Two pricing approaches for carbon emissions allowances

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Issues and goals

Carbon markets as part of instruments to foster reduction of carbon dioxide emissions in order to respond to climate changes issues:

▶ To what extent are carbon market truly efficient to mitigate CO2 emissions?
▶ How to establish a good market design enabling mitigation?

Our two approaches(*):

▶ Quantitative : An indifference price from the producers view point in European Union Emission Trading Scheme (EU ETS)
▶ Qualitative : A game theory approach for the penalty design in a cap and trade scheme.

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The European Union Emission Trading Scheme

EU ETS : Exchange for allowances involving specific industrial sectors (power generation, cement, iron and steel, paper...).

The Kyoto phase (2008-2012) covers almost the half of the overall GHG emissions in Europe

(source: BlueNext)

Third phase (Strengthening) : 2013-2020
- setting an overall EU cap : a 20% cut in EU economy-wide emissions relative to 1990 levels
- a move from allowances for free to auctioning
EU ETS market rules

▶ Each phase: divided in yearly compliance periods

▶ At the beginning of the period: the state/EU decides how it distributes allowances to producers.

▶ At the end of the period: each agent must own as much allowances as its yearly CO2 (eq) emission quotas.

If excess of emissions: the agent pays a penalty: 100€ per ton CO2 (eq).

▶ In between: agent may sell/buy allowances on organized exchanges (ECX, BlueNext, EEX) or over the counter.
CO2 indifference price: general settings

- The agent control is the production strategy: \( (\pi_t)_{0 \leq t \leq T} \)
  \[ \pi_t = (\pi^1_t, \ldots, \pi^n_t) \in \mathbb{A} := \{ \eta \in \mathbb{R}^n; 0 \leq \eta_i \leq p_{\text{max}}^i \} \]
- The processes under control are:
  - \( E^\pi_t \): CO2 emissions at time \( t \)
  - \( W^\pi_t \): wealth at time \( t \)
- The market parameters:
  - \( \Theta_0 \): allocated allowances at time \( t = 0 \)
  - \( P (\cdot) \): the penalty function, increasing and vanishing on \( \mathbb{R}^- \)

Criterion for optimal control process \( \Pi^* \):

\[
E \left[ U(W^\Pi^*_T - P(E^\Pi^*_T - \Theta_0)) \right] = \sup_{\pi \in \text{Adm}} E \left[ U(W^\pi_T - P(E^\pi_T - \Theta_0)) \right]
\]

\( U \) is a strictly increasing and concave utility function.

Indifference price \( P^{\text{CO2}}(q) \) for buying/selling \( q \) allowances, at \( t = 0 \):

\[
P^{\text{CO2}}(q) = \inf \{ p \in \mathbb{R}; \sup_{\pi} E \left[ U(W^\pi_T - qp - P(E^\pi_T - \Theta_0 - q)) \right] \}
- \sup_{\pi} E \left[ U(W^\pi_T - P(E^\pi_T - \Theta_0)) \right] > 0 \}
\]

For simplicity: interest rate is set to zero
We focus on the electricity production sector

Huge part of the market: 70% of the allocation plan (phase 1).

More flexibility for emissions: fuel switching

Inelastic demand: the spot market price captures the stochastic demand

Global demand, production (mix), and emission on 2012-26-04, (from RTE France)
The wealth dynamics for the electricity producer

- **Stochastic input**: the electricity spot price $S$.
- **A 3D state space** $(w, e, s)$ for $(W, E, S)$.
- **The control** $\pi = (\pi^1, \ldots, \pi^n)$, a progressively measurable process valued in $A$: generated power, plants portfolio (coal, gas, oil, hydro, wind, PV, ...)
- **The dynamics of the wealth process**: for all $t \geq \theta$

$$
\begin{align*}
    dW_{t}^{\pi; \theta, w, s} &= \left\{ (\pi_{t} \cdot 1 - Q_{t}^{OTC}) S_{t}^{\theta, s} + Q_{t}^{OTC} P(t) - C(t, \pi_{t}) \right\} dt, \\
    h\left(t, S_{t}^{\theta, s} ; \pi_{t} \right)
\end{align*}
$$

$W_{\theta} = w$,

- **Deterministic production costs**: $C(t, \pi) = \sum_{i=1}^{n} \int_{0}^{\pi} C_{i}^{m}(t, p) dp$,

