Implied Volatility, Fundamental solutions, asymptotic analysis and symmetry methods

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Main aims of this contribution

Outline of talk

- A Mathematician's toolkit (partial) for quasi-closed form solutions in vanilla option pricing.
- Implied Volatility, local volatility, mimicking behaviour. Practitioners like closed form formulas for calibration.
- Symmetry and transformation methods for determining closed form solutions.
- Heat Kernel approach to solving stochastic volatility models. Hagan-Lesniewski, Henry-Labordère.
- Lie Group Symmetry approach to parabolic differential equations in one spatial variable (joint with Peter Carr and Tai-Ho Wang).
- Finding generalized Sabr Models, new classes of stochastic volatility models, using Lie Symmetry Analysis(in collaboration with Tai-Ho Wang, and Shen-Ling Wang)
- Interaction between symmetry and heat kernel approach.

Quick Overview 1

The goal, from PDE point of view, is to solve parabolic equations in one or several spatial dimensions.

- On $[0, T]$, where T is the maturity of European option, solve:

$$u_t + a_{ij}(\mathbf{x})u_{x_i x_j} + b_i u_{x_i} - ru = 0, \mathbf{x} \in \Omega \subset \mathbb{R}^n,\ t \in [0, T]$$
$$u(\mathbf{x}, T) = \psi(\mathbf{x}) \quad \text{final condition}$$

- The matrix $\{a_{ij}\}$ is usually degenerate, so the operator above is often not uniformly parabolic.

Researchers in PDE are more used to seeing the equation expressed as initial value (rather than final value) problem, and this can be achieved, by making the change of variables: $\tau = T - t$, so the problem now reads:

$$u_\tau - a_{ij}(\mathbf{x})u_{x_i x_j} - b_i u_{x_i} + ru = 0, \mathbf{x} \in \Omega \subset \mathbb{R}^n,\ t \in [0, T]$$
$$u(\mathbf{x}, 0) = \psi(\mathbf{x}) \quad \text{initial condition}$$

Fundamental solution

We are interested in finding the fundamental solution of such parabolic equations:

$$F(\mathbf{x}, t, \xi, T), \mathbf{x}, \xi \in \Omega, \ t \in [0, T]$$

Often Ω is $\mathbb{R}^n$ or $\mathbb{R}^n_+$.
Here F satisfies the parabolic equation in the variables $\mathbf{x}$ and t and has a delta function final condition:

$$F(\mathbf{x}, T, \xi, T) = \delta_\xi(\mathbf{x})$$

We may wish to add additional boundary conditions, such as in the case of the valuation of barrier options, and in this case, we seek the Green's function, rather than the Fundamental solution.

Finance in the News

Quote from article by Mike Giles and Ronnie Sircar in Siam NEWS, October 2007:

- “ The major challenges in computational finance arise not from difficult geometries, as in many physical problems, but from the need for rapid calculation of an EXPECTATION or the solution of its associated Kolmogorov partial differential equation.”
- “ Efficiency is at the forefront, because models are re-estimated as new market data arrives and calibration (or “marking to market”) embeds the expectation/PDE calculation in an iterative solution to an inverse problem”’

  Interpretation of last sentence: from traded market prices back out parameters in coefficients of parabolic operator or back out the functional form of these coefficients.

Numerical methods for solving parabolic PDE: (Partial List)

- Finite difference methods. Assessment: Extremely useful, but suffer “Curse of dimensionality”.
- Monte-Carlo Methods
- Finite Element methods (fairly recent)

All of the above are essential but are not the subject of today’s talk!

Recent years have seen a surge in attempts to find closed form or quasi-closed form solutions to certain parabolic problems arising in option pricing.
In today's talk, we will concentrate on two of these. The first is:

1)

Changes of Variables, Transformation Groups and Lie Symmetry Analysis.
Literature (Partial List):

- Albanese and Kusnetsov: Reducing time homogeneous one dimensional diffusions to standard form and solving via special functions.
  Transformations of Markov Processes and Classification Scheme for Solvable Driftless Diffusions. Preprint 2005
- Carr, Laurence and Wang: Reducing time *inhomogeneous* diffusions to standard form. Via Lie symmetry considerations.
  Comptes Rendus de l'Académie des Sciences, 2006.
- Linetsky: Time homogeneous one dimensional diffusions. Approach via eigenfunction expansions. Int. J. Theor. Appl. Finance 7 (2004).
- Ait-Sahalia: Annals of Statistics, 2007 "Closed-Form Likelihood Expansions for Multivariate Diffusions". Reduction method to heat equation with lower order terms.

Mathematician's Toolbox II, ct'd

Lie

• Concerning item I in toolbox we will concentrate on Lie symmetry group methods. An important objective: Determination of the fundamental solution of the parabolic differential equation. Use symmetry group to determine the fundamental solution, by reducing the dimensionality.

- General principle: When a problem has enough symmetry (sometimes quite hidden) it can be reduced to a solvable or at least more tractable problem.

All parts

- What all parts of this talk have in common is to illustrate methods used in trying to produce closed form solutions or good approximations for a family of stochastic volatility models, without jumps.
- WARNING: In so doing we will *not cover several important approaches to stochastic volatility models:*
  a) stochastic time changes , Yor, Deman, Carr, Wu.
  Also our models will not incorporate any jumps (see for instance Cont-Tankov).
  b) Stochastic Taylor Expansions, weak and strong.
  c) Affine Models and Quadratic models: a broad class which accomodate many important models.

Basic Model: Black-Scholes Equation

Historically the most important model for option pricing is

$$dS_t = rS_t dt + \sigma S_t dW_t$$
$$S(0) = S$$

Leading to the following PDE for the price of a call option on S:

$$C_t + \frac{1}{2}\sigma^2 C_{SS} + SC_S - rC = 0 \quad \text{for all} \quad S \in \mathbb{R}_+, 0 \leq t \leq T$$
$$C(S,T) = (S-K)^+ \quad \text{final condition}$$

with closed form solution:

$$C(S,t) = SN(d_1) - Ke^{r(T-t)}N(d_2)$$
$$d_1 = \frac{\log(\frac{S}{K}) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T-t}} \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

Implied Volatility

Suppose the stock process is **not** a geometric Brownian motion and instead is given by:

- $$dS_t = S_t \sigma(S,t) dW_t \quad \text{called " local volatility model"}$$

- $$dS_t = b(S_t) S_t y_t dW_{1t}, \quad dy_t = y_t c(y_t) dW_{2t}, \qquad \text{called "stochastic volatility model"}$$

  Then the option price $C^{\text{my model}}$ will be given by the solution of a parabolic equation with variable coefficients (even after $\log S$ transformation).

- The implied volatility is that value $\sigma = I(K,T)$ such that

  $$C^{\text{Black-Scholes}}(S,t,K,I(K,T)) = C^{\text{my model}}(S,t,K,T,\text{parameters}) =$$

  ie. the constant parameter σ for which we match our favourite model's price for a *given maturity T and a given strike K if we were to use Black-Scholes.*

Implied Volatility II

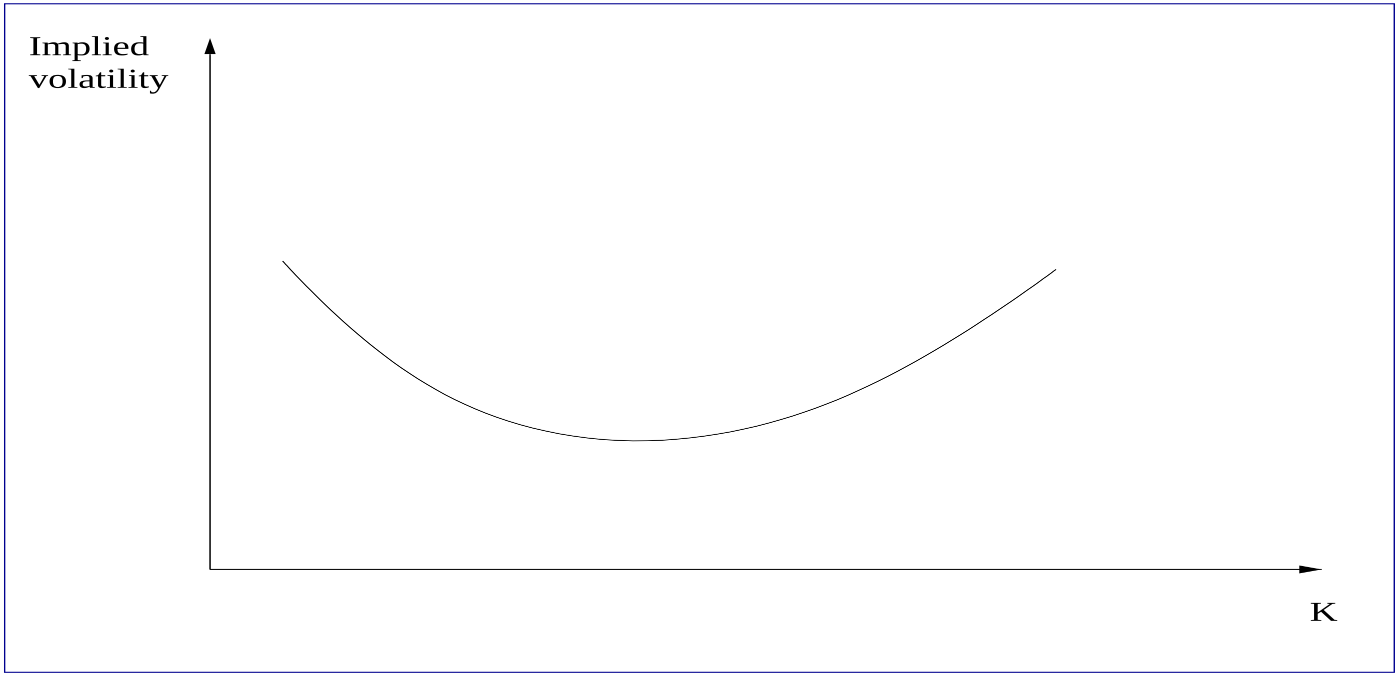
(Repeating):

Definition 1 *The BS implied volatility is the value* $\sigma = \sigma^{BS}(S,t,K,T)$ *such that if we denote by* $C^{BS}(S,t,K,T,\sigma)$ *the Black-Scholes call option price as a function of*

$$C^{BS}(S,t,K,T,\sigma^{BS}(S,t,K,T)) = C^{\text{another model}}(S,t,K,T)$$

$\longmapsto$ If the Black-Scholes model were true we would would have $\sigma(S,t,K,T) \equiv C$ (where C is constant !), for fixed S,t as a function of strike K and T. Instead we see a smile or a skew, as illustrated on next slide.

