Decay of Correlations for Certain Quadratic Maps[1]

L.-S. Young[2]

Abstract. We prove exponential decay of correlations for (f, μ), where f belongs in a positive measure set of quadratic maps of the interval and μ is its absolutely continuous invariant measure. These results generalize to other interval maps.

Consider a dynamical system generated by a map $f : M \to M$ preserving a probability measure μ, and let $\varphi, \psi : M \to \mathbb{R}$ be observables. Mixing properties of the dynamical system are reflected in the decay of correlations between φ and $\psi \circ f^n$ as $n \to \infty$. More precisely, we say that (f, μ) has *exponential decay of correlations* for functions belonging in a certain class X if there is a number $\tau < 1$ s.t. for every $\varphi, \psi \in X$, there is a constant $C = C(\varphi, \psi)$ s.t.

$$\left| \int \varphi \cdot (\psi \circ f^n) d\mu - \int \varphi d\mu \cdot \int \psi d\mu \right| \le C \tau^n \qquad \forall n \ge 0 .$$

The main result of this paper is the following:

Theorem. *Consider $f_a : [-1, 1] \circlearrowleft$ defined by $f_a(x) = 1 - ax^2$, $a \in [0, 2]$. Then there is a positive Lebesgue measure set Δ in parameter space s.t. if $f = f_a$ for $a \in \Delta$, then*

(1) *f has an absolutely continuous invariant measure μ (this is a well known theorem first proved by Yakobson* [J]);

(2) *(f, μ) has exponential decay of correlations for functions of bounded variation*;

(3) *the central limit theorem holds for $\{\varphi \circ f^n\}_{n=1,2,\ldots}$, $\varphi \in BV$.*

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These results generalize to certain open sets of 1-parameter families of unimodal maps.

Exponential decay of correlations has been proved for primarily two types of dynamical systems: piecewise uniformly expanding maps of the interval with their absolutely continuous invariant measures, and Axiom A diffeomorphisms with their Gibbs states. (See e.g. [HK], [Ry1], [Ru1], [Ru2]). These are by no means the only results. (See e.g. [BS], [Z].)

The main feature that is new here is that the maps we consider have singularities, and that these singularities return arbitrarily close to themselves. We reduce the problem to the expanding case by constructing a uniformly expanding map $\tilde{f}$ that factors over f, $\tilde{f}$ acting on a set J that is the disjoint union of a countable number of intervals. We then consider a function space on J with a norm that is a weighted combination of the L^∞, L^1 and total variation norms, and prove the existence of a gap in the spectrum of the Perron-Frobenius operator associated with $\tilde{f}$.

The maps for which our results hold are those studied by Benedicks and Carleson in [**BC2**]. They have a very simple description, and are non-uniformly expanding in a controlled way. We will recall in detail all of the relevant material from [**BC2**] – except the proof of the theorem which says that these maps form a positive measure set in parameter space.

This paper is organized as follows. Precise statements of our results are given in Section 1. Section 2 contains some background material for 1-dimensional maps. In Section 3 we prove the existence of absolutely continuous invariant measures and their mixing properties. Sections 4 and 5 contain the proof of the decay of correlation result.

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§1. Statements of results

We first state our results for the quadratic family $x \mapsto 1 - ax^2$ before proceeding to discuss generalizations to other 1-parameter families of interval maps.

Consider $f_a : [-1, 1] \circlearrowleft$ defined by $f_a(x) = 1 - ax^2$, $a \in [0, 2]$. Let $\alpha > 0$ be a very small number that remains fixed throughout, say $\alpha = \frac{1}{10^6}$. For $\varepsilon > 0$, let

$$\Delta_\varepsilon = \left\{ a \in [2-\varepsilon, 2] : |f_a^n 0| \geq e^{-\alpha n} \quad \text{and} \quad |(f_a^n)'(f_a 0)| \geq (1.9)^n \quad \forall n \geq 0 \right\}.$$

Benedicks and Carleson proved that $\text{Leb}(\Delta_\varepsilon) > 0 \quad \forall \varepsilon > 0$ [BC2]. These are the parameters to which our results apply.

Theorem 1 (Existence of invariant measures). $\exists\, \varepsilon_1 > 0$ s.t. $\forall\, a \in \Delta_{\varepsilon_1}$, $f = f_a$ *has an invariant probability measure* μ *with a density* ρ *that can be written as*

$$\rho = \rho_1 + \rho_2$$

where ρ_1 *has bounded variation and*

$$0 \leq \rho_2(s) \leq \text{const.} \sum_{j=1}^{\infty} \frac{(1.9)^{-j}}{\sqrt{|s - f^j 0|}}.$$

The existence of absolutely continuous invariant measures under similar conditions has been proved many times. See e.g. [J], [BC1], [CE] plus [N], [Ry2], [BY] etc. The nature of the density can be understood as follows. The dynamics of f consists essentially of an expanding part and a contraction due to the quadratic singularity. The expanding part gives rise to ρ_1, while the contraction together with its subsequent iterates account for the inverse square-root singularities in ρ_2.

Theorem 2 (Properties of the absolutely continuous invariant measure). *$\exists\varepsilon_2 > 0$, possibly $< \varepsilon_1$, s.t. $\forall a \in \Delta_{\varepsilon_2}$, $f = f_a$ has the following properties:*

(a) *μ in Theorem 1 is the only absolutely continuous invariant measure;*

(b) *(f, μ) is exact;*

(c) $\text{supp}\mu = [f^2 0,\ f0]$ *and* $\inf_{x \in [f^2 0, f0]} \rho(x) > 0.$

We remark that $(f,\ \mu)$ being exact in this setting is equivalent to $(f^n,\ \mu)$ being ergodic $\forall n > 0$. It is also equivalent to the natural extension of $(f,\ \mu)$ being isomorphic to a Bernoulli shift. (See [L].)

Theorem 3 (Decay of correlations). *Let f be as in Theorem 2. Then $\exists\tau \in (0,\ 1)$ s.t. $\forall\varphi,\ \psi : [-1,\ 1] \to \mathbb{R}$ with bounded variation, there is a constant $C = C(\varphi,\ \psi)$* s.t.

$$\left| \int \varphi \cdot (\psi \circ f^n) d\mu - \int \varphi d\mu \cdot \int \psi d\mu \right| \leq C\tau^n \quad \forall n \geq 0.$$

Exponential decay of correlations has been proved for various classes of interval maps. The piecewise uniformly expanding case is treated by Hofbauer and Keller [HK] and extended to allow for infintely many branches by Rychlik [Ry2]. Ziemian [Z] proved the same result for a class of maps satisfying what is sometimes called "the Misiurewicz condition" [M]. Our aim is to relax this condition sufficiently so that our results hold for a positive measure set of parameters in the quadratic and other families.