- **Deterministic quantity and price for contractual production**: $Q_{t}^{OTC}$ and $P(t)$, bounded
The model for an electricity producer

Merit order of the marginal production cost

\[ C(t, \pi) = \sum_{i=1}^{n} \int_{0}^{\pi} C_i^m(t, p) dp \]
CO2 emissions dynamics

\[ d\mathcal{E}_t^\pi = \sum_{i=1}^{n} \left( \int_{0}^{\pi_t^i} \alpha_m(p)dp \right) dt, \quad \mathcal{E}_0 = e, \alpha(p) \text{ bounded on } A; \]

\[ \alpha_{\text{coal}} = 0.96, \quad \alpha_{\text{gas}} = 0.36, \quad \alpha_{\text{oil-gas}} = 0.60, \quad \alpha_{\text{oil}} = 0.80. \]

Marginal cost in €/Mwh, with the CO2 penalty of 100 €
Electricity spot price and emissions dynamics

A diffusion process for electricity spot market price \((S_t, t \geq 0)\)

\[
\begin{align*}
\begin{cases}
    dS^{t,s}_t &= b\left(t, S^{t,s}_t\right) \, dt + \sigma\left(t, S^{t,s}_t\right) \, dB_t, \quad \forall t \geq \theta \\
    S^{t,s}_{\theta} &= s
\end{cases}
\end{align*}
\]

- \(b\) and \(\sigma\) Lipschitz in \(s\) uniformly in \(t\)
- \((B_t, t \geq 0)\) a Brownian motion, \(|b(t, s)| + |\sigma(t, s)| \leq \kappa_t + K|s|\), with \(\int_0^T \kappa_s^2 \, ds < \infty\)

Epex auction prices in 2011
The model for an electricity producer

The spot price calibration \( S_t = s(\exp(X_t) - a) \)

Calibration of \( X_t \) as

- an Ornstein Uhlenbeck (OU) process: \( dX_t = \theta (\mu - X_t) \, dt + \sigma dB_t, \)
- an OU square process: \( X_t = Y_t^2 \) and \( dY_t = \theta (\mu - Y_t) \, dt + \sigma dB_t, \)
- a CIR process: \( dX_t = \theta (\mu - X_t) \, dt + \sigma \sqrt{X_t} dB_t, \)
- an OU-Variance Gamma process: \( dX_t = \alpha (\mu - X_t) \, dt + dZ_t \) with \( Z_t = mt + \theta G_t + \sigma B_{G_t(\kappa)} \)

on the Epex spot market data
The value function of the producer

Criterion for optimal control process $\Pi^*$:

$$
\mathbb{E} \left[ U(W_T^{\Pi^*} - \mathbb{P}(\mathcal{E}_T^{\Pi^*} - \Theta_0)) \right] = \sup_{\pi \in \text{Adm}} \mathbb{E} \left[ U(W_T^{\pi} - \mathbb{P}(\mathcal{E}_T^{\pi} - \Theta_0)) \right] \tag{1}
$$

**Theorem**

Assume that $U$ is increasing and concave, $\mathbb{P}$ increasing, derivable or convex and that $\pi^* = \pi^*(t, x_2, x_3)$ is an optimum of the problem

$$
\sup_{\pi \in A} \{ h(t, x_3, \pi) - \alpha(\pi) \mathbb{P}'(x_2 - \Theta_0) \}
$$

then the control $\Pi^*$ defined by, $\forall i \in \{1, \ldots, n\}$ and $\forall \theta \in [t, T]$,

$$
\Pi^*_\theta = \pi^*(\theta, \mathcal{E}_\theta^{t,e,\Pi^*}, S_{\theta}^{t,s}),
$$

is an optimal control for (1).

Proof: use the adjoint backward equation for (1) with

$$
Y_T^{\pi} = D_x U(W_T^{\pi} - \mathbb{P}(\mathcal{E}_T - \Theta_0))
$$

and the concavity of $U$. 
The value function of the producer

Corollary

Assume in addition that for all \(i \in \{1, \ldots, n\}\), \(\alpha^i_m\) and \(C^i_m\) are constants. Then \(\Pi^*\) defined by

\[
\Pi^*_{\theta,i} = p_i, \max \left\{ S^i_{\theta} - C^i_m - \alpha^i_m \mathbb{P}'(\mathcal{E}^{t,e;\Pi^*} - \Theta_0) > 0 \right\}, i \in \{1, \ldots, n\}, \theta \in [t, T]
\]

is an optimal control of the value function problem (1).

