1. Validity of the Airy expansion in an arc of radius $2\pi - \epsilon$

We are interested in the Airy equation:

$$\frac{d^2 u}{dz^2} + zu = 0$$

We recall that using a formal argument, we derived the following asymptotic series expansion for two solutions of the Airy equation, $\tilde{u}_+$ and $\tilde{u}_-$.

$$\tilde{u}_\pm = \frac{1}{z^{\frac{1}{4}}} \exp \left( \pm \frac{2}{3} \frac{z^{\frac{3}{2}}}{z^{\frac{3}{2}}} \left[ 1 + \frac{c_1}{z^{\frac{1}{2}}} + \cdots + \frac{c_N}{(z^{\frac{3}{2}})^N} \right] \right)$$

Our goal is to show that for each fixed $N > 0$, we may choose $|z|$ sufficiently large so that there is a solution $u$ to the Airy equation (1) which agrees to order $N$ with the series for $\tilde{u}_+$ truncated to order $N$.

Let $N > 0$ be a fixed integer. Let $u_+$ and $u_-$ denote the truncations of the series $\tilde{u}_+$ and $\tilde{u}_-$, respectively.

Our argument will be as follows: we will consider the second order differential equation satisfied by $u_+$ and $u_-$. It will be of the form $L_\text{truncated} = L_\text{Airy} + \text{high order perturbation terms}$. We will then look for a solution, not to the original Airy equation, but instead, to this perturbed “truncated” differential equation. In other words, we will try to solve $L_\text{truncated} u = \text{perturbation terms}$ for $u$. The essential trick is to solve the ODE problem in a well-chosen function space: we will use the space of all functions whose order $N$ asymptotic series agrees with $u_+$.

We will show this solution exists by using Picard iterations on the associated integral equation. Most of the work will be in showing that the integral operator we construct is a contraction on the space we will have chosen.

Writing this more explicitly, we have

$$L_\text{truncated} = u'' + zu + z^{-n}(P(z^{-\frac{3}{2}})u' + Q(z^{-\frac{3}{2}})u)$$

In this expression, $P$ and $Q$ are polynomials and $n \geq N$ is some fraction. We can write down explicitly what they are, but do not need that extra information.

From this, we obtain the following integral equation for solutions to the Airy equation (by using the variation of constants formula):

$$u(z) = u_+(z) + u_-(z) \int_{\infty}^{z} \frac{u_-(s)F(s)}{W(u_+, u_-)} =: T(u)$$

where

$$W(u_+, u_-) = u_+ u_-' - u_+' u_-$$ the Wronskian of $u_+$ and $u_-$. $F(s) = z^{-n}(P(z^{-\frac{3}{2}})u' + Q(z^{-\frac{3}{2}})u)$.

Observe that in the general variation of constants formula, we would have had an expression symmetric in $u_+$ and $u_-$. We have left out the terms dominated by
$u_-$ since these would contribute terms with the wrong behaviour at infinity. This reflects the fact that, in essence, we are applying “initial conditions at infinity”.

Furthermore, for this to make sense, we need to specify the path of integration from $\infty$ to $z$ so that the integral is finite along this path.

The key point in this argument, the point that allows us to get validity on an angular sector of width $2\pi$, is that we are able to get control of the exponential term on that sector. The higher order terms don’t really matter for this part of the argument.

We wish to make sense of the following:

$$
\int_{\infty}^{z} \frac{(u_+(z)u_-(s))F(s)}{W(u_+, u_-)} \, ds
$$

We note that $W(u_+, u_-)$, $G$ and $H$ are bounded functions of $s$ as $|s| \to \infty$. The problem term comes from the exponential factor in $u_-(s)$. We need to choose a path along which $\exp(i\frac{3}{2}z^{2} - i\frac{3}{2}s^{2})$ is integrable. In other words, we need to find a path along which we can control the real part of $i\frac{3}{2}z^{2} - i\frac{3}{2}s^{2}$. Now, if we write $z = \rho_0 \exp(i\theta_0)$ and $s = \rho \exp(i\theta)$, we end up wanting to control

$$
-\sin\left(\frac{3}{2}\rho_0\right)\rho_0^\frac{3}{2} + \sin\left(\frac{3}{2}\rho\right)\rho^\frac{3}{2}.
$$

If we restrict ourselves to the angular sector $(\frac{-2\pi}{3} + \epsilon, \frac{4\pi}{3} - \epsilon)$, we can do so by careful choose of our arcs. We discussed this point (in a rotated picture) in the problem session. Your notes from lecture should also contain a discussion of how to do this. I will also be providing a picture to illustrate this — check my website for this. Analytically, the idea is that we are now able to get behaviour that is like $-\delta \rho$ when $\rho$ is big and controllable behaviour for small $\rho$.