Our proof of Theorem 3 consists of constructing an extension of $f : ([-1,\ 1],\ \mu) \circlearrowleft$ and analyzing the spectral properties of the Perron-Frobenius operator associated with this extended map. As noted in [K], this knowledge of the spectrum gives us immediately

Theorem 4 (Central limit theorem). *Let f be as in Theorem 2, and let $\varphi : [-1,\ 1] \to$*

$\mathbb{R}$ *be a function with bounded variation and* $\int \varphi d\mu = 0$. *Then*

$$\sigma := \frac{1}{\sqrt{n}} \left[\int \left(\sum_{i=0}^{n-1} \varphi \circ f^i \right)^2 d\mu \right]^{\frac{1}{2}}$$

is well defined, and if $\sigma > 0$, *then* $\forall x \in \mathbb{R}$

$$\mu \left\{ \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \varphi \circ f^i < x \right\} \to \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-\frac{u^2}{2\sigma^2}} du$$

as $n \to \infty$.

Theorem 4 is proved in a similar setting in an unpublished work of Collet [C] using the approach of [BS]. [C] also contains a slightly weaker version of Theorem 3.

We now mention some generalizations of our results. Let I be an interval, and let $f_* : I \circlearrowleft$ be a unimodal map satisfying the Misiurewicz condition. That is, f_* has negative Schwarzian, it has no sinks, and if c is the critical point, then $\inf_{n>0} |f^n c - c| > 0$. Let $\mathrm{End}^3(I)$ denote the space of C^3 endomorphisms of I. Then there exist constants $\alpha_* > 0$ and $\lambda_* > 1$ (depending only on f_*) s.t. for "most" smooth 1-parameter families $\eta : (-1, 1) \to \mathrm{End}^3(I)$, if

$$\begin{aligned} \Delta_\varepsilon(\eta) = \big\{ a \in (-\varepsilon, \varepsilon) \big| \ \text{ if } f = \eta(a), \text{ then } |f^n c - c| \geq e^{-\alpha_* n} \\ \text{and } |(f^n)'(fc)| \geq \lambda_*^n \quad \forall n \geq 1 \big\}, \end{aligned}$$

then $\mathrm{Leb}(\Delta_\varepsilon(\eta)) > 0 \quad \forall \varepsilon > 0$. (See [TTY] for details.) Our results carry over directly to η. That is, Theorem 1 above holds (with different constants), and if f_* satisfies (P3) in §2, then Theorems 2, 3 and 4 are valid also.

In the rest of this paper we will consider only the quadratic family $x \mapsto 1 - ax^2$. This avoids carrying around constants that depend only on f_*.

§2. Some properties of 1-dimensional maps

In this section we try to isolate those properties of $f = f_a$ that are relevant to our discussion. These are essentially the only properties that will be used.

The specific formula of f is immaterial to us, but we will use the fact that it is a C^3 unimodal map with negative Schwarzian and nondegenerate critical point. The fact that f is symmetric about 0 simplifies our notation a little, but this is a totally unnecessary assumption. What is important is that f has certain expanding properties, which we summarize below as properties (P1) and (P2).

Let $(-\delta,\ \delta)$ be a small neighborhood of the critical point. We distinguish between the dynamics of f outside of $(-\delta,\ \delta)$ and that of orbits beginning in $(-\delta,\ \delta)$. (P1) concerns the former:

(P1) *$\exists M \in \mathbb{Z}^+$ and $\lambda > 1$ s.t.*
  (i) *if $x,\ fx, \dots, f^{M-1}x \notin (-\delta,\ \delta)$, then $|(f^M)'x| \geq \lambda^M$;*
  (ii) *for any $k \in \mathbb{Z}^+$, if $x, \dots, f^{k-1}x \notin (-\delta,\ \delta)$ and $f^k x \in (-\delta,\ \delta)$, then $|(f^k)'x| \geq \lambda^k$.*

(P1) holds for f_a for all a sufficiently near 2. First notice that it is satisfied by f_2. One way of seeing this is to conjugate f_2 to its piecewise linear model. (For a more general approach see [M] and the appendix of [CE].) The rest follows because (P1) is an open condition.

For $x \in (-\delta,\ \delta)$, $f'x$ can, of course, be arbitrarily small. (P2) guarantees a period during which $(f^j)'x$ recovers. Suppose $\delta = e^{-k_0}$. Let $I_k = (e^{-(k+1)},\ e^{-k})$ for $k \geq k_0$, and $-I_{-k}$ for $k \leq -k_0$.

(P2) *There is a function $p : (-\delta,\ \delta) - \{0\} \to \mathbb{Z}^+$, constant on I_k and increasing with $|k|$,* *s.t.*

(i) $\frac{1}{2}k \leq p(x) \leq 4k$ *for* $x \in I_{\pm k}$;

(ii) $|(f^j)'(fx)| \approx \text{const}|(f^j)'(f0)| \geq \text{const}(1 \cdot 9)^j \ \forall j < p(x)$;

(iii) $|(f^p)'x| \geq \lambda^p$ *for some* $\lambda > 1$ (*independent of* x).

(P2) holds for all f_a with $a \in \Delta_\varepsilon$, ε sufficiently small. This property is due to Benedicks and Carleson, and the assertion in the last sentence is proved in [BC1] and [BC2]. Since the main construction in our paper is based on this property, let us recall the ideas in their proof:

For $x \neq 0 \in (-\delta, \delta)$, let $p_0(x) = \max\{p \in \mathbb{Z} : |f^j x - f^j 0| < e^{-10\alpha j} \ \forall j < p\}$. First we claim that $\exists C_0$ (independent of δ) s.t. $\forall y \in [fx, f0]$ and $j < p(x)$,

$$\frac{1}{C_0} \leq \left| \frac{(f^j)'y}{(f^j)'(f0)} \right| \leq C_0.$$

This is true because the quotient in question is $\sim \exp \sum_{k=1}^{j} \theta(k)$ where

$$|\theta(k)| = \left| \frac{f'(f^{k-1}y) - f'(f^k 0)}{f'(f^k 0)} \right| \leq \frac{e^{-10\alpha k}}{e^{-\alpha k}}.$$

Using this distortion estimate, we see that

$$ax^2 \cdot \frac{1}{C_0}(1 \cdot 9)^{p_0 - 1} \leq |f^{p_0}[0, \ x]| \leq 1,$$

giving $p_0(x) \leq 4k$ for $x \in I_{\pm k}$ provided k is sufficiently large. The lower bound for $p_0(x)$ is obtained from

$$ax^2 \cdot 4^{p_0 - 1} \geq e^{-10\alpha p_0}.$$

Finally, at time p_0, we must have

$$ax^2 \cdot |(f^{p_0 - 1})'y| \geq e^{-10\alpha p_0}$$

for some $y \in [fx,\ f0]$. This gives

$$|(f^{p_0})'x| \geq 2a|x| \cdot \frac{1}{C_0}\frac{e^{-10\alpha p_0}}{ax^2} \geq \frac{2}{C_0}e^{\frac{p_0}{4}-10\alpha p_0},$$

which is $\geq \lambda^{p_0}$ for some $\lambda > 1$ provided p_0 is reasonably large. For $x \in I_k$, let

$$p(x) = \inf_{y \in I_k} p_0(y).$$

The number $p(x)$ is called the "bound period" of x. In §4 we will use the notion of "total bound period", which is defined as follows: For $x \in (-\delta,\ \delta)$, $\hat{p}(x)$ is the smallest $j > 0$ s.t. $\forall i$ with $0 \leq i < j$, if $f^i x \in (-\delta,\ \delta)$, then $p(f^i x) < j$. It is easy to see that $\hat{p}(x) \leq 6k$ for $x \in I_{\pm k}$, because if $f^i x \in (-\delta,\ \delta)$ for some $i < p(x)$, then

$$\begin{aligned}|f^i x| &\geq |f^i 0| - |f^i 0 - f^i x| \\ &\geq |f^i 0 - f^i x|^{\frac{1}{9}} \\ &\geq \left[ax^2 \cdot \frac{1}{C_0}(1 \cdot 9)^{i-1}\right]^{\frac{1}{9}} \geq x^{\frac{1}{4}},\end{aligned}$$

so $p(f^i x) \leq k$.