Smile

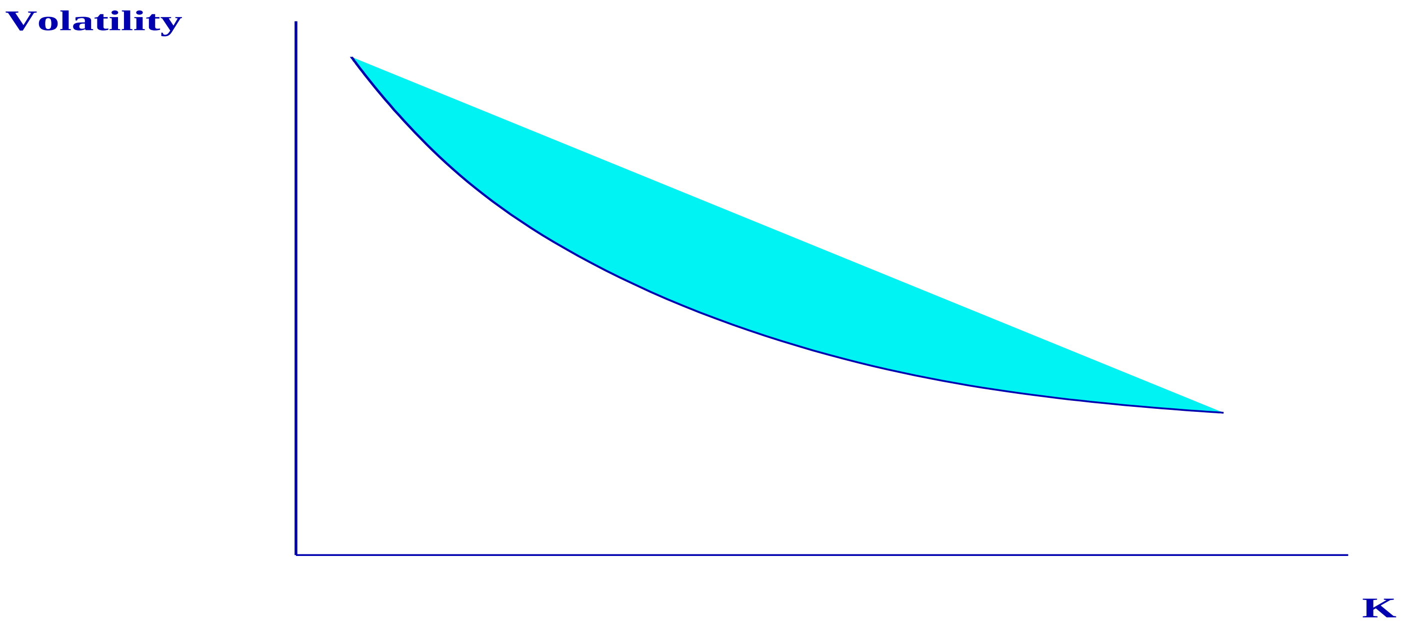


Foreign Currency smile

Market: Foreign Currency Market; Example of Model with a symmetric smile (Renault-Touzi, Mathematical Finance 1997): Stochastic Volatility Model with zero correlation.

$$dS_t = b(S_t) y_t dW_{1t}, dy_t = \nu(y_t) dW_{2t}, < dW_{1t}, dW_{2t} >= 0$$

Skew

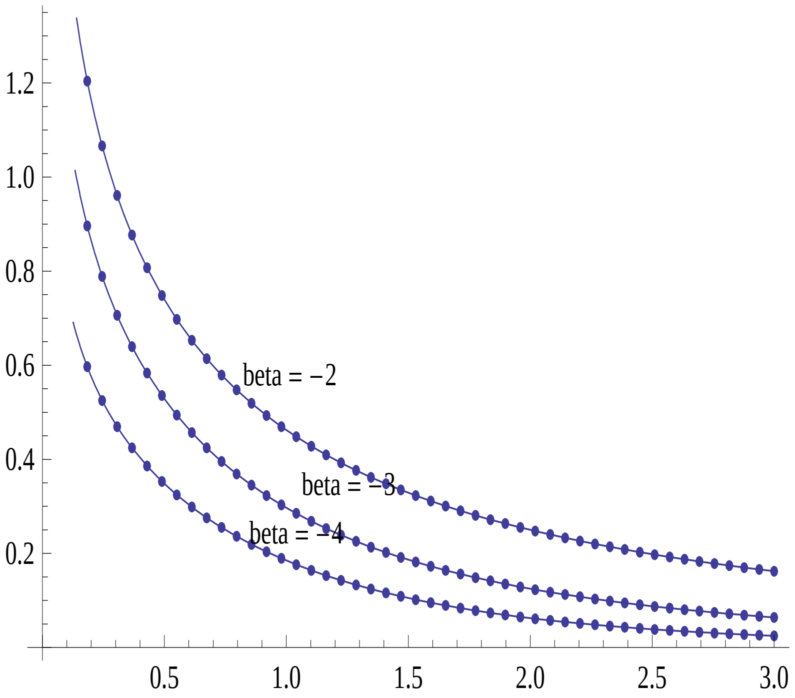


Equity Smile

Market: Equity; example CEV MODEL (Cox and Ross, Rubinstein, Schroeder, Delbaen-Schachermayer, Delbaen-Shirakawa,):

$$dS_t = \sigma S_t^{\beta+1} dW_t, \beta < 0$$

The smaller β the larger the skew.



From BBF harmonic mean approximation, which in CEV case gives: $I(F, K, \beta) = -((\beta(FK)^{\beta} \log[F/K])/(-F^{\beta} + K^{\beta}))$, using $F = 2$.

$$\frac{\partial C}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC = 0$$

$$\frac{\partial C}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC = 0$$

- $\tau = T - t$

$$\frac{\partial C}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC$$

$$\frac{\partial C}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0$$

- $\tau = T - t$

$$\frac{\partial C}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC$$

- $\xi = \ln S$

$$\frac{\partial C}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 C}{\partial \xi^2} + \left(r - \frac{\sigma^2}{2}\right) \frac{\partial C}{\partial \xi} - rC$$

Solving Black-Scholes

- $c(\xi,\tau) = e^{r\tau} C(\xi,\tau)$

$$\frac{\partial c}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 c}{\partial \xi^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial c}{\partial \xi}$$

Solving Black-Scholes

- $c(\xi, \tau) = e^{r\tau} C(\xi, \tau)$

$$\frac{\partial c}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 c}{\partial \xi^2} + \left(r - \frac{\sigma^2}{2} \right) \frac{\partial c}{\partial \xi}$$

- $x = \xi + \left(r - \frac{\sigma^2}{2} \right) \tau$

$$\frac{\partial c}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 c}{\partial x^2}$$

Solving Black-Scholes

In total, we have done the transformation

$$\tau = T - t \quad \text{change of time variable}$$

$$x = \ln S + \left(r - \frac{\sigma^2}{2}\right)(T - t) \quad \text{change of space variable}$$

$$c(x, \tau) = e^{r(T-t)} C(S, t) \quad \text{change of dependent variable}$$

which transforms Black-Scholes equation to heat equation.

Black-Scholes formula

- The existence of a mapping sending the Black-Scholes equation to the Heat equation could have been anticipated out of symmetry considerations.
- Indeed both the Black-Scholes equation and the heat equation have a 6 dimensional group of infinitesimal symmetry generators.
- Precise definition of symmetry generators later. Intuitively a symmetry generator is infinitesimal version of a transformation of dependent and independent variables which when evaluated on a solution of equation gives new solution of the equation. Historical folklore: After discovering their equation, it took Black and Scholes six additional months to solve it.

Preliminaries on implied and other volatilities

- Implied volatility: already defined. Local volatility models: $dS_t = S_t\sigma(S_t,t)dt + rSdt$
- Dupire's formula: From traded option prices to parametric form of $\sigma(F,t)$.

$$\sigma^2_{\text{loc}}(S,t,K,T) = \frac{\frac{\partial C}{\partial T} + rK\frac{\partial C}{\partial K}}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}$$

- From local volatility to implied volatility and vice-versa (fully non-linear PDE)

$$\sigma(\log(\overbrace{\frac{S}{K}}^{x}),T) = \frac{2TI\frac{\partial I}{\partial T} + I^2}{(1 - x\frac{\frac{\partial I}{\partial x}}{I})^2 + TI\frac{\partial^2 I}{\partial x^2} - \frac{1}{4}T^2I^2\frac{\partial^2 I}{\partial x^2}}, \quad \text{(take r =0)}$$

Approximate relationship (Berestycki-Busca-Florent, QF 2002) as $\tau = T - t \to 0$:

$$\lim_{\tau\to 0} I(\log(\frac{F}{K}),\tau) = \frac{1}{\int_0^1 \frac{1}{\sigma(s\log(\frac{F}{K}),0)}ds},$$

uniformly as $\tau \to 0$. Ie. Implied volatility is the harmonic mean of local volatility, in small time limit.