Now, in order to find our solution, we want to find a fixed point of this integral operator $T$. Let us now consider the following metric space:

$$
\mathcal{X} = \left\{ f \in C^1(\Omega_R, \mathbb{C}) : \left| z^{\frac{1}{2}} \frac{f - u_+}{u_+} \right| \to 0 \text{ as } |z| \to \infty \text{ and } \left| z^{\frac{1}{2}} \frac{f' - u'_+}{u'_+} \right| \to 0 \text{ as } |z| \to \infty \right\}
$$

with the metric $\|f\| = \sup \left( \sup \left| \frac{f}{u_+} \right| , \sup \left| \frac{f'}{u'_+} \right| \right)$

and the region $\Omega_R = \{ z \in \mathbb{C} : |z| \geq R \text{ and } \frac{-2\pi}{3} + \epsilon < \arg(z) < \frac{4\pi}{3} - \epsilon \}$

We are at liberty to choose $R$ large. We observe that this is a complete metric space with our choice of metric.

We will show that $T$ maps $\mathcal{X}$ into itself, and furthermore that $\left\| \frac{T u - u_+}{u_+} \right\| \leq C < 1$. Since $Tu - u_+$ is linear, we will simultaneously establish that $\mathcal{X}$ is invariant under iterations of $T$ and furthermore that $T$ is a contraction.

In order to establish this, we need to control the quantities:

$$
\frac{Tu - u_+}{u_+} = \int_{\infty}^{z} \frac{u_-(z)(P(z^{\frac{3}{2}})u' + Q(z^{\frac{3}{2}})u)}{u_+u'_+ - u_-u'_+} \, dz
$$

$$
\frac{(Tu')' - u'_+}{u'_+} = \frac{(Tu) - u_+}{u_+} + \frac{u_+F(z_0, u(z_0))u_-(z_0)}{u'_+}
$$
I want to show that by choosing $R$ large (as a function of the coefficients of $P$, $Q$, $u_+$, $u_-$, but not of $u$), I can make this quantity be bounded below 1. I also need to show that they each go to zero faster than $|z_0|^\frac{1}{4}$. 

Now, we observe that $W(u_+, u_-)$ is a polynomial in $z^{-\frac{1}{4}}$. Thus, for $|z|$ large, it is bounded. We also note that $P(z^{-\frac{1}{2}})$ and $Q(z^{-\frac{1}{2}})$ are bounded for $|z|$ sufficiently large. Our integrand then behaves like $z^{-n}$ times the following expression:

\[(*) \quad u_- u' - u_- u = u_- u'_+ + u_- u'_+ \frac{u' - u'_+}{u'_+} - u_- u_+ - u_- u_+ \frac{u - u_+}{u_+}.
\]

To estimate these terms, we need only use that $u_+ = \exp(-i\frac{2}{3}z^{\frac{1}{2}})$ (bounded terms).

We observe that $u_- u_+$ and $u_- u'_+$ are polynomials of degree $2N$ in $z^{-\frac{1}{2}}$. In particular, we may take these as bounded for $R$ sufficiently large. The terms involving $\frac{u' - u'_+}{u'_+}$ and $\frac{u - u_+}{u_+}$ go to zero as $z \to \infty$ like a (definite) negative power of $z$ (by the definition of $X$, so again, for $R$ sufficiently large, these are bounded (by a bound independent of $u$).

Thus, we obtain that our first integral behaves roughly like the integral of $z^{-k}$ for some $k > 1$. Thus, for $R$ large, our expression can be made $\leq C < 1$.

In order to get the rate at which the expressions go to zero, I am saved by my $z^{-n}$ term. Since I am free to take $N$ large, I can force $n - \frac{1}{4} > 1$ and so I can get that my integral goes to zero even after multiplying by $|z|^\frac{1}{4}$.

We now want to estimate the quantity $\frac{(Tu)' - u'_+}{u'_+}$.

We note

\[
\left| \frac{(Tu)' - u'_+}{u'_+} \right| \leq \left| \frac{(Tu) - u_+}{u_+} \right| + \left| \frac{u_+}{u'_+} F(z, u(z)) u_-(z) \right|
\]

All that remains to estimate is the second term. We note that $\frac{u_+}{u'_+}$ is bounded and, by the same re-arrangement as we did in $(*)$, the remaining terms are also bounded by a constant times $z^{-n}$.

