MARKOV EXTENSIONS AND DECAY OF CORRELATIONS FOR CERTAIN HÉNON MAPS

by

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Abstract. — Hénon maps for which the analysis in [**BC2**] applies are considered. Sets with good hyperbolic properties and nice return structures are constructed and their return time functions are shown to have exponentially decaying tails. This sets the stage for applying the results in [**Y**]. Statistical properties such as exponential decay of correlations and central limit theorem are proved.

Résumé. — Dans cet article, on consider les applications de Hénon pour lesquelles l'analyse de [**BC2**] est valable. On construit des ensembles munis de bonnes propriétés hyperboliques et de bonnes structures de retour, et on montre que leurs fonctions de temps de retour ont des restes à décroissance exponentielle. Ceci permet d'appliquer les résultats de [**Y**]. Des propriétés statistiques tels que décroissance exponentielle des corrélations et le théorème central limite sont établies.

0. Introduction and statements of results

Let $T_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T_{a,b}(x, y) = (1 - ax^2 + y, bx).$$

In [**BC2**], Carleson and the first named author developed a machinery for analyzing the dynamics of $T_{a,b}$ for a positive measure set of parameters (a, b) with $a < 2$ and b small. For lack of a better word let us call these the "good" parameters. The machinery of [**BC2**] is used in [**BY**] to prove that for every "good" pair (a, b), $T = T_{a,b}$ admits a Sinai-Ruelle-Bowen measure ν. The significance of ν is that it describes the asymptotic orbit distribution for a positive Lebesgue measure set of points in the phase space, including most of the points in the vicinity of the attractor. The aim of the present paper is to show that (T, ν) has a natural "Markov extension" with

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an exponentially decaying "tail", and to obtain via this extension some results on stochastic processes of the form $\{\varphi \circ T^n\}_{n=0,1,2,\ldots}$, where $\varphi : \mathbb{R}^2 \mapsto \mathbb{R}$ is a Hölder continuous random variable on the probability space $(\mathbb{R}^2, \nu)$.

Consider in general a map $f : M \circlearrowleft$ preserving a probability measure ν. By a *Markov extension* of (f, ν) we refer to a dynamical system $F : (\Delta, \tilde{\nu}) \circlearrowleft$ and a projection map $\pi : \Delta \mapsto M$; F is assumed to have a Markov partition (with possibly infinitely many states), F and π satisfy $\pi \circ F = f \circ \pi$, and $\pi_* \tilde{\nu} = \nu$. We do not require that π be 1-1 or onto.

Let (f, ν) be as in the last paragraph, and let X be a class of functions on M. We say that (f, ν) has *exponential decay of correlations* for functions in X if there is a number $\tau < 1$ such that for every pair $\varphi, \psi \in X$, there is a constant $C = C(\varphi, \psi)$ such that

$$\left| \int \varphi(\psi \circ f^n) d\nu - \int \varphi d\nu \int \psi d\nu \right| \leq C\tau^n \qquad \forall n \geq 0.$$

Also, we say that (f, ν) has a *central limit theorem* for φ with $\int \varphi d\nu = 0$ if the stochastic process $\varphi, \varphi \circ f, \varphi \circ f^2, \ldots$ satisfies the central limit theorem, i.e. if

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \varphi \circ f^i \stackrel{\text{dist}}{\to} \mathcal{N}(0, \sigma)$$

for some $\sigma \geq 0$. For $\sigma > 0$ this means that $\forall t \in \mathbb{R}$,

$$\nu \left\{ \frac{1}{\sqrt{n}} \sum_{0}^{n-1} \varphi \circ f^i < t \right\} \to \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{t} e^{-u^2/2\sigma^2} du$$

as $n \to \infty$.

For $f = T_{a,b}$, (a, b) "good" parameters, we have the following results:

***Theorem* 1 ([BY]).** — *f admits an SRB measure ν.* (See Section 1.7 for the precise definition.)

***Theorem* 2 ([BY]).** — *ν is the unique SRB measure for f^n for every $n \geq 1$. This implies in particular that (f^n, ν) is ergodic $\forall n \geq 1$.*

By the general theory of SRB measures, the ergodicity of (f^n, ν) for all $n \geq 1$ is equivalent to (f, ν) having the mixing property, or that it is measure-theoretically isomorphic to a Bernoulli shift, see [**L**].

For $\gamma > 0$, let $\mathcal{H}_\gamma$ be the space of Hölder continuous functions on $\mathbb{R}^2$ with Hölder exponent γ.

***Theorem* 3.** — *(f, ν) has exponential decay of correlations for functions in $\mathcal{H}_\gamma$. The rate of decay, τ, may depend on γ.*

Theorem 4. — *(f,ν) has a central limit theorem for all $\varphi \in \mathcal{H}_\gamma$ with $\int \varphi d\nu = 0$; the standard deviation $\sigma > 0$ iff $\varphi n^o = \psi \circ f - \psi$ for some $\psi \in L^2(\nu)$.*

Theorems 1 and 2 are proved in [**BY**], while theorems 3 and 4 are new and are proved in this paper. But since an SRB measure is constructed in the process of proving Theorem 3, this paper also contains an independent proof of Theorem 1. Questions of ergodicity or uniqueness of SRB measures, however, are of a different nature. We will *assume* Theorem 2 for purposes of the present paper.

As mentioned earlier on, our proof of theorems 3 and 4 are carried out using a Markov extension with certain special properties. The second named author has since extended this scheme of proof to a wider setting. We will refer to [**Y**] for certain facts not specific to the Hénon maps, but will otherwise keep the discussion here as self-contained as possible.

The following is a comprehensive summary of what is in this paper, section by section.

In Section 1 we recall from [**BC2**] and [**BY**] some pertinent facts about f.

The aim of Section 2 is to clean up the notion of distance to the "critical set" previously used in [**BC2**] and [**BY**]. We prove that the various distances used before are equivalent.

Section 3 is devoted to organizing the dynamics of f in a coherent fashion. We focus on a naturally defined Cantor set Λ with a product structure defined by local stable and unstable curves and with Λ intersecting each local unstable curve in a positive Lebesgue measure set. The dynamics on Λ is analogous to that of Smale's horseshoe, except that there are infinitely many branches with variable return times. A precise description of Λ is given in Propostion 3.1 in Section 3.1.

In Section 4 we study the return time function $R : \Lambda \to \mathbb{Z}^+$, i.e. $z \in \Lambda$ returns to Λ after $R(z)$ iterates in the representation above. (Note that $R(z)$ is not necessarily the first return time.) We prove that the measure of $\{R > n\}$ decays exponentially fast as $n \to \infty$. This estimate is stated in Lemma 5 in Section 4.1; it plays a crucial role in the subsequent analysis.

In Section 5 we consider the quotient space $\bar{\Lambda}$ obtained by collapsing Λ along W^s_{loc}-curves. We prove, modifying standard arguments for Axiom A systems where necessary, that $\bar{\Lambda}$ has a well defined metric structure and that the Jacobians of the induced quotient maps have a "Hölder"-type property. This step paves the way for the introduction of a Perron-Frobenius operator. The results are stated in Proposition 5.1 in Section 5.1.

Let $\overline{f^R} : \bar{\Lambda} \circlearrowleft$ denote the return map to Λ. In Section 6 we construct a tower map $F : \Delta \circlearrowleft$ over $f^R : \Lambda \circlearrowleft$ with height R (see Section 6.1). F is clearly an extension of f. A Perron-Frobenius operator is introduced for $\overline{F} : \overline{\Delta} \circlearrowleft$, the object obtained by collapsing $W^s_{\rm loc}$-curves in Δ. At this point we appeal to a theorem in [**Y**] on the spectral properties of certain abstractly defined Perron-Frobenius operators. We explain briefly how a gap in the spectrum of this operator implies exponential decay of correlation for f, referring again to [**Y**] for the formal manipulations, and finish with a proof of the Central Limit Theorem.

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1. Dynamics of certain Hénon maps

The purpose of this section is to review some of the basic ideas in [**BC2**] and [**BY**], and to set some notations at the same time. We would like to make the main ideas of this paper accessible to readers without a thorough knowledge of [**BC2**] and [**BY**], but will refer to these papers for technical information as needed. The summary in Section 1 of [**BY**] may be helpful.

1.1. General description of attractors. — In this paper we are interested in the parameter range $a < 2$ and near 2, $b > 0$ and small. The facts in Section 1.1 are elementary and hold for $f = T_{a,b}$ for an open set of parameters (a, b).

There is a fixed point located at approximately $\left(\frac{1}{2}, \frac{1}{2}b\right)$; it is hyperbolic and its unstable manifold, which we will call W, lies in a bounded region of $\mathbb{R}^2$. Let Ω be the closure of W. Then Ω is an attractor in the sense that there is an open neighborhood U of Ω with the property that $\forall z \in U$, $f^n z \to \Omega$ as $n \to \infty$.

Away from the y-axis, f has some hyperbolic properties. For example, let $\delta \gg b$ and let $s(v)$ denote the slope of a vector v. Then

(i) on $\{|x| \geq \delta\}$, Df preserves the cones $\{|s(v)| \leq \delta\}$;
(ii) $\exists M_0 \in \mathbb{Z}^+$ and $c_0 > 0$ such that if $z, fz, \ldots, f^{M-1}z \in \{|x| \geq \delta\}$ and $M \geq M_0$, then

$$\left|Df^M_z v\right| \geq e^{c_0 M}|v| \quad \forall v \text{ with } |s(v)| \leq \delta.$$

It is easy to show, however, that Ω is not an Axiom A attractor.

In contrast to Section 1.1, the statements in Section 1.2 – 1.6 hold only for a positive measure set of parameters. For the rest of this paper we fix a pair of "good" parameters (a, b) and write $f = T_{a,b}$.

1.2. The critical set. — A subset $\mathcal{C} \subset W$, called the *critical set*, is designated to play the role of critical points for 1-dimensional maps. Points in $\mathcal{C}$ have x-coordinates ≈ 0; they lie on $C^2(b)$ segments of W (a curve is called $C^2(b)$ if it is the graph of a function $y = \varphi(x)$ with $|\varphi'|$, $|\varphi''| \leq 10b$); and they have "homoclinic" behaviour in the sense that if τ denotes a unit tangent vector to W, then for $z \in \mathcal{C}$, $|Df_z^j \tau| \leq (5b)^j \; \forall j \geq 0$.

Other important properties of $z \in \mathcal{C}$ are that $\forall n \geq 1$:

(i) $\left|Df_z^n \binom{0}{1}\right| \geq e^{c(n-1)}$ for some $c \approx \log 2$;

(ii) "dist" $(f^n z, \mathcal{C}) > e^{-\alpha n}$ for some small $\alpha > 0$. (The precise meaning of "dist" will be given shortly.)

The idea in [**BC2**], roughly speaking, is that when an orbit of $z_0 \in \mathcal{C}$ comes near $\mathcal{C}$, there is a near-interchange of stable and unstable directions (hence a setback in hyperbolicity); but then the orbit of z_0 follows that of some $\tilde{z} \in \mathcal{C}$ for some time, regaining some hyperbolicity on account of (i). To arrange for (i), it is necessary to keep the orbits of $\mathcal{C}$ from switching stable and unstable directions too drastically too soon; hence (ii).

We now give the precise meaning of "dist" $(\cdot, \mathcal{C})$. Consider $z \in \mathcal{C}$ and let $n_1 > 0$ be the first time its orbit returns to $(-\delta, \delta) \times \mathbb{R}$. It is arranged that there is $\tilde{z}_1 \in \mathcal{C}$ of an earlier generation (see below) with respect to which $f^{n_1} z$ is in *tangential position*, i.e. $\tilde{z}$ lies in a $C^2(b)$ segment of W extending $> 4\,|f^{n_1} z - \tilde{z}_1|$ to each side of $\tilde{z}_1$, and the vertical distance between $f^{n_1} z$ and this segment is $< |f^{n_1} z - \tilde{z}_1|^4$; see e.g. [**BY**], Subsection 1.4.1. Here "dist"$(f^{n_1}, \mathcal{C})$ means $|f^{n_1} z - \tilde{z}_1|$.

We say that $f^{n_1} z$ is "bound" to $\tilde{z}_1 \in \mathcal{C}$ for the next p_1 iterates, where p_1 is the smallest j s.t. $|f^{n_1 + j} z - f^j \tilde{z}_1| > e^{-\beta j}$ for some fixed $\beta > \alpha$. At time $n_1 + p_1$, we say that the orbit of z is "free", and it remains free until the first $n_2 \geq n_1 + p_1$ when it returns again to $(-\delta, \delta) \times \mathbb{R}$. The binding procedure above is then repeated, with bound period p_2 etc.

It is convenient to modify slightly the above definitions of p_i so that the bound periods become "nested", i.e. if a bound period is initiated in the middle of another one, it also expires before the first one does. (See Section 6.2 of [**BC2**].)

We return to the notion of "generations" to which we referred a few paragraphs back. There is a unique $z_0 \in \mathcal{C}$ lying in the roughly horizontal segment of W containing our fixed point. The part of W between $f^2 z_0$ and $f z_0$ is denoted by W_1 and called the leaf of generation 1. Leaves of higher generations are defined inductively by $W_n \equiv f^{n-1} W_1 - W_{n-1}$, and a critical point is of generation n if it is in W_n.

1.3. Dynamics on W. — In Proposition 1 of [**BY**], it is shown that the orbit of every $z \in W$ can be controlled using those of $\mathcal{C}$. More precisely, consider z in a local unstable manifold of our fixed point, and let $n_1 > 0$ be the first time its orbit goes into $(-\delta, \delta) \times \mathbb{R}$. It is shown that there is a "suitable" $\tilde{z}_1 \in \mathcal{C}$ to which we will regard $f^{n_1}z$ as bound for some period of time. "Suitable" here means that (1) $f^{n_1}z$ is in *generalized tangential position* wrt $\tilde{z}_1$ (generalized tangential positions are slight generalizations of tangential positions; see Section 1.6 of [**BY**]); and (2) the angle between $\tau(f^{n_1}z)$, the tangent vector to W at $f^{n_1}z$ and a certain vector field about $\tilde{z}_1$ is "correct"; this will be explained in Section 1.5. After a bound period as defined in Section 1.2, the orbit of z then becomes free until it gets into $(-\delta, \delta) \times \mathbb{R}$ again, finds another suitable point $\tilde{z}_2 \in \mathcal{C}$ to bind with, and the story repeats itself.

Not only do suitable binding points always exist ([**BC2**], Section 7.2), it is shown in [**BY**], Lemma 7, that one could systematically assign to each maximal free segment γ intesecting $(-\delta, \delta) \times \mathbb{R}$ a critical point $\tilde{z}(\gamma)$ that is suitable for binding for all $z \in \gamma$. The picture is as follows:

(i) If γ contains a critical point $\tilde{z}$, then $\tilde{z}(\gamma) = \tilde{z}$; this is always the case if neither end point of γ lies in $(-\delta, \delta) \times \mathbb{R}$.
(ii) If only one end point of γ lies in $(-\delta, \delta) \times \mathbb{R}$, say the left end point γ_-, and γ does not contain a critical point, then $\tilde{z}(\gamma)$ is taken to be the binding point of γ_- (note that γ_- is also in bound state); $\tilde{z}(\gamma)$ always lies to the left of γ_-, away from γ.
(iii) If both end points $\gamma_\pm$ of γ are in $(-\delta, \delta) \times \mathbb{R}$ and γ does not contain a critical point then the binding point of at least one of $\gamma_\pm$ lies on the opposite side of $\gamma_\pm$ as γ and can be taken to be $\tilde{z}(\gamma)$.

We state some estimates for $|Df_z^n \tau|$, $z \in W$, that are consequences of the behaviour of the critical set and the binding process above. Unless otherwise referenced, these estimates are proved in Corollary 1 of [**BY**]:

(I) *Free period estimates.*
  (i) Every free segment γ has slope $< 2b/\delta$, and $\gamma \cap (-\delta, \delta) \times \mathbb{R}$ is a $C^2(b)$ curve (Lemmas 1 and 2, [**BY**]).
  (ii) There is $M_0 \in \mathbb{Z}^+$ and $c_0 > 0$ s.t. if z is free and $z, fz, \ldots, f^{M-1}z \notin (-\delta, \delta) \times \mathbb{R}$ for $M \geq M_0$, then $\left|Df_z^M \tau\right| \geq e^{c_0 M}$.

(II) *Bound period estimates.*
The following hold for some $c \approx \log 2$: if $z \in (-\delta, \delta) \times \mathbb{R}$ is free and is bound at this time to $\tilde{z} \in \mathcal{C}$ with bound period p, then
  (i) if $e^{-\nu-1} \leq |z - \tilde{z}| \leq e^{-\nu}$, then $\frac{1}{2}\nu \leq p \leq 5\nu$;
  (ii) $|Df_z^j \tau| \geq |z - \tilde{z}|\, e^{cj}$ for $0 < j < p$;
  (iii) $|Df_z^p \tau| \geq e^{c\frac{p}{3}}$.

(III) *Orbits ending in free states.*
There exists $c_1 > \frac{1}{3}\log 2$ s.t. if $z \in W \cap (-\delta, \delta) \times \mathbb{R}$ is in a free state, then

$$\left|Df_z^{-j}\tau\right| \le e^{-c_1 j} \qquad \forall j \ge 0$$

(Lemma 3, [**BY**]).

1.4. Bookkeeping, derivative and distortion estimates. — Let $\mathcal{P}$ be the following partition of the interval $(-\delta, \delta)$: first we write $(-\delta, \delta)$ as the disjoint union $\bigcup\{I_\nu \colon |\nu| \ge \text{ some } \nu_0\}$ where $I_\nu = (e^{-(\nu+1)}, e^{-\nu})$ for $\nu > 0$ and $I_\nu = -I_\nu$ for $\nu < 0$; then each I_ν is further subdivided into ν^2 intervals $\{I_{\nu,j}\}$ of equal length.

For $x_0 \in \mathbb{R}$, we let $\mathcal{P}_{[x_0]}$ denote a copy of $\mathcal{P}$ with 0 "moved" to x_0. Similarly, if γ is a roughly horizontal curve in $\mathbb{R}^2$ and $z_0 \in \gamma$, we let $\mathcal{P}_{[z_0]}$ denote the obvious partition on γ. Once γ and z_0 are specified, we will use $I_{\nu,j}$ to denote the corresponding subsegment of γ. Also, if $J \in \mathcal{P}_{[\cdot]}$, we let nJ denote the segment n times the length of J centered at J.

The following derivative estimate is very similar to the derivative estimates in the proof of Lemma 7.2 in [**BC2**].

Derivative estimate. — *Suppose that the point z belongs to a free segment of W and satisfies dist$(f^j z, \mathcal{C}) \ge \delta e^{-\alpha j}$ $\forall j < n$ for some integer n. Then there is a constant $c_2 > 0$ such that*

$$|Df_z^n \tau| \ge \delta e^{c_2 n}. \tag{1.1}$$

Since the proof follows step by step the proof of Lemma 7.2 in [**BC2**], it is omitted. The only difference is that in the present situation the allowed approach rate to the critical set is much slower than that in Lemma 7.2 of [**BC2**]: $\operatorname{dist}(f^n z, \mathcal{C}) \ge \delta e^{-\alpha n}$ $\forall n \ge 0$, versus $\operatorname{dist}(f^n z, \mathcal{C}) \ge e^{-36n}$ $\forall n \ge 0$. This leads to the the expansion estimate of (1.1).