From stochastic volatility to local volatility

Stochastic volatility models:

$$dF_t = \alpha_t b(F_t) dW_{1t}$$
$$d\alpha_t = g(\alpha_t) dW_{2t}$$
$$F_0 = F, \alpha(0) = \alpha \quad \text{initial conditions}$$
$$< dW_{1t}, dW_{2t} >= \rho dt$$

Obtaining a local volatility model with same F marginals:
The “equivalent” local volatility function is given by:

$$\sigma^2_{loc}(K, T) = b^2(K) E[\alpha_T^2 \,|\, F_T = K]$$

Gyongi, Dupire, Atlan, Piterbarg

One can actually show a more general result, giving rise to the concept of "mimicking":

$$
\begin{aligned}
&dS_t = c(S_t, \nu_t, t)dt + b(S_t, t)g(\nu(t), t)dW_{1t} \\
&d\nu_t = \zeta(\nu_t)dt + \beta(\nu_t)dW_{2t}dt \\
&< dW_{1t}, dW_{2t} >= \rho dt \\
&S(0) = S, \quad \nu(0) = \nu,
\end{aligned}
$$

yields the same marginal distributions with respect to the S variable as the following sde:

$$
\begin{aligned}
&dS_t = \sigma(S_t, t)d\bar{W}_t + \gamma(S, t)dt, \\
&S(0) = S
\end{aligned}
$$

where, the effective parameters are:

$$
\begin{aligned}
&\sigma^2(K, T) = b(K, T)E\left[g^2 \,|\, S_T = K\right] \\
&\gamma(K, T) = E\left[c \,|\, S_T = K\right]
\end{aligned}
$$

Proof (time permitting)

Proof: Dupire and Derman and Kani

Following is heuristic. See Klebaner for rigour. From Breeden-Litzenberger (assume $r = 0$, for simplicity).

$$\frac{\partial^2 C(F,t,K,T)}{\partial K^2} = E[\delta(F_T - K)]$$

$$d(F_t - K)^+ = \mathbf{1}_{[K,+\infty)}(F_t)dF_t + \frac{1}{2}\alpha_t^2 b^2(F_t)\delta(F_t - K)dt$$

$$(F_T - K)^+ = \int_0^T \mathbf{1}_{[K,+\infty)}(F_t)dF_t + \frac{1}{2}\int_0^T \alpha_t^2 b^2(F_t)\delta(F_t - K)dt$$

$$= \int_0^T \mathbf{1}_{[K,+\infty)}(F_t)dF_t + b^2(K)\frac{1}{2}\int_0^T \alpha_t^2 \delta(F_t - K)dt$$

Proof ct'd

Taking expectations:

$$C(K,T) = \frac{1}{2} b^2(K) E\left[\int_0^T \alpha_t^2 \delta(F_t - K) dt\right]$$

Take the partial derivative with respect to upper limit T, get:

$$\frac{\partial C(K,T)}{\partial T} = \frac{1}{2} b^2(K) E\left[\alpha_T^2 \;\&\; F_T = K\right]$$

$$\frac{\partial C(K,T)}{\partial T} = \frac{1}{2} b^2(K) E\left[\alpha_T^2 \,|\, F_T = K\right] P[F_T = K]$$

$$\frac{\partial C}{\partial T} = \frac{1}{2} b^2(K) E\left[\alpha_T^2 \,|\, F_T = K\right] \frac{\partial^2 C}{\partial K^2},$$ to conclude that

$$\to \frac{\frac{\partial C}{\partial T}}{\frac{\partial^2 C}{\partial K^2}} = \frac{1}{2} b^2(K) E\left[\alpha_T^2 \,|\, F_T = K\right]$$

and notice that left hand side is the local volatility, using Dupire's formula.

PDE view: From stochastic volatility model to local volatility

Consider a stochastic volatility model:

$$dF_t = b(F_t)F_t y_t dW_{1t}, dy_t = y_t c(y_t) dW_{2t} \; F(0) = \bar{F}, y(0) = \bar{y}$$

Recall Gyongi formula: There exist a local volatility model (ie. a model with one less state variable)

$$dF_t = \sigma(F_t, t) F_t dW_{1t}$$

which has the same marginals with respect to the F_t process, given by

$$\sigma^2(F, t) = E\left[b^2(F_t)F_t^2 y^2 \mid F_t = F\right]$$

Let $\mathcal{F}(\bar{F}, \bar{y}, F, y, t)$ be the corresponding fundamental solution: Then in PDE language we have

$$(\sigma^2)^{\bar{F},\bar{y}}(\hat{F}, T) = \frac{F^2 b^2(F) \int y^2 \mathcal{F}(\bar{F}, \bar{y}, \hat{F}, y, T) dy}{\int \mathcal{F}(\bar{F}, \bar{y}, \hat{F}, y, T) dy}$$

So, if we knew the Fundamental solution in closed form or quasi-closed form, for small t, we could recover the asymptotic value of the local volatility and then the implied volatility, as a function of strike K and maturity T.

Tool II in toolbox

- Heat kernel approach, born as study of small time behaviour of fundamental parabolic differential equations.

Literature

- Lesniewski 2001, Hagan-Lesniewski-Woodward 2004.
- Henry-Labordère, 2005. Henry-Labordère Quantitative Finance 2007.

• Let $p(t, \mathbf{x}, \mathbf{y})$ be the fundamental solution corresponding to the non-degenerate diffusion with infinitesimal generator:

$$a_{ij}(\mathbf{x}) p_{x_i x_j}, \quad \mathbf{x} \in \mathbb{R}^n$$

and the time homogeneous diffusion (Heat flow) on $\mathbb{R}^n$.

$$\mathcal{H}p = p_t - L_{\mathbf{x}} p = 0$$

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$$\mathcal{H}p = p_t - L_{\mathbf{x}}p = 0$$

• The main theorem concerning the small time behaviour of the fundamental solution of this equation is due to Varadhan:

$$\lim_{t \to 0} 4t \log(p_t) = -d^2(\mathbf{x}, \mathbf{y}),$$

holds *uniformly for* $\mathbf{x}, \mathbf{y}$ *in compact subsets of* $\mathbb{R}^n$.

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$d(\mathbf{x}, \mathbf{y})$ *is the Riemannian distance, associated to* $\{g_{ij}\}$, *inverse of* $\{a_{ij}\}$, $ds^2 = g_{ij} ds_i ds_j$.

Riemannian distance

$\{g_{ij}\}$ the inverse of diffusion matrix $\{a_{ij}\}$, the Riemannian distance $d(\mathbf{x},\mathbf{y})$ is defined by:

$$d(\mathbf{x},\mathbf{y}) = \inf_{\mathbf{z}(\cdot):\mathbf{z}(0)=\mathbf{x},\mathbf{z}(1)=\mathbf{y}} \int_0^1 g_{ij}\dot{z}_i\dot{z}_j dt$$

Heston model, original SABR model have negative curvature, so don't need to worry about normal neighborhood and cut-locus.

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Note that $\Gamma(\mathbf{x},\mathbf{y})$, the square of the Riemannian Distance satisfies the Hamilton-Jacobi equation

$$a_{ij}\Gamma_{x_i}\Gamma_{x_j} = 4\Gamma$$

Inside a so-called “normal neighborhood” (Milnor (1969)) around a given point $\mathbf{x}_0$ the solution is C^∞.

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Inside a so-called “normal neighborhood” (Milnor (1969)) around a given point $\mathbf{x}_0$ the solution is C^∞. A notion of solution in the large requires viscosity solutions framework. To get intuition concerning Varadhan’s theorem, suppose that we had an analogue of fundamental solution for Euclidean heat equation, then we would have

$$p_t \sim \frac{1}{(4\pi\tau)^{n/2}} e^{-\frac{d^2(\mathbf{x},\mathbf{y})}{4\tau}}, \tau \to 0$$

Heston model, original SABR model have negative curvature, so don’t need to worry about normal neighborhood and cut-locus.

Varadhan 2

$$p_\tau \sim \frac{1}{(4\pi\tau)^{n/2}} e^{-\frac{d^2(\mathbf{x},\mathbf{y})}{4\tau}}, \tau \to 0 \quad \text{(repeated from last slide)}$$

Take the logarithm to get Compare with Varadhan

$$4\tau \log p_\tau - d^2(\mathbf{x},\mathbf{y}) = O(\frac{n}{2}\tau \log \tau), \quad \tau \to 0 \tag{1}$$

$$\lim_{\tau \to 0} 4\tau \log(p_\tau) = -d^2(\mathbf{x},\mathbf{y}),$$

holds *uniformly for* $\mathbf{x}, \mathbf{y}$ *in compact subsets of* $\mathbb{R}^n$*.*

So, (1) yields a bound on rate convergence in Varadhan's theorem. Special case of results by Molchanov using probabilistic methods and Berger et al using PDE and diff. geom.

PDE: Historical perspective

But who was the Father of it all?

Especially on PDE side?

Hadamard

ard Portraits http://www-groups.dcs.st-and.ac.uk/~history/PictDisplay/Hadamard.html






Jacques Hadamard

- Hadamard's contribution:
  He determined the fundamental solution for linear elliptic and hyperbolic equations and discovered the connection with the natural associated Riemannian metric.
  Lectures on Cauchy's Problem in Linear Partial Differential Equations
  Full of Jewels, even from contemporary point of view.
- Minakshisundaram-Pleijel discovered how to generalize to the case of parabolic equations.

Back to finance

Examples of Riemannian distances arising in finance

- Local Volatility Models $dF_t = F_t\sigma(F_t)$, $u_t - \frac{1}{2}F^2\sigma^2(F)u_{FF} = 0$.

$$d(F_1, F_2) = \int_{F_1}^{F_2} \frac{1}{F\sigma(F)} dF$$

Back to finance

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- SABR stochastic (alpha-beta-rho) volatility model, with $\beta = 0$, in normalized form:

$$dx_t = -\frac{1}{2}y_t^2 dt + y_t dW_{1t},\ dy_t = y_t dW_{2t}, \quad < dW_{1t}, dW_{2t} >= 0$$

Back to finance

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Think of $x = \log F$.