Lemma

Assume that \(U\) is increasing and concave, if \(\mathbb{P}\) is convex, if \(p \mapsto pC^m(t,p)\) is concave and \(p \mapsto p\alpha^m(p)\) is convex. Then \(\nu\) is increasing in \((w,e,s)\) and concave in \((w,e)\).

\[
W^t_{r,w-w',s,\pi} = W^t_{r,w,s,\pi} - w', \ \forall r \geq t
\]

\[
\mathcal{E}^t_{r,e-e',\pi} = \mathcal{E}^t_{r,e;\pi} - e', \ \forall r \geq t.
\]
The value function of the producer

The distribution of the producer state variables at the end of the compliance period

Main input: the electricity prices

The four spot models prices distribution at time $T = 1$ year, calibrated on the Epex 2011 historical data.
The value function of the producer

The total carbon emission for various penalty (0, 20, 40, 60, 80, 100 €)

The total electricity production in terms of penalty
Free allocation of 80% of the production without penalty
The value function of the producer

Distribution of the penalty paid (exp OU-VG)

Distribution of the penalty paid (exp CIR)

Total penalty paid for various penalty (0, 20, 40, 60, 80, 100 €)

Wealth at the end of the compliance period (exp OU-VG)

Wealth at the end of the compliance period (exp CIR)

Wealth at the end of the compliance period
The Hamilton-Jacobi-Bellman PDE

for \( x = (w, e, s) \), \( \nu(t, x) = \sup_{\pi \in \text{Adm}} \mathbb{E}_t, x \{ \mathcal{U}(W^{\pi, t, w, s}_T - \mathbb{P}(\mathcal{E}^{\pi, t, e}_T - \Theta_0)) \} \), \( \text{(2)} \)

\[
\begin{cases}
\frac{\partial \nu}{\partial t} + b(t, s) \frac{\partial \nu}{\partial s} + \frac{\sigma(t, s)^2}{2} \frac{\partial^2 \nu}{\partial s^2} + \sup_{\pi \in \mathbb{A}} \left\{ h(t, s; \pi) \frac{\partial \nu}{\partial w} + \alpha(\pi) \frac{\partial \nu}{\partial e} \right\} = 0 \\
\nu(T, w, e, s) = \mathcal{U}(w - \mathbb{P}(e - \Theta_0))
\end{cases}
\]

By considering the operator \( \mathcal{H} \)

\[
\mathcal{H}(t, x, p, M) = \frac{1}{2} \text{Tr}(\Sigma \Sigma^t M)(t, x) + \sup_{\pi \in \mathbb{A}} \{ \lambda(t, x, \pi) \cdot p \}
\]

\[
\Sigma(t, x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma(t, x_3) \end{pmatrix} \quad \lambda(t, x, \pi) = \begin{pmatrix} h(t, x_3; \pi) \\ \alpha(\pi) \\ b(t, x_3) \end{pmatrix}
\]

\[
\partial_t \nu + \mathcal{H}(t, x, D_x \nu, D_x^2 \nu) = 0, \ \forall (t, x) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^*_+
\]
Well-posedness of the HJB equation

\( b \) and \( \sigma \) Lipschitz uniformly in time.
\( P \) with linear growth
\[ v(t, w, e, s) = o \left( \exp \left( C(2 + |w| + |e|) \right) \right). \]

**Theorem**

- With above assumptions the value function of the stochastic control problem is a continuous (viscosity) solution to the HJB equation.

- Assume further
  \[ \exists \kappa, \quad \| b(t, s) \| \leq \kappa (1 + s) \]
  \[ \| \sigma(t, s) \| \leq \kappa \left( 1 + \sqrt{s} \right), \quad \forall s > 0. \]

Then the HJB equation no more than one viscosity solution.