The following is proved in Proposition 2 of [**BY**]:

Distortion estimate. — *Let $\gamma \subset (-\delta, \delta) \times \mathbb{R}$ be a segment of W. We assume that the entire segment has the same itinerary up to time N in the sense that*

(i) *all $z \in \gamma$ are bound or free simultaneously at any one moment in time;*
(ii) *if $0 = t_0 < t_1 < \cdots < t_q$ are the consecutive free return times before N, then $\forall k \le q$ the entire segment $f^{t_k}\gamma$ has a common binding point $z^{(k)} \in \mathcal{C}$ and $f^{t_k}\gamma \subset 5J_k$ for some $J_k \in \mathcal{P}_{[z^{(k)}]}$.*

Then $\exists C_1$ independent of γ or N s.t. $\forall z_1, z_2 \in \gamma$,

$$\frac{\left|Df^N_{z_1}\tau\right|}{\left|Df^N_{z_2}\tau\right|} \le C_1.$$

1.5. Fields of contracted directions. — First we state a general perturbation lemma for matrices. Given $A_1, A_2, \dots$, we write $A^n := A_n \dots A_1$. The following is a slight paraphrasing of Lemma 5.5 and Corollary 5.7 in [**BC2**]. All the matrices below are assumed to have $|\det| = b$.

Matrix perturbation lemma. — *Given $\kappa \gg b$, $\exists\lambda$ with $b \ll \lambda < \min(1,\kappa)$ s.t. if $A_1, \dots A_n$, $A'_1, \dots A'_n \in GL(2,\mathbb{R})$ and $v \in \mathbb{R}^2$ satisfy*

$$\left|A^i v\right| \ge \tfrac{1}{2}\,\kappa^i \qquad \text{and} \qquad \|A_i - A'_i\| < \lambda^i \qquad \forall i \le n,$$

then we have, for all $i \le n$:

(i) $\left|A'^i v\right| \ge \frac{1}{2}\,\kappa^i$;

(ii) $\sphericalangle(A^i v, A'^i v) \le \lambda^{\frac{i}{4}}$.

If $A \in G(2,\mathbb{R})$ is s.t. $|Av|/|v| \ne$ const, let $e(A)$ denote one of the two unit vectors most contracted by A. We will write $e_n(z) := e\left(Df^n_z\right)$ wherever it makes sense. From the perturbation lemma above, it follows that if $\left|Df^j_{z_0} v\right| \ge \kappa^j$, $0 \le j \le n$, for some κ and some v, then there is a ball B_n of radius $(\lambda/5)^n$ about z_0 on which e_n is defined and has the property that $|Df^n e_n| \le 2(b/\kappa)^n$.

Assuming that κ is fixed and e_n is defined in a neighborhood of z_0 as above, the following hold (Section 5, [**BC2**]):

(i) e_1 is defined everywhere and has slope $= 2ax + \mathcal{O}(b)$;

(ii) $|e_n - e_m| \le \mathcal{O}(b^m)$ for $m < n$;

(iii) for (x_1, y_1), $(x_2, y_2) \in$ some B_n with $|y_1 - y_2| \le |x_1 - x_2|$,

$$|e_n(x_1,y_1) - e_n(x_2,y_2)| = (2a + \mathcal{O}(b))\,|x_1 - x_2|\,.$$

The perturbation lemma above applies in particular to critical points; see Section 1.2. Indeed, every $z_0 \in \mathcal{C}$ is constructed as the limit of a sequence $\{z_n\}$ where z_n is the unique point in the $C^2(b)$ segment of W containing z_0 with $\tau(z_n) = e_n(z_n)$. Going back to the notion of "suitability" of binding points at the beginning of Section 1.3, a formulation of requirement (2) could be that

$$3\,|\tilde{z}_1 - f^{n_1}z| \le \sphericalangle\left(\tau\left(f^{n_1}z\right), e_{\ell_1}\left(f^{n_1}z\right)\right) \le 5\,|\tilde{z}_1 - f^{n_1}z|$$

where $\ell_1 \approx -\varepsilon \cdot \log|f^{n_1}z - \tilde{z}_1|$ is small enough that e_{ℓ_1} is defined on a neighborhood of $\tilde{z}_1$ containing $f^{n_1}z$.

1.6. More on the geometry of the critical set. — The following facts about the relative locations of critical points are used in sections 2 and 3 of this paper.

***Fact* 1**. — (Lemma 5, [**BY**]). *Let $\tilde{z} \in \mathcal{C}$ be contained in a $C^2(b)$ curve $\gamma \subset W$. Assume that γ extends to $> 2d$ on each side of $\tilde{z}$, and let $\zeta \in \gamma$ be s.t. $|\tilde{z} - \zeta| = d$. Then there are no critical points z with $|z - \zeta| < d^2$.*

***Fact* 2**. — (Existence of critical points, [**BC2**] Section 6.2, [**BY**] Subsection 1.3.1). *There is a number ρ, $b \ll \rho \ll 1$, s.t. the following holds: if $z = (x, y)$ lies in a $C^2(b)$ segment $\gamma \subset W$ of generation n with γ extending $> 2\rho^n$ to each side of z, and there is a critical point $\tilde{z} = (\tilde{x}, \tilde{y}) \in \mathcal{C}$ s.t.*

(i) $x = \tilde{x}$,
(ii) *$\tilde{z}$ is of generation $< n$,*
(iii) $|z - \tilde{z}| < b^{n/540}$,

then there is a unique critical point $\hat{z} = (\hat{x}, \hat{y}) \in \gamma$ with $|x - \hat{x}| < |y - \tilde{y}|^{\frac{1}{2}}$.

One way to get a sense of the relative location of a point to the critical set is to do the "capture" procedure introduced in [**BC2**], sections 6.4 and 7.2. This procedure guarantees that near every free $z \in W$ there are many long $C^2(b)$ segments of W some of which will contain critical points. The picture is as follows (for a precise statement see [**BY**] Subsection 2.2.2):

If $z \in W$ is free, then there is a family of $C^2(b)$ subsegments of W labeled $\{\gamma_i\}_{i=1,2,\ldots,i(z)}$, where $i(z)$ is the last integer i with $3^{i+1} < \text{gen}(z)$, s.t.

(i) $m \leq$ generation of $\gamma_i \leq 3m$, $m = 3^i$,
(ii) γ_i is centered at $\approx z$, and has length $\approx 10\rho^m$,
(iii) $\text{dist}(z, \gamma_i) < (Cb)^m$.
(iv) if $z^{(i)}$ is the point on γ_i with the same x-coordinate as z then $|\tau(z) - \tau(z^{(i)})| \leq (Cb)^{m/6}$.

There are, in fact, two such families, one above and one below z.

One may assume that γ_1 contains a critical point. If this critical point is sufficiently near the middle of γ_1, then by Fact 2, γ_2 would also contain a critical point. This may continue all the way down the stack, or there may exist an i s.t. $\tilde{z}_i \in \mathcal{C} \cap \gamma_i$ is so far to one side that no critical point lies in γ_{i+1}. If this happens z is in tangential position with respect to $\tilde{z}_i$.

1.7. SRB measures. — In this article (and also in [**BY**]), an f-invariant Borel probability measure ν is called an *SRB measure* if f has a positive Lyapunov exponent ν-a.e. and the conditional measures of ν on unstable manifolds are absolutely continuous with respect to the Riemannian measures on these leaves. The following are proved in [**BY**]:

(i) f admits an SRB measure ν;
(ii) ν is unique (i.e. f admits no other SRB measure); hence (f, ν) is ergodic;
(iii) (ii) is in fact true for f^n for all $n \geq 1$.

It follows from general nonuniform hyperbolic theory that (iii) is equivalent to (f, ν) having the mixing property, the K-property, and in fact to its being isomorphic to a Bernoulli shift (see e.g. [**L**]).

2. Preliminaries: cleaning up the notion of $\mathrm{dist}(\cdot, \mathcal{C})$ in [BC2] and [BY]

In this section as in the rest of the paper, it is assumed that $f = T_{a,b}$ where (a, b) are "good parameters" as discussed in Section 1.

Two notions of the distance to the critical set for a point z on a free segment of W have been used in [**BC2**] and [**BY**]. The first is a pointwise definition, in which we think of $\mathrm{dist}(z, \mathcal{C})$ as $|z - \tilde{z}(z)|$, where $\tilde{z}$ is a certain critical point captured by z. We will call this distance $d_{\mathrm{cap}}(z, \mathcal{C})$. The second notion is more globally defined. It is shown in [**BY**] that one could systematically assign a critical point $\tilde{z}(\tilde{\gamma})$ to every maximal free segment $\tilde{\gamma}$. Let us define $d_\gamma(z, \mathcal{C})$ to be $|\tilde{z}(\tilde{\gamma}) - z|$, where $\tilde{\gamma}$ is the maximal free segment containing γ. Precise definitions of $d_{\mathrm{cap}}(\cdot, \mathcal{C})$ and $d_\gamma(\cdot, \mathcal{C})$ are given below. The main purpose of this section is to prove

***Lemma* 1**. — *For each point belonging to a free segment of W, we have*

$$\frac{d_{cap}(z, \mathcal{C})}{d_\gamma(z, \mathcal{C})} = 1 + \mathcal{O}(\max(b, d^2)),$$

where $d = \min(d_{cap}(z, \mathcal{C}), d_\gamma(z, \mathcal{C}))$.

We state also a related fact which is, in some ways, more basic:

Lemma 1′. — *Suppose that z is a point that is in tangential position to two different critical points z_1 and z_2. Let $d_1 = |z - z_1|$, $d_2 = |z - z_2|$ and $d = \min(d_1, d_2)$. Then*

$$\frac{d_1}{d_2} = 1 + \mathcal{O}(\max(b, d^2)).$$

We now begin to justify these claims. The following technical sublemma along with Lemma 5 in [**BY**] (see Fact 1 Section 1.6) will be used repeatedly to rule out the presence of critical points in certain regions. Part (b) has independent interest; it plays an important role, for instance, in the proof of Lemma 1.

***Sublemma* 1**. — *Let γ_0 and γ be two free $C^2(b)$ segments of W. Suppose that γ_0 contains a critical point z_0, and that there exist two points $\zeta_0 \in \gamma_0$ and $\zeta \in \gamma$ with the same x-coordinate. Let $d_0 = |\zeta_0 - z_0|$.*

(a) *If $|\zeta - \zeta_0| < d_0^4$ and $|\tau(\zeta) - \tau(\zeta_0)| < d_0^2$, then for all $z \in \gamma$ with $|z - z_0| = d \geq d_0$, there can be no critical point at a distance $< d^2$ from z.*

(b) *The assumptions in part (a) are satisfied if ζ is (say) the left end point of γ, it is in a bound state, and its binding point z_0 lies to the left of γ.*

In the situation of part (b), Sublemma 1 allows us to essentially regard γ as a continuation of γ_0 (which may not be very long compared to γ).

Proof. — The proof of (a) is a slight modification of that of Lemma 5 in [**BY**] and will be omitted.

To prove (b) let us first briefly review the binding procedure. For a detailed account see [**BC2**], sections 6 and 7 and [**BY**], subsections 1.6.2 and 2.2.2. Let n be the generation of γ, and assume that attached to the left endpoint ζ of γ is a bound segment B. Recall that there is a hierarchy of bindings associated with B. We let $\tilde{z}_0$ be the critical point with the property that at this time, i.e. at time n, B is bound to $\tilde{z}_m = f^m \tilde{z}_0$ and $\tilde{z}_m$ is free. Let $\tilde{z}^*$ denote the new binding point acquired by $\tilde{z}_m$ at this time. Then $\tilde{z}^*$ is located on a segment of generation $m_1 < m$. The capture procedure resulting in $\tilde{z}^*$ calls for $\tilde{z}_{m-m_1}$ to be in a favorable position (in particular out of all fold periods); $\tilde{z}_{m-m_1}$ then draws in a segment γ' of W_1 and $\tilde{z}^*$ lies on $f^{m_1}\gamma'$.

We claim that $f^{-m_1}\zeta$ is outside of all fold periods. First, it cannot be in a fold period initiated in the time interval $[n-m, n]$, since bindings and the corresponding fold periods initiated in this time interval are the same as those of $\tilde{z}_0$. Suppose then $f^{-m_1}\zeta$ is in a fold period initiated before time $n-m$. The corresponding bound period in this case would have to last $> (C\log(1/b))m$ iterates beyond time $n - m_1$, contradicting our assumption that ζ is free.

Having established that $f^{-m_1}\zeta$ lies in a segment of W sufficiently parallell to W_1, the estimates in (a) follow immediately from capture arguments and the Matrix Perturbation Lemma in Section 1.5. ☐

Definition of $d_{cap}(z, \mathcal{C})$. Let $\{\gamma_i\}_{i=1,\ldots,k}$ be a stack of leaves captured by z. We let i_* be the largest integer i such that γ_i contains a critical point.

Case 1. $i_* < k$. Let $\tilde{z}(z)$ be the critical point on γ_{i_*}.

Case 2. $i_* = k$. We will show in this case that there is a critical point on $\tilde{\gamma}$, the maximal free segment containing γ. This critical point is unique (this follows e.g. from Lemma 5 in [**BY**]); it will be our $\tilde{z}(z)$. The existence of $\tilde{z}(z)$ follows readily from Fact 2, Section 1.6, once we verify that γ extends $\geq 2\rho^g$ on both sides of z and

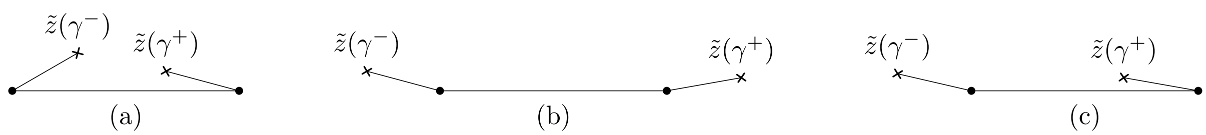

FIGURE 1

z_*, g being the generation of z and z_* the critical point on γ_{i_*}. We leave this as an exercise.

Definition of $d_\gamma(z, \mathcal{C})$. Let γ be a maximal free segment. In Lemma 7 of [**BY**], we established a rule for assigning a critical point $\tilde z(\gamma)$ to each γ. See Section 1.3 for what is proved. Given that the binding points of all critical orbits are selected and fixed, the only situation for which there might be some ambiguity in the choice of $\tilde z(\gamma)$ is when both end points of γ are in $(-\delta, \delta) \times \mathbb{R}$, i.e. case (iii) in Section 1.3. Figure 1 shows all possible configurations of the locations of the binding points $\tilde z_+$ (resp. $\tilde z_-$) relative to γ_+ (resp. γ_-).

In [**BY**] we ruled out (a); (b) will be eliminated in Sublemma 2 below. What is left is (c) (and its mirror image). If (c) occurs , $\tilde z(\gamma) = \tilde z_-$.

Proof of Lemma 1. — Let $z \in \gamma$, where γ is a maximal free segment of W. The idea of our proof is as follows. Look at the contractive fields centered at $\tilde z(\gamma)$ and $\tilde z(z)$. We will show that for a suitable choice of m, z lies in the domains of the e_m-fields induced by both points. Since the angle between $e_m(z)$ and $\tau(z)$ is supposed to reflect the distance between z and the respective critical points, we must have $|\tilde z(\gamma) - z| \approx |\tilde z(z) - z|$.

We consider the case where $\tilde z(\gamma) \notin \gamma$. (The proof is slightly simpler when γ contains a critical point.) Let $d_\gamma = |\tilde z - z|$, $d_c = |\tilde z(z) - z|$. First we observe the weaker estimate $d_\gamma^2 \le d_c \le d_\gamma^{\frac{1}{2}}$; to see that $d_c \ge d_\gamma^2$, use Sublemma 1 (both (a) and (b)); to see that $d_\gamma \ge d_c^2$, use Lemma 5 in [**BY**]. Let m_γ and m_c be defined by

$$\begin{cases} \left(\dfrac{\lambda}{5}\right)^{2m_c} \le d_c \le \left(\dfrac{\lambda}{5}\right)^{2m_c - 1}, \\ \left(\dfrac{\lambda}{5}\right)^{2m_\gamma} \le d_\gamma \le \left(\dfrac{\lambda}{5}\right)^{2m_\gamma - 1}, \end{cases}$$

and let $m = \min(m_\gamma, m_c)$. Then the above relation between d_γ and d_c implies that

$$\frac{1}{2} \leq \frac{m}{m_\gamma}, \frac{m}{m_c} \leq 1.$$

Thus z lies well inside the balls $B_{(\lambda/5)^m}(\tilde{z}(z))$ and $B_{(\lambda/5)^m}(\tilde{z}(\gamma))$, the domains of e_m around these points.

Let $\hat{z}$ be the point on $\tilde{\gamma}$, the $C^2(b)$-segment containing $\tilde{z}(z)$, having the same x-coordinate as z. Since $|\tau(\tilde{z}(z)) - e_m(\tilde{z}(z))| = \mathcal{O}(b^m)$ and $\tilde{\gamma}$ is $C^2(b)$, it follows that

$$|\tau(\hat{z}) - e_m(\hat{z})| = (2a + \mathcal{O}(b))d_c. \tag{2.1}$$

We would like to duplicate the estimate in the last paragraph with z playing the role of $\hat{z}$ and $\tilde{z}(\gamma)$ instead of $\tilde{z}(z)$, except that z does not lie on γ_0, the $C^2(b)$ segment containing $\tilde{z}(\gamma)$, and in any case we do not know how long γ_0 is. To get around this, note that

$$\sphericalangle(\tau(\tilde{z}(\gamma)), \tau(z)) \leq \sphericalangle(\tau(\tilde{z}(\gamma)), \tau(\zeta_0)) + \sphericalangle(\tau(\zeta_0), \tau(\zeta)) + \sphericalangle(\tau(\zeta), \tau(z)),$$

where ζ is the end point of γ closer to $\tilde{z}(\gamma)$ and ζ_0 is the point on γ_0 with the same x-coordinate as ζ. Part (b) of Sublemma 1 then gives

$$|\tau(z) - e_m(z)| = \left(2a + \max\left(\mathcal{O}(b), \mathcal{O}\left(|\zeta - \tilde{z}(\gamma)|^2\right)\right)\right) \cdot d_\gamma. \tag{2.2}$$

Finally, since $\tilde{\gamma}$ is obtained by capturing we have $|\tau(\hat{z}) - \tau(z)| \ll d_c^8$, say. Also, $|e_m(\hat{z}) - e_m(z)| \leq 10 d_c^4$ (apply Property (iii) of Section 1.5 twice). These together with (2.1) and (2.2) give the desired result.

□

We omit the proof of Lemma 1', which is very similar to that of Lemma 1.

***Remark* 1**. — This proof shows that the intuitive definition of the distance to the critical set for a point $z \in W$ really ought to be the angle between $\tau(z)$ and $e_m(z)$, where e_m is the contractive field of a "suitable" order.