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2), \quad y \geq 0$$

We recognize this as the *distance in hyperbolic plane in the Poincaré model.*

Hyperbolic Space

$$\mathcal{H} : ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$$

Space of constant negative Gaussian curvature Gc equal to -1:

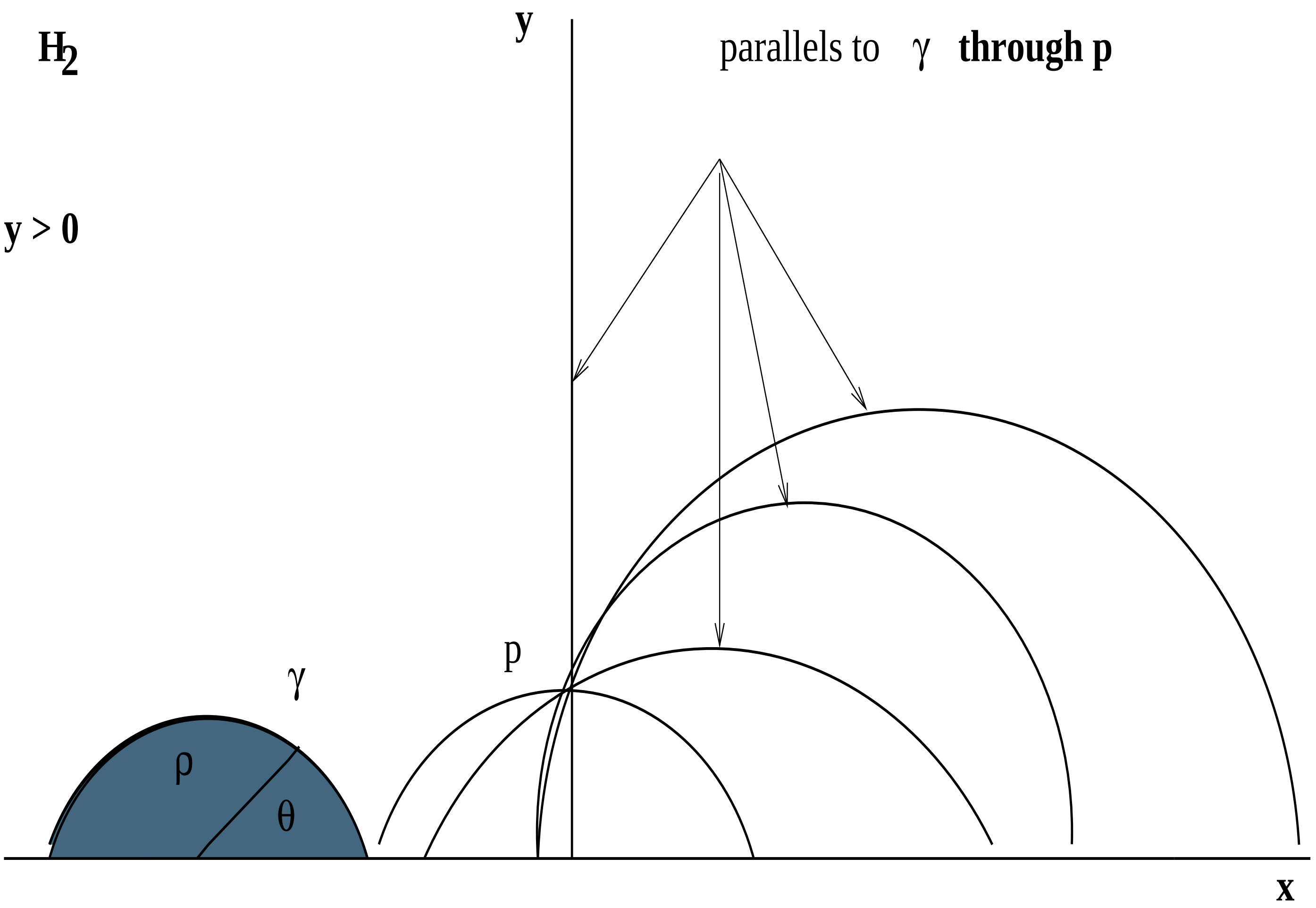
$$Gc = \frac{1}{2H}\left\{\frac{\partial}{\partial u}\left[\frac{F}{EH}\frac{\partial E}{\partial v} - \frac{1}{H}\frac{\partial G}{\partial u}\right] + \frac{\partial}{\partial v}\left[\frac{2}{H}\frac{\partial F}{\partial u} - \frac{1}{H}\frac{\partial E}{\partial v} - \frac{F}{EH}\frac{\partial E}{\partial u}\right]\right\}$$

where

$$ds^2 = Edx^2 + 2Fdxdy + Gdy^2 \quad H = \sqrt{EG - F^2}$$

and in our case: $E = G = \frac{1}{y^2}, \quad F = 0$

geodesics



Geodesics in the hyperbolic plane

So we need to find the geodesics in the hyperbolic plane. These are given by:

$$(x-a)^2 + y^2 = c^2 \quad \text{semicircles centered on } x \text{ axis}$$

Boundary $y = 0$ is never reached, because metric blows up there in non-integrable way.

Distance in hyperbolic plane and elsewhere

- Setting $z = (x, y)$, we have one can then go on to show that:

$$d(z_1, z_2) = \cosh^{-1}\left(1 + \frac{|z_1 - z_2|^2}{2y_1 y_2}\right)$$

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- Heston Model (usually written in terms of variance a^2 to get square root model)

$$\begin{aligned} df_t &= a_t f_t dW_{1t} \\ da_t &= -(\frac{\eta^2}{8a_t} + \frac{\lambda a}{2}(1 - (\frac{\bar{a}}{a})^2)dt + \frac{\eta}{2}dW_{2t} \end{aligned}$$

Since W_{1t} and W_{2t} are correlated, infinitesimal generator has mixed derivative. But can get rid of with suitable transformation: Let $x = \frac{1}{2}\sigma \log f - \frac{a^2}{2}$, $y = \frac{1}{2}a^2$. Associated Riemnannian metric (non-constant negative curvature, infinite curvature at $y = 0$) and

$$ds^2 = \frac{4}{\eta^2}\frac{1}{y}(dx^2 + dy^2) = \underbrace{\frac{4y}{\eta^2}}_{\text{conformal factor}} \underbrace{ds^2_{H^2}}_{\text{hyperbolic metric}}$$

So Heston model, in the new coordinates is in the same conformal class as hyperbolic plane.

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Heat kernel Series solution for fundamental sc

Seek solution in the form of a series:

$$F(x,y,\tau) = \frac{\sqrt{g(x)}}{(2\pi\tau)^{n/2}}\sqrt{\Delta(x,y)}\mathcal{P}(x,y)e^{-\frac{d^2(x,y)}{2\tau}}\sum_{n=1}^{+\infty} u_n(x,y)\tau^n, \quad \tau \to 0$$

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where,

- $d(x,y)$ is the geodesic distance between x and y, i.e., minimizer of the functional

$$\int_0^1 g_{ij}\frac{d\bar{x}^i}{dt}\frac{d\bar{x}^j}{dt}dt$$
$$\bar{x}(0) = x \quad \bar{x}(1) = y,$$

where recall:

$$g = a^{-1}, \quad \text{where } a = \{a_{ij}\} \text{ is principle part of elliptic operator } a_{ij}\frac{\partial^2}{\partial x^i \partial x^j}$$

Heat kernel ct'd

- $$f_\tau - a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} f - b_i \frac{\partial}{\partial x_i} f = 0$$

  Solution in the form :

  $$\frac{\sqrt{g(x)}}{(4\pi\tau)^{n/2}}\sqrt{\Delta(x,y)}\mathcal{P}(x,y)e^{-\frac{d^2(x,y)}{4\tau}}\sum_{n=1}^{+\infty} a_n(x,y)\tau^n, \quad \tau \to 0$$

- $$\Delta(x,y) = |g(x)|^{-1/2}\text{det}\left(\frac{\partial \frac{d^2}{2}}{\partial x \partial y}\right)|g(y)|^{-1/2}$$ Van-Vleck-DeWitt determinant

- $\mathcal{P} =$ exponential of work done by field $\mathcal{A}$, $e^{\int_{C(x,y)} \mathcal{A}\cdot \mathbf{dl}}$
- $\mathcal{A}$ is constructed from PDE, using two ingredients: principle part and from the drift b, i.e.

$$\mathcal{A}^i = b^i - \text{det}(g)^{-1/2}\frac{\partial}{\partial x^j}\left(\text{det}(g)^{1/2}g^{ij}\right)$$

Heat kernel

$$\frac{\sqrt{g(x)}}{(4\pi\tau)^{n/2}}\sqrt{\Delta(x,y)}\mathcal{P}(x,y)e^{-\frac{d^2(x,y)}{4\tau}}\sum_{n=1}^{+\infty}u_n(x,y)\tau^n, \quad \tau\to 0$$

Characterization of heat kernel coefficients

Obtained via a recursive scheme:

$$u_0(x,y)=1$$

$$(1+\frac{1}{k}\left[\nabla^i d^2\right])\nabla_i)u_k = \mathcal{P}^{-1}\Delta^{-1/2}L_S\Delta^{1/2}\mathcal{P}u_{k-1}$$

ordinary differential equations along geodesics-transport equations

$$\nabla_i = \partial_i + \mathcal{A}_i$$

.

Simplest Case

$$\frac{\sqrt{g(x)}}{(4\pi\tau)^{n/2}}\sqrt{\Delta(x,y)}\mathcal{P}(x,y)e^{-\frac{d^2(x,y)}{4\tau}}\sum_{n=1}^{+\infty}a_n(x,y)\tau^n, \quad \tau \to 0$$

Laborious calculations by a host of mathematicians and physicists characterize the on-diagonal form of the heat kernel coefficients, ie. we have

- $$u_0(x,x) = 1$$
  $$u_1(x,x) = P = \frac{1}{6}\text{Scalar Curvature} + \underbrace{g^{ij}(\mathcal{A}_i\mathcal{A}_j - b_j\mathcal{A}_i - \frac{\partial}{\partial x_j}\mathcal{A}_i)}_{Q}$$
  $$u_2(x,x) = \frac{1}{180}\left(|\text{Riemann Tensor}|^2 - |\text{Ricci Tensor}|^2\right) + \frac{1}{2}a_1^2 + \frac{1}{2}|\mathcal{R}|^2 + \frac{1}{30}\Delta_{\text{Bel}}R + \frac{1}{6}Q$$

Discussion

- First few terms in the series very effective as revealed by numerical experiments comparing approximating solution to numerically computed solution.
- However, expansions not rigorously justified (-able ?) without suitable adjustment.