(P1) and (P2) together are sufficient for proving Theorem 1. For Theorems 2-4 we need an additional topological mixing condition:

(P3) *For every interval* $I \subset [-1,\ 1]$, $\exists n = n(I)$ *s.t.* $f^n I \supset [f^2 0,\ f0]$.

Lemma 2.1. (P3) *holds for all* f_a *with* $a \in \Delta_\varepsilon$, ε *sufficiently small.*

Proof. Let $f = f_a$, $a \in \Delta_\varepsilon$. First we prove that $\forall I \subset [-1,\ 1]$, $\exists n_0 = n_0(I)$ s.t. $f^{n_0} I \supset I_{k_0}$ or I_{-k_0}. In light of (P1), some $f^j I$ must intersect $(-\delta,\ \delta)$. If $f^j I$ is completely contained in two adjacent I_k's, keep iterating, and note using (P2) that $|f^p(f^j I)| >> |f^j I|$, $p = p(x)$

for $x \in f^j I$. After finitely many returns to $(-\delta,\ \delta)$, there must exist j_1 and $k_1 \in \mathbb{Z}^+$ s.t. $f^{j_1} I \supset I_{k_1}$ or I_{-k_1}. Consider $f^j I_{k_1}$, $j = 1, 2, \dots$, and let j_2 be the first time (after the bound period of $x \in I_{k_1}$) s.t. $f^j I_{k_1} \supset$ some I_k. Since $|f^{j_2} I_{k_1}| >> |I_{k_1}|$, $f^{j_2} I_{k_1}$ must contain some I_{k_2} or I_{-k_2} with $0 < k_2 < k_1$. We then consider $f^j I_{k_2}$ and repeat the argument until some $f^j I_{k_n} \supset I_{k_0}$ or I_{-k_0}.

Next we argue that there is an $n_1 \in \mathbb{Z}^+$ s.t. for all a sufficiently near 2, if $\hat{x}_a$ is the fixed point of f_a in $(0,\ 1)$, then $f_a^{n_1} I_{k_0} \ni \hat{x}_a$. This is obviously true for $a = 2$ and is an open condition. Finally, observe that for $f =$ any f_a if $\hat{I}$ is an interval containing $\hat{x}$, then $\exists n_2 = n_2(\hat{I})$ s.t. $f^{n_2} \hat{I} \supset [f^2 0,\ f0]$. This completes the proof. ∎

§3. Existence and properties of absolutely continuous invariant measures: Proofs of Theorems 1 and 2

Our first goal in this section is to prove Theorem 1 assuming (P1) and (P2). The proof is straightforward: Extend p to $p : [-1,\ 1] \to \mathbb{Z}^+$ by defining $p(x) = 1$ for $x \notin (-\delta,\ \delta)$. Consider the return map $R : [-1,\ 1] \circlearrowright$ given by $R(x) = f^{p(x)}(x)$. Then some power of R is piecewise uniformly expanding. Piecewise uniformly expanding maps have absolutely continuous invariant measures (acim) because their Perron-Frobenius operators do not increase total variation by very much. An acim for f is then constructed from that for R.

More formally, let β be the coarsest partition of $[-1,\ 1]$ into intervals on which p is constant. Define $g : [-1,\ 1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} \left|\frac{1}{R'(x)}\right| & \text{if } x \in \text{interior}(B),\ \text{some } B \in \beta \\ 0 & \text{otherwise.} \end{cases}$$

The Perron-Frobenius operator associated with R, written P_R or simply P when there is no ambiguity, is defined to be

$$P(\varphi)(x) := \sum_{y \in R^{-1}x} g(y)\varphi(y).$$

Note that if $\beta^n := \beta \vee R^{-1}\beta \vee \cdots \vee R^{-n+1}\beta$, and $g_n(x) := g(R^{n-1}x)\cdots g(x)$, then

$$P^n(\varphi)(x) = \sum_{y \in R^{-n}x} g_n(y)\varphi(y).$$

Lebesgue measure is denoted by m throughout this paper. L^1, $\|\cdot\|_1$ etc. in this section refer to $L^1([-1,\ 1],\ m)$. If I is an interval, let $V_I(\varphi)$ denote the total variation of φ on I. If $I = [-1,\ 1]$, then we simply write $V(\varphi)$. Say $\varphi \in BV$ if $V(\varphi) < \infty$. It is notationally convenient to assume that each $B \in \beta$ or β^n is closed, for this allows us to write

$$V(P^n\varphi) \leq \sum_{B \in \beta^n} V(\varphi g_n) = V(\varphi g_n).$$

Lemma 3.1. (1) $\|g_n\|_\infty \to 0$ *as* $n \to \infty$.

(2) $Vg_n < \infty \quad \forall n$.

Proof. (1) follows immediately from (P1) and (P2). We prove (2) with $n = 1$. Let $B = [b_1,\ b_2] \in \beta$. Since $Sf < 0$, $g|_{(b_1,b_2)}$ has exactly one minimum. So

$$\begin{aligned} V_B g &\leq 2\|g1_B\|_\infty + \lim_{x \to b_1^+} g(x) + \lim_{x \to b_2^-} g(x) \\ &\leq 4\|g1_B\|_\infty. \end{aligned}$$

Recall that on $\{p = j\}$, $|R'| \geq \lambda^j$ for some $\lambda > 1$. So

$$Vg \leq V_{[-1,\ -\delta]}g \ + \ V_{[-\delta,\ 1]}g \ + \ 8\sum_{j=1}^{\infty} \lambda^{-j} \ < \infty.$$

It is easy to check inductively that $Vg < \infty \Rightarrow Vg_n < \infty \ \forall n$. ■

The idea that the Perron-Frobenius operator shrinks variation for piecewise uniformly expanding maps is due to Lasota and Yorke [LY]. Assuming the additional condition that $Vg < \infty$, Rychlik [Ry1] extended their result to the case where f may have an infinite number of uniformly expanding pieces.