In order to make the definition of $d_\gamma(z, \mathcal{C})$ unambiguous it remains to prove

***Sublemma* 2**. — *The configuration in Figure1 (b) does not occur.*

Proof of Sublemma 2.. — The proof is based on the same ideas as that of Lemma 1. Fix an arbitrary $z \in \gamma$. Then Sublemma 1 applied to $\tilde{z}_\pm$ tells us that

$$|z - \tilde{z}_+|^2 \leq |z - \tilde{z}_-| \leq |z - \tilde{z}_+|^{\frac{1}{2}}.$$

Let $d = \max(|z - \tilde{z}_+|, |z - \tilde{z}_-|)$ and m an integer defined by $(\lambda/5)^{2m} \approx d$ as in the proof of Lemma 1. Then z lies well inside the balls $B_{(\lambda/5)^m}(\tilde{z}_-)$ and $B_{(\lambda/5)^m}(\tilde{z}_-)$, the domains of e_m around these points, and we obtain a contradiction since the field arround $\tilde{z}_-$ says $e_m(z)$ must have positive slope and the one around $\tilde{z}_+$ says that the slope is negative. □

3. Construction of a "horseshoe" with positive measure

3.1. Goal of this section. — "Horseshoes" are well known to be building blocks of uniformly hyperbolic systems. We will show in this section that f can be viewed as the discrete time version of a special flow built over a "horseshoe". In order to have positive SRB measure, the "horseshoe" here must necessarily have infinitely many branches with unbounded return times. This picture will be made precise in the statement of Propostion 3.1.

We begin with some formal definitions.

***Definition* 1**. —

(a) Let Γ^u and Γ^s be two families of C^1 curves in $\mathbb{R}^2$ such that

 (i) the curves in Γ^u, respectively Γ^s, are pairwise disjoint;

 (ii) every $\gamma^u \in \Gamma^u$ meets every $\gamma^s \in \Gamma^s$ in exactly one point; and

 (iii) there is a minimum angle between γ^u and γ^s at the point of intersection.

Then the set

$$\Lambda : = \{\gamma^u \cap \gamma^s : \gamma^u \in \Gamma^u, \gamma^s \in \Gamma^s\}$$

is called the *lattice* defined by Γ^u and Γ^s.

(b) Let Λ and Λ' be lattices. We say that Λ' is a *u-sublattice* of Λ if Λ' and Λ have a common defining family Γ^s and the defining family Γ^u of Λ contains that of Λ'; *s-sublattices* are defined similarly.

(c) Given a lattice Λ, $Q \subset \mathbb{R}^2$ is called the *rectangle spanned by* Λ if $\Lambda \subset Q$ and ∂Q is made up of two curves from Γ^u and two from Γ^s.

In Propostion 3.1 we will assert the existence of two lattices Λ^+ and Λ^- with essentially identical properties. For notational simplicity let us agree to the following convention: statements about "Λ" will apply to both Λ^+ and Λ^-. For example, "let Γ^u and Γ^s be the defining families of Λ" means there are four families of curves; the families $(\Gamma^u)^+$ and $(\Gamma^s)^+$ define Λ^+ while $(\Gamma^u)^-$ and $(\Gamma^s)^-$ define Λ^-.

There are two lattices Λ^+ and Λ^- in $\mathbb{R}^2$ with the following properties. Let Γ^u and Γ^s be the defining families of Λ; for $z \in \Lambda$, let $\gamma^u(z)$ denote the γ^u-curve in Γ^u containing z. Then:

(1) *(Topological structure)* Λ is the disjoint union of s-sublattices $\Lambda_i, i = 1, 2, \dots,$ where for each i, $\exists R_i \in \mathbb{Z}^+$ s.t. $f^{R_i}\Lambda_i$ is a u-sublattice of Λ^+ or Λ^-.

(2) *(Hyperbolic estimates)*

(i) Every $\gamma^u \in \Gamma^u$ is a $C^2(b)$ curve; and $\exists \lambda_1 > 1$ s.t.

$$\left|Df_z^{R_i}\tau\right| \geq \lambda_1^{R_i}$$

$\forall z \in \gamma^u \cap Q_i$, τ being a unit tangent vector to γ^u at z and Q_i being the rectangle spanned by Λ_i.

(ii) $\forall z \in \Lambda$ and $\forall \zeta \in \gamma^s(z)$ we have

$$d\left(f^j z, f^j \zeta\right) < Cb^j \qquad \forall j \geq 1.$$

(3) $\text{Leb}(\Lambda \cap \gamma^u) > 0 \quad \forall \gamma^u \in \Gamma^u$.

(4) *(Return time estimates)* Let $R\colon \Lambda \to \mathbb{Z}^+$ be defined by $R(z) = R_i$ for $z \in \Lambda_i$. Then $\exists C_0 > 0$ and $\theta_0 < 1$ s.t. on every γ^u,

$$\text{Leb}\left\{z \in \gamma^u \colon R(z) \geq n\right\} \leq C_0 \theta_0^n \qquad \forall n \geq 1.$$

The rest of this section is devoted to proofs of Assertions (1) and (2) in Propostion 3.1; Assertions (3) and (4) are proved in Section 4. There are slight (and totally harmless) inaccuracies in the above formulation of Propostion 3.1. They are noted in Remarks 2 and 4 in Section 3.4.

3.2. Some preliminary constructions. — First we assume f is a 1-dimensional map and construct for f a Cantor set that would play the role of Λ in Propostion 3.1. Then we carry this construction over to W_1, the top leaf of W (see Section 1.2). We will address certain technical problems in 2-d that are not present in 1-d, and conclude that the two Cantor sets we have constructed have identical geometric estimates.

Temporarily then, we think of f as a map of $[-1, 1]$ given by $f(x) = 1 - ax^2$ for some $a \lesssim 2$, and let Ω_0 be one of the two outermost intervals in the partition $\mathcal{P}$ defined in Section 1.4. We define inductively $\Omega_0 \supset \Omega_1 \supset \Omega_2 \supset \ldots$ as follows. Let ω be a connected component of Ω_{n-1}. First we delete from ω the interval $f^{-n}\left(-\delta e^{-\alpha n}, \delta e^{-\alpha n}\right)$; and if the f^n-image of a component of what is left of ω does not contain some $I_{\nu j}$, then we delete that also. What remains goes into Ω_n. Our desired Cantor set is $\Omega_\infty \equiv \bigcap_n \Omega_n$.

We assume the following is true: if M_1 is the minimum time it takes for $x \in (-\delta, \delta)$ to return to $(-\delta, \delta)$, then $e^{\alpha M_1} \geq 10$. We assume also the corresponding fact for our 2-d map. This is easily arranged since α is fixed before we choose a or δ.

Returning to 2-d, we let Ω_0 be the corresponding segment in W_1 and try to construct Ω_n using the same rules and same notations as in 1-d. Let ω be a connected component of Ω_{n-1}. We assume for the moment the following geometric fact:

$(*)$ if part of $f^n\omega$ is bound and part is free, then the bound part lies at one or both ends of $f^n\omega$.

If all of $f^n\omega$ is in the bound state, or if $f^n\omega \cap (-\delta, \delta) \times \mathbb{R} = \emptyset$, do nothing; i.e. put $\omega \subset \Omega_n$. If not, let γ be the free part of $f^n\omega$, and let $\tilde{\gamma}$ be the maximal free segment containing γ. We will use as binding point $\tilde{z}(\tilde{\gamma})$, where $\tilde{z}(\cdot)$ is as defined in Section 2. Deletions are then made with respect to this binding point, and $\Omega_\infty = \bigcap \Omega_n$ as before.

To justify $(*)$, consider the function t defined on $f^n\omega$ where $t(z)$ is the time to expiration of all bound periods at z (counting only the ones initiated before this step). We take as our induction hypotheses not only $(*)$ but that $t|f^n\omega$ has the following profile: it is either decreasing (by which we mean non-increasing) or it decreases from one end to its minimum and increases from there to the other end. It is easy to check that this type of profile is maintained on each component of Ω_n even if new bindings are imposed.

We note also that the bound part at each end of $f^n\omega$ (if it exists) is small relative to $|I_{\nu j}|$, where $I_{\nu j}$ is the element of the partition determined by $\tilde{z}(\tilde{\gamma})$ that meets it. We know from 1-d or [**BC2**] that this is true wrt the partition determined by some binding point and Lemma 1 assures us that all binding points are essentially the same. This reasoning also gives us that no bound part is ever deleted.

3.3. Stable curves. — The purpose of this subsection is to construct Γ^s, which will consist of a family of local stable manifolds through $\Omega_0 \subset W_1$. We noted in Section 1.4 that $\exists c_2 > 0$ s.t. $|Df_z^n \tau| \geq \delta e^{c_2 n}$ $\forall z \in \Omega_n$. From Section 1.5 then it follows that e_n, the field of most contracted directions of Df^n, is defined in a neighborhood of every $z \in \Omega_n$. To construct Γ^s, however, we need to know that the domain of e_n is larger than this.

As noted in Section 1.5, e_1 is defined everywhere. We integrate e_1, and let $Q_0 = \bigcup_{z \in \widetilde{\Omega}_0} \gamma_1(z)$ where $\gamma_1(z)$ is the integral curve segment of length $10b$ centered at z, and $\widetilde{\Omega}_0$ is the (Cb)-neighborhood of Ω_0 in W_1. We will not need this for some time, but the γ_1-curve in Q_0 have slopes $\approx \pm 2a\delta$ depending on whether we are working with Ω_0^+ or Ω_0^-.

Suppose that at step n, corresponding to every connected component ω of Ω_{n-1} we have a strip Q_ω foliated by integral curves of e_n. More precisely, $Q_\omega = \bigcup_{z \in \tilde{\omega}} \gamma_n(z)$ where $\gamma_n(z)$ is the integral curve segment of length $10b$ centered at z and $\tilde{\omega}$ is the $(Cb)^n$-neighborhood of ω in W_1. We think of γ_n as temporary stable manifolds of order n.

Let $\omega' \subset \omega$ be a component of Ω_n. We want to show that $Q_{\omega'}$ is well defined and is contained in Q_ω. For $z \in \omega'$, let $U_n(z)$ be the $(Cb)^n$-neighborhood of $\gamma_n(z)$ in $\mathbb{R}^2$. First we claim that e_{n+1} is defined on all of $U_n(z)$. Since $|Df_z^j \tau| \geq \kappa^j$, $\forall j \leq n+1$, it suffices, by Section 1.5, to check that $d(f^j\zeta, f^jz) < \lambda^j$, $\forall j \leq n+1$, $\forall \zeta \in U_n(z)$. Let

ζ' be the point in $\gamma_n(z)$ nearest to ζ. Then

$$\begin{aligned} d(f^j\zeta, f^jz) &\le d(f^j\zeta, f^j\zeta') + d(f^j\zeta', f^jz) \\ &< (Cb)^n \cdot 5^j + Cb^j < \lambda^j. \end{aligned}$$

Next we claim that $\gamma_{n+1}(z)$ is well defined and lies inside $U_n(z)$. We see this in two steps: first we use the Lipschitzness of e_n (Property (iii), Section 1.5) and a Gronwall type inequality to see that $e_n|U_n(z)$ can be mapped diffeomorphically onto $\partial/\partial y$ on $\mathbb{R}^2$ via a diffeomorphism Φ_n with $||D\Phi_n|| \le e^{5\cdot 10b}$; then use $|e_{n+1}-e_n| < (Cb)^n$ and the "straightened out" coordinates of e_n to conclude that the Hausdorff distance between $\gamma_{n+1}(z)$ and $\gamma_n(z)$ is $\lesssim 10b \cdot (Cb)^n < \frac{1}{2}(Cb)^n$. Finally, observe that these arguments are easily extended to γ_{n+1}-curves through points in $(Cb)^{n+1}$-neighborhoods of ω', proving $Q_{\omega'} \subset Q_\omega$.

Taking limits of these "temporary stable manifolds", we obtain genuine stable manifolds for points in Ω_∞.

***Lemma* 2**. —

(1) $\forall z \in \Omega_\infty$, *there is a* C^1 *curve* $\gamma_\infty(z)$ *of length* $10b$ *and centered at* z *s.t.* $\forall \zeta \in \gamma_\infty(z)$,

$$d(f^j\zeta, f^jz) < Cb^j \ \forall j \ge 1;$$

(2) $\forall z, z' \in \Omega_\infty$, $zn^o = z' \Longrightarrow \gamma_\infty(z) \cap \gamma_\infty(z') = \emptyset$;

(3) *if* $f^n\gamma_\infty(z) \cap \gamma_\infty(z')n^o = \emptyset$, *then* $f^n\gamma_\infty(z) \subset \gamma_\infty(z')$.

Proof of (1). — Let z_n be the right end point of ω_{n-1}, the component of Ω_{n-1} containing z. Since $\bigcap_n \omega_n = \{z\}$ (reason: $|Df^n\tau| \ge \delta e^{c_2 n}$ on ω_{n-1} and $f^n\omega_{n-1}$ has length < 2), it follows from the estimates above that as $n \to \infty$, $\gamma_n(z_n)$ converges uniformly to a curve which we will call $\gamma_\infty(z)$. Because the e_n's have a uniform Lipschitz constant (Section 1.5, property (iii)), $\gamma_n(z_n)$ in fact converges in the C^1 sense to $\gamma_\infty(z)$. Thus the contractive estimates for $f^j \mid \gamma_n(z_n)$ carry over to $f^j \mid \gamma_\infty(z)$. $\square$

Before proving (2) and (3) we need to do some preparatory work. Consider a $C^2(b)$ curve γ lying in Q_0 and joining $\partial^s Q_0$, the two boundary components of Q_0 that are not part of W. For each connected component ω of Ω_{n-1}, we let γ_ω denote $\gamma \cap Q_\omega$. Note that every point in γ_ω is connected to some point in $\tilde\omega$ by an integral curve γ_n, and also that if $\omega' \subset \Omega_n$ is contained in ω, then $\gamma_{\omega'} \subset \gamma_\omega$. For $z \in \gamma$ we will use $\tau(f^jz)$ to denote the unit tangent vector to $f^j\gamma$ at f^jz.

***Sublemma* 3**. — *Let* γ *be as above. Then if* $\omega \subset \Omega_{n-1}$ *is s.t. part of* $f^n\omega$ *is free and intersects* $(-\delta, \delta) \times \mathbb{R}$, *then the binding point* $\hat z$ *for* $f^n\omega$ *selected earlier is also suitable for* $f^n\gamma_{\omega'}$ *where* $\omega' = \omega \cap \Omega_n$ *(see Section 1.3 and Section 1.5 for the meaning of "suitable"). It follows from this that for all* $\omega \in \Omega_{n-1}$, $f^j \mid \gamma_\omega$, $j \le n$, *has the bound/free estimates expressed in Section 1.3 and the distortion estimate in Section 1.4.*

Proof of Sublemma 3. — We fix $\zeta \in \gamma_\omega$ and investigate the suitability of $\hat{z}$ as a binding point for $(f^n\zeta, Df_\zeta^n\tau(\zeta))$. Let $z \in \tilde{\omega}$ be s.t. $\zeta \in \gamma_n(z)$. Then $d(f^n z, f^n\zeta) < Cb^n$. This cannot jeopardize the generalized tangential position part of the requirement since Cb^n is totally insignificant compared to $d(f^n z, \hat{z})$, which is $> e^{-\alpha n}$. As to the angle part of the requirement, write

$$\sphericalangle(Df_z^n\tau(z), Df_\zeta^n\tau(\zeta)) \leq \sphericalangle(Df_z^n\tau(z), Df_z^n\tau(\zeta)) + \sphericalangle(Df_z^n\tau(\zeta), Df_\zeta^n\tau(\zeta)).$$

The first term is $< 20b{\cdot}Cb^n$ because both $\tau(z)$ and $\tau(\zeta)$ have slopes $< 10b$, $\|Df_z^n\| > 1$, and both $\tau(z)$ and $\tau(\zeta)$ make angles $\approx 2a\delta$ with $e_n(z)$. The second term is $< Cb^{\frac{n}{4}}$ by the matrix perturbation lemma in Section 1.5. The difference between $\tau(f^n z)$ and $\tau(f^n\zeta)$, therefore, are insignificant relative to $(2a \pm 1) \cdot d(f^n z, \hat{z})$, the size of the angle they are supposed to make with the relevant contracting field about $\hat{z}$.

The estimates in Section 1.3 depend on the pair (ζ, τ), $\zeta \in \gamma_\omega$ being "controlled". (For the precise definition see 1.4.2 and 1.5.1 of [**BY**].) The distortion estimate in Section 1.4 holds for $C^2(b)$ segments all of whose points and tangent vectors are controlled. $\square$

Proof of Lemma 2 (continued). — We prove (3); the proof of (2) is similar. We will try to derive a contradiction assuming $f^n\gamma_\infty(z) \not\subset \gamma_\infty(z')$. Let N be a sufficiently large number to be specified. Let η and η' be points in $f^n\gamma_\infty(z)$ and $\gamma_\infty(z')$ respectively s.t.

(i) η and η' are joined by a horizontal line segment $\gamma \subset Q_0$ and

(ii) $\eta \in \partial Q_\omega$ where ω is the component of Ω_{N-1} containing z'. Since η and η' lie in some γ_ω, Sublemma 3 tells us that if $f^N\omega$ is free (our 1st requirement on N), then

$$\left|f^N\eta - f^N\eta'\right| \geq e^{cN}\left|\eta - \eta'\right| \gtrsim e^{cN} \cdot \frac{1}{2}(Cb)^N.$$

On the other hand, if $q \in f^n\gamma_\infty(z) \cap \gamma_\infty(z')$, then

$$\begin{aligned}\left|f^N\eta - f^N\eta'\right| &\leq \left|f^N\eta - f^N q\right| + \left|f^N q - f^N\eta'\right| \\ &< \frac{Cb^{N+n}}{\left(\frac{b}{5}^n\right)} + Cb^N = (5^n + 1)Cb^N.\end{aligned}$$

These two estimates of $\left|f^N\eta - f^N\eta'\right|$ are clearly incompatible for $N \gg n$. $\square$

3.4. Definition of Λ and return times. — We now specify the two families Γ^u and Γ^s that define Λ.

Definition of Γ^s: we let $\Gamma^s = \{\gamma_\infty(z)\colon z \in \Omega_\infty\}$ where $\gamma_\infty(\cdot)$ is as in Lemma 2.

Definition of Γ^u: we let $\widetilde{\Gamma}^u = \{\gamma \subset W\colon \gamma$ is a $C^2(b)$ segment connecting the two components of $\partial^s Q_0\}$, and let $\Gamma^u = \{\gamma\colon \gamma$ is the pointwise limit of a sequence in $\widetilde{\Gamma}^u\}$.

***Remark* 2**. — (1) We have not proved that the curves in Γ^u are pairwise disjoint. However, since every $\gamma \in \Gamma^u$ is the monotone limit of curves in $\widetilde{\Gamma}^u$, there are at most countably many pairs that intersect. It is easy to see that they play no role.

(2) Without further analysis, we also cannot conclude that $\gamma \in \Gamma^u$ is better than $C^{1+1}(b)$, since they are uniform limits of $C^2(b)$ curves. This also is inconsequential.

Having completed the definition of Λ, we now proceed to define the s-sublattices that make up Λ and their return times R. Let us remind ourselves again that in actuality we are interested in the set $\Lambda^+ \cup \Lambda^-$, where $\Lambda^\pm$ correspond to the lattices we have constructed near $\Omega_0^\pm \times [-b, b]$, Ω_0^+ and Ω_0^- being the two outermost intervals in the partition $\mathcal{P}$ introduced in Section 1.4. When we speak about return times, we are referring to return times from the set $\Lambda^+ \cup \Lambda^-$ to itself, i.e. a point in Λ^+ may return to Λ^+ or Λ^-. To keep the notations simple we will continue to write just "Λ".