Adjustments

- PDE approach due to Minakshisundaram-Pleijel(See Berger-Gauduchon-Mazet). Idea: construct a parametrix, via a series in two stages (assume Laplace-Beltrami for simplicity) Stage 1): Geometric Stage , for close points: Essentially same as above-mentioned asymptotic ansatz.

$$\text{Fundamental Solution } F = \frac{1}{(4\pi\tau)^{n/2}} e^{-d^2(x,y)/4\tau} \sum_{i=0}^{\infty} u_i(x,y)\tau^n$$

Ie. Use transport equns to determine coeffts. But now

- Stage 2) To define globally, ie. for distant points, 1) truncate series for any $k > \frac{n}{2}$ (using geometrically determined coefficients for $n < k/2$) and 2) use cut-off function away from the diagonal:
  I.e: Let ρ be smooth cut-off with $\rho(0, \epsilon/4) = 1$, $\rho(\epsilon/2, \infty) = 0$. Consider

$$\mathcal{H}_k = \rho(d(x,y)) \frac{1}{(4\pi\tau)^{n/2}} e^{-d^2(x,y)/4\tau} \sum_{i=0}^{k} u_i(x,y)\tau^n$$

.

Parametrix

- Can show that this is parametrix ie.
$$(\partial_\tau - L_S)\left\{\rho(d(x,y))\left[\frac{1}{(4\pi\tau)^{n/2}}e^{-\frac{d^2(x,y)}{4\tau}}\sum_{i=0}^{k}u_i(x,y)\tau^i\right]\right\} = O(t^{k-\frac{n}{2}})e^{-d^2(x,y)/2t}\mathcal{G}_k$$
where $\mathcal{G}_k$ is smooth.
- Use Levy parametrix idea (iterated convolution) to push error off to infinity. Ie.

$$\text{Fundamental Solution} = \mathcal{H}_k + \mathcal{H}_k * \mathcal{F}$$

where

$$F * G(x,y,t) = \int_0^t \int_M F(x,z,\tau)G(z,y,t-\tau)dV(z)$$

and where, letting $L = \partial_t - L_S$,

$$\mathcal{F} = \sum_{l=1}^{\infty}(\mathcal{L}\mathcal{H}_k)^{*l}$$

Iterated (infinitely) convolution.

Adjustments 2

- Can give precise estimates for the remainder. If $k > \frac{n}{2}$ then :

$$\text{Fundamental Solution} = \frac{e^{\frac{-d^2(x,y)}{2t}}}{(2\pi t)^{n/2}} \rho(d(x,y)) \left(\underbrace{\sum_{j=1}^{k} t^j u_j(x,y)}_{\mathcal{H}_k} + O(t^{k+1}) \right)$$

- Molchanov approach: Use transition probability for close points and arrive at distant points by piecing together using Chapman-Kolmogorov equations.
- Evans-Fleming-Soner-Souganidis: Viscosity solution approach. (To our knowledge) Applied mainly to Exit time problems, so far.
- Perhaps, right question practically minded math-finance researchers?: which of the above adjustments is most efficient while remaining highly accurate?

Tool III in toolbox (sort of)

“Ad Hoc Methods”: Fundamental solution in hyperbolic space Literature: McKean : Journal of Differential Geometry 1970; Grigoryan: Bulletin of the London Mathematical Society, 1998 (using results by Helgason); Matsumoto, Bull. Sci. Math. (2001).

In finance the importance of the case of hyperbolic space derives from it’s connection with one of the three leading stochastic volatility models, the Sabr model:

SDE’s:

$$dS_t = S_t^{\beta} y_t dW_{1t} \quad dy_t = \alpha y_t dW_{2t}, \quad < dW_{1t}, dW_{2t} >= \rho dt$$

$$u_t + \frac{1}{2} S^{2\beta} y^2 u_{SS} + \alpha S^{\beta} y u_{Sy} + \frac{1}{2}\alpha^2 y^2 u_{yy} = 0 \quad \text{PDE}$$

Hyperbolic space arises in the case $\beta = 0$.

hyperbolic plane and symmetries

Hyperbolic plane possesses a rich class of isometries, determined by the so-called Moebius Transformations:

$$L(z) = \frac{az+b}{cz+d}, \quad (z = x + iy, a, b, c, d \in \mathbb{R}, ad - bc > 0)$$

Moebius transformations preserve distances and angles and send geodesics to geodesics.
Also: the restriction $ad - bc > 0$ guarantees that the transformation maps the upper half plane to the upper half plane. Indeed:

$$\frac{az+b}{cz+d} = \frac{(ax+b)(cx+d) + acy^2 + \overbrace{(ad-bc)y}^{>0}\, i}{(cx+d)^2 + y^2}$$

Mc Kean Kernel

Turns out that the fundamental solution in hyperbolic space depends only on the geodesic distance and time. Recall from earlier slide that this geodesic distance was given by:

$$d(z, z_0) = \cosh^{-1}\left(1 + \frac{|z - z_0|^2}{2yy_0}\right) \qquad z = (x, y)$$

Mc Kean (1970) discovered that the heat kernel is given by the following expression:

$$\begin{aligned} G(z, z_0, t) &= \tilde{G}(d(z, z_0), t) \\ &= \frac{\sqrt{2}}{(4\pi t)^{3/2}} e^{-\frac{1}{4}t} \int\limits_{d(z,z_0)}^{+\infty} \frac{s e^{-\frac{s^2}{4t}}}{(\cosh s - \cosh d)^{1/2}} ds \end{aligned}$$

Heat kernel on hyperbolic space; conclusion

- Once we know that the fundamental solution depends *only* on the geodesic distance and time, we have done half the work, because in essence we have reduced the dimension by one!
- So question is : how to determine by some a priori method that the fundamental solution depends only on the geodesic distance or on some other key (reduced) variable ?
- Lie symmetry analysis, that we will discuss later on, provides a systematic method for achieving this.

Original Sabr model

F_t the futures price.

$$dF_t = \alpha_t F_t^\beta dW_{1t}$$
$$d\alpha_t = v\alpha_t dW_{2t}$$
$$F(0) = F, \alpha(0) = \alpha$$

with two correlated Brownian motions.
Here F_t and α_t are stochastic and v and $\beta \in [0, 1]$ are constants. ρ is constant!

λ Sabr model

Henry-Labordère:

$$dF_t = \alpha_t F^\beta dW_{1t}$$
$$d\alpha_t = \lambda(\alpha_t - \bar{\lambda})dt + v\alpha_t dW_{2t}$$
$$F(0) = F, \alpha(0) = \alpha$$

with two correlated Brownian motions.

$$< dW_{1t} dW_{2t} >= \rho dt,$$

Here F_t and α_t are stochastic and v and $\beta \in [0,1]$ are constants, $\lambda, \bar{\lambda}$ are constants. Also ρ is constant!

Uses of Sabr model

- Historically, Sabr model used especially to model the evolution of the forward rates $F_i, i = 1, \ldots, n$

$$dF_i(t) = v_t F_i^{\beta}(t) dW_{1,i}(t)$$
$$dv_t = v_t dW_{2,i}$$

where the forward rate F_j, for the j-th period satisfies

$$F_j(t)P(t,T_j) = \frac{P(t,T_{j-1}) - P(t,T_j)}{\tau_j}$$

τ_j is year fraction corresponding to period T_{j-1}, T_j. Hagan et al. then can price caplets.

- Evolution of swap rates. Hence pricing of swaptions.
- To some extent in equity markets.

Back to local volatility: Berestycki-Busca-Flor

$$\sigma_{loc}(f,\sigma_0,K,T) = \overbrace{E}^{\text{joint process}} \left[F_T^{2\beta}\Sigma_T^2 \mid F(0)=f, F_T=K, \Sigma(0)=\Sigma_0\right]$$

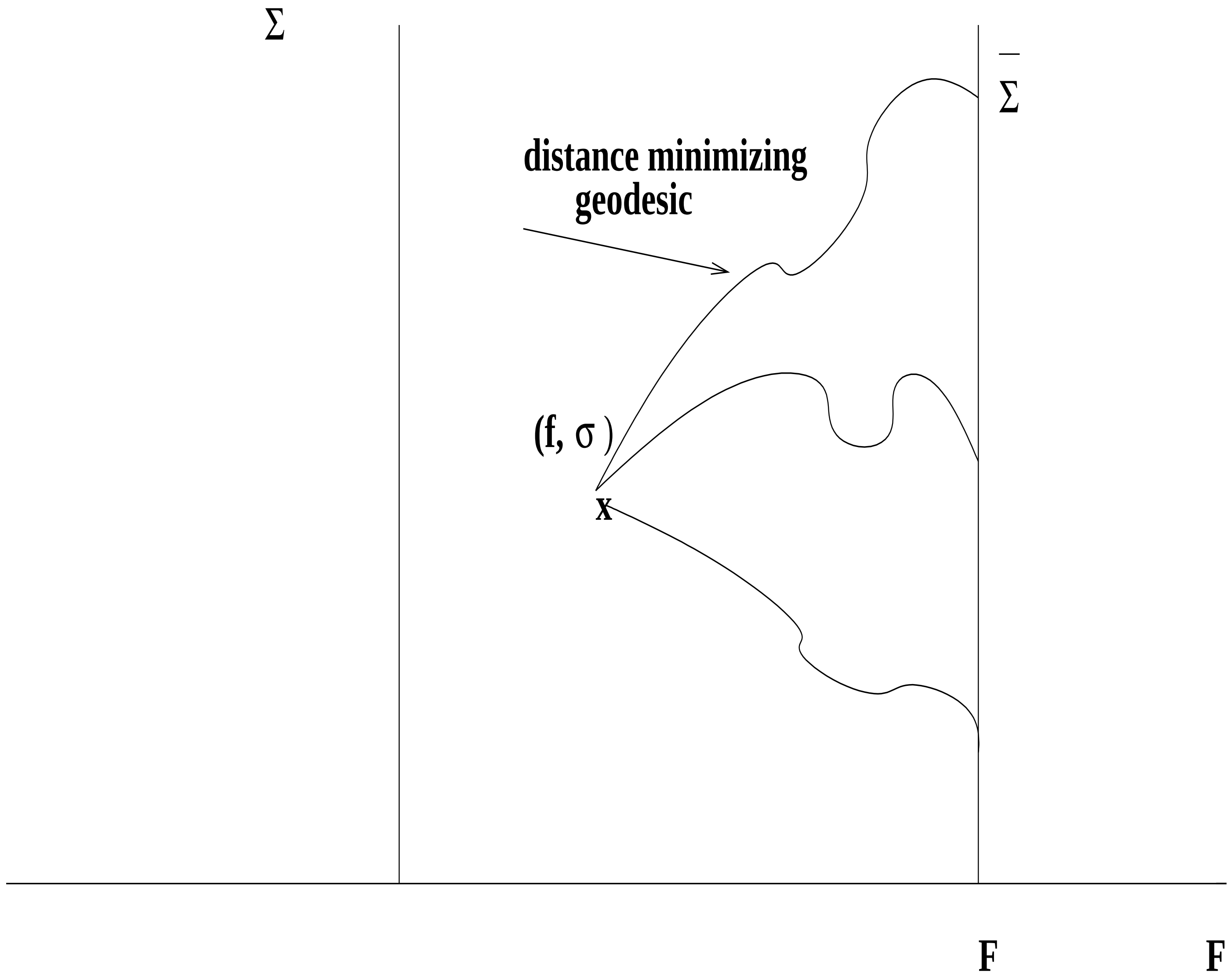
$$= K^{2\beta}\frac{\int_0^\infty \sigma^2 G_(f,\sigma,F,\Sigma,T)d\Sigma}{\int_0^\infty G_(f,\Sigma_0,F,\Sigma,T)d\Sigma}$$

