gamma$. Proposition 2 tells us that for all $z_1, z_2 \in \gamma$,

$$\frac{\rho_n(z_1)}{\rho_n(z_2)} \leq C_1.$$

Integrating over $W^u_{\Delta^+}$-curves in each element of $\mathcal{Q}_n$, we obtain

$$\frac{1}{C}|J| \leq E_{\hat{\lambda}}((J \times \mathbb{R})|\mathcal{Q}_n) \leq C\,|J|$$

for some C independent of J. Our assertion follows from the martingale convergence theorem. □

3.4.4. Completing the construction.

First we observe that $(\lambda|X)_z^{\mathcal{Q}}$ is equivalent to $m_z^{\mathcal{Q}}$ for $\hat{\lambda}$-a.e. z. This is true because the σ-algebra of T-invariant measurable sets is contained in the σ-algebra of measurable sets made up of entire W^u-leaves, and that for every ergodic measure ν, the conditional measures of ν on local W^u-manifolds are either equivalent to m on a.e. leaf, or they are singular to m on a.e. leaf.

Let

$$X' = \{z : \frac{d\,\hat{\lambda}}{d\,\lambda} > 0\}.$$

and let λ' be the saturation of $\lambda|_{X'}$ under T. That is, let $r : X' \to \mathbb{Z}^+$ be the first return time to X' under T and let

$$\lambda' = \sum_{n=0}^{\infty} T_*^n\,(\lambda|(X' \cap \{r > n\})).$$

We noted above that $\lambda|X'$ has absolutely continuous conditional measures on W^u-leaves. Hence the same is true for λ'. Moreover, an invariant measure with absolutely continuous conditional measures on W^u-leaves is the sum of at most a countable number of ergodic measures with the same property (see e.g. [**L**]). We may therefore assume for the rest of this paper that λ^* is one of these ergodic components, normalized to give $\lambda^*(\Lambda) = 1$. This is our SBR-measure.

4. Properties of SBR measures on Λ.

Let λ^* be the T-invariant ergodic probability measure we constructed in Section 3. The purpose of this section is to prove

(1) the support of λ^* is the entire attractor Λ (Section 4.2);
(2) T does not admit any other SBR measures (Section 4.3);
(3) (T, λ^*) is Bernoulli (Section 4.4);

and to indicate how the existence of λ^* implies the corollary in the introduction (see Section 4.1.2).

4.1. Some known facts from the general theory of nonuniformly hyperbolic systems.

The material in this subsection is not particular to the Hénon maps. We consider an arbitrary C^2 diffeomorphism $F : M \circlearrowleft$ of a finite dimensional Riemannian manifold preserving an ergodic Borel probability measure μ with compact support. It will be assumed throughout that (F, μ) has no zero Lyapunov exponents.

4.1.1. Stable and unstable manifolds.

In [**P1**] Pesin proved the existence of stable and unstable manifolds in this nonuniform setting and studied their properties. (Pesin assumed that μ is equivalent to Lebesgue. The case of arbitrary invariant measures is considered by Ruelle [**Ru2**].) We recall here a couple of their results, giving precise statements only of what we will use and leaving out much more that is proved.

We write $T_xM = E^u(x) \oplus E^s(x)$ wherever it makes sense, and for $\delta > 0$, we let $B^u_\delta(x)$ and $B^s_\delta(x)$ denote the balls of radius δ about 0 in $E^u(x)$ and $E^s(x)$ respectively. Let $B_\delta(x) = B^u_\delta(x) \times B^s_\delta(x)$. It is sometimes convenient geometrically to introduce a new inner product $\langle\cdot,\cdot\rangle'_x$ on T_xM : under $\langle\cdot,\cdot\rangle'_x$, $E^u(x)$ and $E^s(x)$ are perpendicular; whereas restricted to $E^u(x)$ and $E^s(x)$, $\langle\cdot,\cdot\rangle'_x$ agrees with the given Riemannian metric. Let $\|\cdot\|'_x$ denote the corresponding norm. The following is true:

There exist Borel subsets $\Gamma_1 \subset \Gamma_2 \subset \ldots$ of M with $\mu(\bigcup \Gamma_i) = 1$ and sequences of positive numbers δ_n, ϵ_n and θ_n, possibly $\downarrow 0$ as $n \uparrow \infty$, such that (1) and (2) below hold for every $x \in \Gamma_n$. (Think of the Γ_n's as uniformly hyperbolic sets that are not necessarily invariant, with the strength of hyperbolicity deteriorating as $n \to \infty$.)

