Sharp bounds and optimal hedge ratios for basket options

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Main aims of this contribution

- To review some work done on distribution free bound for option prices under a variety of different constraints
- To describe recent joint work with **Tai-Ho Wang** on **obtaining sharp bounds and optimal hedge ratios in a static no arbitrage one period setting.**

Basket Options

Basket Options

The Payoff of a basket option:

$$\psi(S_1, \cdots, S_n) = (\sum_i w_i S_i - K)^+$$

- Price weighted –> $w_i = \frac{1}{I(t_0)}.cst.$
- Capitalization Weighted,

$$w_i = Cst.\frac{\text{nb. } S_i \text{ shares outstanding}}{\text{Total capitalization}}.$$

w_i are readjusted periodically.

- S&P 500, S&P 100, Dow Jones 100.

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Black-Scholes

The most popular model: **Multidimensional Black-Scholes**

$$dS_t^i = S_t^i(r - d_i)dt + \sigma_i S_t^i dZ_i^t$$
$$S^i(0) = S_0^1$$
$$< dZ_i, dZ_j >= \rho_{ij} dt$$

European Basket Options:

$$u_t + S_i S_j \rho_{ij} \sigma_i \sigma_j u_{S_i S_j} + (r - d_i) S_i u_{S_i} - ru = 0$$
$$u(S_1, \cdots, S_n, T) = (\sum_{i=1}^{n} w_i S_i - K)^+$$

Closed form solution B-S

Solution in closed form

Solution at time T in closed form

$$e^{-rT}\frac{1}{(2\pi)^{n/2})(detV)^{1/2}}\int(\sum w_i e^{(r-d_i-\frac{\sigma_i^2}{2})T-\sqrt{T}X}e^{\frac{1}{2}X^tV^{-1}X}dX$$

where V is the variance-covariance matrix $\{\sigma_i\sigma_i\rho_{ij}\}_{i,j=1}^n$.

Optimization Problem 1

An optimization Problem

For European Option, it suffices, to value the option, to know the time T distribution of the asset prices. So one may consider the question: *Minimize*/*Maximize* the price of the basket option at time 0 subject to suitable constraints. Which constraints? Some Possibilities (Considered by Others)

- Prescribed Marginals.

Denoting by μ_T the time T distribution, the price of the basket option is given by

$$e^{-rT}\int(\sum w_i S_i - K)d\mu_T$$

where $\mu_T = \mu_T(S_1, S_2, \cdots, S_n)$ and the marginal with respect to the i-th variable is denoted μ_i.

Copulas

Prescribed Marginals

For instance one might maximize or minimize the basket option price under the constraint that the marginals are all log-normal.

The solution to this problem (for arbitrary prescribed marginals) is best expressed in terms of copulas.

Reminder: A copula is joint distribution

$$C(x_1, x_2, \cdots, x_n) = P\left(X_1 \leq x_1, \cdots, X_n \leq x_n\right),$$

with uniform marginals and obeying certain natural structure conditions.

Sklar

Sklar's theorem Any joint distribution with continuous marginal distribution functions $F_i, i = 1, \cdots, n$, can be expressed as

$$C(F_1^{-1}, F_2^{-1}, \cdots, F_n^{-1})$$

where $C(x_1, \cdots, x_n)$ is a copula and where F^{-1} is the gen- eralized inverse of F, ie $F^{-1}(t) = \inf\{x \in \Re | F(x) \geq t$

Copula2

So minimization problem with fixed marginals closely related to problem of finding *optimal copula*.
This problem was solved **in the case** $n = 2$ by Rapuch and Roncalli (Crédit Lyonnais web site) based on earlier results of Muller and Scarsini and Chen.
The Frechet Copulas $C^-(u_1, u_2)$ and $C^+(u_1, u_2)$ given by

$$C^- = \max(u_1 + u_2 - 1, 0)$$
$$C^+ = \min(u_1, u_2)$$

Let $C^-(\mathcal{M}_1, \mathcal{M}_2)$ and $C^+(\mathcal{M}_1, \mathcal{M}_2)$ be the corresponding call option prices. Then for a generic basket option on two assets with the same marginals we have

$$C^-(\mathcal{M}_1, \mathcal{M}_2) \leq C(\mathcal{M}_1, \mathcal{M}_2) \leq C^+(\mathcal{M}_1, \mathcal{M}_2)$$

Frechet ct'd

Their result is deduced from the following more general result of Muller and Scarsini.

Theorem Let F_1 and F_2 be the probability distribution functions of X and X_2. Let $E_C[f(X_1, X_2)]$ denote the expectation of the function $f(X_1, X_2)$ when the copula of the random vector (X_1, X_2) is C. If $C_1 \prec C_2$ (concordance order, same as ptwise in 2-D) then $E_{C_1}[f(X_1, X_2)] \leq E_{C_2}[f(X_1, X_2)]$ for all supermodular functions f such that the expectations exist.

Supermodular is a natural generalization of non-negative mixed secon derivative $\frac{\partial^2 f}{\partial x_1 \partial x_2} \geq 0$. Ie. a C^2 supermodular function is convex.

Supermodular $\Leftrightarrow$

$$\Delta^{(2)} f = f(x_1 + \epsilon_1, x_2 + \epsilon_2) - f(x_1 + \epsilon_1, x_2)$$

$$- f(x_1, x_2 + \epsilon_2) + f(x_1, x_2) \geq 0$$

Rapuch-Roncalli, Ct'd

Basket Option Payoff is supermodular.

Intuitive Proof:

Use the fact that

$$\frac{\partial^2 (w_1 S_1 + w_2 S_2 - K)^+}{\partial S_1 \partial S_2} = w_1 w_2 \delta(\{w_1 S_1 + w_2 S_2 = K\})$$

is a positive distribution.

So can apply theorem.

Above discussion for two assets. Higher dimensional generalizations

According to Rapuch and Roncalli **Open Problem**

Alternative: Market Prices of traded Options

- Alternative to prescribing the marginals $\mathcal{M}_i$: Require that basket option prices be compatible with observed option prices and with observed forward prices of a given maturity. Indeed the marginals are no known, whereas we can observe option prices and these give us *partial information about these margina*
Problem:
- Given spot prices $S_0^1, S_0^2, \cdots, S_0^n$ at time 0.
-Given European option prices $C_1, \cdots, C_n$ where C_i is an option on S^i. Here all the option prices hav the same maturity T and assume constant interest rates so the forward prices are simply $e^{rT} S_0^i, i = 1, \cdots n$.
- $K_i, i = 1, \cdots n$ is the strike of the i-th option.
- K is the strike of the basket option.
Among all probability distributions compatible with these "constraints" , minimize and maximize the pri of a basket option with weights $w_i, i = 1, \cdots n$.

Mathematical Formulation
Fixed option and forward prices

Let $d\mu(S_T^1, \cdots, S_T^n)$ be the probability density associated to the distribution of the $S_T^i, i = 1, \cdots n$ at time T . Let μ_i be the marginal of μ in the i-th stock.
Then we require that

- **Prescribed Option Prices**

$$e^{-rT} \int (S_T^i - K)^+ d\mu_i = C_i, \cdots i = 1, \cdots, n$$

- **Prescribed Forward Prices**

$$e^{-rT} \int S_T^i d\mu_i = S_0^i, i = 1, \cdots n$$

Formulation Ct'd

We seek to minimize and maximize the basket option pric
Basket Option Price

$$\mathcal{B} = e^{-rT} \int (\sum_{i=1}^{n} w_i S_i - K)^+ d\mu_{\mathcal{B}}(S_1, \cdots, S_n)$$

Thus obtaining *Distribution Free* Bounds

$$\mathcal{B}_L \leq \mathcal{B} \leq \mathcal{B}_U$$

Super and Subreplicating portfolio

Upper bound, resp. Lower Bound (made explicit later on) can be expressed as

$$\sum_{i=1}^{n} \alpha_i^U C_i + \beta_i^U S_0^i + \gamma^U B_{0,T} = \mathcal{B}_U$$

$$\sum_{i=1}^{n} \alpha_i^L C_i + \beta_i^L S_0^i + \gamma^L B_{0,T} = \mathcal{B}_L$$

ie.

$$(\alpha_1^U, \cdots, \alpha_n^U, \beta_1^U, \gamma^U, \cdots, \beta_n^U)$$

is the least expensive super-replicating portfolio

Super and Subreplicating

$$(\alpha_1^L, \cdots, \alpha_n^L, \beta_1^L, \cdots, \beta_n^L)$$

the most expensive sub-replicating portfolio.