We stipulate ahead of time that $\forall z \in \Lambda$, $R(z) = R(z')$ $\forall z' \in \gamma^s(z)$, so R need only be defined on $\Lambda \cap \Omega_0$. We will construct partitions on subsets of Ω_0 and use 1-dimensional language. For example, $f^n x = y$ for $x, y \in \Lambda \cap \Omega_0$ means that $f^n x \in \gamma^s(y)$. Similarly, for subsegments $\omega, \omega' \subset \Omega_0$, $f^n\omega = \omega'$ means that $f^n\omega \cap \Lambda$, when slid along γ^s-curves to Ω_0, gives exactly $\omega' \cap \Lambda$. (We caution that "$f^n\omega = \omega'$" does not imply $f^n(\omega \cap \Lambda) = \omega' \cap \Lambda$!) For $\omega \subset \Omega_{n-1}$, $\mathcal{P} \mid f^n\omega$ refers to $\mathcal{P}_{[\tilde z]}$ where $\tilde z$ is the binding point for $f^n\omega$ selected earlier.

We will construct below sets $\widetilde{\Omega}_n \subset \Omega_n$ and partitions $\widetilde{\mathcal{P}}_n$ on $\widetilde{\Omega}_n$ so that $\widetilde{\Omega}_0 \supset \widetilde{\Omega}_1 \supset \widetilde{\Omega}_2 \supset \dots$ and $z \in \widetilde{\Omega}_{n-1} - \widetilde{\Omega}_n$ iff $R(z) = n$. As usual, we think of points belonging to the same element of $\widetilde{\mathcal{P}}_n$ as having indistinguishable trajectories up to time n. We augment $\mathcal{P}$ defined in Section 1.4 to $\mathcal{P} = \{\omega \in$ original $\mathcal{P}\} \cup \{[-1, -\delta), (\delta, 1]\}$, and let $\widehat{\mathcal{P}}$ be the partition on $\Omega_0 - \Omega_\infty$ dividing this set into connected components. The symbol "$\vee$" refers to the join of two partitions, i.e. $\mathcal{A} \vee \mathcal{B} \equiv \{A \cap B\colon A \in \mathcal{A}, B \in \mathcal{B}\}$.

An interval $\omega \subset \Omega_n$ is said to make a *regular return* to Ω_0 at time n if

(i) all of $f^n\omega$ is free;
(ii) $f^n\omega \supset 3\Omega_0$.

Rules for defining $\widetilde{\Omega}_n, \widetilde{\mathcal{P}}_n$ *and* R:

(0) $\widetilde{\Omega}_0 = \Omega_0$, $\widetilde{\mathcal{P}}_0 = \{\widetilde{\Omega}_0\}$.

Consider $\omega \in \widetilde{\mathcal{P}}_{n-1}$.

(1) If ω does not make a regular return to Ω_0 at time n, put $\omega \cap \Omega_n$ into $\widetilde{\Omega}_n$, and let
$$\widetilde{\mathcal{P}}_n \left|(\omega \cap \Omega_n) = (f^{-n}\mathcal{P})\right| (\omega \cap \Omega_n)$$
with the usual adjoining of end intervals (this is always done with or without our saying so explicitly).

(2) If ω makes a regular return at time n, we put $\omega' \equiv (\omega - f^{-n}\Omega_\infty) \cap \Omega_n$ in $\widetilde{\Omega}_n$, and let $\widetilde{\mathcal{P}}_n \left|\tilde{\omega} = (f^{-n}\mathcal{P} \vee f^{-n}\widehat{\mathcal{P}})\right| \omega'$. For $z \in \omega$ s.t. $f^n z \in \Omega_\infty$, we define $R(z) = n$.

(3) We require that $R \geq n_0$ for some n_0 to be specified in Section 6.1. To comply with this, if $\omega \in \widetilde{\mathcal{P}}_{n-1}$ makes a regular return at time n with $n < n_0$, then we treat ω according to Rule (1) and not Rule (2).

(4) For $z \in \bigcap_n \widetilde{\Omega}_n$, set $R(z) = \infty$.

***Remark* 3**. — We digress to make the following adjustments in our definitions of $\tilde{\mathcal{P}}_n$; they will simplify the proofs in Section 4. Let us say that a $C^2(b)$ segment $\gamma \subset (-\delta, \delta)$ is of "full length" if $\exists \nu, j$ s.t. $\gamma \cap I_{\nu j} n^{\circ} = \emptyset$ and $\ell(\gamma) \approx \ell(I_{\nu j})$. Recall that in Section 3.2 we made sure that when something is deleted from $f^n\omega$, $\omega \in \Omega_{n-1}$, no "short" segment is left behind. We wish to do the same for $f^n\omega$ for every $\omega \in \tilde{\mathcal{P}}_{n-1}$. For definiteness suppose that $\omega \in \tilde{\mathcal{P}}_{n-1}$ was created at step $k \leq n-1$, and that not all of ω will remain in Ω_n. We distinguish between the cases where $f^k\omega \approx$ some $I_{\nu j}$ and where $f^k\omega$ is a gap of Λ.

If $f^k\omega$ is a gap of Λ, then $\text{dist}(f^n(\partial\omega), \mathcal{C}) \geq \delta e^{-\alpha(n-k)} \geq 10\delta e^{-\alpha n}$, so the end points of $f^n\omega$ straddle the forbidden interval $(-\delta e^{-\alpha n}, \delta e^{-\alpha n})$ by wide margins and no problem will arise.

Suppose $f^k\omega \approx$ some $I_{\nu j}$. Since deletions occur only at free returns, we have $\ell(f^n\omega) \gg \delta e^{-\alpha n}$. The only problematic scenario is when a tiny part of $f^n\omega$ sticks out, say, to the left of $(-\delta e^{-\alpha n}, \delta e^{-\alpha n})$. If this awkward bit remains in Ω_n, then there must be something in Ω_{n-1} that is mapped by f^n to the left of it. The reasoning of the last paragraph rules out the possibility that the left end point of $f^n\omega$ is a limit of infinitely many small segments coming from the gaps of Λ. Thus $\exists\, \omega' \in \tilde{\mathcal{P}}_{n-1}$ that shares this relevant end point with ω. Hence $\exists \omega'' \in \tilde{\mathcal{P}}_k$ with this property and $f^k\omega'' \approx$ some $I_{\nu' j'}$. We now retroactively move the boundary between ω and ω'' so that the awkward bit in question belongs to the image of ω' or ω''. It is easy to see that no boundary is moved more than once, for a gap beween the adjacent elements appear immediately thereafter.

It is now clear what the sublattices in Propostion 3.1 (1) are: each Λ_i is an s-sublattice corresponding to a subset of $\Omega_0 \cap \Lambda$ of the form $f^{-n}\Omega_\infty \cap \Lambda \cap \omega$, $\omega \in \widetilde{\mathcal{P}}_{n-1}$

making a regular return at time n. Note that if Λ_i is one of the s-sublattices, and $\gamma = Q_i \cap \gamma^u$, $\gamma^u \in \Gamma^u$, $Q_i =$ the rectangle spanned by Λ_i, then $f^n\gamma \in \Gamma^u$. To see this, first assume $\gamma \subset W$. Then $f^n\gamma$ is $C^2(b)$ because it is free; hence it is in $\widetilde{\Gamma}^u$. This property clearly passes on to curves in Γ^u, proving $f^n\Lambda_i \subset \Lambda$. Note also that the hyperbolic estimates in Propostion 3.1 are simply Estimate III in Section 1.3 and Lemma 2 (1).

To complete our objective of proving Assertions (1) and (2) in Propostion 3.1 then, it remains only to show that $f^{R_i}\Lambda_i$ is a u-sublattice. This requires proving that the Cantor set $f^{R_i}\Lambda_i$ somehow matches completely with Λ in the horizontal direction. We claim that this is a consequence of our construction but defer the proof to Section 3.5.

***Remark* 4**. — We have not proved that $R(z) < \infty$ for every $z \in \Lambda$. Indeed, the assertion in Propostion 3.1 (1) that $\Lambda = \bigcup \Lambda_i$ is inaccurate and should be ammended to read "for every $\gamma^u \in \Gamma^u, \mathrm{Leb}\,((\Lambda - \bigcup \Lambda_i) \cap \gamma^u) = 0$". That $R < \infty$ a.e. on $\Lambda \cap \gamma^u$ will follow from the Main Lemma in Section 4.

3.5. Matching of Cantor sets. — To complete the proof of Propostion 3.1 (1), we need to show that whenever Rule (2) in the previous subsection is applied,

$$f^n(\omega \cap \Omega_\infty) \supset \Omega_\infty.$$

We formulate this as

***Lemma* 3**. — Let $\omega \in \Omega_{n-1}$ be s.t. $f^n\omega$ crosses Q_0 completely. Then $\forall z \in \Lambda$, $\exists z' \in \omega \cap \Lambda$ s.t. $f^n z' \in \gamma^s(z)$.

Let us first explain the central idea of the proof assuming that f is a 1-dimensional map. Given $z \in \Omega_\infty$, there is (by hypothesis) $z' \in \omega$ with $f^n z' = z$; what is at issue is whether $z' \in \Omega_\infty$. First, $z' \in \Omega_n$ because $\omega \subset \Omega_{n-1}$ and $f^n z' \in \Omega_0$. It suffices therefore to show that $|f^{j+n}z'| > 2\delta e^{-(j+n)\alpha}$ $\forall j > 0$. This is true because $|f^{j+n}z'| = |f^j z| > \delta e^{-\alpha j}$, which is $> 10\delta e^{-\alpha(j+n)}$ since $e^{\alpha n} \geq 10$.

For the 2-dimensional situation at hand, what complicates matters is that different layers of W require different binding points, and that the binding point at step $n+j$ for $f^{n+j}z'$ may not be vertically aligned with the binding point at step j for $f^j z$. Our aim in this subsection is to dispel with these technicalities so that the 1-d argument prevails.

Lemma 3′. — *Let ω be a connected component of Ω_{n-1}, and suppose that f^nQ_ω crosses Q_0 completely in the horizontal direction. Then $\forall j \geq 1$, if ω_j is a component of Ω_j, then there is a component ω_{n+j} of Ω_{n+j} s.t. $Q_{\omega_j} \cap f^nQ_\omega \subset f^nQ_{\omega_{n+j}}$.*

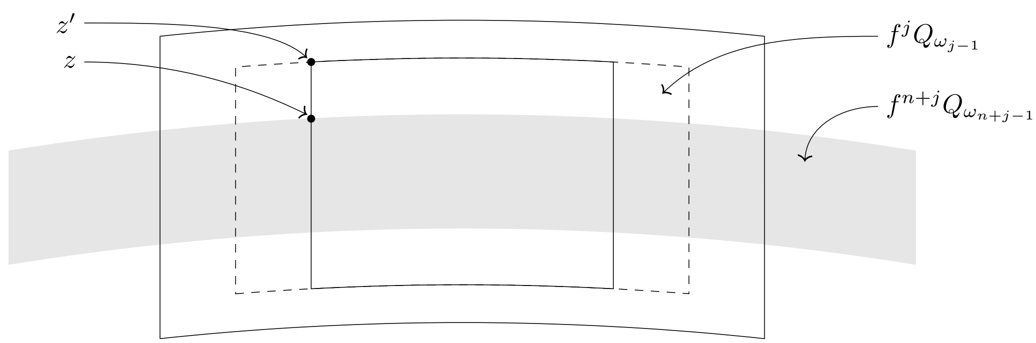

FIGURE 2

In this subsection we will regard Q_ω as foliated by temporary stable curves through ω, ignoring the slight discrepancies between the temporary curves of various generations or their slightly different domains of definition. Those matters were dealt with in Section 3.3. We remark that if Lemma 3′ holds, then it will follow that $(\bigcup_{\omega_j} Q_{\omega_j}) \cap f^nQ_\omega \subset \bigcup_{\omega_{n+j}} f^nQ_{\omega_{n+j}}$ for all $j \geq 1$. Taking the limit as $j \to \infty$, we will obtain $(\bigcup_{z\in\Omega_\infty} \gamma^s(z)) \cap f^nQ_\omega \subset \bigcup_{z\in\Omega_\infty} f^n\gamma^s(z)$, which gives Lemma 3.

Proof of Lemma 3′. — Let n and ω be fixed, and assume the conclusion of Lemma 3′ for all components of Ω_{j-1}. We pick one ω_{j-1}, and let ω_{n+j-1} be as in the lemma, see Figure 2. We will examine what is deleted from $f^jQ_{\omega_{j-1}}$ at step j versus what is deleted from $f^{n+j}Q_{\omega_{n+j-1}}$ at step $n+j$.

Let $z \in f^{n+j}\omega_{n+j-1} \cap f^jQ_{\omega_{j-1}}$ be such that z is deleted at step $n+j$, and let $z' \in \gamma^s_j(z) \cap f^j\omega_{j-1}$. We will show that z' is deleted at step j. The notations and results of Section 2 will be used heavily in the next few lines. First if $d_\gamma(z', \mathcal{C}) < \delta e^{-\alpha j}$, we are done. Suppose not. Then since $|z-z'| < (Cb)^j$,

$$|z - \tilde{z}(z')| \geq |z' - \tilde{z}(z')| - |z-z'| \geq \tfrac{9}{10}\delta e^{-\alpha j},$$

and z is in tangential position wrt $\tilde{z}(z')$. But we also have

$$|z - \tilde{z}(z)| \leq 2\delta e^{-(n+j)\alpha} < \tfrac{2}{10}\delta e^{-\alpha j},$$

and this is incompatible with our estimate on $|z - \tilde{z}(z')|$. $\square$

4. Return time estimates

The goal of this section is to prove Assertions (3) and (4) in Propostion 3.1.

4.1. Statement of lemmas and ideas of proofs. — Let $\Omega_0 \subset W_1$ be as in Section 3, and recall that there are sets $\Omega_0 = \widetilde{\Omega}_0 \supset \widetilde{\Omega}_1 \supset \widetilde{\Omega}_2 \supset \ldots$ and partitions $\widetilde{\mathcal{P}}_n$ on $\widetilde{\Omega}_n$ so that for all $z \in \Omega_\infty$, $z \in \widetilde{\Omega}_n$ iff $R(z) > n$, and points in the same element of $\widetilde{\mathcal{P}}_n$ are viewed as having the same itinerary up to time n. (See Section 3.2 and Section 3.4.) Let $|\cdot|$ denote the Lebesgue measure on γ^u-curves.

***Lemma* 4**. — $|\Omega_\infty| > 0$.

Proof. — This is a 1-d argument using estimates in [**BC1**]. In the construction of $\{\Omega_n\}$, let $\omega \subset \Omega_k$ be a component that is formed at step k, and suppose that some part of it will be deleted at step n. Since $f^k\omega \supset$ some $I_{\nu j}$, we are guaranteed that $|f^n\omega| \geq \delta^{3\beta}e^{-3\alpha\beta k} \geq \delta^{3\beta}e^{-3\alpha\beta n}$. But the subsegment of $f^n\omega$ to be deleted has length $\leq 4\delta e^{-\alpha n}$. Taking distorsion into consideration when pulling back to Ω_0, we have that

$$\frac{|\Omega_{n-1} - \Omega_n|}{|\Omega_{n-1}|} \leq C_1\delta^{(1-3\beta)}e^{-\alpha(1-3\beta)n}$$

and the statement of the lemma follows since

$$\prod_{n=M_1}^{\infty}\left(1 - C_1\delta^{(1-3\beta)}e^{-\alpha(1-3\beta)n}\right) > 0,$$

where M_1 is the minimum time for a point in $(-\delta, \delta)$ to return to $(-\delta, \delta)$; see Section 3.2. □

***Lemma* 5**. — (Main Lemma) $\exists C_0 > 0$ *and* $\theta_0 < 1$ *s.t.*

$$\left|\widetilde{\Omega}_n\right| < C_0\theta_0^n \qquad \forall n \geq 1.$$

***Remark* 5**. — We have stated lemmas 4 and 5 for Ω_n on W_1, but the corresponding statements are true for every $\gamma^u \in \Gamma^u$ with uniform estimates (independent of γ^u). The proofs are in fact identical through the use of Lemma 2. A related fact that will not be needed till later is in fact the absolute continuity of $\{\gamma^s\}$ as a "foliation"; see Sublemma 10 in Section 5. It says in particular that $\exists\ C > 0$ s.t. for all γ, $\gamma' \in \Gamma^u$, if $\Psi : \gamma \cap \Lambda \mapsto \gamma'$ is defined by $\Psi(z) = \gamma(z) \cap \gamma'$, then $|\Psi(A)| \leq C|A|$ for every Borel subset A of $\Lambda \cap \gamma$.

We postpone the proof of Lemma 5 for later, but use instead the remainder of this subsection to discuss the main ideas behind this tail estimate for R. Consider a segment $\omega \subset \Omega_k$.

(1) It is easy to see that once $f^n\omega$ becomes sufficiently long, then it will make a regular return to Ω_0 within a finite number of iterates. Our situation is as follows:

as we iterate f, ω grows in length — except when it comes near $\mathcal{C}$, at which time it may lose a piece in the middle and it may get subdivided into $I_{\nu j}$'s for distortion control; these components are then iterated individually. An unfortunate component of ω may get cut faster than it has the chance to grow, but our contention is that because of the estimates in Section 1.3 the *general tendency* is for a component to grow long.

(2) When a regular return occurs, small pieces corresponding to the gaps of Λ are created, and these small pieces are handled individually as they move on. We must therefore carry out the large deviation estimate in (1) simultaneously for the entire collection of gaps; such an estimate will involve the distribution of gap sizes.

(3) As has already been suggested in (2), it is not quite the end of the story when a component of ω grows long, for at regular returns only a (fixed) percentage of the long segment gets absorbed into the Cantor set Λ. To estimate distribution of return times we must estimate the frequencies with which the components containing typical points makes regular returns.

These ideas are made rigorous in Sections 4.2, 4.3 and 4.4. We remark also that (1) is essentially dealt with in [**BC2**] in the slightly different context of parameter exclusions, and that we learned some of the estimates for (2) and (3) from [**C**].

4.2. Growth of components of a segment to a fixed size: a large deviation estimate. — In this and the next subsections we will be studying the time evolution of a curve γ which is contained in $f^j\widetilde{\Omega}_j$ for some $j \geq 0$. It is convenient to think of points as being in γ at time 0, so let us introduce the following notations: $\Omega_k^{(j)} \equiv \{z \in \gamma\colon f^{-j}z \in \Omega_{j+k}\}$, similarly for $\widetilde{\Omega}_k^{(j)}$, and $\widetilde{\mathcal{P}}_k^{(j)}(z) \equiv \{\eta \in \gamma\colon f^{-j}\eta \in \widetilde{\mathcal{P}}_{j+k}(f^{-j}z)\}$.