(1) Let $\Gamma_n(x) = \{y \in \Gamma_n : d(x, y) < \epsilon_n\}$. For $y \in \Gamma_n(x)$, let $W^u_x(y)$ denote the connected component of $(\exp_x^{-1} W^u(y)) \cap B_{\delta_n}(x)$ that contains $\exp_x^{-1} y$. Then for all $y \in \Gamma_n(x)$, $W^u_x(y)$ is the graph of a function $\varphi : B^u_{\delta_n}(x) \to B^s_{\delta_n}(x)$ with $\|D\varphi\|'_x \le \frac{1}{100}$. Moreover, as a C^1 embedded disk, $W^u_x(y)$ varies continuously with y. An analogous statement holds for $W^s_x(y)$.

(2) For $i = 1, 2$, let Σ_i be either $W^u_x(y)$ for some $y \in \Gamma_n(x)$ or a plane in $B_{\delta_n}(x)$ parallel to $E^u(x)$. Let $\Sigma'_1 = \Sigma_1 \cap \bigcup_{y \in \Gamma_n(x)} W^s_x(y)$, and let π be the map that takes $z \in \Sigma'_1$ to Σ_2 by sliding along $W^s_z(\cdot)$. Then for every Borel $A \subset \Sigma'_1$,
$$\mathrm{Leb}(\pi A) \ge \theta_n \ \mathrm{Leb}(A).$$

Property (2) is the precise statement of what is called the "absolute continuity of the W^s-foliation". It is proved for "dissipative" systems in [**PS**].

4.1.2 Generic points for SBR measures.

Let $F : (M, \mu) \circlearrowleft$ be as above. A point $x \in M$ is said to be *future-generic with respect to* μ or simply μ-generic if for every continuous function $\varphi : M \to \mathbb{R}$, $n^{-1}\Sigma_{i=0}^{n-1} \varphi \circ F^i(x) \to \int \varphi d\mu$ as $n \to \infty$. In particular, if μ is ergodic, then μ-a.e. x is μ-generic, and if x is μ-generic, then every $y \in W^s(x)$ is μ-generic as well.

The corollary stated in the introduction is an immediate consequence of the following general fact.

PROPOSITION 3. *Let $F : (M, \mu) \circlearrowleft$ be as above. If μ is an ergodic SBR measure with no zero Lyapunov exponents, then there is a Borel subset $Y \subset M$ with positive Riemannian measure such that every $y \in Y$ is μ-generic.*

PROOF: Since μ is an SBR measure, there is a piece of unstable manifold γ and a set $A \subset \gamma$ with $mA > 0$ such that every $x \in A$ is μ-generic. (As usual, m denotes the induced Riemannian measure on γ.) Using the absolute continuity of the W^s-foliation discussed in the last subsection, we see that

$$Y := \bigcup_{x \in A} W^s(x)$$

has the desired properties. $\square$

4.1.3. Ergodic properties of SBR measures.

The following is proved by Pesin [**P2**] when μ is equivalent to Riemannian volume and generalized by Ledrappier [**L**] to the situation where μ has absolutely continuous conditional measures on W^u:

Let μ be as above and (F, μ) be ergodic. Then there are pairwise disjoint Borel sets $A_1, \dots, A_n \in M$, such that

(1) $F(A_i) = A_{i+1}$ for $i < n$, $F(A_n) = A_1$;

and

(2) $(F^n|A_i, \mu|A_i)$ is Bernoulli for all i.