- The former therefore corresponds to a reasonable *ask price* for the seller of the basket option.
- The latter is a reasonable *bid price*.

History

History

Important parameter

$$D = K - \sum_{i=1}^{n} w_i K_i$$

- **Merton** in his 1973 paper on Rational Bounds for Option Prices, was the first to consider the question of upper bounds.
- Also many papers on distribution free bounds on an option on a single asset: Merton, Lo, Ritchken, Perrakis, Ryan, Bertsimas and Popescu, Boyle-Lin.

Merton

- When $D = 0$ he showed that

$$\mathcal{B} \leq \sum_{i=1}^{n} w_i C_i$$

But he did not show that his bound was *optimal*.

- **Laurence- Wang**(2003) show that this bound *is optimal* in the case $D \geq 0$, but show that it is not optimal in the case $D < 0$.

Elementary Proof, in the case $D = 0$

$$(\sum_{i=1}^{n} w_i S_T^i - K)^+ = (\sum_{i=1}^{n} w_i S_T^i - \sum w_i K_i)^+$$

$$= \left(\sum_{i=1}^{n} w_i (S_T^i - K)^i \right)^+$$

$\Longrightarrow$ Taking expectations

$$e^{-rT} E_\mu[(\sum_{i=1}^{n} w_i S_T^i - K)^+] \leq e^{-rT} \sum_{i=1}^{n} w_i E_\mu[(S_t^i - K)^+]$$

$$= \sum_{i=1}^{n} w_i c_i$$

Elementary

- Same proof works when $D > 0$ to show that $\sum w_i c_i$ is an upper bound but does clarify why upper bound is optimal. The latter is proved by optimization argument.
- Conditions characterizing when upper bound in Case $D \neq 0$ is attained is an open problem.

Related Work Ct'd

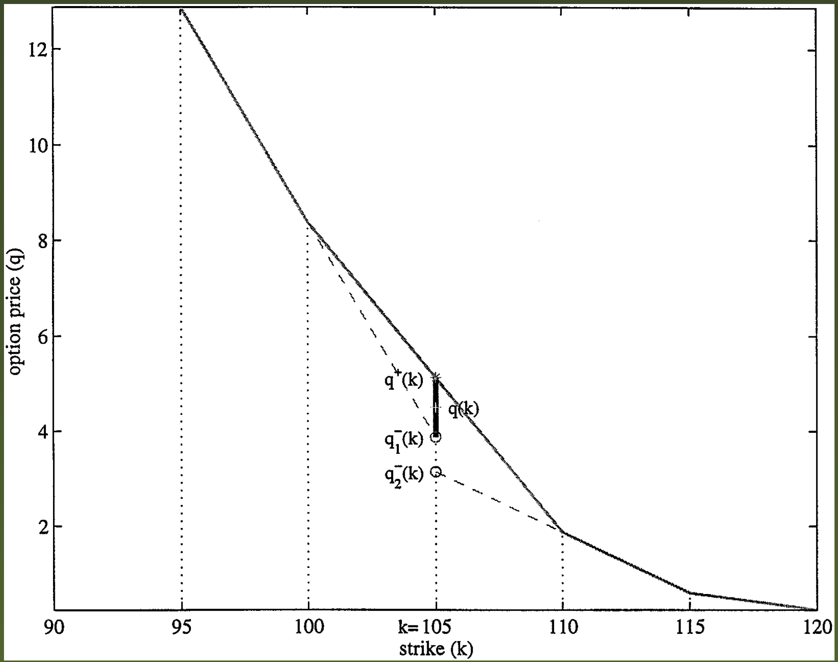
Bertsimas and Popescus use a LP approach to derive bounds on assets under various constraints.

Given prices $C_i(K_i)$ of call options with strikes $0 \leq K_1 \leq .. \leq K_n$ on a stock X, the range of all possible prices for a call option with strike K where $K \in (K_j, K_{j+1})$ for some $j = 0, \cdots, n$ is $[C^-(K), C^+(K)]$ where

$$\begin{aligned} & C^-(K) \\ & = \max\left(C_j \frac{K - K_{j-1}}{K_j - K_{j-1}} + C_{j-1} \frac{K_j - K}{K_k - K_{j-1}}, \right. \\ & \qquad \left. C_{j+1} \frac{K_{j+2} - K}{K_{j+2} - K_{j+1}} + C_{j+2} \frac{K - K_{j+1}}{K_{j+2} - K_{j+1}} \right) \\ & C^+(K) = \frac{K_{j+1} - K}{K_{j+1} - K_j} + C_{j+1} \frac{K - K_j}{K_{j+1} - K_j} \end{aligned}$$

GraphicRepBertsimasPopescu

Figure 1. The optimal upper and lower bounds on the price of a call option, given prices of calls on the same stock, with different strikes and the same maturity date.



Notes. (Actual data quoted from *The Wall Street Journal*, July 7, 1998: Microsoft July '98 call options with: $k_i = [95, 100, 110, 115, 120]$, $q_i = [12\frac{7}{8}, 8\frac{3}{8}, 1\frac{7}{8}, \frac{5}{8}, \frac{1}{4}]$; $k = 105$, $q(k) = 4\frac{1}{2}$). Note that the bounds are derived

Main Results

Let

$$\mathcal{M}_1 = \text{all probability measures}$$

$$\mathcal{M} = \{\mu \in \mathcal{M}_1 : \int \left(\sum_{i=1}^{n} w_i S_i - K \right)^+ d\mu < +\infty$$

and satisfies option and forward constraints$\}$

Also assume that the following compatibility constraints on the option and forward data are satisfied:

Let $S_0 = (S_0^1, \cdots, S_0^n)$ and $c = (c_1, \cdots, c_n)$ then

$$\mathcal{C} = \{(S_0, c) \in \Re_+^{2n} : S_0^i - c_i \geq 0, S_0^i - c_i - K_i e^{-rT} \leq 0, \forall i = 1, \cdot, n\}$$

(due to Merton).