We will also use the following language. For $z \in \gamma$, we say that z makes an *essential free return* ($\equiv$ e.f.r.) to $(-\delta, \delta) \times \mathbb{R}$ at time k if $z \in \widetilde{\Omega}_{k-1}^{(j)}$ and $f^k\widetilde{\mathcal{P}}_{k-1}^{(j)}(z)$ is free and contains some $I_{\nu j}$. We say z makes a *regular return* to Ω_0 at time k if $z \in \widetilde{\Omega}_{k-1}^{(j)}$ and $f^k\widetilde{\mathcal{P}}_{k-1}^{(j)}(z)$ makes a regular return. We define the stopping time

$$\begin{aligned} E(z) \equiv &\text{the smallest } k \in \mathbb{Z}^+ \text{ s.t. either } z \notin \Omega_k^{(j)} \text{ or } z \text{ makes} \\ &\text{a regular return at time } k \end{aligned}$$

and let

$$\gamma_n = \{z \in \gamma : E(z) > n\}.$$

***Sublemma* 4.** — (c.f. Section 2, [**BC2**]) *There exist $D_1' > 0$ and $\theta_1' < 1$ for which the following holds. Let $\gamma = f^j\omega$ for some $\omega \in \widetilde{\mathcal{P}}_j$, and let $n \in \mathbb{Z}^+$. We assume that*

γ is free and is $\approx$ some $I_{r,\ell}$ with $|r| \le \frac{n}{6}$. Then

$$|\gamma_n| \le D_1' \theta_1'^{\,n} |\gamma|.$$

Proof. — We will prove that

$$(*) \qquad |\gamma_n| \le D_1' e^{-\frac{1}{6}n + \frac{9}{10}|r|} |\gamma|$$

for some D_1' independent of γ or n. This implies the Sublemma immediately: for $|r| \le \frac{n}{6}$, $D_1' e^{-\frac{1}{6}n+\frac{9}{10}|r|} \le D_1' e^{-\frac{1}{6}n+\frac{9}{10}\frac{n}{6}} = D_1' e^{-\frac{1}{60}n}$, so it suffices to take $\theta_1' = e^{-\frac{1}{60}}$.

Consider $z \in \gamma$ with $E(z) > n$, and suppose that z makes exactly s e.f.r.'s in the first n iterates, at times $0 = t_0 < t_1 < \cdots < t_s \le n$. It follows from Remark 2 in Section 3.4 that $f^{t_i} \widetilde{\mathcal{P}}^{(j)}_{t_i}(z) \approx$ some $I_{r_i \ell_i}$ for each i. A slightly extended version of our estimates in Section 1.3 gives for all $i < s$:

$$t_{i+1} - t_i \le 4\,|r_i|$$

and

$$\left| f^{t_{i+1}-t_i} I_{r_i \ell_i} \right| \ge e^{-3\beta |r_i|}.$$

This second inequality can be used to estimate the fraction

$$\varphi(r_1, \ldots, r_s) \equiv \frac{1}{|\gamma|} \cdot |\{z \in \gamma \colon \mathcal{F}(z) = (r_1, \ldots, r_s)\}|$$

where $\mathcal{F}(z)$ denotes the r_i-locations of the e.f.r.'s of z. Letting C_1 be the distortion constant in Section 1.4 and writing $r_0 = r$, we have

$$\begin{aligned} \varphi(r_1, \ldots, r_s) &< C_1^s \prod_{i=1}^{s} \exp\{ -|r_i| + 3\beta\,|r_{i-1}|\,\} \\ &< C_1^s \exp\Big\{ -\tfrac{7}{8} \sum_{i=1}^{s} |r_i| + 3\beta |r| \Big\}. \end{aligned}$$

Next we make, for fixed s and R, the purely combinatorial estimate on the number of all possible s-tuples $(r_1, \ldots, r_s)$, $r_i \in \mathbb{Z}$, with $\sum_{i=1}^{s} |r_i| = R$. This is clearly $< 2^s \binom{R+s-1}{s-1}$. For us, since the time between consecutive e.f.r.'s is $> \Delta \equiv \log \frac{1}{\delta}$, the number of feasible $(r_1, \ldots, r_s)$ as locations of e.f.r.'s is in fact $\le 2^{\frac{R}{\Delta}} \cdot \binom{R+\frac{R}{\Delta}-1}{\frac{R}{\Delta}}$, which by Sterling's formula is $< 2^{\frac{R}{\Delta}} (1 + \sigma(\delta))^R$ with $\sigma(\delta) \to 0$ as $\delta \to 0$.

Let

$$A^n_{s,R} \equiv \big\{ z \in \gamma_n \colon z \text{ makes exactly } s \text{ e.f.r.'s up to time } n \text{ and } \sum_{i=1}^{s} |r_i| = R \big\}.$$

We may then estimate $|\gamma_n|$ by

$$|\gamma_n| = \sum_{\substack{\text{all relevant}\\ s,R}} \big|A^n_{s,R}\big| \le \sum_{R=\frac{n}{4}-|r|}^{\infty} \sum_{s=1}^{\frac{R}{\Delta}} \begin{pmatrix} \#(r_1,\ldots,r_s) \\ \text{with } \sum_1^s |r_i| = R \end{pmatrix} \cdot \varphi(r_1,\ldots,r_s)\cdot|\gamma|.$$

The lower limit of summation for R comes from the fact that $|r|+4R$ must be $> n$, otherwise the $(s+1)^{st}$ e.f.r. or a regular return, whichever happens first, would have taken place by time n. We do not need to concern ourselves with $s=0$ because γ must make an e.f.r. by time $n/2$ (because $e^{-\frac{n}{6}} e^{c_1 \frac{n}{2}} \gg 1$). Note that this is an overestimate also in the sense that some of the $(r_1,\ldots,r_s)$-configurations are forbidden due to the $I_{r_i\ell_i}$'s being too close to $\mathcal{C}$.

Plugging our earlier estimates into this last inequality, we obtain

$$|\gamma_n| \le C \sum_{R=\frac{n}{4}-|r|}^{\infty} 2^{\frac{R}{\Delta}} (1+\sigma(\delta))^R \cdot C_1^{\frac{R}{\Delta}} e^{-\frac{7}{8}R + 3\beta|r|}\cdot|\gamma|,$$

which is less than the right side of $(*)$ provided δ is sufficiently small. $\square$

We now re-state Sublemma 4 in anticipation of how it will be used.

Corollary to Sublemma 4. — *There exist $D_1>0$ and $\theta_1<1$ for which the following holds. Let γ be contained in a free $(C^2(b))$ segment of W. We assume that either*

(i) $\gamma = f^j\omega$ *for some* $\omega \in \widetilde{\mathcal{P}}_j$, γ *need not be of "full length"; or*

(ii) $\gamma = \bigcup_i f^j\omega_i$ *where for each* i, $\omega_i \in \widetilde{\mathcal{P}}_j$ *and* $f^j\omega_i \approx$ *some* $I_{r_i\ell_i}$.

Then

$$|\gamma_n| \le D_1\theta_1^n \qquad \forall n \ge 1.$$

Proof. — First we prove (ii). For fixed n, we have by Sublemma 4 that

$$\big|\{z \in \gamma - \big(-e^{-\frac{n}{6}}, e^{-\frac{n}{6}}\big) : E(z) > n\}\big| \le D_1'\theta_1'^n|\gamma| \le D_1'\theta_1'^n,$$

and also that

$$\big|\gamma \cap \big(-e^{-\frac{n}{6}}, e^{-\frac{n}{6}}\big)\big| \le 2e^{-\frac{n}{6}}.$$

To prove (i), fix n and observe as above that we may assume $|\gamma| \geq e^{-n/6}$, and that γ makes an e.f.r. at time $j_0 < \frac{n}{2}$. Suppose that this is not a regular return, and let $\gamma' = f^{j_0}\gamma$. Then $|\gamma_n| \leq |f^{j_0}\gamma_n| \leq |\{z \in \gamma' : E(z) > n/2\}|$, and this last quantity is estimated as in the proof of (ii). $\square$

4.3. Growth of "gaps" to a fixed size. — First we prove a sublemma about the distribution of gap sizes.

***Sublemma* 5**. — *There exist $C > 0$ and $\sigma > 0$ s.t. for all $\gamma^u \in \Gamma^u$, if $\mathcal{G} = \{$components of $\gamma^u - \Lambda\}$, then*

$$\sum_{\gamma \in \mathcal{G}\colon |\gamma| \leq \ell} |\gamma| \leq C\ell^\sigma.$$

Proof. — In view of Sublemma 3, it suffices to consider $\gamma^u = \Omega_0$.

Observe first that all the gaps of Ω_∞ created at step n have length

$$> {C_1}^{-1} 4\delta e^{-\alpha n} e^{-c_1 n}.$$

(If $f^n\omega$ partially crosses $(-\delta e^{-\alpha n}, \delta e^{-\alpha n})$, then the part deleted is attached to an earlier gap, making it even bigger.) Given ℓ, let N_0 be s.t. $\ell \approx e^{-(\alpha + c_1)N_0}$. Then

$$\sum_{\substack{\gamma \in \mathcal{G} \\ |\gamma| \leq \ell}} |\gamma| \leq \sum_{n \geq N_0} \sum_{\substack{\omega = \text{comp.} \\ \text{of } \Omega_{n-1}}} C_1 4 e^{-(1-3\beta)\alpha n} |\omega|$$

(cf. Lemma 4). Thus

$$\sum_{\substack{\gamma \in \mathcal{G} \\ |\gamma| \leq \ell}} |\gamma| \leq C e^{-\alpha N_0 (1-3\beta)} \leq C\ell^\sigma,$$

some $\sigma > 0$. $\square$

***Sublemma* 6**. — *Let $\Lambda^c \equiv \Omega_0 - \Lambda$, and for $z \in \Lambda^c$, define $E(z)$ as in Section 4.2 with $\gamma =$ the component of Λ^c containing z. Let $\Lambda^c_n \equiv \{z \in \Lambda^c\colon E(z) > n\}$. Then $\exists\, D_2 > 0$ and $\theta_2 < 1$ s.t.*

$$|\Lambda^c_n| \leq D_2 \theta_2^n \qquad \forall n \geq 1.$$

Proof. — Let $\mathcal{G}$ be the set of all components of Λ^c. For each $n \in \mathbb{Z}^+$, let $\mathcal{G}'_n \equiv \{\omega \in \mathcal{G}\colon |\omega| \leq D_1\theta_1^n\}$ where D_1 and θ_1 are as in the Corollary to Sublemma 4, and let $\mathcal{G}''_n = \mathcal{G} - \mathcal{G}'_n$.

By Sublemma 5,

$$\sum_{\omega \in \mathcal{G}'_n} |\omega| \leq C(D_1\theta_1^n)^\sigma \equiv D'_2 \theta_1^{\sigma n}.$$

For $\omega \in \mathcal{G}''_n$, we know from the Corollary to Sublemma 4 that $\omega_n := \{z \in \omega : E(z) > n\}$ has length $|\omega_n| \le D_1\theta_1^n$, and from Sublemma 5 that if $N_k = \#\{\omega\colon D_1\theta_1^k \le |\omega| < D_1\theta_1^{k-1}\}$, then

$$N_k \le \frac{C(D_1\theta_1^{k-1})^\sigma}{D_1\theta_1^k} \equiv C'\theta_1^{(k-1)\sigma-k}.$$

Thus

$$\sum_{\omega\in\mathcal{G}''_n} |\omega_n| \le D_1\theta_1^n \cdot \sum_{k\le n} N_k \le D''_2\theta_1^{\sigma n}.$$

□

4.4. Frequencies of regular returns and Proof of Lemma 5. — We define a sequence of stopping times $T_0 < T_1 < \cdots$ on subsets of Ω as follows. Let $T_0 \equiv 0$, and assuming that $T_{k-1}(z)$ is defined, let $T_k(z)$ be the smallest $j > T_{k-1}$ s.t. $\tilde{\mathcal{P}}_{j-1}(z)$ makes a regular return to Ω_0 at time j. Let $\Theta_k = \{z \in \Omega_0 : T_k(z) \text{ is defined}\}$. It follows from the Corollary to Sublemma 4 that $\Theta_k \supset \Omega_\infty$ a.e. for each k. Observe that Θ_k is the disjoint union of a countable number of segments $\{\omega\}$ with the property that each ω is an element of some $\tilde{\mathcal{P}}_{j-1}$, $T_k|\omega \equiv j$, and a certain proportion of ω is absorbed back into Λ at time j. This is to say, $\exists\varepsilon_0 > 0$ such that for all $\omega \subset \Theta_k$ as above,

$$\frac{|\omega \cap \{R = T_k\}|}{|\omega|} \ge \varepsilon_0.$$

This implies inductively that for every k,

$$|\{z \in \Theta_k : R(z) > T_k\}| \le (1-\varepsilon_0)^k.$$

Let $\varepsilon_1 > 0$ be a small number to be determined. Then for all n,

$$\tilde{\Omega}_n \subset \{z \in \tilde{\Omega}_n : T_{[\varepsilon_1 n]}(z) > n\} \cup \{z \in \Theta_{[\varepsilon_1 n]} : R(z) > T_{[\varepsilon_1 n]}(z)\}.$$

The measure of the second set on the right has already been estimated. It remains therefore to prove

***Sublemma* 7.** — *$\exists D_3 > 0$, $\theta_3 < 1$, and $\varepsilon_1 > 0$ such that*

$$|\{z \in \tilde{\Omega}_n : T_{[\varepsilon_1 n]}(z) > n\}| < D_3\theta_3^n \qquad \forall n \ge 1.$$

Proof. — Let $1 \le n_1 < n_2 < \cdots < n_\ell \le n$ be fixed for the time being. For $k \le n$, we define $A_k \equiv A_k(n_1, \ldots, n_\ell)$ to be

$$A_k \equiv \{z \in \widetilde{\Omega}_k\colon \text{the regular return times of } z \text{ up to time } k \text{ are}$$
$$\text{exactly those } n_i\text{'s with } n_i \le k\},$$

and we estimate $|A_n|$ following these steps:

(i) $|A_{n_1-1}| \le D_1 \theta_1^{n_1-1}$ by Sublemma 4 applied to $\gamma = \Omega_0$.

(ii) Note that A_{n_1-1} is a union of elements of $\widetilde{\mathcal{P}}_{n_1-1}$, and that A_{n_1-1} could be seen as

$$A_{n_1} = \{\omega = \omega' \cup \omega'' \colon \omega \in \widetilde{\mathcal{P}}_{n_1-1} \mid A_{n_1-1},\ \omega \text{ making a regular return at time } n_1\},$$

where $\omega' = \omega \cap f^{-n_1}\Lambda^c$ and $\omega'' = (\omega - f^{-n_1}\Omega_0) \cap \Omega_{n_1}$.

(iii) Using Sublemma 4 to deal with ω' and Sublemma 6 to deal with ω'' we obtain

$$\frac{|A_{n_2-1}|}{|A_{n_1-1}|} \le \frac{D_3' \theta_3'^{\,n_2-n_1-1}}{|\Omega_0|}$$

for some D_3' and θ_3' independent of the n_i's.

(iv) Proceeding inductively, we obtain

$$\begin{aligned} |A_n| &= \frac{|A_n|}{|A_{n_\ell-1}|} \cdot \frac{|A_{n_\ell-1}|}{|A_{n_{\ell-1}-1}|} \cdots \frac{|A_{n_2-1}|}{|A_{n_1-1}|} \cdot |A_{n_1-1}| \\ &\le \left(\frac{D_3'}{|\Omega_0|\theta_3'}\right)^{\ell} \theta_3'^{\,n}. \end{aligned}$$

We may now choose $\varepsilon_1 > 0$ small enough that

$$\left(\frac{D_3'}{|\Omega_0|\theta_3'}\right)^{\varepsilon_1} \cdot \theta_3' \equiv \theta_3'' < 1,$$

and conclude that

$$\begin{aligned} \left|\{z \in \widetilde{\Omega}_n \colon T_{[\varepsilon_1 n]} > n\}\right| &= \sum_{\ell=0}^{\varepsilon_1 n} \sum_{\substack{(n_1,\dots,n_\ell): \\ 1 \le n_1 < \cdots < n_\ell \le n}} |A_n(n_1, \dots, n_\ell)| \\ &< \sum_{\ell=0}^{\varepsilon_1 n} \binom{n}{\ell} (\theta_3'')^n \\ &< D_3 \theta_3^n \end{aligned}$$

provided ε_1 is sufficiently small. □

5. Reduction to expanding maps

5.1. Purpose of this section. — From Assertion (1) in Propostion 3.1, we know that $f^R \colon \Lambda \circlearrowleft$ sends γ^s-fibers to γ^s-fibers, so that *topologically* a quotient map is well defined. More precisely, let $\bar\Lambda = \Lambda/\approx$ where $\approx$ is the equivalence relation defined by $z \approx z'$ iff $z' \in \gamma^s(z)$. Then $\overline{f^R} \colon \bar\Lambda \circlearrowleft$ makes sense, and with $\bar\Lambda_i$ having the obvious meaning, $\overline{f^R}$ maps each one of the Cantor sets $\bar\Lambda_i$ homeomorphically onto $\bar\Lambda$.

The aim of this section is to study the *differential* properties of $\overline{f^R} \colon \bar\Lambda \circlearrowleft$ in the sense of the Jacobian of $\overline{f^R}$ with respect to a certain reference measure. Let $T \colon (X_1, m_1) \to (X_2, m_2)$ be a measureable bijection between two measure spaces. We say that T is *nonsingular* if T maps sets of m_1-measure 0 to sets of m_2-measure 0. For a nonsingular transformation T, we define the Jacobian of T with respect to m_1 and m_2, written $J_{m_1,m_2}(T)$ or simply JT, to be the Radon-Nikodym derivative $d(m_2 \circ T)/dm_1$.

There is a measureable family of reference measures $\{m_\gamma, \gamma \in \Gamma^u\}$ with the following properties:

(1) Each m_γ is supported on $\gamma \cap \Lambda$; it is a finite measure equivalent to the restriction of 1-dimensional Lebesgue measure on γ to $\gamma \cap \Lambda$.
(2) m_γ is invariant under sliding along γ^s, i.e. if $\theta \colon \gamma \cap \Lambda \to \gamma' \cap \Lambda$ is defined by $\{\theta(z)\} = \gamma' \cap \gamma^s(z)$, then for $E \subset \gamma \cap \Lambda$, $m_{\gamma'}(\theta E) = m_\gamma(E)$.
(3) For $z \in \gamma \cap \Lambda_i$, let $Jf^R(z)$ denote the Jacobian of $f^R \mid (\gamma \cap \Lambda_i)$ at z with respect to our reference measures on the respective γ^u-curves (we know that $f^R \mid (\gamma \cap \Lambda_i)$ is nonsingular on account of (1)). Then
$$Jf^R(z) = Jf^R(z')$$
for all $z' \in \gamma^s(z)$.
(4) $\exists\, \lambda > 1$ s.t. $Jf^R(z) \geq \lambda^R$ a.e.
(5) Restricted to each $\gamma \cap \Lambda_i$, $(\log Jf^R) \circ (f^R)^{-1}$ is "Hölder" in the sense to be made precise in Section 5.4, with uniform estimates independent of γ or i.

Property (2) above tells us that $\{m_\gamma\}$ defines a reference measure $\bar m$ on our quotient space $\bar\Lambda$. Property (3) says that $\overline{f^R} \colon \bar\Lambda \circlearrowleft$ is nonsingular w.r.t. $\bar m$; we will call its Jacobian $J\overline{f^R}$. Properties (4) and (5) allow us to view $\overline{f^R} \colon (\bar\Lambda, \bar m) \circlearrowleft$ as a piecewise uniformly expanding map whose derivative has a certain "Hölder" property. We use "" for "Hölder" because it is not the usual Hölder condition; the relevant condition here is dynamically defined and will be explained in Section 5.4.

Our proof of Proposition 5.1 is essentially an adaption of some ideas used in the construction of Gibbs states. See e.g. [**B**] for an exposition.