Lemma 3.2. (following [Ry1]). *Let* $\varphi \in BV$. *Then*

$$\sup_n V(P^n \varphi) < \infty.$$

Proof. Choose N s.t. $\|g_N\|_\infty < \frac{1}{10}$. Since the local oscillation of g_N at each point is $< \frac{1}{5}$ and $Vg_N < \infty$, we can choose a finite partition β' s.t.

$$\max_{B \in \beta'} V_B g_N < \frac{1}{4}.$$

Then

$$\begin{aligned} V(P^N \varphi) &\leq V(\varphi g_N) \\ &= \sum_{B \in \beta'} V_B(\varphi g_N) \\ &\leq \sum_{B \in \beta'} [V_B \varphi \cdot \|g_N\|_\infty + \|\varphi 1_B\|_\infty \cdot V_B g_N]. \end{aligned}$$

Using the fact that

$$\|\varphi 1_B\|_\infty \leq \frac{1}{m(B)} \left| \int_B \varphi \, dm \right| + V_B \varphi,$$

we get

$$V(P^N \varphi) \leq \sum_{B \in \beta'} \left[V_B \varphi \cdot (\|g_N\|_\infty + V_B g_N) + \frac{V_B g_N}{m(B)} \left| \int_B \varphi \, dm \right| \right],$$

which is $< \frac{1}{2}V(\varphi) + C\|\varphi\|_1$ with $C = \max_{B\in\beta'} \frac{V_B g_N}{m(B)}$. Since $\|P\varphi\|_1 \le \|\varphi\|_1$, it follows from this that

$$V(P^{kN}\varphi) \le \left(\frac{1}{2}\right)^k V(\varphi) + \text{const}\|\varphi\|_1$$

for all k. ∎

Proof of Theorem 1. Let $\varphi \equiv \frac{1}{2}$. Since $V\left(\frac{1}{n}\sum_{i=0}^{n-1} P^n(\varphi)\right) \le$ some C $\forall n$, a subsequence of these functions converges in L^1 to some $\varphi_1 \in BV$. Clearly, $\varphi_1 dm$ is an R-invariant probability measure. Let P_f denote the Perron-Frobenius operator associated with f, and let

$$\varphi_0 := \sum_{k=0}^{\infty} P_f^k\left(\varphi_1 \cdot 1_{\{p>k\}}\right).$$

Then $\varphi_0 dm$ is a finite f-invariant measure, finite because $\sum_{k=1}^{\infty} \|\varphi_1\|_\infty m\{p > k\} < \infty$. (See (P2).) Normalizing $\varphi_0 dm$, we obtain our desired acim μ. To see that the density of μ has the properties as claimed, let $\rho_1 = \varphi_1 / \int \varphi_0 dm$. Since

$$P_f\left(\varphi_1 \cdot 1_{\{p>1\}}\right)(s) \le \frac{\text{const}\|\varphi_1\|_\infty}{\sqrt{|s-1|}},$$

(P2) (ii) tells us that for $k > 1$,

$$P_f^k\left(\varphi_1 \cdot 1_{\{p>k\}}\right)(s) \le \frac{\text{const}\|\varphi_1\|_\infty (1\cdot 9)^{-(k-1)}}{\sqrt{|s - f^k 0|}}.$$

∎

Next we prove Theorem 2 assuming (P1) - (P3). Note that (P1) and (P2) imply that f has a positive Lyapunov exponent at $\mu-$ a.e. x. First we recall a theorem of Ledrappier:

Theorem [L]. *Let $h : [-1, 1] \circlearrowleft$ be a $C^{1+\varepsilon}$ piecewise monotonic map with nonflat critical points, and let ν be an acim with a positive Lyapunov exponent a.e. Then a.e. ergodic component of h is an acim. Moreover, if (h^n, ν) is ergodic $\forall n > 0$, then the natural extension of (h, ν) is isomorphic to a Bernoulli shift.*

Our next proposition probably has some independent interest. It is a corollary to the proofs in [L].

Proposition 3.3. *Let $h : [-1, 1] \circlearrowleft$ be a $C^{1+\varepsilon}$ piecewise monotonic map with k critical points, all of which nonflat. Let ν be an acim with a positive Lyapunov exponent a.e. Then*

(1) *ν is the sum of at most k ergodic acim's;*

(2) *each ergodic component of ν is supported on a finite number of intervals, on which the density is > 0 a.e.;*

(3) *if h is topologically mixing, then (h, ν) is measure-theoretically mixing, and hence Bernoulli.*

Proof of Proposition 3.3. Let $J_0, \ldots, J_k$ be the intervals of monotonicity of h. It is shown in [L] that if we view the natural extension $(\bar{h}, \bar{\nu})$ of (h, ν) as living on

$$\Omega = \{(x, \underline{a}) \in [-1, 1] \times \{0, \ldots, k\}^{\mathbb{N}} | \exists x_0, x_1, \ldots \in [-1, 1] \text{ with } x_0 = x, \\ h(x_i) = x_{i-1} \text{ and } x_i \in J_{a_i}\},$$

then local unstable manifolds of $\bar{h}$ are canonically identified with subintervals of $[-1, 1]$ and acim's on $[-1, 1]$ with positive Lyapunov exponents correspond to measures on Ω with smooth coniditional measures on unstable manifolds.

Let ν_i be an ergodic component of ν. It follows from the discussion above that at ν_i- a.e. x, there is an interval J^i_x containing x on which the density of ν_i is > 0 a.e. Let $J^i_1, J^i_2, \ldots$ be the maximal intervals on which the density of ν_i is > 0 a.e. By the ergodicity of ν_i, there can only be a finite number of these. Moreover, it is not possible for $h|J^i_j$ to be $1-1$ for all j, because h has a positive Lyapunov exponent and the J^i_j's cannot grow indefinitely in length. So at least one of them must get "folded", i.e. $\exists j = j(i)$ s.t. J^i_j contains a critical point in its interior. This puts a bound on the number of ergodic components.

Assertion (3) follows from what we have said and the last statement in the theorem we quoted from [L]. ■

Proof of Theorem 2. Since $\rho_1 \in BV$, there is an interval $I \subset [-1,\ 1]$ s.t. $\inf_{x \in I} \rho(x) > 0$. By virtue of (P3), we have $\inf_{x \in [f^2 0, f 0]} \rho(x) > 0$. The rest follows immediately from Proposition 3.3 and (P3). ■

§4. Decay of correlations: Main steps in the Proof of Theorem 3

Let f be as in Theorem 2.

Step I. Construction of an extension of $f : ([-1,\ 1],\ \mu) \circlearrowleft$.

Recall that there is a function $p : [-1,\ 1] - \{0\} \to \mathbb{Z}^+$ with the following properties:

- $\star$ $p(x) \equiv 1 \quad \forall x \notin (-\delta,\ \delta)$,
- $\star$ $\frac{1}{2}k \le p(x) \le 4k \quad \forall x \in I_{\pm k}$,
- $\star$ $P(x) \uparrow$ as $|x| \downarrow 0$,
- $\star$ $\exists \lambda > 1$ s.t. $\forall x \neq 0 \in (-\delta,\ \delta)$, $|(f^p)' x| \ge \lambda^p$.