Proof: following Barles, Buckdahn and Pardoux 96, Crandall Ishii and Lions 92, Pham 2007.
CO2 indifference price

Buying/selling $q$ allowances at time 0 at price $p$:

$$v(0, w - qp, e - q, s) = \sup_{\pi \in \text{Adm}} \mathbb{E} \left[ \mathcal{U} \left( W_{T}^{\pi;t,w,s} - qp - P(E_{T}^{\pi;t,e} - q - \Theta_{0}) \right) \right]$$

As $v$ is continuous in $w, e$, the indifference price for $q$ allowances is $P^{\text{CO2}}(q)$ such that

$$v(0, w - qP^{\text{CO2}}(q), e - q, s) = v(0, w, e, s)$$

- $P^{\text{CO2}}(q; T, w, e, s) = \lim_{t \to T} P^{\text{CO2}}(q; t, w, e, s) = \frac{P(e - \Theta_{0}) - P(e - q - \Theta_{0})}{q}$

and for $P(x) := \lambda x^{+}$ (EU ETS $\lambda = 100\text{€}$)

$$P^{\text{CO2}}(q; T, w, e, s) = \lambda \mathbb{1}_{\{e - \Theta_{0} > q\}} + \frac{\lambda e - \Theta_{0}}{q} \mathbb{1}_{\{q \leq e - \Theta_{0} \leq 0\}}.$$  

- If $(S_{t})_{t \geq 0}$ is deterministic

$$P^{\text{CO2}}(q; t, w, e, s) = \frac{P(E_{T}^{t,e;\pi^{*}}) - P(E_{T}^{t,e-q;\pi^{*}})}{q}.$$
CO\textsubscript{2} indifference price in stochastic environment

- Existence of a non trivial value $0 < P^{CO2} < \lambda$ ?

If so

- What sensitivity to the design:
  - of the penalty $P$ ?
  - of the initial allocation $\Theta_0$ ?

- What regularity of the value function in terms of $\lambda$ and $\Theta_0$ ?
Regularity of the value function

A FBSDE representation?

the classical framework (see e.g. Pardoux Tang 99, Delarue 02)

\[
\begin{align*}
\forall t \in [0, T], \\
X_r^{t,x} &= x + \int_t^r f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) \, ds + \int_t^r \sigma(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) \, dB_s, \\
Y_r^{t,x} &= h(X_T^{t,x}) + \int_r^T g(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) \, ds - \int_r^T Z_s^{t,x} \, dB_s, \\
E \int_0^T \left( |X_t|^2 + |Y_t|^2 + |Z_t|^2 \right) \, dt &< \infty
\end{align*}
\]

\[Lu(s, x, y, z) := \frac{1}{2} Tr(\sigma \sigma' (s, x, y) D_x^2 u) + f(s, x, y, z) \cdot D_x u\]

Under condition of well posedness for the FBSDE system,

\[u(t, x) := Y_t^{t,x} \quad \text{is the viscosity solution of :}\]

\[
\begin{align*}
\frac{\partial u}{\partial t} + (Lu)(t, x, u, D_x u \cdot \sigma) + g(t, x, u, D_x u \cdot \sigma) &= 0, \quad \forall (t, x) \in [0, T) \times \mathbb{R}^n, \\
u(T, x) &= h(x)
\end{align*}
\]

In our case

\[
\begin{align*}
\mathcal{H}(t, x, p, M) &= \frac{1}{2} Tr(\Sigma \Sigma^t M)(t, x) + \lambda(t, x, \pi^*(t, x, p)) \cdot p \\
\text{where } \pi^*(t, x, p) &= \text{Argsup}_{\pi \in \mathbb{A}} \{ \lambda(t, x, \pi) \cdot p \}
\end{align*}
\]

obstacle: the volatility \( \Sigma(t, x) \) is not invertible and thus it is impossible to write \( \pi^*(t, x, p) = \tilde{\pi}^*(t, x, \Sigma \cdot p) \).
Regularity of the value function

\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{\partial v}{\partial t} + b(t, s) \frac{\partial v}{\partial s} + \frac{\sigma(t, s)^2}{2} \frac{\partial^2 v}{\partial s^2} + \sup_{\pi \in \Pi} \left\{ h(t, s; \pi) \frac{\partial v}{\partial w} + \alpha(\pi) \frac{\partial v}{\partial e} \right\} = 0 \\
v(T, w, e, s) = U(w - P(e - \Theta_0))
\end{array} \right.
\end{aligned}
\]