Compatibility

These requirements derived from

- I.

$$c_i = e^{-rT} \int\limits_{S \in \Re^n_+} (S^i - K_i)^+ d\mu$$

$$\geq e^{-rT} \int\limits_{S \in \Re^n_+} (S^i_T - K_i) d\mu$$

$$= S^i_0 - K_i e^{-rT}$$

Compatibility

- II

$$
\begin{aligned}
c_i &= e^{-rT} \int_{\Re_+} (S_T^i - K_i)^+ d\mu \\
&\leq e^{-rT} \int_{\Re_+} (S_T^i)^+ d\mu \\
&= e^{-rT} \int_{\Re_+} (S_T^i) d\mu \\
&= S_0^i
\end{aligned}
$$

Feasibility

$$S = (S^1, \cdots S^n) \quad K = (K_1, \cdots, K_n), \quad C = (c_1, \cdots, c_n)$$

$$S = \begin{cases} e^{rT}(2C) + K & \text{prob } 1/2 \\ 2e^{rT}(S_0 - C) - K & \text{prob } 1/2 \end{cases}$$

Notice that

$$2e^{rT}(S_0 - C) - K \leq K \Leftrightarrow Q = S_0 - C - Ke^{-rT} \leq 0$$

Upper Bound

Sharp Upper Bound for all $n \geq 2$ Let $(S_0, c) \in \mathcal{C}^o$. The upper bound subject to the constraint $\mu \in \mathcal{M}$ is given by

$$\sum_{i=1}^{n} w_i c_i \quad \text{if} \quad D \geq 0$$

Upper Bound Ct'd

Upper bound in case $D < 0$. Introduce

$$F_i = e^{rT}\frac{S_0^i - c_i}{K_i}, i = 1, \cdots n$$

F_i is an important parameter throughout. Reorder the indices so that

$$F_1 \leq F_2 .. \leq F_n \leq 1$$

and let $\bar{i}$ be the (newly ordered) smallest index such that

$$w_1 K_1 + w_2 K_2 + \cdots w_{\bar{i}-1} K_{\bar{i}-1} \leq -D$$

and

$$w_1 K_1 + w_2 K_2 + \cdots + w_{\bar{i}} K_{\bar{i}} \geq -D$$

Upper Bound $D < 0$

Upper Bound for $D < 0$

The upper bound in this case is

$$\sum_{i=1}^{n} w_i c_i + e^{-rT} \sum_{i=1}^{\bar{i}-1} w_i K_i F_i - e^{-rT} (K - \sum_{i=\bar{i}}^{n} w_i K_i) F_{\bar{i}}$$

This shows that the upper bound is *larger* in the case $D < 0$. Indeed, by the definition of $\bar{i}$ we have

$$K - \sum_{i=\bar{i}}^{n} w_i K_i = w_1 K_1 + \cdots + w_{\bar{i}-1} K_{\bar{i}-1} + D \leq 0$$

Upper bound $D < 0$ ct'd

Another way to express the upper bound

$$\sum_{i=\bar{i}+1}^{n} w_i c_i + \sum_{i=1}^{\bar{i}-1} w_i S_i^0 - \underbrace{\left(K - \sum_{i=\bar{i}}^{n} w_i K_i\right)}_{\leq 0} w_{\bar{i}} S_{\bar{i}}^0$$

$$+ \underbrace{\left(K - \sum_{i=\bar{i}+1}^{n} w_i c_i\right)}_{\geq 0} c_{\bar{i}}$$

Hence the optimal hedging strategy consists of being lon options and stock and has no cash component.

Maximizers when $D = 0$

When is upper bound attained ?

"Condition P" In the case $D = 0$ any probability measure supported on the set

$$\mathcal{S}_1 \cup \mathcal{S}_2$$

attains the upper bound. Here

$$\mathcal{S}_1 = \{x \in \Re_n^+ : x = y + \bar{K}, y \in \Re_+^n\}$$
$$\mathcal{S}_2 = \{x \in \Re_n^+ : x = \bar{K} - y, y \in \Re_+^n\}$$

where $\bar{K} = (K_1, K_2, .., K_n)$, attains the upper bound.

For $n = 2$ and $D < 0$ we can also give an example of a discrete dis-

tribution attaining the upper bound. In the case $D > 0$ of a discrete e

Sharp Lower Bound for $n = 2$

Introduce the quantities

$$A_i = w_i c_i - \frac{K - w_i K_i}{K_i}(S_0^i - c_0^i) \quad \text{for} \quad i = 1,2$$
$$F = F_1 + F_2 - 1$$
$$F^+ = \max(F, 0)$$

then

Proposition 1 *Let $(S_0, c) \in \mathcal{C}^o$. The lower bound subject to the constraint $\mu \in \mathcal{M}$ is given by*

$$\max\big(A_1 + e^{-rT} w_2 K_2 F^+,$$
$$A_2 + e^{-rT} w_1 K_1 F^+, A_1 + A_2 + e^{-rT} K F^+, 0\big)$$
for $\quad D \leq 0$

Lower Bound Ct'd

Proposition Ct'd

$$\max\Big(A_1 + e^{-rT}(K - w_1K_1)F^+,$$
$$A_2 + e^{-rT}(K - w_2K_2)F^+,$$
$$A_1 + A_2 + e^{-rT}KF^+, 0\Big)$$

for $\quad D \geq 0$

Lower Bound all n

Proposition A lower bound for all $n \geq 2$ is given by

$$L_1 = \sum_{i=1}^{n} w_i S_0^i - e^{-rT} K$$

Ie. *price of a forward contract with strike K on basket.* This lower bound follows from

$$\begin{aligned} & E_\mu[w \cdot S - K]^+ \\ & \geq E_\mu[w \cdot S - K] \\ & = w \cdot S_0 e^{rT} - K = w \cdot F - K \end{aligned}$$

This inequality shows that *forward prices* $F = (F_1, \cdots, F_n)$ play a **fundamental role** for lower bound

Optimality LB, $n > 2$

Indeed in case $n = 2$ we find Optimal Lower Bound= $\max(L_1, \cdots, L_4)$.
For $n > 2$ this lower bound is sharp provided (**Condition Q**) there exists a probability measure $\mu \in \mathcal{M}$ (ie. consistent with the constraints) supported in the region

$$\{(x_1, x_2, \cdots, x_n) \in \Re_+^n : \sum_{i=1}^{n} w_i x_i - K \geq 0\}$$

Discrete

Remark In the two dimensional case we can exhibit a discrete distribution satisfying **Condition Q** for $D < 0$. For simplicity take $w_1 = w_2 = 1$ and $r = 0$. It is given by the following 3-point distribution

$$(S_1, S_2) = \left\{ \begin{array}{cc} (K_1, K - K_1) \quad \text{with prob.} \quad p_1 = \frac{Q_2}{D} \text{ in } R_1 \\ (K_1 + \frac{c_1}{1 - \frac{Q_1+Q_2}{D}}, K_2 + \frac{c_2}{1 - \frac{Q_1+Q_2}{D}} \quad \text{in } R_4 \\ (K - K_2, K_2) \quad \text{with prob.}\, p_5 = \frac{Q_1}{D} \end{array} \right\}$$

where

$$Q_i = S_0^i - c_i - K_i$$

Bounds for Puts

- Optimal Bounds for puts on basket follow easily from bounds for calls.
- Use Put-Call parity, applicable for baskets since

$$(\sum w_i S^i - K)^+ - (K - \sum w_i S^i)^+ = S^i - K$$

ie.
Payoff of call on basket – Payoff of Put on basket = Payoff of Forward Contract on basket.

Optimal Super-replicating portfolio

- In the case $D \geq 0$, a super-replicating portfolio $(u^*, \bar{\nu}^*, v^*)$ is given by

$$(u_1^*, \cdots, u_n^*, \bar{\nu}^*, v_1^*, \cdots, v_n^*) = (w_1, \cdots, w_n, 0, 0, \cdots,$$

- In the case $D < 0$, a super-replicating portfolio $(u^*, \bar{\nu}^*, v^*)$ is given by

$$\begin{aligned} & (u_1^*, \cdots, u_{\bar{i}-1}^*, u_{\bar{i}}^*, u_{\bar{i}+1}^*, \cdots, u_n^*, \bar{\nu}^*, v_1^*, \cdots, v_{\bar{i}-1}^*, v_{\bar{i}}^*, v_{\bar{i}+1}^*, \cdots, v_n^*) \\ = \quad & (0, \cdots, 0, \frac{K - \sum_{i=\bar{i}+1}^n w_i K_i}{K_{\bar{i}}}, w_{\bar{i}+1}, \cdots, w_n, 0, w_1, \cdots, w_{\bar{i}-1}, \\ & \frac{\sum_{i=\bar{i}}^n w_i K_i - K}{K_{\bar{i}}}, 0, \cdots, 0). \end{aligned}$$

Super-Replicating: conclusion

Notice that this means that

• In the case $D \geq 0$ we can optimally the basket option using only the options on individual stocks.