For reasons to become obvious shortly, we will choose λ with $\lambda e^{-\frac{1}{4}} < 1$.

Our new space J is the disjoint union of a countable number of intervals $J_1,\ J_2, \dots$, where $J_1 = [-1,\ 1]$ and for $k > 1$, J_k has length $\lambda^{\frac{1}{2}(k-1)} m\{p \ge k\}$. Note immediately that

$$\sum_1^\infty m(J_k) \le 1 + \sum_{k>1} \lambda^{\frac{1}{2}(k-1)} \cdot 2 \min\left\{\delta,\ e^{-\frac{k}{4}}\right\} < \infty.$$

We now define a map $\tilde{f} : J \circlearrowleft$. Let

$$J_1^- = [-1,\ -\delta], \quad J_1^0 = (-\delta,\ \delta), \quad J_1^+ = [\delta,\ 1].$$

On $J_1^- \cup J_1^+$, let $\tilde{f} = f$ in the sense that $f(J_1^{\pm}) \subset J_1$. The interval J_1^0 is mapped affinely onto J_2, with a magnification of $\lambda^{\frac{1}{2}}$. Assume now that $\tilde{f}$ has been defined on $J_1, \dots, J_{k-1}$. We again write $J_k = J_k^- \cup J_k^0 \cup J_k^+$ where $J_k^0 = \tilde{f}^{k-1}\{p > k\}$. The map $\tilde{f}$ then takes J_k^0 affinely onto J_{k+1}, magnifying it by $\lambda^{\frac{1}{2}}$, and takes $J_k^{\pm}$ into J_1 in such a way that $\forall x \in \{p = k\} \subset J_1^0$, $\tilde{f}^p x = f^p x$. Note that on $J_k^{\pm}$, $|\tilde{f}'| \geq \lambda^{\frac{1}{2}k}$.

Clearly, there is a projection $\pi : J \to [-1,\ 1]$ s.t. $\pi \circ \tilde{f} = f \circ \pi$.

Next observe that μ lifts to an $\tilde{f}$-invariant probability measure $\tilde{\mu}$ on J with density

$$\tilde{\rho} := \sum_{k=1}^{\infty} P_{\tilde{f}}^k \left(\rho_1 \cdot 1_{\{p>k\}}\right),$$

where ρ_1 is as in Theorem 1. We mention a few relevant properties of $\tilde{\rho}$. For a function $\varphi : J \to \mathbb{R}$ or $\mathbb{C}$, let us use the notation φ_k to denote $\varphi|_{J_k}$. Then $\tilde{\rho}_k \in BV\ \forall k$ and $\|\tilde{\rho}_k\|_\infty \leq \lambda^{-\frac{1}{2}(k-1)}\|\tilde{\rho}_1\|_\infty$. Since $p > 1$ on $(-\delta,\ \delta)$, the support of $\tilde{\rho}$ is contained in $[f^2\delta,\ f\delta] \cup \bigcup_{k>1} J_k$. Henceforth let us change J_1 to $[f^2\delta,\ f\delta]$. Moreover, if $I \subset J_1$ is an interval, then $\exists n = n(I)$ s.t. $\tilde{f}^n I$ contains the fixed point of $\tilde{f}$ in J_1^+. (This follows from the corresponding statement for f.) So the same argument as in the last section tells us that $\inf_{x\in J_1} \tilde{\rho}_1(x) > 0$.

Lemma 4.1. *$(\tilde{f},\ \tilde{\mu})$ is exact.*

Proof. Let $\tilde{\beta}$ be the partition of J into $\{J_k^{\pm,0}\}$. We have $H_{\tilde{\mu}}(\tilde{\beta}) < \infty$ because const $e^{-2k} \leq \tilde{\mu}(J_k^0) \leq$ const $e^{-\frac{k}{4}}\forall k$, and $\tilde{\mu}(J_k^{\pm})$ is either 0 or has similar bounds. Since $\tilde{f}$ is essentially expanding, $\tilde{\beta}$ is clearly a generator. So if $\tilde{\Gamma} : (\overline{\tilde{f}},\ \overline{\tilde{\mu}}) \to (\tilde{f},\ \tilde{\mu})$ is the natural extension of $(\tilde{f},\ \tilde{\mu})$, and B is the Borel σ-algebra on J, then $\tilde{\pi}^{-1}\left(\bigwedge_{n\geq 0}\tilde{f}^{-n}B\right)$ is contained in the Pinsker σ-algebra of $(\overline{\tilde{\tilde{f}}},\ \overline{\tilde{\tilde{\mu}}})$ ([Ro] 12.3). It therefore suffices to prove the triviality of the Pinsker σ-algebra of $(\overline{\tilde{\tilde{f}}},\ \overline{\tilde{\tilde{\mu}}})$.

We will show that $(\overline{\tilde{\tilde{f}}},\ \overline{\tilde{\tilde{\mu}}})$ is measure-theoretically isomorphic to $(\bar{f},\ \bar{\mu})$, the natural extension of $(f,\ \mu)$. (See Theorem 2.) Let $\underline{\tilde{x}} = (\tilde{x}_0,\ \tilde{x}_1, \dots)$ be a history of $(\tilde{f},\ \tilde{\mu})$ and define $\Phi(\underline{\tilde{x}}) = \underline{x} = (x_0,\ x_1, \dots)$ by letting $x_i = \pi\tilde{x}_i$. Clearly $\Phi_*(\overline{\tilde{\tilde{\mu}}}) = \bar{\mu}$, since $\Phi_*(\overline{\tilde{\tilde{\mu}}})$ is $\bar{f}$-invariant and projects onto μ. So all we have to do is to verify that Φ is $1-1$ a.e.

Let $\underline{x} = (x_0,\ x_1, \dots)$ be a typical element of $(\bar{f},\ \bar{\mu})$. We say that x_i is a "marker" for $\underline{x}$ if $x_i \in (-\delta,\ \delta)$ and $\nexists j > i$ s.t. $\hat{p}(x_j) > j - i$, $\hat{p}$ being the total bound period defined in §2. If x_i is a marker, and $\Phi(\underline{\tilde{x}}) = \underline{x}$, then the only possibility for $\tilde{x}_i$ is that $\tilde{x}_i = x_i \in J_1$. Hence $\Phi^{-1}\{\underline{x}\}$ contains at most one point if $\underline{x}$ has infinitely many markers.

Assume for simplicity that $x_0 \in (-\delta,\ \delta)$. From Theorem 1 we know that $\Sigma_j\ \mu\left(-e^{-\frac{j}{6}},\ e^{-\frac{j}{6}}\right) < \infty$, so by the Borel-Cantelli Lemma, $\exists k(\underline{x}) \in \mathbb{Z}^+$ s.t. $x_j \notin \left(-e^{-\frac{j}{6}},\ e^{-\frac{j}{6}}\right)\ \forall j > k(\underline{x})$. This guarantees that for $j > k(\underline{x})$, $\hat{p}(x_j) < j$. Suppose that x_0 is not a marker. Let $j_1 > 0$ be s.t. $x_{j_1} \in (-\delta,\ \delta)$ and $p(x_{j_1}) > j_1$. Then either x_{j_1} is a marker, or $\exists j_2 > j_1$ s.t. $x_{j_2} \in (-\delta,\ \delta)$ and $p(x_{j_2}) > j_2 - j_1$. If this process continued, there would be a $j_n > k(\underline{x})$ s.t. $\hat{p}(x_{j_n}) > j_n$, contradicting our choice of $k(\underline{x})$. ∎

Step II. The Perron-Frobenius operator.