Note that

\[
\begin{aligned}
&h(t, s; \pi^*) \frac{\partial v}{\partial w} + \alpha(\pi^*) \frac{\partial v}{\partial e} = \sup_{\pi \in \Pi} \left\{ h(t, s; \pi) \frac{\partial v}{\partial w} + \alpha(\pi) \frac{\partial v}{\partial e} \right\},
\end{aligned}
\]

where

\[
\pi^*(t, x) = \left( P^i_{\max} \mathbb{1}_{\{ (s - C^i_m) \partial_w v(t, x) + \alpha^i_m \partial_e v(t, x) > 0 \}} \right)_{1 \leq i \leq n}.
\]

and

\[
\frac{\partial v}{\partial w} (t, x) = -P'(e - \Theta_0).
\]
Regularity of the indifference price function

We set $\tilde{P}_{CO_2}(q; \cdot) = qP^{CO_2}(q; \cdot)$.

**Proposition**

If the first order derivatives of $v$ exist, then

\[
\begin{align*}
\partial_q \tilde{P}_{CO_2} &= - \frac{\partial_e v}{\partial_w v} (t, w - \tilde{P}_{CO_2}, e - q, s) \\
\partial_t \tilde{P}_{CO_2} &= \frac{\partial_t v (t, w - \tilde{P}_{CO_2}, e - q, s) - \partial_t v (t, w, e, s)}{\partial_w v (t, w - \tilde{P}_{CO_2}, e - q, s)} \\
\partial_w \tilde{P}_{CO_2} &= \left( 1 - \frac{\partial_w v (t, w, e, s)}{\partial_w v (t, w - \tilde{P}_{CO_2}, e - q, s)} \right) \\
\partial_e \tilde{P}_{CO_2} &= \frac{\partial_e v (t, w - \tilde{P}_{CO_2}, e - q, s) - \partial_e v (t, w, e, s)}{\partial_w v (t, w - \tilde{P}_{CO_2}, e - q, s)} \\
\partial_s \tilde{P}_{CO_2} &= \frac{\partial_s v (t, w - \tilde{P}_{CO_2}, e - q, s) - \partial_s v (t, w, e, s)}{\partial_w v (t, w - \tilde{P}_{CO_2}, e - q, s)}
\end{align*}
\]
The case of the EU ETS penalty function $\mathcal{P}(x) = \lambda x^+$. 

**Proposition**

The derivatives $\frac{\partial v}{\partial t}$, $\frac{\partial v}{\partial e}$, $\frac{\partial v}{\partial s}$ are well defined and

$$
\frac{\partial v}{\partial e}(t, w, e, s) = -\lambda 1\{e - \Theta_0 > 0\} \mathbb{E} \left[ U' \left( W_{T}^{t,w,s;\pi^*} - \lambda (\mathcal{E}_{T}^{t,e;\pi^*} - \Theta_0)^+ \right) \right]
$$

$$
\frac{\partial v}{\partial t}(t, w, e, s) = \mathbb{E} \left\{ (\mathcal{E}_{T}^{t,e;\pi^*} - \Theta_0)^+ U' (W_{T}^{t,w,s;\pi^*} - \lambda (\mathcal{E}_{T}^{t,e;\pi^*} - \Theta_0)^+) \right\}
$$

For a model price $S_t = s(e^{X_t - a})$

$$
\frac{\partial v}{\partial s}(t, w, e, s) = \mathbb{E} \left[ U' (W_{T}^{t,w,s;\pi^*} - \lambda (\mathcal{E}_{T}^{t,e;\pi^*} - \Theta_0)^+) \right] \\
\times \int_{t}^{T} \sum_{i=1}^{n} (\pi_{u,i}^* - Q_u^{OTC}) \exp(X_u) du .
$$
Solve the HJB equation

- Numerical scheme for fully non linear PDE
  - Implicit-Explicit scheme
  - Optimal control computation algorithm
  - Consistency, Stability, Monotonicity, Convergence (see Barles and Souganidis 91, Barles and Jakobsen 07, Forsyth and Labahn 08)

- Artificial boundary condition
  - Restrict to a compact the computational domain