5.2. The reference measures. — In this subsection we define $\{m_\gamma, \gamma \in \Gamma^u\}$ and prove Properties (1) and (2) in Proposition 5.1. For simplicity of notation we will write m instead of m_γ when there is no ambiguity about γ. The Jacobian w.r.t. m will be denoted $J(\cdot)$, while the one w.r.t. 1-dimensional Lebesgue measure on γ^u-curves will be denoted $(\cdot)'$, i.e. $f'(z) = |Df_z\tau(z)|$ for $z \in \gamma^u$.

We pick and fix an arbitrary γ^u-curve in the definition of Λ and call it $\hat\gamma$. For $z \in \Lambda$, let $\hat z$ denote the point in $\hat\gamma \cap \gamma^s(z)$, and let $\varphi(z) = \log f'(z)$. We define for $n = 1, 2, \ldots$

$$u_n(z) \equiv \sum_{i=0}^{n-1} \left(\varphi(f^i z) - \varphi(f^i \hat z)\right).$$

***Sublemma* 8.** — *$\exists C' > 0$ and b' with $b < b' \ll 1$ s.t. $\forall n > k \geq 0$,*

$$\sum_{i=k}^{n} \left(\varphi(f^i z) - \varphi(f^i \hat z)\right) \leq C'(b')^k$$

Proof. — First we write

$$\varphi(f^i z) - \varphi(f^i \hat z) = \log \frac{f'(f'z)}{f'(f^i \hat z)} \leq \frac{|f'(f^i z) - f'(f^i \hat z)|}{f'(f^i \hat z)}.$$

Then letting $\tau_i = \tau(f^i z)$ and $\hat\tau_i = \tau(f^i \hat z)$, we have

$$\left|f'(f^i z) - f'(f^i \hat z)\right| \leq \left|Df_{f^i z}\tau_i - Df_{f^i \hat z}\tau_i\right| + \left|Df_{f^i \hat z}\tau_i - Df_{f^i \hat z}\hat\tau_i\right|.$$

The first term above is clearly $\leq C'b^i$ since $d(f^i z, f^i \hat z) \leq Cb^i$ (Lemma 2 (1)). The second term is $\leq 5|\tau_i - \hat\tau_i|$, which we estimate by

$$\begin{aligned} \sphericalangle(\tau_i, \hat\tau_i) &\leq \sphericalangle(Df^i_z\tau_0, Df^i_{\hat z}\tau_0) + \sphericalangle(Df^i_{\hat z}\tau_0, Df^i_{\hat z}\hat\tau_0) \\ &\leq Cb^{\frac{i}{4}} + Cb^i, \end{aligned} \tag{5.1}$$

the first because of the Matrix Pertubation Lemma in Section 1.5 and the second because $Df^i_{\hat z}$ is hyperbolic and τ_0 and $\hat\tau_0$ are bounded away from their most contracted direction.

To complete the proof, observe that $f'(f^j \hat z) \geq \frac{b}{5}\ \forall j$, and that $f'(f^j \hat z) \geq \delta$ for the first few j's. The desired conclusion follows easily with, say, $b' > b^{\frac{1}{8}}$. ∎

It follows from Sublemma 8 that $u = \lim_n u_n$ exists for all $z \in \Lambda$. We know in fact that $|u|$ can be made arbitrarily small for b small. On each γ, let m be the measure whose density w.r.t. Lebesgue measure on γ is $\chi_{\Lambda\cap\gamma} \cdot e^u$, $\chi_{(\cdot)}$ being the characteristic function. Property (1) of Proposition 5.1 is immediate. Next follows a lemma, which gives a Lipschitz estimate for the tangential derivative at free return times: $(f^n)'z$. This estimate is used in several places: in the proofs of the Hölder regularity of the Jacobians (Section 5.4), and the invariance of the reference measures m_γ (Assertion (2) of Proposition 5.1).

For $z_1, z_2 \in \gamma$, let $[z_1, z_2]$ denote the segment of γ between z_1 and z_2.

***Sublemma* 9**. — *$\exists C_2'$ depending on δ s.t. the following holds for each γ^u and every $n \geq 0$. Let $\omega \subset \gamma^u$ be a segment in Ω_n, and suppose that*

(i) *for each $i \in n$, $f^i\omega \subset 3I_{\mu j}$ for some μ, j;* *and*

(ii) *$f^n\omega$ is free.*

Then $\forall z_1, z_2 \in \omega$ we have

(1) $\log \dfrac{(f^n)'z_1}{(f^n)'z_2} \leq C_2' e^{\alpha n} |f^n[z_1, z_2]|$;

(2) $\log \dfrac{(f^n)'z_1}{(f^n)'z_2} \leq C_2' |f^n[z_1, z_2]|$ *if $f^n\omega \supset \Omega_0$.*

Proof. — We follow the proof of Proposition 2 in [**BY**] p. 562–564, but make an improvement in the estimates. As in this proof we obtain

$$T \overset{def}{=} \log \frac{(f^n)'z_1}{(f^n)'z_2} \leq C \sum_{k=0}^{q} \frac{|f^{t_k}[z_1, z_2]|}{e^{-\nu_k}}$$

where $\{t_k\}_{k=0}^q$ are the free return times, $t_q = n$, and $f^{t_k}\omega \overset{\subset}{\approx} I_{\nu_k}$. We then define

$$m(\nu) = \max\{t_k \colon \nu_k = \nu\},$$

and using the fact that $|f^{t_{k+1}}[z_1, z_2]| \geq 2|f^{t_k}[z_1, z_2]|$ we have

$$T \leq C' \sum_{\nu \in S} \frac{|f^{m(\nu)}[z_1, z_2]|}{e^{-\nu}}$$

where S is the set of ν_k's not counted with multiplicity. Since $f^i\omega$ lies $> \delta e^{-\alpha i}$ from the critical set for each i, and the $m(\nu)$'s are distinct for different ν's, we obtain

$$\sum_{\nu \in S} \frac{|f^{m(\nu)}[z_1, z_2]|}{e^{-\nu}} \leq e^{\alpha n} \left(\sum_{k=0}^{q} 2^{-k} \right) |f^n[z_1, z_2]|$$

proving (1). To prove (2) we have as in the proof of Proposition 2 in [**BY**]

$$T \leq C' \sum_{\nu \in S} \frac{|f^{m(\nu)}[z_1, z_2]|}{e^{-\nu}} \leq C' \sum_{\nu} \frac{1}{\nu^2} = C_1,$$

where C_1 is the usual distortion constant (see Section 1.4). Now for each ν apply this to points in $f^m(\nu)\omega$ for the time interval $[m(\nu), n]$ to obtain

$$\frac{|f^{m(\nu)}[z_1,z_2]|}{|f^{m(\nu)}\omega|} \le C_1 \frac{|f^n[z_1,z_2]|}{|f^n\omega|},$$

and conclude that

$$\begin{aligned} T &\le C' \sum_{\nu\in S} \frac{|f^{m(\nu)}\omega|}{e^{-\nu}} \cdot \frac{|f^{m(\nu)}[z_1,z_2]|}{|f^{m(\nu)}\omega|} \\ &\le C' \cdot \left(\sum_{\nu\in S} \frac{|f^{m(\nu)}\omega|}{e^{-\nu}} \right) \cdot C_1 \frac{|f^n[z_1,z_2]|}{|f^n\omega|} \\ &\le \frac{C_1^2}{|\Omega_0|} |f^n[z_1,z_2]|. \quad \square \end{aligned}$$

$\square$

Let γ and γ' be arbitrary curves in Γ^n, and let $\theta : \gamma \cap \Lambda \to \gamma'$ be defined by $\theta(z) \in \gamma^s(z) \cap \gamma'$. Property (2) of Proposition 5.1 follows from the following sublemma:

***Sublemma* 10**. — *Temporarily let μ_γ and $\mu_{\gamma'}$ denote the Lebesgue measures on γ and γ' respectively. Then $\theta_*^{-1}\mu_{\gamma'}$ is absolutely continuous wrt μ_γ, written $\theta_*^{-1}\mu_{\gamma'} \prec\prec \mu_\gamma$, and*

$$\frac{d\theta_*^{-1}\mu_{\gamma'}}{d\mu_\gamma}(z) = e^{u(z)-u(\theta z)} \qquad for \qquad \mu_\gamma \ a.e.\, z \in \gamma \cap \Lambda.$$

Absolutely continuity arguments are well known in dynamical systems (see e.g. [**PS**]), but since our setting is a little nonstandard let us include a proof. Observe that the m-measures are designed precisely so that θ takes m-measures to m-measures.

Proof. — Let $\omega \subset \widehat{\omega}$ be subsegments of γ with the property that $\widehat{\omega}$ makes a regular return to Ω_0 at time $\hat{k}$, ω is free at time $k > \hat{k}$, and all points in ω have the same itinerary up to time k in the usual sense. We require that $0 \ll \hat{k} \ll k$ and that $|f^{\hat{k}}\omega| \gg (Cb)^{\frac{\hat{k}}{4}}$. (The second condition requires that k not be too much larger than $\hat{k}$; it is not a serious imposition.) Let $\omega' \subset \widehat{\omega}'$ be the corresponding subsegments of γ'. In what follows "$a \approx b$" means that a/b is very near 1 and tends to 1 as all the "$\gg$" tend to ∞.

Let z be an arbitrary point in $\omega \cap \Lambda$, and let $z' = \theta z$. We claim that

$$|\omega'| \approx \frac{|f^{\hat{k}}\omega'|}{(f^{\hat{k}})'z'} = \frac{|f^{\hat{k}}\omega'|}{|f^{\hat{k}}\omega|} \cdot \frac{|f^{\hat{k}}\omega|}{(f^{\hat{k}})'z} \cdot \frac{(f^{\hat{k}})'z}{(f^{\hat{k}})'z'} \approx |\omega| \cdot e^{u(z)-u(z')}.$$

For the first "$\approx$" use the fact that $(f^{\hat{k}})'$ is roughly identical at all points in ω'. This is true by Sublemma 9 provided that $k - \hat{k}$ is sufficiently large. The same argument is used for ω in the second "$\approx$". Additionally we need the fact that $|f^{\hat{k}}\omega'| \approx |f^{\hat{k}}\omega|$, which is true because the two curves are so short they can be regarded as straight lines, and their lengths are $\gg (Cb)^{\frac{\hat{k}}{4}}$ while their slopes are $< (Cb)^{\frac{\hat{k}}{4}}$ apart (by the Matrix Perturbation Lemma in Section 1.5) In the third "$\approx$" we use Sublemma 8 and the fact that $\hat{k}$ is large.

Let $\widetilde{\Lambda} = \{z \in \Lambda : z \ \text{ makes infinitely returns to}\, \Lambda\}$. We leave it as an excercise for the reader to verify that there is a cover $\mathcal{U}$ of $\widetilde{\Lambda}$ by pairwise disjoint sets of the type ω above with M arbitrarily large. To prove $\theta_*^{-1}\mu_{\gamma'} \prec\prec \mu_\gamma$, let A be a closed subset of $\gamma \cap \widetilde{\Lambda}$. Choosing a subcover $\{w_i\}$ of $\mathcal{U}$ s.t. $\mu_\gamma(\bigcup \omega_i) < \mu_\gamma(A) + \varepsilon$, we have that $\mu_{\gamma'}(\theta A) \lesssim e^{2\max|u|} \sum_i \mu_{\gamma'}(\omega_i) < e^{2\max|u|}(\mu_\gamma(A) + \varepsilon)$. To prove the statement on Radon-Nikodym derivatives, consider a Lebesgue density point z of $\gamma \cap \widetilde{\Lambda}$ and choose ω containing z with $|\omega \cap \Lambda| \approx |\omega|$, $|\omega' \cap \Lambda| \approx |\omega'|$. $\square$

5.3. The Jacobians. — For a.e. $z \in \Lambda \cap \gamma$, $\gamma \in \Gamma^u$, we have

$$J(f^R)(z) = (f^R)'z \cdot \frac{e^{u(f^R z)}}{e^{u(z)}}.$$

Proof of Property (3) in Proposition 5.1. We will verify that $J(f^R)(z)$ depends only on $\hat{z}$ and not on z:

$$\begin{aligned} \log J(f^R)z = & \sum_{i=0}^{R-1} \varphi(f^i z) \ + \sum_{i=0}^{\infty} \left(\varphi\left(f^i\left(f^R z\right) - \varphi\left(f^i\left(\widehat{f^R z}\right)\right)\right)\right) \\ & \qquad - \sum_{i=0}^{\infty} \left(\varphi\left(f^i z\right) - \varphi\left(f^i \hat{z}\right)\right) \\ = & \sum_{i=0}^{R-1} \varphi\left(f^i \hat{z}\right) + \sum_{i=0}^{\infty} \left(\varphi\left(f^i\left(f^R \hat{z}\right)\right) - \varphi\left(f^i\left(\widehat{f^R z}\right)\right)\right). \quad \square \end{aligned}$$

Proof of Property (4) in Proposition 5.1. — As observed earlier on, $|u|$ can be made arbitrarily small; it is in fact of order b. Our Jacobian $J(f^R)z$ is therefore a small pertubation of $(f^R)', z$, which is $\geq e^{c_1 R}$ for some $c_1 > \frac{1}{3}\log 2$ (see Section 1.3). $\square$

5.4. Regularity of the Jacobian. — Having established that $\bar{m}$ and $J(\overline{f^R})$ make sense on $\bar{\Lambda}$, we now introduce a dynamically defined notion of "Hölderness" satisfied by $J(\overline{f^R}) \circ (\overline{f^R})^{-1}$.

For $z_1, z_2 \in \bar{\Lambda}$, define their *separation time* $s(z_1, z_2)$ to be the smallest n s.t. $f^n z_1$, $f^n z_2$ do not lie in three continuous $I_{\nu j}$'s. Here we have taken the liberty to confuse $z \in \bar{\Lambda}$ with the representative on some $\gamma \in \Gamma^u$, and to include $[-1, -\delta)$ and $(\delta, 1]$ when we speak about $I_{\nu j}$'s.

***Definition* 2.** — A function $\psi\colon \bar{\Lambda} \to \mathbb{R}$ is said to be *Hölder with respect to the separation time* $s(\,\cdot\,,\,\cdot\,)$ if $\exists\, C > 0$ and $\beta < 1$ s.t. for m-a.e. $z_1, z_2 \in \bar{\Lambda}$, $|\psi z_1 - \psi z_2| \leq C\beta^{s(z_1,z_2)}$.

The following lemma gives the precise statement of Property (5) in Proposition 5.1.

***Lemma* 6.** — *$\exists C_2 > 0$ and $\beta < 1$ s.t. for every i and $\forall z_1, z_2 \in \bar{\Lambda}_i$,*

$$\left| \frac{J(\overline{f^R})(z_1)}{J(\overline{f^R})(z_2)} - 1 \right| \leq C_2 \beta^{s(\overline{f^R} z_1, \overline{f^R} z_2)}.$$

This can be rephrased as follows. For each i let $(\overline{f^R})_i^{-1}\colon \bar{\Lambda} \to \bar{\Lambda}_i$ be the inverse of $\overline{f^R} \mid \bar{\Lambda}_i$. Then $z \to J(\overline{f^R}) \circ (\overline{f^R})_i^{-1}(z)$ is Hölder w.r.t. $s(\cdot\,,\,\cdot)$ above with uniform C_2 and β independent of i.

***Remark* 6.** — Some explanations are probably in order here.

(1) *Why would the regularity of the Jacobian involve separation times?* If we were working with a 1-d map $f\colon [-1, 1] \circlearrowleft$, then $x \mapsto \log f'(x)$ is Hölder in the usual sense — provided that near the critical point 0, we compare two points only if they are much closer to each other than to 0, e.g. if they lie in 3 contiguous $I_{\nu j}$'s. In particular, two points on opposite sides of 0 cannot be compared. In the present situation $\bar{\Lambda}$ is obtained by collapsing W^s_{loc}-curves, so that points in $\bar{\Lambda}$ represent not points in $\mathbb{R}^2$ but *futures* of orbits, and $J(\overline{f^R})(z)$ has incorporated into it information on the entire orbit of the point $z \in \Lambda$. Now two points $z_1, z_2 \in \gamma^u$ could be arbitrarily near each other, and be mapped at some future time to opposite sides of the critical set. The sooner this takes place, the less one could expect $J(\overline{f^R})(z_1)$ and $J(\overline{f^R})(z_2)$ to be comparable. Hence separation time enters.

(2) *Why $C\beta^{s(\overline{f^R} z_1, \overline{f^R} z_2)}$?* Built into this formulation is the assumption that the map f is expanding on average. Consider for simplicity a C^2 uniformly expanding map g with $g' \approx \lambda > 1$. Fix $\delta > 0$ small enough that $d(gx, gy) \approx \lambda\, d(x, y)$ whenever $d(gx, gy) < \delta$. Consider x, y and n s.t. $d(g^i x, g^i y) < \delta\ \forall i \leq n$. The following estimate

is standard:

$$\begin{aligned}|\log(g^n)'x - \log(g^n)'y| &\le \sum_{i=0}^{n-1} \left|\log g'(g^i x) - \log g'(g^i y)\right| \\ &\le C\sum_0^{n-1} d(g^i x, g^i y) \le C' d(g^n x, g^n y).\end{aligned}$$

Now if $s(x', y')$ is the first time $d(g^s x', g^s y') > \delta$, then $d(g^n x, g^n y) \approx C\lambda^{-s(g^n x, g^n y)}$, so that

$$|\log(g^n)'x - \log(g^n)'y| \le C'(\lambda^{-1})^{s(g^n x, g^n y)}.$$

Here β plays the rôle of λ^{-1}, and separation may occur long before two orbits move $> \delta$ apart.

We now proceed with the proof of Lemma 6.

***Sublemma* 11**. — *$\exists C_2'' > 0$ and $\beta < 1$ s.t. $\forall z_1, z_2 \in \gamma^u \cap \Lambda$, $|u(z_1) - u(z_2)| \le C_2'' \beta^{s(z_1,z_2)}$.*

Proof. Let $n = s(z_1, z_2)$, and pick $k \in [\frac{n}{3}, \frac{n}{2}]$ s.t. $f^k[z_1, z_2]$ is free. We know that k exists because if $f^{\frac{n}{3}}[z_1, z_2]$ is in bound state, then it was $> e^{-\frac{n}{3}\alpha}$ from the critical set when the last (total) bound period was initiated, which means that this bound period must expire before time $\frac{1}{3}n + 4\alpha\frac{1}{3}n < \frac{1}{2}n$ (see Section 1.2). Write

$$u(z_1) - u(z_2) = \sum_{i=0}^{\infty} \left\{ \left(\varphi(f^i z_1) - \varphi(f^i \hat{z}_1)\right) - \left(\varphi(f^i z_2) - \varphi(f^i \hat{z}_2)\right) \right\}.$$

The part $\sum_{i=0}^{k-1}\{\cdot\}$ is estimated by

$$\Big|\sum_{i=0}^{k-1}\{\cdot\}\Big| \le \Big|\log\frac{(f^k)'z_1}{(f^k)'z_2}\Big| + \Big|\log\frac{(f^k)'\hat{z}_1}{(f^k)'\hat{z}_2}\Big|.$$

The first term on the right is $\le C_2' e^{\alpha k} |f^k[z_1, z_2]|$ by Sublemma 9 (1). Observing that separation can occur only when $f^n[z_1, z_2]$ is free and using the estimates for orbits ending in free states in Section 1.3, we have that $|f^k[z_1, z_2]| \le e^{-c_1(n-k)} |f^n[z_1, z_2]|$ for some $c_1 \gtrsim \frac{1}{3}\log 2$. Altogether this first term contributes $\le C_2' e^{\alpha\frac{n}{2} - c_1\frac{n}{2}} \le C_2' \beta^n$ for some $\beta < 1$. The corresponding term for $\hat{z}_i$, $i = 1, 2$, is handled similarly.