For the rest of this paper let m denote Lebesgue measure on J, and let $\|\cdot\|_1$, $\|\cdot\|_\infty$ denote the L^1 and L^∞ norms wrt m on J. For $\varphi : J \to \mathbb{C}$, we define

$$V(\varphi) = \sum_k V(\varphi_k),$$

where $V(\varphi_k)$ is the total variation of φ on J_k. Consider the function space

$$X = \{\varphi : J \to \mathbb{C} | V(\varphi) < \infty,\ \|\varphi\|_1 < \infty,\ \|\varphi\|_\infty < \infty\}$$

with norm

$$\|\varphi\| := V(\varphi) + \|\varphi\|_1 + \varepsilon \|\varphi\|_\infty,$$

where $\varepsilon > 0$ is a small number to be determined later. $(X, \ \|\cdot\|)$ is a Banach space.

As usual, the Perron-Frobenius operator $\tilde{P}$ associated with $\tilde{f}$ is defined to be

$$\tilde{P}(\varphi) = \sum_{y \in \tilde{f}^{-1} x} \varphi(y) \tilde{g}(y)$$

where

$$\tilde{g}(y) = \frac{1}{|\tilde{f}' y|}$$

except possibly at the end points of $\tilde{\beta} = \{J_k^{\pm,0}\}$. For purposes of estimating variation it is convenient to adopt the following convention. For $B \in \tilde{\beta}^n$ and $a \in \partial B$, we consider a as belonging in B if for some j, $0 \le j < n$, $\tilde{f}^j B \subset$ some $J_k^{\pm}$ and $\tilde{f}^j a \in \partial J_k^{\pm}$; otherwise we say $a \notin B$. The advantage of this convention is that if we let $\tilde{g} = 0$ on $\partial J_k^{\pm}$ and $\tilde{g}_n(x) = \tilde{g}(\tilde{f}^{n-1} x) \cdots \tilde{g}(x)$, then we can estimate $V(\tilde{P}^n \varphi)$ by

$$V(\tilde{P}^n \varphi) \le \sum_{B \in \tilde{\beta}^n} V_B(\varphi \tilde{g}_n)$$

(which is clear), and at the same time have

$$\sum_{B \in \tilde{\beta}^n} V_B(\tilde{g}_n) < \infty.$$

(See Sublemma 5.1 and cf. [Ry1].)

The next lemma contains the main estimates in this paper. Its proof is postponed to §5.

Lemma 4.2. *$\tilde{P} : X \to X$ is a bounded linear operator whose spectrum*

$$\sigma(\tilde{P}) = \sigma_0 \cup \{\xi_1, \cdots, \xi_n\}$$

where $\sigma_0 \subset \{|z| \leq \tau_0\}$ for some $\tau_0 < 1$ and $\xi_i \in S^1$ $\forall i$. Moreover, each ξ_i is a smiple pole, and the corresponding projection has a finite dimensional range.

Step III. Finishing the proof.

First we use the exactness of $(\tilde{f},\ \tilde{\mu})$ to prove that $\sigma(\tilde{P}) \cap S^1 = \{1\}$. Let $\varphi \in X$ be s.t. $\tilde{P}(\varphi) = \xi\varphi$ for some $\xi \in S^1$. Since $\tilde{\rho} > 0$, we can write $\varphi = \theta\tilde{\rho}$ for some θ. Note that $\theta \in L^2(\tilde{\mu})$, for

$$\|\theta_k\|_\infty \leq \frac{\|\varphi\|_\infty}{\min \tilde{\rho}_k} \leq \|\varphi\|_\infty \cdot \frac{\lambda^{\frac{1}{2}(k-1)}}{\min \tilde{\rho}_1},$$

so that

$$\int \theta^2 d\tilde{\mu} \leq \text{const} \sum_k \lambda^k e^{-\frac{k}{4}},$$

which is $< \infty$ by the upper bound we imposed on λ.

The rest of this argument is quite standard (see e.g. [HK]). We let $U = U_{\tilde{f}}$ be the operator on $L^2(\tilde{\mu})$ defind by $U(\psi) = \psi \circ \tilde{f}$. Then $U^*\theta = \xi\theta$ (in the sense of L^2) because

$$\begin{aligned}\langle U^*\theta,\ \psi\rangle &= \int (\bar{\psi} \circ \tilde{f})\theta\tilde{\rho}dm \\ &= \int \bar{\psi} \cdot \tilde{P}(\theta\tilde{\rho})dm = \left\langle \frac{\tilde{P}(\theta\tilde{\rho})}{\tilde{\rho}},\ \psi \right\rangle = \langle \xi\theta,\ \psi\rangle\end{aligned}$$

for every $\psi \in L^2(\tilde{\mu})$. From this and $|\xi| = 1$ we deduce that $U\theta = \bar{\xi}\theta$, which means that $\theta = \xi^n U^n \theta$ is measurable wrt $\tilde{f}^{-n}B$. Hence $\theta =$ some constant c a.e. by Lemma 4.1. Thinking of $\tilde{P}$ as an operator on $L^1(m)$, one sees immediately that $\xi = 1$.

We have shown that $\varphi = c\tilde{\rho}$ a.e. To see that $\varphi = c\tilde{\rho}$ as elements of X, use Lemma 5.1. This together with Lemma 4.2 proves that $\tilde{P} = \tilde{P}_0 + \tilde{P}_1$, where $\tilde{P}_0\tilde{P}_1 = \tilde{P}_1\tilde{P}_0 = 0$, $\sigma(\tilde{P}_0) \subset \{|z| \le \tau_0\}$, and $\tilde{P}_1(\varphi) = c_\varphi \tilde{\rho}$ for some $c_\varphi \in \mathbb{C}$. In fact, $c_\varphi = \int \varphi dm$, because $\int \tilde{P}_1 \varphi dm = \int \tilde{P}_1^n \varphi dm \overset{\text{large } n}{\approx} \int \tilde{P}^n \varphi dm = \int \varphi dm$.