• In the case $D < 0$, an optimal super-replicating portfolio consists not only of options but also of the stocks.

Optimal Sub-Replicating Portfolio

Sub-replicating portfolio

A subreplicating portfolio in 2-asset case is given by $(u_*, \bar{\nu}_*, v_*)$, where $u_* = (u_{*1}, u_{*2})$ and $v_* = (v_{*1}, v_{*2})$. Cases where the lower bound is reached is equal to one of A_1, A_2, $A_1 + A_2$, $A_1 + A_2 + e^{-rT}KF$, $A_1 + e^{-rT}(K - w_1K_1)F$, $A_2 + e^{-rT}(K - w_2K_2)F$, $A_1 + e^{-rT}w_2K_2F$, $A_2 + e^{-rT}w_1K_1F$, we refer to as Cases $1 : 8$, where we recall that lower bound was given by:

$$\max\Big(A_1 + e^{-rT}w_2K_2F^+,$$

$$A_2 + e^{-rT}w_1K_1F^+, A_1 + A_2 + e^{-rT}KF^+, 0\Big)$$

$$\text{for} \quad D \leq 0$$

Lower Bound Ct'd

and

$$\max\Big(A_1 + e^{-rT}(K - w_1 K_1)F^+,$$
$$A_2 + e^{-rT}(K - w_2 K_2)F^+, A_1 + A_2 + e^{-rT} K F^+, 0\Big)$$
$$\text{for} \quad D \geq 0$$

Explicit Sub-Replicating Strategies

A subreplicating portfolio in 2-asset case is given by $(u_*, \bar{\nu}_*, v_*)$, where $u_* = (u_{*1}, u_{*2})$ and $v_* = (v_{*1}, v_{*2})$.
u —–> Options, $\bar{\nu}$ —–> Cash, v —–> Stocks

Case	u_{*1}	u_{*2}	$\bar{\nu}_*$	v_{*1}	v_{*1}	I
1	$\frac{K}{K_1}$	0	0	$-\frac{K-w_1K_1}{K_1}$	0	a
2	0	$\frac{K}{K_2}$	0	0	$-\frac{K-w_2K_2}{K_2}$	a
3	$\frac{K}{K_1}$	$\frac{K}{K_2}$	0	$-\frac{K-w_1K_1}{K_1}$	$-\frac{K-w_2K_2}{K_2}$	a
4	0	0	$-e^{-rT}K$	w_1	w_2	a
5	w_1	$-\frac{K-w_1K_1}{K_2}$	$-e^{-rT}(K-w_1K_1)$	0	$-\frac{K-w_1K_1}{K_2}$	po
6	$-\frac{K-w_2K_2}{K_1}$	w_2	$-e^{-rT}(K-w_2K_2)$	$-\frac{K-w_2K_2}{K_1}$	0	po
7	$w_1+\frac{D}{K_1}$	$-w_2$	$-e^{-rT}w_2K_2$	$-\frac{D}{K_1}$	w_2	ne
8	$-w_1$	$w_2+\frac{D}{K_2}$	$-e^{-rT}w_1K_1$	w_1	$-\frac{D}{K_2}$	ne

In the above we assume that $K > \max(w_1K_1, w_2K_2)$. If not, bounds and hedge ratios take on a different form.

Remark Semi-Static

Note that since the composition of the optimal replicating portfolio depends on the spot prices S_0^i, which determine which of the four expressions attain the max, one may think of the above result as being a semi-static hedge, ie. static and then change, static then change, etc.
Hedge may change at intermediate time $[T_1, \cdots, T_{n-1}]$ for which expression attaining max. changes.

Distributions attaining Lower and Upper Bounds when $n = 2$

Since our problem involves, for $n = 2$, $2n + 1 = 5$ constraints on the measure μ it is reasonable to try finding a discrete distribution that attains the optimal bounds. But can one find an optimal discrete measure?

Yes,..

but, start from scratch! Will Discuss later on if time permits,

- Large Literature on optimal bounds.
- Very *scarce literature on explicit optimizers*.

A famous example of an optimal solution: Andrew Lo's adaptation of Scarf's **two point** solution of t

inventory problem. Lo uses this in his famous paper where he derived the following distribution free upp

bound for the prices C and P of a European call option on one asset give that the variance V^* and the sho

rate r (assumed constant over the period under consideration) are known.

Lo Ct's, two point

$$C \leq \begin{cases} \frac{S(T)-Ke^{-rT}+S_0V^*e^{-2rT}}{1+V^*e^{-2rT}} \\ \text{if} \quad \frac{S(T)}{K} \geq \frac{2e^{-rT}}{1+V^*e^{-2rT}} \\ \\ \frac{1}{2}\left[S(T) - Ke^{-rT} + \sqrt{(Ke^{-rT} - S(T))^2 + S^2(T)V^*e^{-2rT}}\right] \\ \text{if} \quad \frac{S(T)}{K} < \frac{2e^{-rT}}{1+V^*e^{-2rT}} \end{cases}$$

Equality attained by probability distribution *concentrated at two points*.
In the case $n = 2$ of Laurence-Wang *all* discrete solutions discovered so far are **three of four** point solutions.

The approach to finding sharp bounds

We start from the dual variational problems:

The Dual Problems

The dual problem to respectively $\mathcal{P}_U$ and $\mathcal{P}_L$ is as follows.

Dual to $\mathcal{P}_U$

$$\inf_{u,\bar{\nu},v} \{\sum_{i=1}^{n} c_i u_i + \bar{\nu} + \sum_{i=1}^{n} v_i S_0^i \quad : u_i, \bar{\nu}, v_i \in \Re, i = 1, \cdots$$

subject to the constraints

$$\sum_{i=1}^{n} u_i (S_i - K_i)^+ + e^{rT}\bar{\nu} + \sum_{i=1}^{n} v_i S_i \geq (\sum_{i=1}^{n} w_i S_i - K_i)^+, \quad \forall S \in \Re_+^n. \quad ($$

Dual Ct'd

Dual to $\mathcal{P}_L$

$$\sup_{u,\bar{\nu},v} \{\sum_{i=1}^{n} c_i u_i + \bar{\nu} + \sum_{i=1}^{n} v_i S_0^i \quad : u_i, \bar{\nu}, v_i \in \Re, i = 1, \cdots$$

subject to the constraints

$$\sum_{i=1}^{n} u_i (S_i - K_i)^+ + e^{rT}\bar{\nu} + \sum_{i=1}^{n} v_i S_i \leq (\sum_{i=1}^{n} w_i S_i - K)^+, \quad \forall S \in \Re_+^n. \qquad ($$

Note that since inequalities (2) and (4) are required to hold for *all* non-negative n-vectors S, they impose implici constraints on the coefficients u ,$\bar{\nu}$ and v.