Input
- Data for the producer model
- Calibration of the spot price
HJB : dimension reduction with exponential utility

\[ U_{\text{exp}}(w) = \frac{1 - \exp(-\rho w)}{\rho}, \text{ for } \rho > 0 \]

\[ W_r^{\pi; t, w, s} = w + W_r^{\pi; t, 0, s}, \, r \geq t \]

\[ \nu(t, w, e, s) = U(w) g(t, e, s) \]

where \( g \) solve the following HJB, \( z = (e, s) \):

\[
\begin{cases}
    g_t + G(t, z, g, D_z g, D_{z^2} g) = 0 \\
    g(T, (e, s)) = \exp(\rho \mathbb{P}(e - \Theta_0))
\end{cases}
\]

with

\[ G(t, z, a, p, M) = \frac{1}{2} \text{Tr} \left\{ \Sigma \Sigma^t M \right\}_{(t, z)} + \inf_{\pi \in A} \left\{ B(t, z, \pi) \cdot p - m(t, z, \pi) a \right\} \]

\[ \tilde{\Sigma}(t, z) = \begin{pmatrix} 0 & 0 \\ 0 & \sigma(t, z_2) \end{pmatrix} \quad B(t, z, \pi) = \left( \sum_{i=1}^{n} \int_{0}^{\pi_i} \alpha_m^i(p) dp \right) \]

\[ m(t, z, \pi) = \rho h(t, z_2, \pi) \]

\[ g(t, (e, s)) = \inf_{\pi \in \mathbb{E}} \mathbb{E} \left[ \exp \left( -\rho \left( \int_{t}^{T} dW_u^{\pi; t, 0, s} - \mathbb{P} \left( \mathcal{E}_{T; t, e}^{\pi} - \Theta_0 \right) \right) \right) \right] \]
Numerical Example

\[ \mathcal{U}(x) = -\exp(-\rho x), \text{ then} \]

\[
\mathcal{P}_{\text{CO2}}(q; 0, w, e, s) = \frac{1}{\rho q} \log \left( \frac{v(0, w, e - q, s)}{v(0, w, e, s)} \right)
\]

Spot price of the form \( S_t = s(\exp(X_t) - a) \) with \( X \) CIR.

Penalty \( \lambda = 100 \)

Model data for the producer: \( n = 4 \) plants

Marginal costs:

\[ C_{\text{coal}}^m = 40, \ C_{\text{gas}}^m = 50, \]
\[ C_{\text{oil-gas}}^m = 100, \ C_{\text{oil}}^m = 150 \]

Marginal emission rates:

\[ \alpha_{\text{coal}}^m = 0.96, \ \alpha_{\text{gas}}^m = 0.36, \]
\[ \alpha_{\text{oil-gas}}^m = 0.50, \ \alpha_{\text{oil}}^m = 0.80 \]
Computation of the indifference price

Value function and indifference prices (exp(CIR) case)

Indif. price in terms of the initial spot price

Indif. price in terms of the initial allocation
A game approach for cap and trade scheme

We consider now $J$ electricity producers.

- The market regulator fixes a global level of cumulated CO$_2$ eq on a trading period, over all the market participants, a cap $\Lambda$ that one may not globally exceed.

- $(\mathcal{E}_t^{(j)}, t \in [0, T], j = 1, \ldots, J)$ are the cumulated emissions processes of each player, that we suppose observable by all the players. During the period $[0, T]$, the players may buy/sell/exchange allowances to covert their own exceeded emissions of a final penalty if the global emissions exceed the cap;

\[
\mathcal{E}_T := \sum_{j=1}^{J} \mathcal{E}_T^{(j)}, \quad \text{Penalty} = \mathbb{P}(\mathcal{E}_T^{(j)}, \mathcal{E}_T) = \mathbb{P}_{\text{local}}(\mathcal{E}_T^{(j)})\mathbb{P}_{\text{global}}(\mathcal{E}_T)
\]

\[
\mathbb{P}_{\text{EU EST}}(\mathcal{E}_T^{(j)}) = \lambda(\mathcal{E}_T^{(j)} - \Theta_T^{(j)})^+
\]

\[
\mathbb{P}(\mathcal{E}_T^{(j)}, \mathcal{E}_T) = \lambda(\mathcal{E}_T^{(j)} - \Theta_T^{(j)})^+ 1\{\mathcal{E}_T > \Lambda\}
\]

or \[
\mathbb{P}(\mathcal{E}_T^{(j)}, \mathcal{E}_T) = \lambda(\mathcal{E}_T^{(j)} - \Theta_T^{(j)})^+ \frac{(\mathcal{E}_T - \Lambda)^+}{\Lambda}
\]
The Cap & Trade scheme