Finally we return to the original dynamical system $f : ([-1, 1], \mu) \circlearrowleft$. Observe that $\varphi \in BV([-1, 1])$ lifts to $\tilde{\varphi} : J \to \mathbb{R}$ with $\tilde{\varphi}\tilde{\rho} \in X$:

$$\begin{aligned} V(\tilde{\varphi}\tilde{\rho}) &\le \sum_k \left[V(\tilde{\varphi}_k)\|\hat{\rho}_k\|_\infty + V(\tilde{\rho}_k)\|\tilde{\varphi}_k\|_\infty \right] \\ &\le \sum_k \left[2V(\varphi) \cdot \lambda^{-\frac{1}{2}(k-1)}\|\tilde{\rho}_1\|_\infty + \lambda^{-\frac{1}{2}(k-1)} V(\tilde{\rho}_1) \cdot \|\varphi\|_\infty \right] < \infty. \end{aligned}$$

We have thus shown that $\forall \varphi,\ \psi \in BV([-1, 1])$,

$$\begin{aligned} & \left| \int (\psi \circ f^n)\varphi d\mu - \int \varphi d\mu \cdot \int \psi d\mu \right| \\ = & \left| \int (\tilde{\psi} \circ \tilde{f}^n)\tilde{\varphi} d\tilde{\mu} - \int \tilde{\varphi} d\tilde{\mu} \cdot \int \tilde{\psi} d\tilde{\mu} \right| \\ = & \left| \int \tilde{\psi} \left[\tilde{P}^n(\tilde{\varphi}\tilde{\rho}) - \left(\int \tilde{\varphi}\tilde{\rho} dm \right) \tilde{\rho} \right] dm \right| \\ \le & \quad m(J) \cdot \|\psi\|_\infty \cdot C(\varphi,\ \psi)\tau^n \qquad\qquad \text{for some} \quad \tau_0 < \tau < 1. \end{aligned}$$

This completes the proof. ■.

§5. Spectral properties of the Perron-Frobenius operator

The purpose of this section is to prove Lemma 4.2. Recall that our norm consists of a total variation part, an L^1-part, and an L^∞-part. Roughly speaking, the variation part is contracted by $\tilde{P}$; the L^1-part does no harm; and the L^∞-part is needed to control what happens on the J_k's for k large.

¿From here on we will be working exclusively with $\tilde{f} : J \circlearrowleft$. So for notational simplicity let us drop all the $\sim$'s in $\tilde{f},\ \tilde{\beta},\ \tilde{P}$ etc. The following line of argument is also fairly standard:

Lemma 5.1. (1) *P is a bounded operator on X;*

(2) $\exists N \in \mathbb{Z}^+$ *and* $R \geq 0$ *s.t.* $\forall \varphi \in X$,

$$\|P^N \varphi\| \leq \frac{1}{5}\|\varphi\| + R\|\varphi\|_1.$$

Using Lemma 5.1, and remembering that $\|P\varphi\|_1 \leq \|\varphi\|_1$, one deduces immediately that $\sup_n \|P^n\| < \infty$. This gives $\sigma(P) \subset \{|z| \leq 1\}$.

Lemma 5.2. *For N sufficiently large, there is a finite rank operator Q with $\|P^N - Q\| < 1$.*

This second lemma tells us that outside of some disk of radius < 1, $\sigma(P)$ consists of at most a finite number of points $\eta_1, \ldots, \eta_\ell$, and that the projection corresponding to each η_i has a finite dimensional range ([DS] VIII 8). No $\eta_i \in S^1$ can be a pole of order > 1, for that would violate $\sup_n \|P^n\| < \infty$. This completes the proof.

Sublemma 5.1a. $\sup_{B \in \beta^N} V_B(g_N) < \infty\ \forall N$.

Proof. Since $|f'x| \geq \delta\ \forall x$ and f maps each element of β to at most 3 elements of β, we have

$$\sum_{\substack{B \in \beta^N \\ B \subset J_1}} V_B(g_N) \leq 3^N \cdot 4\delta^{-N}.$$

(See the proof of Lemma 3.1.) Next consider $B \in \beta^N$ with $B \subset J_k,\ k > 1$. If $f^j B \subset$

$\left(\bigcup_{i>1} J_i\right) \forall j \leq N$, then $V_B g_N = 0$. If not, then $|(f^N)'x| \geq \delta^N \lambda^{\frac{1}{2}(k-1)} \forall x \in B$, giving

$$\sum_{\substack{B\in\beta^n\\ B\subset J_k}} V_B(g_N) \leq 3^N \cdot 4\delta^{-N} \cdot \lambda^{-\frac{1}{2}(k-1)}.$$

∎

We now choose and fix an $N \in \mathbb{Z}^+$ with $\|g_N\|_\infty \leq \frac{1}{100}$ and $\lambda^{-\frac{N}{2}} \leq \frac{1}{10}$.

Sublemma 5.1b. *For every $\varepsilon > 0$, $\exists R_1 = R_1(N, \ \varepsilon)$ s.t. $\forall \varphi \in X$,*

$$V(P^N\varphi) \leq \frac{1}{10}V(\varphi) + R_1\|\varphi\|_1 + \frac{1}{10}\varepsilon\|\varphi\|_\infty.$$

Proof.

$$\begin{aligned} V(P^N\varphi) &\leq \sum_{B\in\beta^N} V_B(\varphi g_N) \\ &\leq \sum_B \|g_N\|_\infty V_B(\varphi) + \sum_B \|\varphi 1_B\|_\infty V_B(g_N). \end{aligned}$$

The first term is $\leq \|g_N\|_\infty V(\varphi)$, and the second term is estimated as follows. For $K \in \mathbb{Z}^+$,

$$\begin{aligned} &\sum_{\substack{B\in\beta^N\\ B\subset\bigcup_{k\leq K} J_k}} V_B(g_N) \cdot \|\varphi 1_B\|_\infty \\ &\leq \sum V_B(g_N)\left\{\frac{1}{mB}\left|\int_B \varphi dm\right| + V_B(\varphi)\right\} \\ &\leq \left(\max_B V_B(g_N)\right) \cdot \left(\sum_B V_B(\varphi)\right) + \left(\max_B \frac{V_B(g_N)}{mB}\right) \cdot \left(\sum_B \left|\int_B \varphi\right|\right) \\ &\leq \left(4\|g_N\|_\infty\right) \cdot V(\varphi) + R_{1,K}\|\varphi\|_1 \end{aligned}$$

where $R_{1,K}$ is a constant depending on K. Choose K s.t.

$$\sum_{\substack{B\in\beta^N \\ B\subset\bigcup_{k>K} J_k}} V_B(g_N) < \frac{1}{10}\varepsilon.$$

$\blacksquare$

Sublemma 5.1.c. $\forall$ *sufficiently small* $\varepsilon > 0$, $\exists R_2 = R_2(N, \varepsilon)$ s.t.

$$\varepsilon\|P^N\varphi\|_\infty \le \frac{1}{10}V(\varphi) + R_2\|\varphi\|_1 + \frac{1}{10}\varepsilon\|\varphi\|_\infty.$$

Let us explain the idea of the lemma, assuming $N = 1$. A formal proof will follow. We wanted to say that $\|P\varphi\|_\infty < \|\varphi\|_\infty$. Clearly, $\|(P\varphi)\cdot 1_{\bigcup_{k>1} J_k}\|_\infty \le \lambda^{-\frac{1}{2}}\|\varphi\|_\infty$, but for $(P\varphi)_1$ all we can say is that $\|(P\varphi)_1\|_\infty \le C\|\varphi_1\|_\infty + 2\sum_{k>1}\lambda^{-\frac{1}{2}(k-1)}\|\varphi_k\|_\infty$ for some C. Choose K_0 s.t. $2\sum_{k>K_0}\lambda^{-\frac{1}{2}(k-1)}$ is small. For $1 < k \le K_0$, we can write $\|\varphi_k\|_\infty \le V(\varphi_k) + |\int\varphi_k|/m(J_k)$. Together this gives us

$$\|P\varphi\|_\infty \le \text{ const}\cdot(V(\varphi) + \|\varphi\|_1) + \left(\lambda^{-\frac{1}{2}} + \text{ small}\right)\cdot\|\varphi\|_\infty,$$

which explains the ε-weight in the L^∞-part of the norm $\|\cdot\|$.