Thus, we have established that $T$ is a contraction on our function space. The Contraction Mapping Principle then gives us the existence of a unique fixed point. This fixed point is then the solution to the Airy equation whose $N$th order asymptotic expansion matches that of $u_+$.

A similar argument gives the result for $u_-$. We note that in this case, we have to take a different branch cut. Thus, the only region in which both asymptotic series represent solutions is in the angular sector of width $\frac{2\pi}{3}$. This is the region that would have been predicted by the general abstract theory of asymptotic expansions to ODE.

2. Expansion with respect to a large parameter

We consider the following problem:

\[u'' + \lambda^2 f(x)u = 0 \quad \text{for } x \in [-1, 1] \quad \text{where } f(x) > 0\]

Our asymptotic expansions are then of the form:

\[
\exp(\pm i\lambda \int \sqrt{f(x)} dx) \left( 1 + \frac{a_1(x)}{\lambda^2} + \frac{a_2(x)}{\lambda^4} + \cdots + \frac{a_n(x)}{\lambda^{2n}} + \cdots \right)
\]
Let $v_\pm$ denote the truncation of this expansion. Then $v_\pm$ are the solutions of the following second order equation:

$$v'' + \lambda^2 f(x)v = G(x)\frac{v'}{\lambda^k} + H(x)\frac{v}{\lambda^l} =: F(x,v)$$

where $G$ and $H$ are continuous functions on $[-1,1]$.

By the same arguments as in the previous problem, we are looking then for a solution to the inhomogenous problem

$$v'' + \lambda^2 f(x)v + F(x,v) = G(x)\frac{v'}{\lambda^k} + H(x)\frac{v}{\lambda^l} =: F(x,v)$$

We obtain then the integral formulation:

$$u = v_+ \int \frac{v_- F(x,u)}{W(v_+,v_-)} - v_- \int \frac{v_+ F(x,u)}{W(v_+,v_-)} =: Tu$$

Our goal is again to find the a fixed point of our integral operator $T$ in a suitable function space. We need only show this exists for large $\lambda$. So, we will show that $T$ is a contraction on the complete metric space $(C^2([-1,1],\mathbb{R}), |w| = \sup(\sup_{[-1,1]} |w(x)|, \sup_{[-1,1]} |w'(x)|))$. We observe that $T$ is linear in $u$, so it suffices to show that its norm is less than 1. Thus, we need only control $|Tu|$ and $|(Tu)'|$ for large values of $\lambda$. We also note that $G$ and $H$ are bounded on $[-1,1]$. Take $M \geq |G| + |F|$ on $[-1,1]$.

$$|T(u)| \leq |v_+ \int \frac{v_- F(x,u)}{W(v_+,v_-)} - v_- \int \frac{v_+ F(x,u)}{W(v_+,v_-)}|$$

now, we observe:

$$|F(x,u)| \leq \left| \frac{G(x)}{\lambda^k} u' \right| + \left| \frac{H(x)}{\lambda^l} u \right|$$

$$\leq \frac{2M}{\lambda^k} \sup(\sup_{[-1,1]} |u'|, \sup_{[-1,1]} |u|)$$

$$\leq \frac{2M}{\lambda^k} \|u\|$$

so we obtain

$$\sup_{[-1,1]} |T(u)| \leq \frac{2M}{\lambda^k} \|u\| \left( v_+ \int \frac{v_-}{W(v_+,v_-)} + v_- \int \frac{v_+}{W(v_+,v_-)} \right)$$

now, we recall the actual definitions of $v_\pm$ to verify that:

$$v_+ \int \frac{v_-}{W(v_+,v_-)} + v_- \int \frac{v_+}{W(v_+,v_-)} \leq C \text{ as } \lambda \to \infty$$
so we may choose \( \lambda \) large enough so

\[
\sup_{[-1,1]} |T(u)| \leq \frac{1}{2} \|u\|
\]

By nearly identical analysis, we are able to establish that \( \sup_{[-1,1]} |(Tu)'| \leq \frac{1}{2} \|u\| \) and so we obtain

\[
\|T(u)\| \leq \frac{1}{2} \|u\|.
\]

This establishes that \( T \) is a contraction on our space, and the result follows.

3. Acknowledgements

I would like to thank Alexey Kuptsov, Sam Stechmann and Saverio Spagnolie for their significant help with these solutions.