Expressing, to leading order, G in terms of the geodesic distance,

$$G_(f,\Sigma_0,F,\Sigma,T) = \frac{1}{4\pi T}e^{-\frac{d^2((f,\sigma),(F,\Sigma))}{4T}},$$

Berestycki, Florent and Busca's analysis, or, at the heuristic level, the Laplace's method asymptotic expansion, again tells us to look at the reduced distance $d_{\mathcal{R}}$, ie. the value of Σ_T for which the point (f,σ) is the closest to the hyperplane $f=F, \Sigma_T=\bar{\Sigma}$ In the case of the SABR model, since we know the geodesic distance explicity, this point can be determined explicity. Thus gives rise to the notion of REDUCED distance ie, distance from (f,Σ_0) to hyperplane $\bar{F}=F$. (see Martin Forde's work).

Picture again



BBF ct's

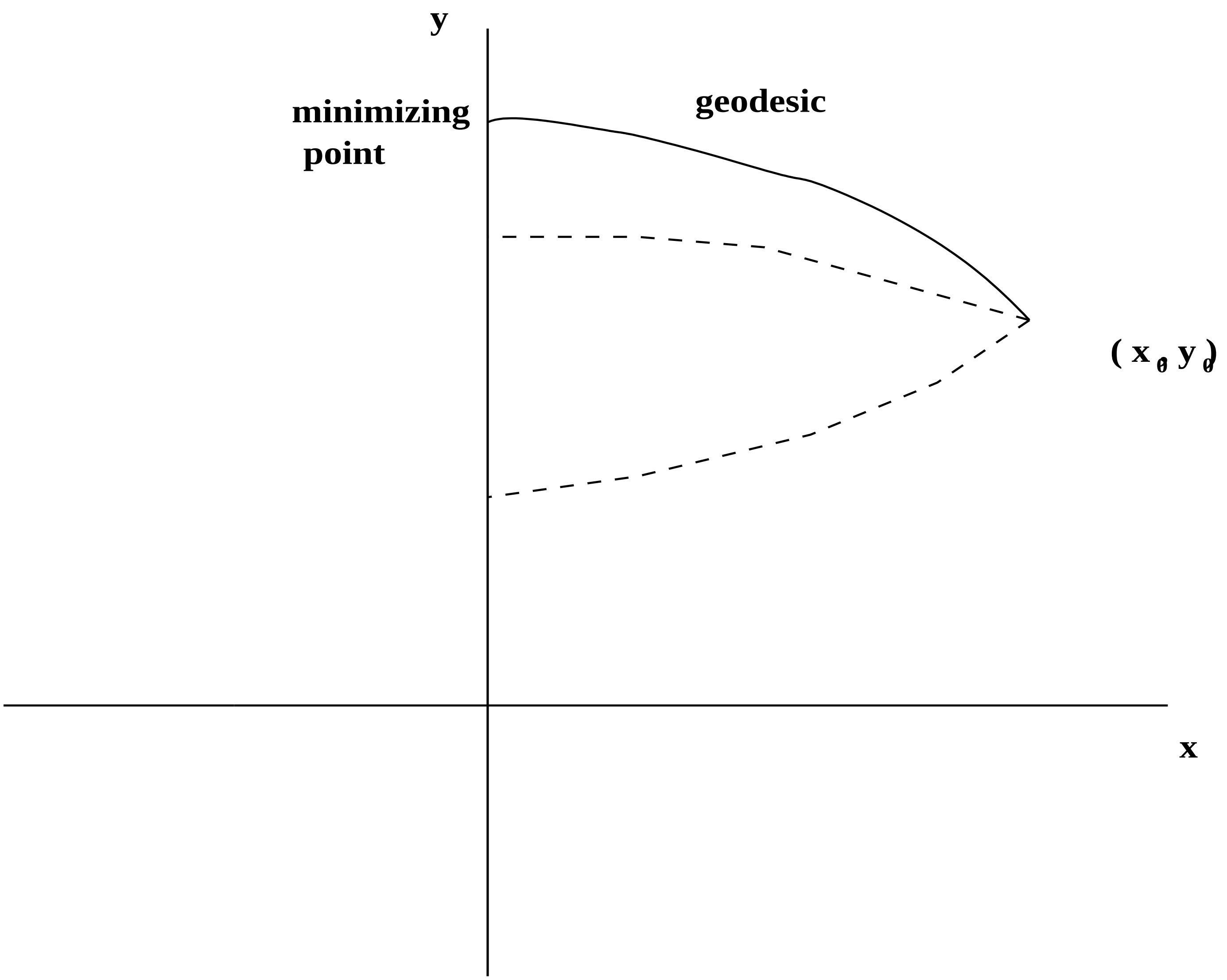
Berestycki-Busca-Florent

Proposition 1 *In the limit $\tau \to 0$, the ATM implied volatility, in the normalized log variables $x = \log(F/K), y = \nu$ is given by*

$$I(x_0, y_0) = \frac{x_0}{d\left((x_0, y_0), \{x = 0\}\right)}$$

where the denominator corresponds to the Riemannian distance of the point (x_0, y_0) to the hyperplane $x = 0$. Here (x_0, y_0) are the initial points for the $2 - D$ diffusion.

Illustration



Examples

Let us give some applications of this formula:

- No stochastic volatility $(dF_t = F_t \sigma(F_t, t) dW_t)$.
  In this case the geodesic distance between F_0 and K is simply:

$$\int_K^F \frac{1}{f\sigma(f)}(f)df \quad \text{inverse of 'matrix' } f\sigma(f) \tag{2}$$

- In the lognormal variable $x = \log(F/K)$ this becomes

$$\int_0^x \frac{1}{\sigma(x')}dx' \quad x' = xs \hookrightarrow$$
$$= x\int_0^1 \frac{1}{\sigma(sx)}dx$$

  Therefore

$$I = \frac{x}{x\int_0^1 \frac{1}{\sigma(sx)}dx} = \frac{1}{\int_0^1 \frac{1}{\sigma(sx)}ds}$$

  and we recover the (earlier) geometric mean result for local vol.

Examples

- **Example 2: Heston Model**

$$dS_t = rS_t + \sqrt{y_t} S_t dW_{1t}$$
$$dy_t = a(\theta - y_t)dt + \kappa\sqrt{y_t} dW_{2t}$$

where a, θ, κ are non-negative constants.
In the Heston model the geodesic distance is the unique solution of

$$yd_x^2 + 2\rho\kappa y d_x d_y + \kappa^2 y d_y^2 = 1 \quad \text{reduced distance, ie distance to } \{x = 0\}$$
$$d(x = 0, y) = 0 \quad d(x, y) > 0, \quad \text{for } x > 0 \quad \text{initial condition}$$

Once d is computed (numerically) we get

$$I(x, y) = \frac{x}{d(x, y)}$$

Sabr according to Hagan and Lesniewski, fro

- Issue: We now would like to pass from *uncorrelated to correlated Sabr models.*

$$dF_t = F_t^{\beta}\Sigma_t dW_{1t}$$
$$d\Sigma_t = \upsilon\Sigma_t dW_{2t}$$
$$< dW_{1t}, dW_{2t} >= \rho dt$$

- *Equation:*

$$G_\tau - \frac{1}{2}f^{2\beta}\sigma^2 G_{ff} - \rho\upsilon\sigma f^{\beta}G_{f\sigma} - \frac{1}{2}\upsilon^2\sigma^2 G_{\sigma\sigma}$$
$$G_\tau - \frac{1}{2}\sigma^2\left(f^{2\beta}G_{ff} - 2\rho\upsilon G_{f\sigma} - \frac{1}{2}\upsilon^2 G_{\sigma\sigma}\right)$$

$$G(0,f,\sigma) = \delta(f-f_0,\sigma-\sigma_0)$$ *initial condition*

Hagan-Lesniewski change of variables

$$G_\tau - \frac{1}{2}\sigma^2\left(f^{2\beta}G_{ff} - 2\rho v G_{f\sigma} - \frac{1}{2}v^2 G_{\sigma\sigma}\right)$$
$$= G_\tau - a_{ij}\frac{\partial^2 G}{\partial x_i \partial x_j}$$

Riemannian metric

$$g = \{a_{ij}\}^{-1} = \frac{1}{\sqrt{1-\rho^2}y^2F^{2\beta}}\begin{pmatrix} 1 & -\rho F^\beta \\ -\rho F^\beta & F^{2\beta}\end{pmatrix} \quad \text{take } v = 1$$

- See a change of variables $(f,\sigma)\overbrace{\mapsto}\phi(x,y)\sigma)$, which maps metric to standard hyperbolic plane with metric $\frac{1}{y^2}(dx^2+dy^2)$.

$$\phi(z) = \left(\frac{1}{\sqrt{1-\rho^2}}\left(\int_0^x \frac{du}{F^\beta} - \rho y\right), y\right)$$