Proof of Sublemma 5.1c. Since

$$\sup_{k>N}\|(P^N\varphi)_k\|_\infty \le \lambda^{-\frac{N}{2}}\|\varphi\|_\infty \le \frac{1}{10}\|\varphi\|_\infty,$$

we only have to consider $(P^N\varphi)_k$ for $k = 1, \ldots, N$. Let N_0 be a large number to be specified shortly. Define

$$M_1(\varphi) = 2\left(\sum_{k>N_0}\lambda^{-\frac{k}{2}}\right)\|\varphi\|_\infty,$$

$$M_2(\varphi) = \max\{\|\varphi_1\|_\infty, \ldots, \|\varphi_{N_0}\|_\infty\}$$

and $$M = \max\{M_1, M_2\}.$$

One verifies inductively that $\forall j,\ k \le N,\ \|(P^j\varphi)_k\|_\infty \le D^j M$ where

$$D = \frac{1}{a\delta} + 2N + 2\sum \lambda^{-\frac{k}{2}}.$$

For instance,

$$\|(P^{j+1}\varphi)_1\|_\infty \le \frac{1}{a\delta}(D^j M) + 2N(D^j M) + \left(2\sum \lambda^{-\frac{k}{2}}\right) M,$$

the three terms being contributions from $(P^j\varphi)_1$, $\overset{j}{\underset{2}{\Sigma}}(P^j\varphi)_i$ and $\underset{i>j}{\Sigma}(P^j\varphi)_i$ respectively.

We have proved that

$$\|P^N\varphi\|_\infty \le \max\left\{\frac{1}{10}\|\varphi\|_\infty,\ D^N M(\varphi)\right\}.$$

If $\frac{1}{10}\|\varphi\|_\infty$ is bigger, we are done. So suppose $D^N M(\varphi)$ dominates. We choose N_0 and ε s.t.

$$D^N\left(2\sum_{k>N_0}\lambda^{-\frac{k}{2}}\right) \le \frac{1}{10} \qquad \text{and} \qquad 100\varepsilon D^N \le \frac{1}{100}$$

and consider the following possibilities:

Case 1: $M = M_1$. We have $\|P^N\varphi\|_\infty \le D^N M_1 \le \frac{1}{10}\|\varphi\|_\infty$.

Case 2: $M \le 100V(\varphi)$. We have $\varepsilon\|P^N\varphi\|_\infty \le \frac{1}{100}V(\varphi)$.

Case 3: $M = M_2$ and $M_2 > 100V(\varphi)$. Let $k \le N_0$ be s.t. $\|\varphi_k\|_\infty = M$. Since

$$M = \|\varphi_k\|_\infty \le \frac{1}{mJ_k}\|\varphi_k\|_1 + V(\varphi),$$

it follows that

$$\frac{99}{100}M \le \frac{1}{mJ_k}\|\varphi_k\|_1$$

and

$$\|P^N\varphi\|_\infty \le \left(\frac{100}{99}\lambda^{\frac{N_0}{2}} D^N\right) \|\varphi\|_1 .$$

∎

Proof of Lemma 5.1. We fix ε acceptable with regard to 5.1c and take $R = R_1 + R_2 + 1$. (Note that if we had defined $\|\cdot\| = V(\cdot) + \|\cdot\|_1$, then P could be unbounded. Take $\varphi^{(k)} = \frac{1}{mJ_k} 1_{J_k}$.) ∎

Proof of Lemma 5.2. Let $N \in \mathbb{Z}^+$ be as in Lemma 5.1, and let $E = E_{\beta^N}$ be the conditional expectation wrt the σ-algebra generated by β^N on J. Define

$$\hat{Q}(\varphi) = P^N(E\varphi)$$

and

$$Q_j(\varphi) = P^N\left(E\left(\varphi \cdot 1_{\cup_{k\le j} J_k}\right)\right).$$

We claim that for sufficiently large j, $Q = Q_j$ has the desired property.

Sublemma 5.2a. $\|P^N - \hat{Q}\| \le \frac{1}{10}$.

Proof. Let $\theta = \varphi - E\varphi$, so that $(P^N - \hat{Q})\varphi = P^N\theta$.

(i) $V(P^N\theta) \le \|g_N\|_\infty \sum_{B\in\beta^N} V_B(\theta) + \max_{B\in\beta^N} V_B(g_N) \cdot \sum_B \|\theta \cdot 1_B\|_\infty$.

Since $V_B(\theta) = V_B(\varphi)$ and $\|\theta 1_B\|_\infty \le V_B(\theta)$, we conclude that

$$V(P^N\theta) \le \frac{5}{100} V(\varphi).$$

(ii) $\|P^N\theta\|_1 \le \sum_{B\in\beta^N} mB \cdot V_B(\theta) \le \frac{1}{100} V(\varphi)$.

(iii) The same argument as in 5.1c (case 3 cannot occur) gives

$$\varepsilon \|P^N \theta\|_\infty \le \max\left\{\frac{1}{10}\varepsilon\|\theta\|_\infty,\ \frac{1}{100}V(\varphi)\right\}.$$

■

To complete the proof of Lemma 5.2, consider an arbitrary $\varphi \in X$. Write

$$\begin{aligned} & \|P^N\varphi - Q_j\varphi\| \\ \le \quad & \left\|(P^N - \hat{Q})\left(\varphi 1_{\cup_{k\le j} J_k}\right)\right\| \quad + \quad \left\|P^N\left(\varphi 1_{\cup_{k>j} J_k}\right)\right\| \quad . \end{aligned}$$

The first term is $\le \frac{1}{10}\|\varphi\|$ by 5.2a, and the second term is

$$\le \quad \frac{1}{5}\|\varphi\| \quad + \quad R\left\|\varphi 1_{\cup_{k>j} J_k}\right\|_1$$

by Lemma 5.1. Since

$$\left\|\varphi 1_{\cup_{k>j} J_k}\right\|_1 \le \left(\sum_{k>j} mJ_k\right) \cdot \|\varphi\|_\infty \le \frac{1}{\varepsilon}\left(\sum_{k>j} mJ_k\right) \cdot \|\varphi\|,$$

we have $\|P^N\varphi - Q_j\varphi\| \le \frac{1}{2}\|\varphi\|$ for sufficiently large j. ■

Note added after completion of manuscript. The author has heard that G. Keller and T. Nowicki recently obtained some related results.

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