Trading constraints

$\Theta_{\text{total}}$ is the finite total amount of allowances in the market. $\left(\Theta_t^{(j)}, t \in [0, T]\right)$ the number of allowances of the player $j$. The producer decides of a rate of buying/selling allowances:

$$Q_t^{(j)} \in [Q_{\text{min}}^{(j)}(t), Q_{\text{max}}^{(j)}(t)]$$

for example:

$$Q_{\text{max}}^{(j)}(t) = \frac{\left(\Theta_{\text{total}} - \sum_k \Theta_t^{(k)}\right)^+}{\Delta t}$$

with $\Delta t=$ one hour; one can buy only if the market have something to sell.

$$Q_{\text{min}}^{(j)}(t) = -\frac{\Theta_t^{(j)}}{\Delta t}$$

this means that, one might not sell immediately all the allowances one has in the portfolio.

We denote by $\mathcal{Y}$ the price process of the CO$_2$ allowances in the market.
Players evaluation

Let \((\Pi^{(j)}, Q^{(j)})\) a production & trading strategy of player \(j\). To the emission processes \((\mathcal{E}^{(j)}, j = 1, \ldots, J, \mathcal{E})\), we add the allowance portfolio processes

\[
\Theta_t^{(j)} = \Theta_0^{(j)} + \int_0^t Q_s^{(j)} ds
\]

and the wealth processes

\[
dW_t^{(j)} = \left\{ (\Pi_t^{(j)} \cdot 1) S_t - \sum_{i=1}^n \int_0^{\Pi_t^{(j),i}} C_m^{(j),i}(t, p) dp \right\} dt + \mathcal{Y}_t Q^{(j)} dt
\]

\[
= h^{(j)}(t, S_t; \Pi_t^{(j)}) dt + \mathcal{Y}_t Q^{(j)} dt,
\]

Evaluation of player \(j\) : depends on \((\Pi^{(j)}, Q^{(j)})\) but also on \((\Pi^{(-j)}, Q^{(-j)})\)

\[
((\Pi, Q)_t; t \in [0, T]) = ((\Pi, Q)^{(j)}_t, t \in [0, T], j = 1, \ldots, J).
\]

\[
\phi^{(j)}((\Pi, Q)) = \phi^{(j)}((\Pi, Q)^{(j)}, (\Pi, Q)^{(-j)})
\]

\[
= \mathbb{E}_{t,e,w,s,q} \left[ \mathcal{U}^{(j)}(W_T^{(j)} - \mathbf{P}(\mathcal{E}_T^{(j)} - \Theta_T^{(j)}), \mathcal{E}_T) \right].
\]
Nash equilibrium

Theorem

Let \((\Pi, Q)^{(-j)}\) be the set of strategies of the others players than \(j\).
Assume that \(\mathcal{U}(j)\) is concave, increasing, \(\mathcal{P}\) is convex or derivable and that 
\((\rho^\diamond, q^\diamond)(t, x_2, x_3, x_4, x_5, x_6)\) is an optimum of: \(\forall \, t \in [0.T]\)

\[
\sup_{\rho \in A(j), q \in [Q(j)_{\min}(t)(Q(j)^{(-j)}), Q(j)_{\max}(t)]} \left\{ h(j)(t, x_3, \rho) + x_5 q - \alpha^{(j)}(\rho)\mathcal{P}'_{\text{local}}(x_2 - x_6)\mathcal{P}_{\text{global}}(x_4) - \alpha(\rho, \Pi_t^{(-j)})\mathcal{P}'_{\text{local}}(x_2 - x_6)\mathcal{P}_{\text{global}}(x_4) \right\}
\]