Isometry

If such a diffeomorphism exists it establishes an isometry (ie. *distance between points is preserved* between the corresponding Riemannian spaces. So we have

$$\mathcal{R} \overbrace{\mapsto}^{\phi} H_2$$

So calling δ the distance in $\mathcal{R}$, we have

$$\cosh(\overbrace{\delta(z,Z)}^{\text{new distance}}) = \cosh\left(\overbrace{d\left(\phi(z),\phi(Z)\right)}^{H_2 \text{ distance}}\right)$$

$$= \cosh\left[1 + \frac{\left(\int_X^x \frac{1}{f^\beta} du)^2\right) - 2\rho(y-Y)\int_X^x \frac{1}{f^\beta} du + (y-Y)^2}{2(1-\rho^2)yY}\right] \qquad z=(x,y), Z=(X,Y)$$

$$= \cosh\left[1 + \frac{\zeta^2 - 2\rho(y-Y)\zeta + (y-Y)^2}{2(1-\rho^2)yY}\right]$$

epilogue

Therefore we see to find the minimum of the distance:

$$\cosh(\overbrace{\delta(z,Z)}^{\text{new distance}}) = \cosh\left[1 + \frac{\zeta^2 - 2\rho(y-Y)\zeta + (y-Y)^2}{2(1-\rho^2)yY}\right]$$

over Σ (recall $Z = (F, \Sigma)$), for fixed (f, σ) (here $z = (f, \sigma)$).
Given the explicit form of the expression above, calculus yields this easily:
Answer:

$$\Sigma_0 = \sigma\sqrt{\zeta^2 - 2\rho\zeta - 1}$$

$$\text{minimum distance= reduced distance} = \log\left(\frac{\sqrt{\zeta^2 - 2\rho\zeta + 1} + \zeta + \rho}{1-\rho}\right)$$

Knowing the above value *explicitly is the key to the Sabr asymptotic formula.*

Back to Tool 1 in toolbox

Lie symmetry approach

How to solve it ?

In order to solve a differential equation, you look at it till a solution occurs to you !

- George Pólya, *How to solve it*

Convenient Variables

The main difficulty in integrating a given differential equation lies in introducing convenient variables, which there is no rule for finding. Therefore we must travel the reverse path and after finding some notable substitution, look for problems to which it can be successfully applied.

- Jacobi, *Lectures on Dynamics*, 1847.

Lie's Symmetry Analysis

From geometric point of view, the great Norwegian mathematician Sophus Lie created a systematic way for analyzing solutions of differential equations. His theory nowadays has been known as Lie's symmetry analysis of differential equations. To this day, Lie's theory has been greatly extended and used to find closed form solutions, to classify equations up to some unknown functions, to find fundamental solutions and so on.

The Lie objective: Classification and Reductio

Symmetry Group A symmetry group of a system $\mathcal{S}$ of differential equations is a local group of transformations G, acting on some open subset

$$M \subset \underbrace{X}_{\text{independent variables}} \times \underbrace{U}_{\text{dependent variables}},$$

with the property that whenever $u = f(x)$ is a solution of $\mathcal{S}$ and whenever $g \cdot f$ is defined for $g \in G$, then $u = g \cdot f(x)$ is also a solution of the system.

- Definition allows arbitrary nonlinear transformations of both the independent and dependent variables in the definition of symmetry.

Symmetry operators: Intuition

A infinitesimal symmetry operator is an infinitesimal version of symmetry-group transformation and takes the form

$$X = \tau(t)\frac{\partial}{\partial t} + \sum_{i=1}^{n} \xi_i(x,t)\frac{\partial}{\partial x_i} + uf(x,t)\frac{\partial}{\partial u},$$

which changes infinitesimally the dependent variable u and the independent variables x is such a way that it takes solutions to solutions. Consider the heat equation in n dimensions:

- Simplest example:

  $$u_t(\mathbf{x},t) - \Delta u(\mathbf{x},t) = 0$$

  Then if $\bar{u}(\mathbf{x},t)$ is a solution, and if U is an orthogonal transformation

  $$v(\mathbf{x},t) = \bar{u}(U\mathbf{x},t) \qquad \text{is also a solution}$$

- Another example: In $\mathbb{R}^n$, if $\bar{u}(x,t)$ is a solution then $\bar{u}^{(4)}(x,t) = \bar{u}(e^{-\epsilon}x, e^{-2\epsilon}t)$ parabolic scaling.

Heat equation, ct’d

Less obvious examples:

- $$
\begin{aligned}
-- \quad & \bar{u}^{(5)}(x,t) = e^{-\epsilon x+\epsilon^2 t}\bar{u}(x-2\epsilon t, t) \\
-- \quad & \bar{u}^{(6)}(x,t) = \frac{1}{\sqrt{1+4\epsilon}}\exp\left(\frac{-\epsilon x^2}{1+4\epsilon t}\right)\bar{u}\left(\frac{x}{1+4\epsilon t}, \frac{t}{1+4\epsilon t}\right)
\end{aligned}
$$

- Mechanism of $\bar{u}^{(5)}(x,t)$ (say) being a solution, when $\bar{u}$ is:

$$
\begin{aligned}
\bar{u}^{(5)}_t &= e^{-\epsilon x+\epsilon^2 t}(\bar{u}_t - 2\epsilon\bar{u}_x) \\
\bar{u}^{(5)}_{xx} &= e^{-\epsilon x+\epsilon^2 t}\,(\bar{u}_{xx} + 2\bar{u}_x - \epsilon\bar{u}_x)
\end{aligned}
$$

so (due to cancellation)

$$
\bar{u}^{(5)}_t - \bar{u}^{(5)}_{xx} = e^{-\epsilon x+\epsilon^2 t}(\bar{u}_t - \bar{u}_{xx}) = 0,
$$

since $\bar{u}$ is a solution.

Symmetry group of the heat equation

All in call it can be shown that the infinitesimal symmetry group of the heat equation is generated by

- Finite (six) dimensional subalgebra

$$
\begin{aligned}
\mathbf{v}_1 &= \partial_x \\
\mathbf{v}_2 &= \partial_t \\
\mathbf{v}_3 &= u\partial_u \\
\mathbf{v}_4 &= x\partial_x + 2t\partial_t \\
\mathbf{v}_5 &= 2t\partial_x - xu\partial_u \\
\mathbf{v}_6 &= 4tx\partial_x + 4t^2\partial_t - (x^2 + 2t)u\partial_u
\end{aligned}
$$

- Infinite dimensional subalgebra $v_\alpha = \alpha(x,t)\partial_u$, where α is an arbitrary solution of the heat equation. ie., the finite dimensional heat equation has a 6 dimensional symmetry group.

CEV: an example with $4D$ group

$$u_t - \frac{1}{2}\sigma^2 x^{2\beta} u_{xx} = 0$$

corresponding to SDE $dx_t = \sigma x^\beta dW_t$.
It can be shown that this equation admits a 4 dimensional infinitesimal symmetry group, unless $\beta = 1$ or $\beta = 0$, in which case the symmetry group is 6 dimensional. For instance,

$$X_3 = t\partial_t + \frac{x}{2(1-\beta)}\partial_x$$

is a symmetry generator and this corresponds to fact that: if $\bar{u}$ is a solution then $\bar{u}^3(x,t) = \bar{u}(e^{-\frac{\epsilon}{2(1-\beta)}}x, e^{-\epsilon}t)$ is also a solution.

Group classification

Lie's group classification of linear second order PDE with one spatial variable: (Lie's work re-discovered by Ibragimov)

$$Pu_t + Qu_x + Ru_{xx} + Su = 0$$

$P, R \neq 0$ and P, Q, R, S are functions of t and x.

- Any parabolic equation admitting generators of trivial symmetries $u\frac{\partial}{\partial u}$ (linearity) and $\phi(t,x)\frac{\partial}{\partial u}$ (linear superposition) . Any equation can be reduced to the form

$$v_\tau = v_{yy} + Z(\tau, y)v$$

by a point transformation, ie.: $y = \alpha(t,x)$, $\tau = \beta(t)$ and $v = \gamma(t,x)u$ with $\alpha_x \neq 0$ and $\beta_t \neq 0$.

Ie. reduce to Schroedinger equation with time independent potential.

Group classification

- An equation admitting one more (ie. $2D$ group) symmetry generator can be transformed into the form

$$v_\tau = v_{yy} + Z(y)v$$

whose symmetry generators are generated by

$$u\frac{\partial}{\partial u}, \quad \phi(t,x)\frac{\partial}{\partial u}, \quad \frac{\partial}{\partial \tau}$$

Group classification: $4D$ group

- An equation admitting three more symmetry generators (ie. 4 D group) can be transformed into the form

$$v_\tau - v_{yy} + \frac{A}{y^2}v = 0,$$ all CEV models can be so reduced

where A is a constant, whose symmetry generators are

$$u\frac{\partial}{\partial u}, \quad \phi(t,x)\frac{\partial}{\partial u}, \quad \frac{\partial}{\partial \tau}, \quad 2\tau\frac{\partial}{\partial \tau} + y\frac{\partial}{\partial y}$$

$$\tau^2\frac{\partial}{\partial \tau} + \tau y\frac{\partial}{\partial y} - \left(\frac{1}{4}y^2 + \frac{1}{2}\tau\right)v\frac{\partial}{\partial v}$$

Group classification: 6 dimensional group

- An equation admits five more symmetry generators can be transformed into the form

$$v_\tau = v_{yy}$$

whose symmetry generators are

$$u\frac{\partial}{\partial u}, \quad \phi(t,x)\frac{\partial}{\partial u}, \quad \frac{\partial}{\partial \tau} \quad 2\tau\frac{\partial}{\partial \tau} + y\frac{\partial}{\partial y},$$
$$\tau^2\frac{\partial}{\partial \tau} + \tau y\frac{\partial}{\partial y} - \left(\frac{1}{4}y^2 + \frac{1}{2}\tau\right)v\frac{\partial}{\partial v}$$
$$\frac{\partial}{\partial y}, \quad 2\tau\frac{\partial}{\partial y} - yv\frac{\partial}{\partial v}$$

All linear parabolic equations with a six dimensional group can be reduced to heat equation

Many potentials not in 6 or 4 D Class

- $$u_t - \frac{1}{2}u_{xx} + \frac{1}{x}u = 0$$

  is not in $4D$ class.
  But

- $$u_t - \frac{1}{2}u_{xx} + \left[\frac{\lambda}{(x - D(t))^2} + A(t)y^2 + B(t)y + C(t)\right]u = 0$$

  is in $4D$ class, *provided that*

  $$B(t) = \ddot{D} - 2AD$$

Fundamental solution: Transformation metho

Thus for parabolic differential equations in one spatial variable, there are 4 different equivalence classes:

- Equations with a six dimensional symmetry group: Fundamental Solution $F^{(1)}$
- Equations with a four dimensional symmetry group: Fundamental Solution $F^{(2)}$
- Equations with a two dimensional symmetry group.
- No non trivial symmetry.