4.2. The support of λ^*.

Recall the sequence of choices leading to the selection of "good" parameters in [**BC2**]: First $\delta > 0$ is fixed. Then $a_0 < a_1 < 2$ are chosen with a_0 very near 2. Next, b is chosen sufficiently small depending on a_0 and a_1; and finally, for fixed b, "good" maps $T_{a,b}$ are selected by varying $a \in (a_0, a_1)$.

Consider first the 1-d situation. Assume that $\delta > 0$ is fixed, so that $\Delta^{\pm}$, the outermost intervals on the partition of $(-\delta, \delta)$ are determined. (See Subsection 3.1 for definitions.) Let $\hat{x}_a$ be the unique fixed point of $f_a : [-1, 1] \circlearrowleft$. For $a = 2$ since f_2 is topologically conjugate to its piecewise linear model, $\bigcup_{n \geq 0} f_2^{-n} \hat{x}_2$ is dense in $[-1, 1]$. So $\exists N_0 \in \mathbb{Z}^+$, intervals $\tilde{\Delta}^{\pm} \subset \Delta^{\pm}$, and a neighborhood V of $\hat{x}_2$, such that $f_2^{N_0} \tilde{\Delta}^{\pm} \supset V$. Moreover, one can choose N_0 , $\tilde{\Delta}^{\pm}$ and V such that for all $x \in \tilde{\Delta}^{\pm}$, $f_2^{N_0} x$ is "free". Chose a_0 such that for all $a \in (a_0, 2)$, this picture persists with the same N_0, $\tilde{\Delta}^{\pm}$ and V.

Returning now to our Hénon maps $T_{a,b}$, we assume b is sufficiently small that the fixed point $\hat{z}_{a,b}$ lies in $V \times \mathbb{R}$, and that if γ is a curve with small Hausdorff distance from either $\Delta^+ \times \{0\}$ or $\Delta^- \times \{0\}$, then $T_{a,b}^{N_0} \gamma$ contains a curve with small Hausdorff distance from $V \times \{0\}$. Moreover, it is clear from our derivative estimates is Section 2 that if, in addition, $\gamma \subset W$ and is a free segment, then $T^{N_0} \gamma$ is a $C^2(b)$ curve, which must then intersect $W^s_{\text{loc}}(\hat{z})$ with an angle $\geq \pi/4$ in at least one point.

¿From our construction of λ^* in Section 3, it follows that the support of λ^* contains a curve γ near $\Delta^+ \times \{0\}$ or $\Delta^- \times \{0\}$ that is the C^1-limit of free segments in W. This guarantees that γ intersects $W^s(\hat{z})$ transversally, which in turn implies that $W^u_{\text{loc}}(\hat{z}) \subset$ supp λ^*. Since $\Lambda = \overline{W}$, it follows that $\Lambda \subset$ supp λ^*. The reverse inclusion is obvious.

4.3 Uniqueness of SBR measures.

Suppose that λ^* is not unique, so that there is another SBR measure μ on Λ. Without loss of generality we may assume that μ is ergodic. Fix $N_1 \in \mathbb{Z}^+$ with $\mu\Gamma_{N_1} > 0$. (See 4.1.1 for definition.) We claim that $\exists z_1 \in \Gamma_{N_1}$ such that in a neighborhood of z_1 we have the following picture: (For notational simplicity let us confuse $A \subset B_{\delta_{N_1}}(z_1)$ with $\exp_{z_1} A$ in the next few paragraphs.)

In $B_{\delta_{N_1}}(z_1)$, there is a "rectangle" two of whose sides, γ_1 and γ_2, are $W^u_{z_1}$-manifolds. Let us assume that $\gamma_1 = W^u_{z_1}(z_1)$. The other two "sides" of this "rectangle" are sets of the form $W^s_{z_1}(A_i)$, $i = 1, 2$, where A_i is a subset of γ_1 and $W^s_{z_1}(A_i) := \bigcup_{z \in A_i} W^s_{z_1}(z)$. The sets A_1 and A_2 are to have the following properties:

(1) $m(A_1), m(A_2) > 0 \qquad (m = \text{Leb on } \gamma_1)$;

(2) every $z \in A_1 \cup A_2$ is *generic* with respect to μ.