For $\sum_{i\geq k}\{\cdot\}$ we have, by Sublemma 8,

$$\Big|\sum_k^\infty\{\cdot\}\Big| \leq \Big|\sum_k^\infty \big(\varphi(f^i z_1)-\varphi(f^i\hat z_1)\big)\Big| + \Big|\sum_k^\infty \big(\varphi(f^i z_2)-\varphi(f^i\hat z_2)\big)\Big|$$
$$\leq 2C'(b')^k \leq 2C'b^{n/8}. \quad \square$$

Proof of Lemma 6. — It suffices to work with one γ^u-curve. We consider $z_1, z_2 \in \gamma^u \cap \Lambda_i$ and let $n = s(f^R z_1, f^R z_2)$. We noted in Section 5.3 that

$$\log \frac{J(f^R)z_1}{J(f^R)z_2} = \log \frac{(f^R)'z_1}{(f^R)'z_2} + \big(u(f^R z_1) - u(f^R z_2)\big) - (u(z_1) - u(z_2))$$
$$\overset{def}{=} (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}).$$

Since z_1 and z_2 lie in a segment that makes a regular return to Ω_0 at time R, we have by Sublemma 9 (2) that (I) $\leq C_2'|f^R[z_1,z_2]|$. Using Section 1.3 again we see that $|f^R[z_1,z_2]| \leq e^{-c_1 n}|f^n[f^R z_1, f^R z_2]| \leq e^{-c_1 n}$. Also, (II) $\leq C_2''\beta^n$ by Sublemma 11, and (III) $\leq C_2'''\beta^{s(z_1,z_2)}$ where $s(z_1,z_2)$ is obviously $> n$. $\square$

6. Proofs of Theorems

To study the rate of mixing of f it is not sufficient to consider $f^R : \Lambda \circlearrowleft$ alone: the return time function R also plays an important role. In this section we construct a tower $\Delta = \bigcup_{\ell=0}^\infty \Delta_\ell$ the bottom level of which is Λ and construct a map $F : \Delta \circlearrowleft$ in a way analogous to that of building a special flow over $f^R : \Lambda \circlearrowleft$ under the function R. This allows us to consider the Perron-Frobenius operator or transfer operator associated with $\overline{F} : \overline{\Delta} \circlearrowleft$, the quotient map of $F : \Delta \circlearrowleft$ obtained by collapsing along local stable leaves. Spectral properties of this operator are summed up in Proposition 6.3 in Section 6.3. We refer the reader to [**Y**] for a proof of this Proposition, and derive from it the results of this paper.

6.1. Construction of a tower. — Let

$$\Delta \overset{\text{def}}{=} \{(z,\ell) : z \in \Lambda, \ell = 0,1,2,\dots,R(z)-1\}.$$

We introduce $F : \Delta \circlearrowleft$ defined by

$$F(z,\ell) = \begin{cases} (z,\ell+1) & \text{if} \quad \ell+1 < R(z) \\ \big(f^R z, 0\big) & \text{if} \quad \ell+1 = R(z). \end{cases}$$

It is clear that there is a projection $\pi : \Delta \to \mathbb{R}^2$ s.t. $\pi|\Delta_0$ is the identity map on Λ and $f \circ \pi = \pi \circ F$.

An equivalent but less formal way of looking at Δ is to view it as the disjoint union $\bigcup_{\ell=0}^{\infty} \Delta_\ell$ where $\Delta_\ell \stackrel{\text{def}}{=} \{z \in \Lambda : R(z) > \ell\}$ denotes the ℓ^{th} level of the tower. Next we subdivide each level into components $\Delta_\ell = \bigcup_i \Delta_{\ell,i}$ in such a way that F has a Markov type property with respect to the partition $\{\Delta_{\ell,i}\}$. Again using 1-d language, a natural subdivision of Δ_ℓ might be the restriction of the partition $\widetilde{\mathcal{P}}_\ell$ constructed in Section 3.4; but this partition is too "big". We introduce instead a sequence of partitions $\mathcal{P}_n$ on $\widetilde{\Omega}_n$ so that $\mathcal{P}_n$ is coarser than $\widetilde{\mathcal{P}}_n$ and each element of $\mathcal{P}_n$ contains no more than finitely many elements of $\mathcal{P}_{n+1}$. This is easily done by following the algorithm in the construction of $\widetilde{\mathcal{P}}_n$, except that when $f^n\omega$ is a regular return, no subdivisions are made on that part of ω that gets mapped onto $\Omega_0 - \Omega_\infty$. (Elements of $\mathcal{P}_n$ are not necessarily intervals; they may have "holes" due to absorption into Ω_∞.) We say that $z_1, z_2 \in \Delta_\ell$ are in the same component $\Delta_{\ell,i}$ if they both lie in the same element of $\mathcal{P}_\ell$.

We summarize the topological properties of $F : \Delta \circlearrowleft$:

(I) Δ is the disjoint union $\bigcup_{\ell=0}^{\infty} \Delta_\ell$ where the ℓ^{th} level Δ_ℓ is a copy of $\{z \in \Lambda : R(z) > \ell\}$; each Δ_ℓ is further subdivided into a finite number of "components" $\Delta_{\ell,i}$ each one of which is a copy of an s-subrectangle of Λ.

(II) Under F, each $\Delta_{\ell,i}$ is mapped onto the union of finitely many components of $\Delta_{\ell+1}$ and possibly a u-subrectangle of $\Delta_0^{\pm}$. Let $\Delta_{\ell,i}^* = \Delta_{\ell,i} \cap F^{-1}\Delta_0$. We think of points in $\bigcup \Delta_{\ell,i}^*$ as "returning to the bottom level" under F, while other points "move upward" to the next level.

From the description above it is clear that the quotient map $\overline{F} : \overline{\Delta} \circlearrowleft$ obtained by collapsing γ^s-curves as in Section 5 is well defined. Let $\bar{m}$ be the reference measure on $\overline{\Lambda}$ or $\overline{\Delta}_0$. Since each $\overline{\Delta}_{\ell,i}$ is a copy of a subset of $\overline{\Delta}_0$, $\bar{m}$ is defined on $\overline{\Delta}_{\ell,i}$ via the natural identification. Let $J(\overline{F})$ denote the Jacobian of $\overline{F}$ with respect to $\bar{m}$; more precisely, if $z \in \overline{\Delta}_{\ell,i}$ and $\overline{F}z \in \overline{\Delta}_{\ell+1,i'}$, then $J\overline{F}(z)$ is the Jacobian of the map $\overline{F} \mid \left(\overline{\Delta}_{\ell,i} \cap \overline{F}^{-1}\left(\overline{\Delta}_{\ell+1,i'}\right)\right)$. Also, given the present setting it is natural to define the separation time of $z_1, z_2 \in \Delta$ to be

$$s(z_1, z_2) \equiv \text{ the smallest } n \geq 0 \text{ s.t. } f^n z_1 \text{ and } f^n z_2 \text{ lie in different } \Delta_{\ell,i}\text{'s.}$$

This definition of $s(\cdot,\cdot)$ will permanently replace the one in Section 5.4. Observe that under the present definition, $z_1, z_2 \in \Delta_0$ separate faster than under the old one. Hence the distortion estimate in Lemma 4 is all the more valid. Again we summarize:

(III) There is a reference measure $\bar{m}$ on $\bar{\Delta}$ uniformly equivalent to the restriction of Lebesgue measure on $\gamma^u \cap \Lambda$ for every γ^u such that with respect to $\bar{m}$, the Jacobian $J\overline{F}$ of $\overline{F}$ satisfies:

(i) $J\overline{F}(z) = 1 \ \forall z \notin \bigcup \bar{\Delta}^*_{\ell,i}$;

(ii) $\exists C_2 > 0$ and $\beta < 1$ s.t. $\forall z_1, z_2 \in \overline{F}^{-1}\left(\bar{\Delta}_0^{\pm}\right)$,

$$\left| \frac{J\overline{F}(z_1)}{J\overline{F}(z_2)} - 1 \right| \leq C_2 \beta^{s\left(\overline{F}z_1, \overline{F}z_2\right)}.$$

Here $\bar{\Delta}_0^+$ and $\bar{\Delta}_0^-$ are the two components of $\bar{\Delta}_0$. Let

$$C_3 = \frac{\bar{m}\left(\bar{\Delta}_0\right)}{\min\left(\bar{m}\left(\bar{\Delta}_0^+\right), \bar{m}\left(\bar{\Delta}_0^-\right)\right)}.$$

We state for the record the following very important tail estimate on the height of the tower $\bar{\Delta}$ (or equivalently the return times to $\bar{\Delta}_0$).

(IV) The height function $R : \bar{\Delta}_0 \to \mathbb{Z}^+$ has the following properties:

(i) $R \geq N$ where N is chosen so that $C_2 e^{C_2} C_3 \beta^N \leq \frac{1}{100}$;

(ii) $\exists C_0 > 0$ and $\theta_0 < 1$ s.t.

$$\bar{m}\{R > n\} \leq C_0 \theta_0^n \qquad \forall n \geq 0.$$

The lower bound for R in (i) is for purposes of guaranteeing a definite amount of contraction for the Perron-Frobenius operator between consecutive returns of an orbit to the base. The feasibility of such a bound was arranged in Section 3.4. Note the order in which the constants in (III) and (IV) are chosen: C_2, C_3 and β, and hence N can be chosen to depend only on the derivatives of f and not on the construction of the tower; whereas C_0 and θ_0 depend on f as well as on N. The tail estimate in (ii) is a slight reformulation of Propostion 3.1 (IV)(ii).

6.2. SRB measures: Proof of Theorem 1. — We construct in this subsection an SRB measure ν for f with $\nu(\Lambda) > 0$. This gives an alternate proof of Theorem 1 to that in [**BY**].

Let $f^R : \Lambda \circlearrowleft$ be the mapping with $f^R \mid \Lambda_i = f^{R_i} \mid \Lambda_i$ for each i, and let μ_0 denote the restriction of 1-dimensional Lebesgue measure on Ω_0 to $\Lambda \cap \Omega_0$. For $n = 1, 2, \ldots$, let

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \left(f^R\right)_*^i \mu_0.$$

Then μ_n is supported on a countable number of γ^u-curves on each one of which it has a density ρ_n. Clearly, $\rho_n = 0$ on $\gamma^u - \Lambda$. The distortion estimate in Sublemma 9 tells us that

$$\frac{\rho_n(x)}{\rho_n(y)} \le C_1 \qquad \text{for a.e. } x, y \in \Lambda \cap \gamma^u.$$

From this and from the absolute continuity of the curves in Γ^s and the boundedness of the Radon-Nikodym derivatives (see Sublemma 10), it follows that if $\tilde{\rho}_n \mid \gamma^u := \rho_n/\mu_n(\gamma^u)$, then $M^{-1} \le \tilde{\rho}_n \le M$ a.e. on $\gamma^u \cap \Lambda$ for some M independent of n and γ^u.

Letting $n \to \infty$, a subsequence μ_{n_k} converges weakly to μ_∞. We have immediately that μ_∞ is f^R-invariant and that it is supported on Λ.

Let $\{\mu_\infty^\gamma\}$ be the conditional measures of μ_∞ on γ^u-curves, and let Q_ω be an s-subrectangle of Q corresponding to an arbitrary subsegment ω of some γ^u. Then for a.e. $\gamma \in \Gamma^u$, we have

$$M^{-1} \min_{\gamma^u \in \Gamma^u} |Q_\omega \cap \gamma^u| \le \mu_\infty^\gamma(Q_\omega \cap \gamma) \le M \max_{\gamma^u \in \Gamma^u} |Q_\omega \cap \gamma^u|,$$

proving (again using the absolute continuity of Γ^s) that μ_∞^γ is uniformly equivalent to the arclength measure $s \mid (\gamma \cap \Lambda)$.

To extract from μ_∞ an f-invariant measure, simply let

$$\nu = \sum_{i=0}^{\infty} f_*^i \left(\mu_\infty \mid \{R > i\}\right).$$

That ν is a finite measure follows from Propostion 3.1 (4) (ii); we may therefore normalize and assume $\nu(\mathbb{R}^2) = 1$. It is clear that ν satisfies the definition of an SRB measure as defined in Section 1.7.

By the same token, we could view μ_∞ as a measure on Δ_0, and construct as above an F-invariant measure $\tilde{\nu}$ on Δ with $\pi_*\tilde{\nu} = \nu$. It is also clear from the discussion above that $\bar{\nu}$, the measure on $\bar{\Delta}$ that is the quotient of $\tilde{\nu}$, is uniformly equivalent to our reference measure $\bar{m}$.

6.3. Definition and properties of the Perron-Frobenius operator associated with $\overline{F} : \bar{\Delta} \circlearrowleft$. — First we introduce the function space on which our operator acts. Fix $\varepsilon > 0$ with the following two properties:

(i) $e^{2\varepsilon}\theta_0 < 1$ where θ_0 is as in Section 6.1 (IV) (ii);
(ii) $\frac{1}{\bar{m}\bar{\Delta}_0} \sum_{\ell,i} \bar{m}\bar{\Delta}^*_{\ell,i} e^{\ell\varepsilon} \le 2$.

Note that property (ii) is consistent with

$$\frac{1}{\bar{m}\bar{\Delta}_0} \sum_{\ell,i} \bar{m}\bar{\Delta}^*_{\ell,i} = 1.$$

We remark for future reference the relative sizes of β and $e^{-\varepsilon}$: from Section 6.1 (IV)(i) we have that β^N times various constants is $\leq \frac{1}{100}$, while (ii) above implies that $e^{\varepsilon N} \leq 2$. Thus β should be thought of as $< e^{-\varepsilon}$.

Our function space X will consist of those $\overline{\varphi} : \bar{\Delta} \to \mathbb{C}$ with $\|\overline{\varphi}\| < \infty$, where $\|\cdot\|$ is a weighted $L^\infty+$ Hölder norm defined as follows: let $\overline{\varphi}_{\ell,i} = \overline{\varphi} \mid \bar{\Delta}_{\ell,i}$, and let $|\cdot|_p$ denote the L^p-norm $(1 \leq p \leq \infty)$ wrt the reference measure $\bar{m}$. We define

$$\left\|\overline{\varphi}_{\ell,i}\right\|_\infty = \left|\overline{\varphi}_{\ell,i}\right|_\infty e^{-\ell\varepsilon}$$

and

$$\left\|\overline{\varphi}_{\ell,i}\right\|_h = \left(\operatorname*{ess\,sup}_{z_1,z_2\in\Delta_{\ell,i}} \frac{|\overline{\varphi}z_1 - \overline{\varphi}z_2|}{\beta^{s(z_1,z_2)}}\right) \cdot e^{-\ell\varepsilon}$$

where ε is as above. Finally let

$$\|\overline{\varphi}\| = \|\overline{\varphi}\|_\infty + \|\overline{\varphi}\|_h$$

where

$$\|\overline{\varphi}\|_\infty = \sup_{\ell,i} \left\|\overline{\varphi}_{\ell,i}\right\|_\infty \qquad \text{and} \qquad \|\overline{\varphi}\|_h = \sup_{\ell,i} \left\|\overline{\varphi}_{\ell,i}\right\|_h.$$

The Perron-Frobenius operator or transfer operator associated with the dynamical system $\overline{F} : \bar{\Delta} \circlearrowleft$ is defined by

$$\mathcal{P}\left(\overline{\varphi}\right)(z) = \sum_{w:\overline{F}w=z} \frac{\overline{\varphi}(w)}{J\overline{F}(w)}.$$

Our choice of $(X, \|\cdot\|)$ was to ensure that $\mathcal{P}$ has the following spectral properties:

(1) $\mathcal{P} : X \to X$ is a bounded linear operator; its spectrum $\sigma(\mathcal{P})$ has the following properties:

- $\sigma(\mathcal{P}) \subset \{|\lambda| \leq 1\}$
- $\exists \tau_0 < 1$ s.t. $\sigma(\mathcal{P}) \cap \{|\lambda| \geq \tau_0\}$ consists of a finite number of points the eigenspaces corresponding to which are all finite dimensional.

(2) $1 \in \sigma(\mathcal{P})$, and $\bar{\rho} \in X$ is an eigenfunction corresponding to the eigenvalue 1. Moreover, if the greatest common divisor (gcd) of $\{R(z) : z \in \Delta_0\} = 1$, then 1 is the only element of $\sigma(\mathcal{P})$ with $|\lambda| = 1$ and its eigenspace is 1-dimensional.

See [**Y**] for a proof of Proposition 6.3.

6.4. Decay of correlations: Proof of Theorem 3. — Recall that we are concerned with a "good" Hénon map $f : \mathbb{R}^2 \circlearrowleft$ which we know has an attractor Σ and a unique SRB measure ν. *We assume for the rest of this paper that (f^n, ν) is ergodic for all $n \geq 1$.* The following conventions will be used: if φ is a function on $\mathbb{R}^2$ or on Σ, then the lift of φ to Δ will be called $\tilde{\varphi}$; and if $\tilde{\varphi} : \Delta \to \mathbb{C}$ is constant on γ^s-curves then we will sometimes confuse it with the obvious function on $\bar{\Delta}$ called $\overline{\varphi}$.

For $\gamma > 0$, let $\mathcal{H}_\gamma = \mathcal{H}_\gamma(\Sigma)$ denote the class of Hölder continuous functions on Σ with exponent γ, i.e.

$$\mathcal{H}_\gamma = \{\varphi : \Sigma \to \mathbb{R} \mid \exists C = C_\varphi \text{ s.t. } \forall x, y \in \Sigma, |\varphi x - \varphi y| \leq C|x-y|^\gamma\}.$$

We will use as shorthand for the correlation between φ and $\psi \circ f^n$ with respect to ν the notation $D_n(\varphi, \psi; \nu)$, i.e.

$$D_n(\varphi, \psi; \nu) = \left| \int (\psi \circ f^n)\, \varphi d\nu - \int \varphi d\nu \int \psi d\nu \right|.$$

We outline below the steps needed to derive from Proposition 6.3 an exponentially small bound for $D_n(\varphi, \psi; \nu)$. Since the derivation is completely formal and (with the exception of one small geometric fact noted below) has nothing to do with the present setting, we will refer the reader to [**Y**] for details of the proofs.

Here are the main steps of the argument:

(1) Observe that $D_n(\varphi, \psi; \nu) = D_n\left(\tilde{\varphi}, \tilde{\psi}; \tilde{\nu}\right)$, and that by considering a power of f if necessary, we may assume gcd $\{R(z) : z \in \Delta_0\} = 1$.