Warm-Up Problem

The problem in 1-asset case is to find

$$\max_{\nu} \int_{\mathbb{R}_+} (x - \bar{K})^+ \nu(dx)$$

subject to the constraints

$$\int_{\mathbb{R}_+} x\nu(dx) = m,$$
$$\int_{\mathbb{R}_+} (x - K)^+ \nu(dx) = c,$$
$$\int_{\mathbb{R}_+} \nu(dx) = 1.$$

Dual

Its dual is

$$\min_{u,\bar{\nu},v} cu + \bar{\nu} + mv$$

subject to the constraints

$$u(x-K)^+ + \bar{\nu} + vx \geq (x-\bar{K})^+, \quad \forall x \in \mathbb{R}_+.$$

1-Option, Ct'd

Introduce the transformation

$$\begin{aligned}\bar{x} &= \frac{x}{K},\\ \lambda &= u + v,\\ \mu &= Kv.\end{aligned}$$

Then the problem is transformed, in the new variables into

$$\min_{\lambda,\mu,\bar{\nu}} c\lambda + \bar{\nu} + F\mu, \qquad (5$$

where $F = \frac{m-c}{K}$, subject to the (implicit) constraints on λ,
$\bar{\nu}$ and μ

$$(K\lambda - \mu)(\bar{x} - 1)^{+} + \bar{\nu} + \mu\bar{x} - (K\bar{x} - \bar{K})^{+} \geq 0, \quad \forall \bar{x} \in \mathbb{R}_{+}$$

1D Ct'd

Define the function $f : \mathbb{R}_+ \to \mathbb{R}$ by

$$f(\bar{x}) \;=\; (K\lambda - \mu)(\bar{x} - 1)^+ + \bar{\nu} + \mu\bar{x} - (K\bar{x} - \bar{K})^+.$$

f is a piecewise affine function, it attains its local extrema at the point $\{0, 1, \frac{\bar{K}}{K}\}$. In order to find conditions on $(\lambda, \bar{\nu}, \mu)$ so $f(\bar{x}) \geq 0$ for all $\bar{x} \in \mathbb{R}_+$, require that f be nonnegative at these points and impose the extra conditions $\lambda \geq 1$ to ensure the functional is bounded below. Thu the dual problem in this subcase is to find

$$\min_{\lambda,\mu,\bar{\nu}} c\lambda + \bar{\nu} + F\mu$$

subject to the constraints

$\lambda \geq 1$

Subcase $\bar{K} \geq K$

(with $f(\bar{x}) = (K\lambda - \mu)(\bar{x} - 1)^+ + \bar{\nu} + \mu\bar{x} - (K\bar{x} - K)^+$

$$\lambda \geq 1 \quad (7$$

$$f(0) = \bar{\nu} \geq 0 \quad (8$$

$$f(1) = \bar{\nu} + \mu \geq 0 \quad (9$$

$$f(\frac{\bar{K}}{K}) = (\bar{K} - K)\lambda + \bar{\nu} + \mu \geq 0 \quad (10$$

Notice that conditions in (10) are redundant because of (7), (9) and the assumptions $\bar{K} \geq K$. Also note that by (7), we have

$$c\lambda + \bar{\nu} + F\mu \geq c + \bar{\nu} + F\mu$$

Case 1, t'd

Ignoring the constant term c temporarily, the problem finding the minimum of

$$\min_{\bar{\nu},\mu} \bar{\nu} + F\mu \tag{11}$$

subject to

$$\bar{\nu} \geq 0$$
$$\bar{\nu} + \mu \geq 0$$

The minimum is zero. So, bringing back c , minimum for original is c.

Remark 1 *This is reasonable for trivial reasons since option price is decreasing in strike. Therefore, in the case $\bar{K} > K$, the option price of strike $\bar{K}$ is at most c.*

Case 2: Warm Up

(Recall we know $C(K)$, are optimizing $C(\bar{K})$).

Subcase $\bar{K} \leq K$ (with $f(\bar{x}) = (K\lambda - \mu)(\bar{x}-1)^+ + \bar{\nu} + \mu\bar{x} - (K\bar{x}-K)^+$), the dua
problem in this subcase is equivalent to finding

$$\min_{\lambda,\mu,\bar{\nu}} c\lambda + \bar{\nu} + F\mu$$

subject to

$$\lambda \geq 1 \quad (1$$

$$f(0) = \bar{\nu} \geq 0, \quad (1$$

$$f(1) = \bar{\nu} + \mu \geq K - \bar{K} \quad (1$$

$$f(\frac{\bar{K}}{K}) = \bar{\nu} + \frac{\bar{K}}{K}\mu \geq 0 \quad (1$$

Notice that (15) is redundant since

$$\mu + \frac{K}{\bar{K}}\bar{\nu} \geq \mu + \bar{\nu} \geq K - \bar{K} \geq 0.$$

Upper Bound: Conclusion

Ignoring λ by using (12), the problem reduces to

$$\min_{\lambda,\mu,\bar{\nu}} c\lambda + \bar{\nu} + F\mu$$

subject to (13) and (14). Note that the region defined by the constraints $\bar{\nu} \geq 0, \bar{\nu} + \mu \geq 0$ (an unbounde polygon) has only one vertex which is $(\bar{\nu}, \mu) = (0, K - \bar{K})$. Hence the value in this case is, bringing back the constant term c, is $c + (K - \bar{K})F$.

● Conclusion: **Upper Bound** is :

$$c + (K - \bar{\boldsymbol{K}})^{+} F$$

Similar analysis yields:

Lower Bound : $(c - (\bar{K} - K)F)^{+}$, if $\bar{K} \geq K$, $c + (KF - \bar{K})^{+}$, if $\bar{K} \leq K$

Check UB > LB

Case 1 $\bar{K} \geq K$

$$UB = c, LB = c - ((\bar{K} - K)F)^+$$

so clear
Case 2 $\bar{K} \leq K$
In this case

$$UB = c + (K - \bar{K})F \geq LB = c + (KF - \bar{K})^+$$
$$(K - \bar{K})F \geq (KF - \bar{K})^+$$

$F \geq 0$ and $F \leq 1$ so true.
End of Warm Up Problem

Back to Real Problem: Normalization of Problem

We can reduce the original problem to a form where $r = 0$ and all $w_i, i = 1, \cdots, n = 1$.

$$x_i = w_i S_i, \quad i = 1, \cdots, n,$$
$$K_i' = w_i K_i, \quad i = 1, \cdots, n,$$
$$\Gamma = \prod_{i=1}^{n} w_i,$$
$$\nu(x_1, .., x_n) = \frac{1}{\Gamma} d\tilde{\mu}(x_1, .., x_n) \in \mathbf{M}_1,$$

From now on work in normalized variables.

S_0^i —-> m_i.

Solution of Dual Problem LB for $n = 2$

Let us introduce the following transformation

$$
\begin{aligned}
u_1 &= \lambda_1 + \lambda_4, \\
u_2 &= \lambda_2 + \lambda_5, \\
u_3 &= \lambda_3, \\
u_4 &= \lambda_4, \\
u_5 &= \lambda_5.
\end{aligned}
$$

Let f be the function

$$
\begin{aligned}
f(x, y) &= \lambda_1 (x - K_1)^+ + \lambda_2 (y - K_2)^+ + \lambda_3 \\
&\quad + \lambda_4 x + \lambda_5 y - (x + y - K)^+ \\
&= (u_1 - u_4)(x - K_1)^+ + (u_2 - u_5)(y - K_2)^+ \\
&\quad + u_3 + u_4 x + u_5 y - (x + y - K)^+
\end{aligned}
$$

Solution of Dual LB for $n = 2$

Then the problem is transformed into find

$$\max_{u} c_1 u_1 + c_2 u_2 + u_3 + (m_1 - c_1) u_4 + (m_2 - c_2) u_5$$

subject to (the implicit constraints on u)

$$\begin{aligned} & f(x, y) \\ & = (u_1 - u_4)(x - K_1)^+ + (u_2 - u_5)(y - K_2)^+ \\ & + u_3 + u_4 x + u_5 y - (x + y - K)^+ \leq 0, \quad \forall x \geq 0, \end{aligned}$$

Piecewise Linear

f is a piecewise affine function, it attains its local extrema at the points $\{(0,0),(K_1,0),(0,K_2),(K,0),(0,K),(K-K_2,K_2),(K_1,K-K_1),(K_1,K_2)\}$. In order to derive conditions on $u=(u_1,u_2,u_3,u_4,u_5)$ such that $f(x,y)\leq 0$ for all $(x,y)\in\mathbb{R}_+^2$, we impose the extra conditions

$$u_1 \leq 1 \quad \text{and} \quad u_2 \leq 1$$

to prevent f from being unbounded from above. We divid the problem of finding the maximimum of f into 8 case according to where the global maximum of f is attained.