Moreover, there is at least one point $w \in \Lambda$ in the "interior" of our "rectangle". (See Figure 1.)

Let us assume this picture for now and complete the proof. We claim that there is a W^u-leaf γ with the following properties:

(3) m-a.e. $z \in \gamma$ is generic with respect to λ^*;

(4) γ connects the "outside" of our "rectangle" to the "inside".

To see that this claim is valid, recall the geometric properties of $\hat{\lambda}$ in Section 3 and the argument in Section 4.2 showing that $W^u_{\text{loc}}(\hat{z})$ is the C^1 limit of curves with property (3). Iterating forward, we see that every compact segment of W is the C^1 limit of curves with property (3). Now $w \in \Lambda = \bar{W}$ is "inside" our "rectangle", whereas we may assume that $\hat{z}$ is "outside". (3) and (4) should now be obvious.

Since γ clearly cannot intersect γ_1 or γ_2, it must intersect $W^s_{z_1}(A_1) \cup W^s_{z_2}(A_2)$. If we show that this intersection has positive m-measures in γ, then we will have proved that a positive m-measure set in γ is generic with respect to both λ^* and μ, forcing $\mu = \lambda^*$.

Two points need to be justified. First, the "rectangle" in our picture. Let $D = \{z \in \mathbb{R}^2 : z \text{ is generic with respect to } \mu\}$. Chose $z_1 \in \Gamma_{N_1}$ such that

(i) $\mu\{z \in \Gamma_{N_1} : d(z, z_1) < \epsilon\} > 0$ for all $\epsilon > 0$,

(ii) $m\{z \in W^u_{\text{loc}}(z_1) : z \in \Gamma_{N_1} \cap D \text{ and } d^u(z, z_1) < \epsilon\} > 0$ for all $\epsilon > 0$. (Here $d^u :=$ dist along $W^u(z_1)$.)

Let $\gamma_1 = W^u_{z_1}(z_1)$. Then A_1 and A_2 can be chosen as subsets of $\gamma_1 \cap \Gamma_{N_1} \cap D$. To complete our "rectangle" and to guarantee that some point of Λ lies inside, it suffices to argue that arbitrarily near γ_1, there are infinitely many $W^u_{z_1}$-curves. If this was not the case, then by (i) above we must have $\mu\gamma_1 > 0$. A standard argument in nonuniform hyperbolic theory then tells us that for some for some $n > 0$, $T^n\gamma \supset \gamma$ and $T^n|_{T^{-n}\gamma}$ is, in suitable coordinates, expanding. (See e.g. [**K**] for more details.) This implies that $\mu\{z_1\} > 0$, contradicting our assumption that μ is SBR.

The other point which perhaps needs some justification is our assertion that

$$m\left(\gamma \cap \left(W^s_{z_1}(A_1) \cup W^s_{z_1}(A_2)\right)\right) > 0.$$

We may assume that A_1 is a Cantor set, and is $= \bigcap_{n=1}^{\infty} E_n$ where each E_n is the disjoint union of a finite number of curve segments $\{E_{n,i}, 1 \le i \le i_n\}$ in γ_1. For each (n, i) let $y^l_{n,i}$ and $y^r_{n,i}$ be the left and right endpoint of $E_{n,i}$, and let $S_{n,i}$ be the strip in $B_{\delta_{N_1}}(z_1)$ bounded by $W^s_{z_1}(y^l_{n,i}), W^s_{z_1}(y^r_{n,i})$ and $\partial B_{\delta_{N_1}}(z_1)$. Then clearly $\bigcap_n \bigcup_i S_{n,i} = W^s_{z_1}(A_1)$. Let us assume that γ connects the two sides of $S_{1,1}$ and let τ_1 be a subsegment of γ joining $W^s_{z_1}(y^l_{1,1})$ to $W^s_{z_1}(y^r_{1,1})$. For each (n, i) with $n > 1$, choose inductively $\tau_{n,i} \subset$ some $\tau_{n-1,j}$ such that $\tau_{n,i}$ joins $W^s_{z_1}(y^l_{n,i})$ to $W^s_{z_1}(y^r_{n,i})$. Then for every (n, i), we have