(2) In preparation for using the Perron-Frobenius operator, we maneuver $D_n(\tilde{\varphi}, \tilde{\psi}; \tilde{\nu})$ into an object describable purely in terms of functions on $\bar{\Delta}$. This is done in two steps:

(i) Fix $\kappa \in \left(0, \frac{1}{2}\right)$, and let $k = \kappa n$. Let $\mathcal{M}$ be the partition of Δ into $\{\Delta_{\ell,j}\}$, and let $\overline{\psi}_k : \bar{\Delta} \to \mathbb{R}$ be the function constant on elements η of $\mathcal{M}_{2k} := \bigvee_{i=0}^{2k-1} F^{-i}\mathcal{M}$ with $\overline{\psi}_k \mid \eta := \tilde{\psi} \circ F^k$ (some selected point in η). Verify that

$$D_n\left(\tilde{\varphi}, \tilde{\psi}; \tilde{\nu}\right) = D_{n-k}\left(\tilde{\varphi}, \tilde{\psi} \circ F^k; \tilde{\nu}\right) \approx D_{n-k}\left(\tilde{\varphi}, \bar{\psi}_k; \tilde{\nu}\right).$$

(ii) Let $\overline{\varphi}_k$ be defined as above, and let $\tilde{\varphi}_k := d\left(F_*^k\left(\overline{\varphi}_k \tilde{\nu}\right)\right)/d\tilde{\nu}$. Verify that

$$D_{n-k}\left(\tilde{\varphi}, \overline{\psi}_k; \tilde{\nu}\right) \approx D_{n-k}\left(\tilde{\varphi}_k, \overline{\psi}_k; \tilde{\nu}\right)$$

and observe that

$$D_{n-k}\left(\tilde{\varphi}_k, \overline{\psi}_k; \tilde{\nu}\right) = \left| \int \overline{\psi}_k \mathcal{P}^n\left(\overline{\varphi}_k \bar{\rho}\right) d\bar{m} - \int \overline{\psi}_k \bar{\rho} d\bar{m} \int \overline{\varphi}_k \bar{\rho} d\bar{m} \right|.$$

(3) Use Proposition 6.3 to prove that

$$\left\| \mathcal{P}^n\left(\overline{\varphi}_k \bar{\rho}\right) - \left(\int \overline{\varphi}_k \bar{\rho} d\bar{m}\right) \overline{\rho} \right\| \leq \text{ const } \cdot \tau_1^{n-2k}$$

for some $\tau_1 < 1$.

In Step (2) above, it is neccessary to translate the Hölder property for $\varphi, \psi \in \mathcal{H}_\gamma$ to a Hölder type condition for $\tilde{\varphi}$ and $\tilde{\psi}$. The following geometric fact is used:

***Sublemma* 12.** — *$\forall z \in \Delta$, diam $\left(\pi F^k\left(\mathcal{M}_{2k}(z)\right)\right) \leq 2C\alpha^k$.*

We leave the proof as an easy exercise.

6.5. Proof of the Central Limit Theorem. — Theorem 4 (the CLT) is also proved in [**Y**] but we prefer to give another proof here. As in [**Y**] this proof is based on a theorem of Gordin [**G**] but we apply Gordin's theorem to test functions in the Banach space X and use an L^2 approximation argument (which in fact uses the decay of correlation) to prove the theorem for Hölder test functions on Δ.

The version of Gordin's theorem we need may be stated as follows:

***Theorem* 5 ([G]).** — *Let $(\Omega, \mathcal{F}, \nu)$ be a probability space, let $T : \Omega \circlearrowleft$ be a non-invertible measure-preserving transformation, and let $\varphi \in L^2(\nu)$ be s.t. $E\varphi = 0$. Suppose that*

$$(*) \qquad \sum_{j\geq 0} |E(\varphi|T^{-j}\mathcal{F})|_2 < \infty$$

Then

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \varphi \circ T^i \overset{\text{dist}}{\mapsto} \mathcal{N}(0, \sigma)$$

where

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n} \int \left(\sum_{i=0}^{n-1} \varphi \circ T^i\right)^2 d\nu.$$

Exponential decay of correlations alone is not sufficient to conclude that $(*)$ holds. Suppose, however, that there is a reference measure m with respect to which T is non-singular, and suppose that $d\nu = \rho\, dm$ for some $\rho \geq c > 0$. Then we have a well-defined Perron-Frobenius operator given by $\mathcal{P}(\varphi) = \psi$, where ψ is the density of $T_*(\varphi m)$, and a gap in the spectrum of $\mathcal{P}$ (with respect to a suitable function space) is sufficient to conclude that $(*)$ converges exponentially. In fact, we have

$$(**) \qquad \int |E(\varphi|T^{-j}\mathcal{M})|^2\, d\nu \leq |\varphi|_\infty \int |\mathcal{P}^j(\varphi\rho)|\, dm$$

see [**K**] or [**Y**]. See also Ruelle's earlier work [**R**].

For $\varphi \in \mathcal{H}_\gamma$ let $\widetilde{\varphi}(z) = \varphi \circ \pi(z)$ be the lift of φ to Δ. For $k \in \mathbb{Z}^+$, we use $\widetilde{\varphi}^{(k)}$ to denote $\widetilde{\varphi} \circ F^k$ and define $\overline{\varphi}_k$ by

$$\overline{\varphi}_k | A = \frac{1}{\tilde{\nu}(A)} \int_A \tilde{\varphi} \circ F^k \, d\tilde{\nu},$$

where the A's are the elements of the partition $\mathcal{M}_{2k}$ defined in Section 6.4.

As in Section 6.3, for functions on $\overline{\Delta}$ we use $|\cdot|_p$ to denote the L^p-norm, $1 \le p \le \infty$, with respect to the reference measure $\overline{m}$ and $||\cdot||$ to denote the norm on the space X. Let $||\cdot||_{\mathcal{H}_\gamma}$ denote the usual γ-Hölder norm for functions $\varphi : \mathbb{R}^2 \to \mathbb{C}$, i.e.

$$||\varphi||_{\mathcal{H}_\gamma} = \sup_{x \in \mathbb{R}^2} |\varphi(x)| + \sup_{x \ne y} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^\gamma}.$$

The following estimates are used in several places.

***Sublemma* 13**. — *There is a constant $C = C(f)$ such that for all functions $\overline{\varphi} \in X$ we have*

$$|\overline{\varphi}|_1 \le C ||\overline{\varphi}||.$$

This is an easy exercise (see [**Y**], Section 3.2).

***Sublemma* 14**. — *For $\varphi \in \mathcal{H}_\gamma$*

$$\sup |\overline{\varphi}_k - \widetilde{\varphi} \circ F^k| \le C(f) ||\varphi||_{\mathcal{H}_\gamma} \lambda^k \qquad \forall k \ge 0,$$

where $\lambda = \lambda(f, \gamma)$ satisfies $0 < \lambda < 1$.

This sublemma is a direct consequence of Sublemma 12.

Let us introduce the notation

$$S_n(\widetilde{\varphi}) = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \widetilde{\varphi} \circ F^j.$$

We now fix $\psi \in \mathcal{H}_\gamma$ with $\int \psi \, d\nu = 0$ and prove the CLT for this function. Observe first that there is a constant $K_0 = K_0(\psi)$ such that for all $\varepsilon > 0$, if $N(\varepsilon) := [K_0 \log(1/\varepsilon)]$, then

$$\sup |\overline{\psi}_{N(\varepsilon)} - \widetilde{\psi} \circ F^{N(\varepsilon)}| \le \varepsilon.$$

This follows immediately from Sublemma 14.

***Lemma* 7**. — *There is a function $r(\varepsilon)$ with $r(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that for each $\varepsilon > 0$, if*

$$\begin{cases} T_n = S_n(\widetilde{\psi} \circ F^{N(\varepsilon)}) \\ U_n = S_n(\overline{\psi}_{N(\varepsilon)}) \end{cases}$$

then

$$\sup_{n \geq 0} \|T_n - U_n\|_{L^2(\tilde{\nu})} \leq r(\varepsilon).$$

For the proof of this lemma we need the following estimates:

***Sublemma* 15**. — *There exist constants $C > 0$ and $0 < \beta_0, \tau < 1$, β_0 and τ depending only on f and $C = C(f, \psi)$, such that for all $j, N \in \mathbb{Z}^+$ the following hold:*

(i) $\left|\int \overline{\psi}_N \overline{\psi}_N \circ F^j d\tilde{\nu}\right| \leq C\beta_0^{-2N}\tau^j$;

(ii) $\left|\int \overline{\psi}_N \widetilde{\psi}^{(N)} \circ F^j d\tilde{\nu}\right| \leq C\beta_0^{-2N}\tau^j$;

(iii) $\left|\int \widetilde{\psi}^{(N)} \overline{\psi}_N \circ F^j d\tilde{\nu}\right| \leq C\beta_0^{-2N}\tau^j$;

(iv) $\left|\int \widetilde{\psi}^{(N)} \widetilde{\psi}^{(N)} \circ F^j d\tilde{\nu}\right| \leq C\beta_0^{-2N}\tau^j$.

Since $\overline{\psi}_N$ can also be viewed as a function on Δ, that the left side of (i)–(iv) above is $\leq C\tau^j$ for some $C = C(f, \psi, N)$ follows from the decay of correlations proof outlined in Section 6.4. The proof of this sublemma consists of re-doing the estimates there and making transparent the dependence on N. We carry this out for (i) and (ii) and leave the rest as exercises.

Proof of Sublemma 15(i) and (ii). — For $\overline{\varphi}, \overline{\eta} \in X$ we have

$$\begin{aligned} D_j(\overline{\varphi}, \overline{\eta}; \tilde{\nu}) &= \left| \int \overline{\eta} \circ F^j \overline{\varphi}\, d\tilde{\nu} - (\int \overline{\varphi}\, d\tilde{\nu})(\int \overline{\eta}\, d\tilde{\nu}) \right| \\ &= \left| \int \overline{\eta} \left[\mathcal{P}^j(\overline{\varphi}\,\overline{\rho}) - \left(\int \overline{\varphi}\,\overline{\rho}\, d\overline{m} \right) \overline{\rho} \right] d\overline{m} \right| \\ &\leq |\overline{\eta}|_\infty \left| \mathcal{P}^j(\overline{\varphi}\,\overline{\rho}) - \left(\int \overline{\varphi}\,\overline{\rho}\, d\overline{m} \right) \overline{\rho} \right|_1 . \end{aligned}$$

From the spectral information of the Perron-Frobenius operator and Sublemma 13 we conclude that

$$\begin{aligned} D_j(\overline{\varphi}, \overline{\eta}; \tilde{\nu}) &\leq C|\overline{\eta}|_\infty \,||\overline{\varphi}\,\overline{\rho}||\, \tau_0^j \\ &\leq C|\overline{\eta}|_\infty \max\{||\overline{\rho}||, |\overline{\rho}|_\infty\} \max\{||\overline{\varphi}||, |\overline{\varphi}|_\infty\} \tau_0^j. \end{aligned}$$

Now replace both $\overline{\varphi}$ and $\overline{\eta}$ by $\overline{\psi}_N$ and note that $||\overline{\psi}_N|| \le C(\psi)\beta^{-N}$. It follows that (i) holds.

As for the proof of (ii) let $\widetilde{\eta}$ be the lift of a function $\eta \in \mathcal{H}_\gamma$, let $\overline{\varphi} \in X$ and consider

$$\begin{aligned} D_j(\overline{\varphi}, \widetilde{\eta}; \tilde{\nu}) &= \left| \int (\widetilde{\eta} \circ F^j)\overline{\varphi}\, d\tilde{\nu} \right| = \left| \int (\widetilde{\eta} \circ F^k) \circ F^{j-k}\overline{\varphi}\, d\tilde{\nu} \right| \\ &\le \left| \int \left[\widetilde{\eta} \circ F^k - \overline{\eta}_k\right] \circ F^{j-k}\overline{\varphi}\, d\tilde{\nu} \right| + \left| \int \overline{\eta}_k \circ F^{j-k}\overline{\varphi}\, d\tilde{\nu} \right| \\ &\le |\overline{\varphi}|_\infty \, ||\widetilde{\eta} \circ F^k - \overline{\eta}_k||_{L^1(\tilde{\nu})} + \left| \int \overline{\eta}_k \mathcal{P}^{j-k}(\overline{\rho}\,\overline{\varphi})\, d\overline{m} \right| \\ &\le C|\overline{\varphi}|_\infty \, ||\eta||_{\mathcal{H}_\gamma}\lambda^k + C|\overline{\eta}_k|_\infty \, ||\overline{\varphi}||\tau_0^{j-k}. \end{aligned}$$

We have here used Sublemma 14 to estimate $||\widetilde{\eta} \circ F^k - \overline{\eta}_k||_{L^1(\tilde{\nu})}$.

Finally we choose $k = [\kappa j]$ for a suitable small κ and substitute $\overline{\psi}_N$ for $\overline{\varphi}$ and $\widetilde{\psi}^{(N)}$ for $\widetilde{\eta}$ above. The conclusion of (ii) with suitable choices of β_0 and τ then follows from the estimates $||\psi \circ f^N||_{\mathcal{H}_\gamma} \le K_1^N||\psi||_{\mathcal{H}_\gamma}$, $K_1 = K_1(f, \gamma)$, and $||\overline{\psi}_N|| \le C(\psi)\beta^{-N}$.

□

Proof of Lemma 7. — We have

$$\begin{aligned} \|T_n - U_n\|_2^2 = \frac{1}{n}\Bigg[& n\left\|\widetilde{\psi} \circ F^N - \overline{\psi}_N\right\|_2^2 \\ & + \sum_{j=1}^{n-1}(n-j)\int \left(\widetilde{\psi}^{(N)} - \overline{\psi}_N\right)\left(\widetilde{\psi}^{(N)} - \overline{\psi}_N\right) \circ F^j d\tilde{\nu}\Bigg]. \end{aligned}$$

We pick $j_0 = j_0(\varepsilon)$. The exact choice of j_0 will be made later. For $1 \le j \le j_0$ we estimate the covariances by Cauchy's inequality

$$\begin{aligned} & \left| \int \left(\widetilde{\psi}^{(N)} - \overline{\psi}_N\right)\left(\widetilde{\psi}^{(N)} - \overline{\psi}_N\right) \circ F^j d\tilde{\nu}\right| \\ & \qquad \le \left\|\widetilde{\psi}^{(N)} - \overline{\psi}_N\right\|_2 \left\|\widetilde{\psi}^{(N)} - \overline{\psi}_N\right\|_2 \\ & \qquad \le \varepsilon \cdot \varepsilon = \varepsilon^2. \end{aligned}$$

For j in the range $j_0 < j \le n-1$ we use the estimates of Sublemma 15. Combining these estimates we obtain

$$\|U_n - T_n\|_2^2 \le \varepsilon^2(1 + j_0) + \frac{4C}{\beta_0^{2N}} \sum_{j=j_0+1}^{n-1} \frac{1}{n}\,(n-j)\, e^{-\theta j}$$

with $\tau = e^{-\theta}$.

The last sum is estimated as

$$\frac{1}{n}\sum_{j=j_0+1}^{n-1}(n-j)e^{-\theta j} \leq e^{-\theta j_0}.$$

Hence $\|U_n - T_n\|_2^2 \leq (1+j_0)\varepsilon^2 + (4C/\beta_0^{2K_0\log\frac{1}{\varepsilon}})e^{-\theta j_0}$. By choosing

$$j_0(\varepsilon) = 4K_0\frac{1}{\theta}\log\frac{1}{\beta_0}\log\frac{1}{\varepsilon}$$

we obtain the estimate

$$\|U_n - T_n\|_2^2 \leq r(\varepsilon) = \mathcal{O}\left(\varepsilon^2\log\frac{1}{\varepsilon}\right).$$

□

Proof of Theorem 4. — We will prove the Central Limit Theorem for $\psi \in \mathcal{H}_\gamma$. More specifically, we will show that $F_n(t) \to \mathcal{N}(0,\sigma)$ in distribution, where

$$F_n(t) = \nu\left(\left\{z : \frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\psi\circ f^j \leq t\right\}\right)$$

$$\sigma^2 = \sum_{j=0}^{\infty}\int\psi\,\psi\circ f^j\,d\nu \geq 0.$$

We will in the following assume $\sigma > 0$. It is in fact true that $\sigma = 0$ iff $\psi = \varphi - \varphi\circ f$ for some function $\varphi \in L^2$. (This fact was communicated to us by Bill Parry.) Hence the CLT is true for $\sigma = 0$ with the Normal Distribution Function $\Phi_\sigma(t)$ interpreted as the unit step function.

We shall first see how we can use Gordin's theorem to conclude that the CLT holds for test functions from the class X.

Let $\overline{\varphi} \in X$ with $\int\overline{\varphi}\,\overline{\rho}\,d\overline{m} = 0$. The spectral properties of the Perron-Frobenius operator $\mathcal{P}$ guaranties that

$$||\mathcal{P}^j(\overline{\varphi}\,\overline{\rho})|| \leq C\tau^j \qquad \forall j \geq 0.$$

Then $|\mathcal{P}^j(\overline{\varphi}\,\overline{\rho})|_1 \leq C\tau^j\ \forall j \geq 0$ and from $(**)$ it follows that condition $(*)$ in Gordin's theorem is valid. We conclude that

$$(***)\qquad \tilde{\nu}\left(\left\{z : \frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\overline{\varphi}\circ F^j(z) \leq t\right\}\right) \overset{\text{dist}}{\to} \mathcal{N}(0,\sigma).$$

Note that from the invariance of the measure $\tilde{\nu}$ under F it follows immediately that

$$\sigma^2(\widetilde{\psi}^{(N)}) = \mathrm{Var}[T_n] = \mathrm{Var}[S_n(\tilde{\psi}^{(N)})] = \mathrm{Var}[S_n(\tilde{\psi})] = \sigma^2(\widetilde{\psi}).$$

We will also use the notation $\sigma_\varepsilon^2 = \sigma^2(\overline{\psi}_N) = \mathrm{Var}[U_n]$.

From Lemma 7 we conclude that

$$\sigma^2\left(\overline{\psi}_{N(\varepsilon)}\right) \to \sigma^2\left(\widetilde{\psi}\right) \qquad \text{as} \qquad \varepsilon \to 0$$

and hence we can assume that $\sigma_\varepsilon > 0$ by chosing ε sufficiently small.

By moving up to the tower Δ and using the invariance of $\tilde{\nu}$ under F we can also write

$$F_n(t) = \tilde{\nu}\left(\left\{z : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \widetilde{\psi}^{(N)} \circ F^j(z) \le t\right\}\right).$$

We wish to compare $F_n(t)$ with the distribution function

$$G_n(t) = \tilde{\nu}\left(\left\{z : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \overline{\psi}_N \circ F^j(z) \le t\right\}\right).$$

It follow from $(***)$ with $\overline{\varphi}$ replaced by $\overline{\psi}_N$ that $G_n(t) \to \mathcal{N}(0, \sigma_\varepsilon)$ in distribution. Now pick $\eta > 0$. By Tjebyshev's inequality

$$\begin{aligned}
F_n(t) &\le G_n(t+\eta) + \tilde{\nu}\left\{|T_n - U_n| \ge \eta\right\} \\
&\le G_n(t+\eta) + \frac{1}{\eta^2}\left\|T_n - U_n\right\|_2^2 \\
&\le G_n(t+\eta) + \frac{1}{\eta^2} r(\varepsilon).
\end{aligned}$$

Letting $n \to \infty$ we conclude that $\overline{\lim_{n\to\infty}} F_n(t) \le \Phi_{\sigma_\varepsilon}(x+\eta) + \frac{1}{\eta^2} r(\varepsilon)$. Here

$$\Phi_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-\frac{u^2}{2\sigma^2}} du.$$

By letting $\varepsilon \to 0$ we have

$$\overline{\lim_{n\to\infty}} F_n(t) \le \Phi_\sigma(x+\eta).$$

Now let $\eta \to 0$. It follows that $\overline{\lim}_{n\to\infty} F_n(t) \le \Phi_\sigma(x)$. The proof that

$$\underline{\lim}_{n\to\infty} F_n(t) \ge \Phi_\sigma(x)$$

is completely analogous.

$\square$

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