The eight Cases

Namely, the cases

Case 1 $\max f(x,y) = f(K_1, K_2)$;

Case 2 $\max f(x,y) = f(K_1, K - K_1)$;

Case 3 $\max f(x,y) = f(K - K_2, K_2)$;

Case 4 $\max f(x,y) = f(0, K)$;

Case 5 $\max f(x,y) = f(K, 0)$;

Case 6 $\max f(x,y) = f(0, K_2)$;

Case 7 $\max f(x,y) = f(K_1, 0)$;

Case 8 $\max f(x,y) = f(0, 0)$.

Finding lower bound in primal by finding upper bound in dual

Sample Case

Case 1 $\max f(x,y) = f(K_1, K_2), D \geq 0$
In this case the problem becomes

$$\max_u c_1u_1 + c_2u_2 + u_3 + (m_1 - c_1)u_4 + (m_2 - c_2)u_5$$

subject to the conditions

$$u_1 \leq 1, \quad (16$$
$$u_2 \leq 1, \quad (17$$
$$K_1u_4 + D \geq 0, \quad (18$$
$$K_2u_5 + D \geq 0, \quad (19$$
$$-(K - K_1)u_1 + K_2u_5 + D \geq 0, \quad (20$$
$$-(K - K_2)u_2 + K_1u_4 + D \geq 0 \quad (21$$

Conditions

$$u_5 \geq 1, \tag{22}$$
$$u_4 \geq 1, \tag{23}$$
$$K_1 u_4 + K_2 u_5 + D \geq 0, \tag{24}$$
$$u_3 + K_1 u_4 + K_2 u_5 + D \leq 0. \tag{25}$$

Solving Dual

The above can be reduced after eliminating redundant equations to:

$$\max_u c_1 u_1 + c_2 u_2 + (m_1 - c_1 - K_1)u_4 + (m_2 - c_2 - K_2)u_5 - D$$

subject to the conditions

$$\begin{aligned}
&u_1 \leq 1, \\
&u_2 \leq 1, \\
&u_4 \geq 1, \\
&u_5 \geq 1, \\
&-(K - K_1)u_1 + K_2 u_5 + D \geq 0, \\
&-(K - K_2)u_2 + K_1 u_4 + D \geq 0.
\end{aligned}$$

Now find dual of this LP problem.

Dual of Dual

Use Table trick to find the dual

$$\min_{Au \leq b} c \cdot U, \Rightarrow \max_{\lambda^t A = c, c \geq 0} b \cdot \lambda,$$

	u_1	u_2	u_4	u_5	b
λ_1	1	0	0	0	1
λ_2	0	1	0	0	1
λ_3	0	0	-1	0	-1
λ_4	0	0	0	-1	-1
λ_5	$K - K_1$	0	0	$-K_2$	D
λ_6	0	$K - K_2$	$-K_1$	0	D
	c_1	c_1	$m_1 - c_1 - K_1$	$m_2 - c_2 - K_2$	obj. functional

Constraints, Case 1

The dual problem of the maximization problem is to find

$$\min_{\lambda \geq 0} \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 + D\lambda_5 + D\lambda_6$$

The dual constraints (in addition to $\lambda \geq 0$) are

$$\lambda_1 + (K - K_1)\lambda_5 = c_1 \quad (2$$

$$\lambda_2 + (K - K_2)\lambda_6 = c_2 \quad (2$$

$$-\lambda_3 - K_1\lambda_6 = m_1 - c_1 - K_1 \quad (2$$

$$-\lambda_4 - K_2\lambda_5 = m_2 - c_2 - K_2 \quad (2$$

Notice that the above linear system of equations has 4 equations with 6 variables, so one can choose

variables to be free (choose λ_1 and λ_2) and solve the others

Case 1, Ct'd

After solving get

$$\lambda_5 = \frac{c_1 - \lambda_1}{K - K_1}$$

$$\lambda_6 = \frac{c_2 - \lambda_2}{K - K_2}$$

$$\lambda_3 = -(m_1 - c_1 - K_1) - \frac{K_1}{K - K_2} c_2 + \frac{K_1}{K - K_2} \lambda_2$$

$$\lambda_4 = -(m_2 - c_2 - K_2) - \frac{K_2}{K - K_1} c_1 + \frac{K_2}{K - K_1} \lambda_1$$

Case 1, Ct'd

Let's temporarily ignore the constant term $-D$ in the objective functional. The dual problem of the maximization problem is to find

$$\min_{\lambda \geq 0} \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 + D\lambda_5 + D\lambda_6$$

subject to the constraints

$$\lambda_2 \geq \max\{c_2 + \frac{K - K_2}{K_1}(m_1 - c_1 - K_1), 0\}$$
$$\lambda_1 \geq \max\{c_1 + \frac{K - K_1}{K_2}(m_2 - c_2 - K_2), 0\}$$
$$\lambda_1 \leq c_1,$$
$$\lambda_2 \leq c_2.$$

This is a trivial problem, whose value turns out to be $m_1 + m_2 - K$.

LB and UB for primal

All eight cases can be solved in a similar way, but with, on average, lengthier computations then above.

Upper Bound The dual problem for the upper bound in n-asset case is to find

$$\min_{u,\bar{\nu},v} \sum_{i=1}^{n} c_i u_i + \bar{\nu} + \sum_{i=1}^{n} m_i v_i$$

subject to the constraints

$$\sum_{i=1}^{n} u_i (x_i - K_i)^+ + \bar{\nu} + \sum_{i=1}^{n} v_i x_i \geq (\sum_{i=1}^{n} x_i - K)^+, \quad \forall x \in \mathbb{R}_+^n.$$

Let us introduce the following transformation, for $i = 1, \cdots, n$,

$$\begin{aligned} \bar{x}_i &= \frac{x_i}{K_i}, \\ \lambda_i &= u_i + v_i, \\ \mu_i &= K_i v_i. \end{aligned}$$

Transformed Problem

Then the problem is transformed, in the new variables, into

$$\min_{\lambda,\mu,\bar{\nu}} \sum_{i=1}^{n} c_i \lambda_i + \bar{\nu} + \sum_{i=1}^{n} \frac{m_i - c_i}{K_i} \mu_i \qquad (3$$

subject to the (implicit) constraints on λ, $\bar{\nu}$ and μ

$$\sum_{i=1}^{n} (K_i \lambda_i - \mu_i)(\bar{x}_i - 1)^+ + \bar{\nu} + \sum_{i=1}^{n} \mu_i \bar{x}_i - (\sum_{i=1}^{n} K_i \bar{x}_i - K)^+ \geq 0 \quad \forall \bar{x} \in \mathbb{R}_+^n. \qquad (3$$

Define the function $f : \mathbb{R}_+^n \longrightarrow \mathbb{R}$ by

$$f(\bar{x}) \quad = \quad \sum_{i=1}^{n} (K_i \lambda_i - \mu_i)(\bar{x}_i - 1)^+ + \bar{\nu} + \sum_{i=1}^{n} \mu_i \bar{x}_i - (\sum_{i=1}^{n} K_i \bar{x}_i - K)^+.$$

Solving n-D problem

f is a piecewise affine function, it attains its local extrema at the points $\{(x_1, \cdots, x_n) \in \mathbb{R}_+^n\}$ describe below which we group into 2 categories as follows.