$$\begin{aligned} m(\tau_{n,i}) &\ge \text{min dist between } W^s_{z_1}(y^l_{n,i}) \text{ and } W^s_{z_1}(y^r_{n,i}) \\ &\ge c \cdot \text{dist}\left(L \cap W^s_{z_1}(y^l_{n,i}), L \cap W^s_{z_1}(y^r_{n,i})\right), \end{aligned}$$

where L is any line in $B_{\delta_{N_1}}(z_1)$ parallel to $E^u(z_1)$ and $c > 0$ is a constant depending only on the angle between $E^u(z_1)$ and $E^s(z_1)$. Using the absolute continuity of the W^s-foliation (see Section 4.1.1) we may then conclude that

$$m(\tau_{n,i}) \ge c \cdot \theta_{N_1} \cdot m(A_1 \cap E_{n,i})$$

and hence

$$m\left(\gamma \cap W^s_{z_1}(A_1)\right) \ge m\left(\bigcap_n \bigcup_i \tau_{n,i}\right) \ge c \cdot \theta_{N_1} \cdot mA_1.$$

4.4 Bernoulliness of (T, λ^*).

In light of Section 4.1.3 it suffices to show that (T^n, λ^*) is ergodic for all $n \ge 1$. Let us fix $n_0 \in \mathbb{Z}^+$ and let μ_0 be one of the ergodic components of (T^{n_0}, λ^*) with absolutely continuous conditional measures on W^u-manifolds. From our construction of λ^* it follows that for some $k \in \mathbb{Z}^+$, $\mu_0(T^k X) > 0$, where X is the set in Section 3.4.3. Using again the fact that points in the same W^u-leaf belong in the same ergodic component of T^{n_0}, we see that $W^u_{\text{loc}}(\hat{z}) \subset \text{supp } \mu_0$, from which it follows that $\text{supp}\, \mu_0 = \Lambda$. Let $\overline{\mu}_0$ be μ_0 normalized.

Repeating the argument in Section 4.3 with $(T^{n_0}, \overline{\mu}_0)$ in place of (T, λ^*), we see that $\overline{\mu}_0$ is the unique SBR measure for T^{n_0}. Thus (T^{n_0}, λ^*) has only one ergodic component and our proof is complete.

4.5 Further properties of (T, λ^*) and an open problem. We mention a couple of facts about (T, λ^*) that follow from general nonuniform hyperbolic theory. Let $\chi_1 > 0 > \chi_2$ be the Lyaponov exponents of (T, λ^*). We have the entropy formula

$$h_{\lambda^*}(T) = \chi_1,$$

where the quantity on the left is metric entropy. (See [**P2**],[**LS**].) Using [**Y1**] and the entropy formula above, we obtain the following formula for the dimension of λ^*:

$$\mathrm{HD}(\lambda^*) = h_{\lambda^*}(T) \cdot \left(\frac{1}{\chi_1} - \frac{1}{\chi_2}\right) = 1 - \frac{\chi_1}{\chi_2} = 1 + \frac{\chi_1}{\chi_1 - \log b}.$$

We finish by mentioning a problem the resolution of which would give a more complete geometric picture of these "good" Hénon maps. Let

$$B = \{z \in \mathbb{R}^2 : d(T^n, \Lambda) \to 0 \text{ as } n \to \infty\}.$$

The set B is called the *basin of attraction* of Λ and is known to contain an open neighborhood of Λ. Proposition 3 in Section 4.1 tells us that a positive Lebesgue measure subset of B consists of points that are λ^*-generic. That is to say, the statistics of these orbits are completely governed by the invariant measure λ^*. It would be nice to know if this property holds not just on a large set but almost everywhere in B.

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