Then the strategy \((\Pi^\diamond, Q^\diamond)^{(j)}\) defined by

\[
\Pi_t^{\diamond(j), i} = \rho^{\diamond}(t, \mathcal{E}_t^{(j)}, S_t, \mathcal{E}_t, \mathcal{Y}_t, \Theta^{(j)}),
\]

\[
Q_t^{\diamond(j)} = q^{\diamond}(t, \mathcal{E}_t^{(j)}, S_t, \mathcal{E}_t, \mathcal{Y}_t, \Theta^{(j)}),
\]

is a dominant strategy: \((\Pi, Q)^{\diamond(j)}\) is the optimal control of

\[
\phi^{(j)}((\Pi^\diamond, Q^\diamond)^{(j)}, (\Pi, Q)^{(-j)}) = \sup_{(\Pi, Q)^{(j)} \in \text{Adm}} \phi^{(j)}((\Pi, Q)^{(-j)}),
\]

whatever are the strategies \((\Pi, Q)^{(-j)}\) of other players.
The Cap & Trade scheme

Allowance prices derivation

Following Carmon Delarue Espinosa Touzi 2010, we switch to a risk neutral framework: Assume that $Y$ is the price of one allowance in one-compliance period and that the market for allowances is frictionless and liquid. Then $(Y, S)$ is a martingale for a measure $Q$ equivalent to the historical measure $\mathbb{P}$:

$$dY_t = Z_t d\tilde{B}_t.$$

Theorem

If $\mathbb{P}$ has Lipschitz derivatives, there exists a unique solution $(S_t, E^j, \Theta^j, Y, Z)$ to

$$
\begin{cases}
    dS_t = \sigma(t, S_t) d\tilde{B}_t \\
    \text{for } j = 1, \ldots, J, \\
    dE^j_t = \sum_{i=1}^{n_j} \alpha_{ij}^i \Pi^{\diamond}(j) dt, \\
    d\Theta^j_t = Q^{\diamond}(j) (Y_t) dt, \\
    dY_t = Z_t d\tilde{B}_t, \quad \text{with } Y_T = \lambda P_{\text{global}}(E_T).
\end{cases}
$$

where $(\Pi^{\diamond}, Q^{\diamond})$ is the set of dominant strategies.
The Cap & Trade scheme

A particular case of penalty \( P(E_T^{(j)}, E_T) = \lambda(E_T^{(j)} - \Theta_T^{(j)}) + (E_T - \Lambda)^+ \)

**Proposition**

There exists a unique solution \( (S_t, E_t^j, E, \Theta_t^j, \mathcal{Y}, \mathcal{Z}) \) to

\[
\begin{align*}
\text{for } j = 1, \ldots, J, \quad &dE_t^j = \sum_{i=1}^{n_j} \alpha_{ij} \Pi_t^{\phi(j)} dt, \quad d\Theta_t^j = Q_t^{\phi(j)} (\mathcal{Y}_t) dt, \\
\text{with } &\mathcal{Y}_T = \lambda \mathbb{P}_{\text{global}}(E_T)
\end{align*}
\]

where \((\Pi_t^{\phi}, Q_t^{\phi})\) is the dominant strategy

\[
\Pi_t^{\phi(j)} = p_{i,\text{max}}^{(j)} \mathbb{I} \left\{ S_t - C_{(m)}^{(j)},i - \alpha_{m}^{(j)},i \lambda \mathbb{I} \{ E_t^{(j)} > \Theta_t^{(j)} \} \mathbb{P}_{\text{global}}(E_t) - \sum_{k=1}^{J} \alpha_{m}^{(k),i} \lambda (E_t - \Theta_t^{(j)}) + \mathbb{P}_{\text{global}}'(E_t) > 0 \right\}
\]

\[
Q_t^{\phi(j)} = Q_{\text{max}}^{(j)}(t) \mathbb{I} \{ \mathcal{Y}_t < \mathbb{P}_{\text{global}}(E_t) \mathbb{I} \{ E_t^{(j)} > \Theta_t^{(j)} \} \} + Q_{\text{min}}^{(j)}(t) \mathbb{I} \{ \mathcal{Y}_t > \mathbb{P}_{\text{global}}(E_t) \mathbb{I} \{ E_t^{(j)} > \Theta_t^{(j)} \} \}
\]
Concluding remarks

- joint work with Nadia Maïzi (Mines ParisTech) and Odile Pourtalier (Inria).

- In preparation: a software on the indifference price approach on a web portal, with the help of El Hadj Ali Dia, Jacques Morice and Selim Karia.