Category I

$$\{(x_1, \cdots, x_n) : x_i = 0, 1\}.$$

There are 2^n points in Category I.

Category II

For $l = 0, \cdots, n$,

$$x_j = \frac{K - \sum_{\alpha=1}^{l} K_{i_\alpha}}{K_j}, \text{ for } j \in \{i_1, \cdots, i_l\} \text{ such that } x_j \text{ is nonnegative,}$$

$$x_i = 1, \quad \text{for every} \quad i \in \{i_1, \cdots, i_l\} \setminus \{j\},$$

$$x_i = 0, \quad \text{otherwise.}$$

N-D Upper Bound

Note that if $D \geq 0$, there are $n2^{n-1}$ points in this category. Therefore, in order to find conditions on $(\lambda, \bar{\nu}, \mu)$ such that $f(\bar{x}) \geq 0$ for all $\bar{x} \in \mathbb{R}^n_+$, we need only to require that f be nonnegative at these poin and impose the extra conditions

$$\lambda_i \geq 1 \quad \text{for} \quad i = 1, \cdots, n$$

to ensure the functional is bounded below. Introducing the quantity

$$I^l = \{\mathbf{i}^l = \{i_1, \cdots, i_l\} : 1 \leq i_1 < \cdots < i_l \leq n\}$$

the problem is to maximize

$$\max_{(\lambda, \bar{\nu}, \mu)} \sum_{i=1}^{n} c_i \lambda_i + \bar{\nu} + \sum_{i=1}^{n} F_i \mu_i$$

Constraints: N asset case

$$\lambda_i \leq 1, \quad i = 1, \cdots, n, \tag{32}$$

$$\bar{\nu} \leq 0, \tag{33}$$

$$\bar{\nu} + \sum_{i \in \mathbf{i}^l} \mu_i \leq \sum_{i \in \mathbf{i}^l} K_i - K, \quad \mathbf{i}^l \in I^l, \tag{34}$$

$$\bar{\nu} + \frac{K}{K_i} \mu_i \leq 0, \quad i = 1, \cdots, n. \tag{35}$$

N-D

Strategy for obtaining the Upper Bound

- We derive the dual problem.
- We find a feasible solution for the primal and a feasible solution for the dual for which the respective objective functions coincide. Hence this feasible solution must be **optimal**.

$$\max_{dual feasible} \text{dual} \leq \min_{prinal feasible} \text{primal}$$

How good are these bounds: Numerical Results

In the case $n = 2$, where we have both optimal upper and lower bounds, we have simulated asset values (S_T^1, S_T^2) from the following three distributions:

- Bivariate correlated geometric Brownian motion.
- Bivariate correlated normal distribution
- Bivariate correlated Student-T distribution.

Results

- Our results indicate a wide gap between the optimal upper bound and the optimal lower bound.
- In the case $D = 0$ an instantaneous correlation ρ close to 1 plus in addition a choice of parameters ensuring that condition P is satisfied implies upper bound is nearly attained. For $D \neq 0$ this is not the case.
- In the case $D = 0$ a negative correlation close to -1

Example relating correlation & Condition P

Example For instance in the lognormal case, as is easily seen, the two stock distributions $S_i(T) = S_i(0)e^{\sigma_i\sqrt{T}N - \frac{1}{2}\sigma_i^2 T}$, $i = 1, 2$, driven by the same standard normal distribution N are co-monotonic at time T since

$$S_1(T) = \varphi(S_2(T)) \quad (\phi \text{ increasing } \equiv const \cdot S_2(T)^{\frac{\sigma_1}{\sigma_2}}),$$

where

$$const = S_1(0)S_2(0)^{-\frac{\sigma_1}{\sigma_2}} e^{-\frac{\sigma_1^2}{2}T + \frac{\sigma_1\sigma_2}{2}T}.$$

These two distributions satisfy Condition P provided that among the pai (K_1, K_2) satisfying $w_1K_1 + w_2K_2 = K$ we make the particular choic $(\bar{K}_1, \bar{K}_2)$ which also satisfies $K_1 = const \cdot K_2^{\frac{\sigma_1}{\sigma_2}}$

Condition P

This is the only choice of a pair (K_1, K_2) satisfying $w_1 K_1 + w_2 K_2 = K$ that leads to distributions possessing property P (at time T). Thus comonotonicity only implies that the upper bound is attained under certain special conditions. To see the above assertion, note that for Condition P to hold we need

$$S_1(T) \geq K_1 \Leftrightarrow S_2(T) \geq K_1$$

But

$$\begin{aligned} S_1(T) = Cst(S_2(T))^{\frac{\sigma_1}{\sigma_2}} &\geq K_1 \\ &\Leftrightarrow \\ S_2(T) &\geq K_2 \end{aligned}$$

$\Rightarrow$

$$K_2 = (\frac{K_1}{Cst})^{\frac{\sigma_2}{\sigma_1}}, \quad K_1 = Cst(K_2)\frac{\sigma_1}{\sigma_2}$$

Discrete Solutions attaining the bounds for $n = 2$

- The positive first quadrant is divided into 6 subregions in the case $D = 0$ as indicated in Figure 5, or into seven subregions, when $D > 0$ and when $D < 0$. These correspond to the subregions $R_i, i = 1, \cdots, 6$, in the case $D = 0$, or $R_i, i = 1, \cdots, 7$, in the cases $D > 0$ or $D < 0$, bounded by th lines $S_1 + S_2 = K$, $S_1 = K_1$, $S_2 = K_2$ and the S_1 and S_2 axes. We denote by p_i the joint probabilit that $(S_1, S_2) = (S_1^i, S_2^i)$ for $(S_1^i, S_2^i) \in R_i$.
  Seek to determine the p_i and the position of the pairs (S_1^i, S_2^i) that yield a solution that attains our bound Ansatz

$$(S_1^i, S_2^i) = (K_1 \pm \alpha_i, K_2 \pm \beta_i)$$

where $i = 1, \cdots 6 \quad \text{if} \quad D = 0$ and $i = 1, \cdots 7, \quad \text{if} \quad D < 0\,,$ or $D > 0$ where $\alpha_i > 0, \beta_i > 0$

Discrete Simplex

Introduce a_i, b_i

$$a_i = \frac{\alpha_i p_i}{K_1} \quad i = 1, \cdots 7$$

$$b_i = \frac{\beta_i p_i}{K_2} \quad i = 1, \cdots 7$$

so that • **Problem now** has become that of determining (a_i, b_i) and p_i for all i. For instance in Case $D \geq 0$:

Constraints

$$a_2 + a_3 + a_4 = \frac{c_1}{K_1} \quad (3$$

$$b_4 + b_5 + b_6 = \frac{c_2}{K_2} \quad (3$$

$$a_2 + a_3 + a_4 - a_1 - a_5 - a_6 - a_7 = \frac{m_1 - K_1}{K_1} \quad (3$$

$$b_4 + b_5 + b_6 + b_7 - b_1 - b_2 - b_3 = \frac{m_2 - K_2}{K_2} \quad (3$$

After Transf -> Linear Objective

Objective Functional

$$K_1 a_3 - K_2 b_3 + K_1 a_4 + K_2 b_4 - K_1 a_5 + K_2 b_5 - K_1 a_7 - K_2 b_7 - D(p_3 + p_4 + p_5 + p_7)$$

Where $a_i = \frac{\alpha_i p_i}{K_1}$ (original functional quadratic).

Discrete

Use the equalities to eliminate a_1, a_4 and b_1, b_4. Note that since these are required to be non-negative th leads to additional inequalities to be satisfied by the remaining variables. The functional may then be written

$$-K_1a_2 - K_1a_5 - K_1a_7 - K_2b_3 - K_2b_6 - K_2b_7 + c_1 + c_2 - D(p_3 + p_4 + p_5 + p_7)$$

and the last two constraints then become

$$a_1 + a_5 + a_6 + a_7 = -\frac{Q_1}{K_1} \quad (4$$

$$b_1 + b_2 + b_3 + b_7 = -\frac{Q_2}{K_2} \quad (4$$

determine the inequalities to be satisfied by $(a_i, b_i), i = 1, \cdots 7$ in order for the corresponding poi

$(S_1^i, S_2^i) = (K_1 + \frac{K_1 a_i}{p_i}, K_2 + \frac{K_2 b_i}{p_i})$ to be contained in the region R_i

Discrete

$$-a_5 - a_6 - a_7 \le \frac{Q_1}{K_1} + p_1 \qquad \text{(comes from} \qquad a_1 \le p_1) \qquad (4$$

$$a_5 + a_6 + a_7 + \gamma b_2 + \gamma b_3 + \gamma b_7 \le -\frac{Q_1}{K_1} - \gamma \frac{Q_2}{K_2} + \frac{D}{K_1} p_1, \qquad (4$$

$$\text{(comes from} \qquad -a_1 - \gamma b_1 - \frac{D}{K_1} p_1 \le 0)$$

$$-b_2 - b_3 - b_7 \le \frac{Q_2}{K_2} + p_1 \quad \text{old} \qquad b_1 \le p_1 \qquad (4$$

$$a_2 - \gamma b_2 \le \frac{D}{K_1} p_2 \qquad (4$$

$$b_2 \le p_2 \qquad (4$$

$$b_3 \le p_3 \qquad (4$$

Discrete

$$\gamma b_3 - a_3 \leq -\frac{D}{K_1} p_3 \tag{48}$$

$$a_5 \leq p_5 \tag{49}$$

$$a_5 - \gamma b_5 \leq -\frac{D}{K_1} p_5 \tag{50}$$

$$a_6 \leq p_6 \tag{51}$$

$$\gamma b_6 - a_6 \leq \frac{D}{K_1} p_6 \tag{52}$$

$$a_7 + \gamma b_7 \leq -\frac{D}{K_1} p_7 \tag{53}$$

Discrete Ct'd

$$a_5 + a_6 + a_7 \leq -\frac{Q_1}{K_1} \tag{54}$$

$$a_2 + a_3 \leq \frac{c_1}{K_1} \tag{55}$$

$$b_2 + b_3 + b_7 \leq -\frac{Q_2}{K_2} \quad (b_1 \geq 0) \tag{56}$$

$$b_5 + b_6 \leq \frac{c_2}{K_2} \quad (b_4 \geq 0) \tag{57}$$

$$\sum_{i=1}^{7} p_i = 1 \tag{58}$$

Graph of Simplex Tableaux

INSERT pdf for Sample Tableaux.

Conclusion

1. Knowing options of one strike and one maturity, it is possible to obtain closed form sharp distribution free upper and lower bounds for basket options on n assets
2. Gap between Upper Bound and Lower Bound is large
3. Gap can be narrowed (but at what rate?) by adding more input instruments, ie. prices of options with other strikes and-or maturities.
4. Successful numerical approach to this problem must circumvent dealing with huge number of LP problems
5. Hedge ratios produced in L- Wang work not usable for hedging in normal market environment but might be useful in protecting against extreme unpredictable moves.

Conclusion Ct'd

- But can be used to give a sanity check on new parametric models. Also closed form solutions as check on numerical solutions.
- Alternative ways to narrow the gap might involve adding additional restrictions such as correlation in a band.
- Stock pays an unknown dividend.
- American Options.

Appendix: Strong Duality

Proposition 2 **(Strong Duality)** *If* $(m, c) \in \mathring{\mathcal{C}}$ *(the interior of* $\mathcal{C}$*) then the values of the dual and primal problems coincide.*

The strong duality follows from Theorem 1, page 187 in Isii's paper. However to ensure that the hypotheses of his theorem are verified we must ensure that $\mathcal{B}_U < \infty$ in the case of the upper bound, and similarly $\mathcal{B}_L > -\infty$. Since our payoff function is nonnegative the condition for the lower bound is trivially satisfied. In the case of the upper bound it follows from the following argument.

$$E_\nu\left[\left(\sum_{i=1}^n x_i - K\right)^+\right] = E_\nu\left[\left(\sum_{i=1}^n (x_i - K_i) + \sum_{i=1}^n K_i - K\right)^+\right]$$
$$\leq \sum_{i=1}^n E_\nu[(x_i - K_i)^+] + (\sum_{i=1}^n K_i - K)^+,$$

where the last inequality follows from the elementary inequality $(a+b)^+ \leq a^+ + b^+$.

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Therefore, for any probability measure ν, satisfying the option and forward constraints, we have that $\left(\sum_{i=1}^{n} x_i - K\right)^+$ is integrable and

$$E_\nu\left[(\sum_{i=1}^{n} x_i - K)^+\right] \leq \sum_{i=1}^{n} c_i + (\sum_{i=1}^{n} K_i - K)^+ < +\infty.$$

Taking the sup over all such ν we have established the necessary condition required in Isii's theorem.

p1 + Q1/K1	0	-1	-1	0	0	0	1	0
- Q1/K1	0	1	1	γ	0	γ	0	1
p1 + Q2/K2	0	0	0	-1	0	-1	0	0
p3	0	0	0	1	0	0	0	0
-D/K1 p3	-1	0	0	γ	0	0	0	0
p5	0	1	0	0	0	0	0	0
- D/K1 p5	0	1	0	0	-γ	0	0	0
-D/K1 p7	0	0	1	0	0	γ	0	0
-Q1/K1	0	1	1	0	0	0	0	0
c1/K1	1	0	0	0	0	0	0	0
-Q2/K2	0	0	0	1	0	1	0	0
c2/K2	0	0	0	0	1	0	0	0
-c1/K1 - c2/K1 + D/K1- D/K1(p1)	0	-1	-1	-γ	0	-γ	0	0
b	A_3	A_5	A_7	B_3	B_5	B_7	s1	s2

p1 + Q1/K1	0	0	0	γ	0	γ	1	1
- Q1/K1 - γ Q2/K2 + D/K1 p1	0	Pivot =1	1	γ	0	γ	0	1
p1 + Q2/K1 + Q1/K1	0	0	0	-1	0	-1	0	0
p3	0	0	0	1	0	0	0	0
-D/K1 p3	-1	0	0	γ	0	0	0	0
p5 + Q1/K1	0	0	0	-γ	0	-γ	0	0
- D/K1 p5 + Q1/K1	0	0	0	-γ	-γ	-γ	0	0
-D/K1 p7	0	0	1	0	0	γ	0	0
0	0	0	1	-γ	0	-γ	0	0
c1/K1	1	0	0	0	0	0	0	0
-Q2/K2	0	0	0	1	0	1	0	0
c2/K2	0	0	0	0	1	0	0	0
-c1/K1 - c2/K1 - Q1/K1 - Q2/K2 + D	0	0	0	0	0	-γ	0	0
b	A_3	A_5	A_7	B_3	B_5	B_7	s1	s2

Table 6

This Table illustrates the application of the simplex method to find a minimizing discrete distribution in the case $n = 2$, $D < 0$ that attains the value $m_1 + m_2 - K = c_1 + c_2 + Q_1 + Q_2 - D$, which is one of the four minimizing values in that case. See (**??**).