Transforming away

- Suppose given an equation we know has a six or four dimensional symmetry group.
  We seek to determine fundamental solution for this equation with pole at ξ at initial time 0 so that

$$F_\xi(x,0) = \delta_\xi(x)\delta_0(t)$$

  Let $\Lambda(x,t) : \mathbb{R}^2 \to \mathbb{R}^2, (y,s) = \Lambda(x,t)$ be the mapping that brings the equation into canonical form $\Lambda(x,t) : (x,t) \to (y,s)$, while mapping $t=0$ to $s=0$ (preservation of initial surface). Then the fundamental solution $F_\xi(x,t)$ can be obtained from the fundamental solutions $F^{(1,2)}$ of the equations in canonical form , via: $F_\xi(x,0) = F^{(1,2)}_{\Lambda(\xi_0,0)}(\Lambda(x,t))\frac{dx}{dy}(\Lambda|^{-1}(x,0)|$

Lie classification and canonical equations

Illustration 1: Potential of driftless CEV

$$u_t - \frac{1}{2}x^{2\beta}u_{xx} = 0, x \in [0, \infty]$$

In canonical form, when $\beta < 1$ this is reduced to the canonical form

$$v_t - \frac{1}{2}v_{ss} + \frac{\beta(\beta-2)}{4(1-\beta)^2 s^2}v = 0, t \in [0, T], x \in [0, \infty]$$

which is in the four dimensional class.

Reduction illustrated in CEV case

- Change of independent variable $y = \int \frac{1}{x^\beta} = \frac{1}{1-\beta} x^{1-\beta}$ With $w(y,t) = u(x,t)$ equation becomes:

$$w_t + \frac{1}{2} w_{xx} + \frac{\beta}{\beta - 1} \frac{1}{x} w_x = 0$$ volatility now is 1, but have added drift

- Remove drift

$$w(y,t) = y^{\frac{\beta}{2(1-\beta)}} v(y,t)$$ this transformation removes the drift

New equation, after changing time to $t = T - t$

$$v_t - \frac{1}{2} v_{yy} + \frac{\beta(2-\beta)}{8(1-\beta)^2} \frac{1}{y^2} v = 0$$

CEV ct'd

The fundamental solution for the equation

$$v_t - \frac{1}{2}v_{yy} + \frac{\lambda}{y^2}v = 0$$

$$F(y, y', \tau) = \frac{1}{\tau}\sqrt{y}(y')^{-\frac{1}{2(1-\alpha)}} e^{-\frac{y^2}{2\tau} - \frac{(y')^2}{2\tau}} \mathcal{I}_{\frac{\sqrt{1+8\lambda}}{2}}\left(\frac{yy'}{\tau}\right)$$

In our case $\frac{\sqrt{1+8\lambda}}{2} = \frac{1}{2(|\alpha-1|}$ and mapping back to original variables we find a fundamental solution for CEV process. Note we are working in the backward variable.

CLW: one dimensional time inhomogeneous d

The main result in the paper by Carr-L. and Wang (CRAS 2006) , consists in giving a PDE which must be satisfied by the volatility of any one dimensional *time inhomogeneous* diffusion in order that it be have a four or six dimensional symmetry group.
This PDE allows one to find new time inhomogeneous one dimensional diffusions, whose diffusion coefficient solves the equation.

- There is no complete classification available.
- There *are* classifications for special models. For instance
- Finkel (2000) finds all time independent potentials $V(x, y, t)$ such that

$$u_t - u_{xx} - u_{yy} + V(x, y)u = 0$$

- L. and Wang (IJTAF 2004) isolate the subset of parabolic equations, within the classes discovered by Finkel, for which the fundamental solution can be found in closed form.

Correlated Generalized Sabr models

- Consider the following SABR-like model

$$dS_t = b(S_t)\sigma_t dW_1(t)$$
$$d\sigma_t = \sigma_t v(\sigma_t) dW_2(t)$$
$$< dW_{1t}, dW_{2t} >= \rho dt$$

where:

- b depends only on the stock price S.
- The two driving Brownian motions are of correlation ρ.
- Assume the interest rate is zero. The classical Sabr model of Hagan-Lewsniewski and Woodward corresponds to the choice $b(S) = S^{\beta}$ $\&$ $v(S) = 1$

Uncorrelated Generalized Sabr models

- For this class of models, with no drift and zero correlation, there exists a *complete* symmetry classification, we will now describe and then explain. The classification, is given in following pages. (due to Wang, L. and Wang).
- Symmetry classification for $\rho \neq 0$, only partial results. Very hard.

Classification result for zero correlation

- $b(x) =$ const and $v(y) =$ const (read lognormal). (corresponds to Sabr $\beta = 0$).
- dimension = 5
- Basis

$$X_1 = \left(\frac{1}{2}x^2 - \frac{b^2}{2v^2}y^2\right)\partial_x + xy\partial_y$$

$$X_2 = x\partial_x + y\partial_y$$

$$X_3 = \partial_t$$

$$X_4 = u\partial_u$$

$$X_5 = \partial_x$$

Classification result

- $b(x) = (b_1 x + b_0)^2$
  $v(y) =$ const read lognormal.
- dimension = 5
- Basis

$$
\begin{aligned}
X_1 &= \partial_t \\
X_2 &= (b_1 x^2 + b_0)^2 \partial_x + b_1^2 x u \partial_u \\
X_3 &= -\left(x + \frac{b_0}{b_1} \right) \partial_x + y \partial_y \\
X_4 &= \frac{-b_1^2 (b_1 x + b_0)^2 y^2 + v^2}{2 b_1^2 v} \partial_x - \frac{y v}{b_1 (b_1 x + b_0)} \partial_y \\
&\quad + \frac{1}{2} \left[\frac{v}{b_1^2 x + b_0 b_1} - \frac{b_1 (b_1 x + b_0) y^2}{v} \right] u \partial_u \\
X_5 &= u \partial_u
\end{aligned}
$$

Classification result

- $b(x) = Ke^{-\int \frac{dx}{k_2x^2+k_1x+k_0}} \left(k_2x^2 + k_1x + k_0\right)$ and $v(y) = \text{const.}$
- dimension = 3
- Basis

$$X_1 = \left(k_2x^2 + k_1x + k_0\right)\partial_x + y\partial_y + k_2xu\partial_u$$
$$X_2 = \partial_t$$
$$X_3 = u\partial_u$$

Classification result

$b(x) = (b_1 x + b_0)^\beta$, $\beta \neq 1, 2$ and $v(y) = \text{const}$

- dimension = 3
- Basis

$$X_1 = \frac{b_1 x + b_0}{b_1(1-\beta)} \partial_x + y\partial_y$$

$$X_2 = \partial_t$$

$$X_3 = u\partial_u$$

Classification result

- $b(x) = Ke^{rx}$ and $v(y) = $ const
- dimension = 3
- Basis

$$X_1 = -\partial_x + ry\partial_y$$
$$X_2 = \partial_t$$
$$X_3 = u\partial_u$$

Classification result

$b(x) = b_2x^2 + b_1x + b_0$ and $v(y)$ arbitrary except NOT a homogeneous polynomial of degree one.

- dimension = 3
- Basis

$$X_1 = \partial_t$$
$$X_2 = (b_2x^2 + b_1x + b_0)\partial_x + b_2xu\partial_u$$
$$X_3 = u\partial_u$$

Classification result

- $b(x) = b_2x^2 + b_1x + b_0$ and $v(y) = By$

Subcase $b_2 = b_0 = 0, b_1 \neq 0$ corresponds to SABR $\beta = 1$.

- dimension = 5
- Basis

$$
\begin{aligned}
X_1 &= \partial_t \\
X_2 &= t\partial_t - \frac{y}{2}\partial_y \\
X_3 &= t^2\partial_t - yt\partial_y + \left(\frac{1}{2B^2y^2} - \frac{3t}{2}\right)u\partial_u \\
X_4 &= (b_2x^2 + b_1x + b_0)\partial_x + b_2xu\partial_u \\
X_5 &= u\partial_u
\end{aligned}
$$

Classification result

$b''(x) \neq$ const, b arbitrary and $v(y) = By$

- dimension = 4
- Basis

$$
\begin{aligned}
X_1 &= \partial_t \\
X_2 &= t\partial_t - \frac{y}{2}\partial_y \\
X_3 &= t^2\partial_t - yt\partial_y + \left(\frac{1}{2B^2y^2} - \frac{3t}{2}\right)u\partial_u \\
X_4 &= u\partial_u
\end{aligned}
$$

Classification

- $b(x) = Ke^{-\int \frac{dx}{k_2x^2+k_1x+k_0}} \left(k_2x^2 + k_1x + k_0\right)$ and $v(y) = By^{\frac{1}{A}}$, $A \neq 1$

- dimension = 3
- Basis

$$X_1 = \partial_t$$
$$X_2 = t\partial_t + \frac{1-A}{2}(k_2x^2 + k_1x + k_0)\partial_x - \frac{A}{2}y\partial_y + \frac{(1-A)k_2}{2}xu\partial_u$$
$$X_3 = u\partial_u$$

Classification

- $b(x) = (b_1 x + b_0)^\beta$, $\beta \neq 1, 2$ and $v(y) = By^{\frac{1}{A}}$, $A \neq 1$
- dimension = 3
- Basis

$$X_1 = \partial_t$$

$$X_2 = t\partial_t + \frac{(1-A)(b_1 x + b_0)}{2b_1(1-\beta)}\partial_x - \frac{Ay}{2}\partial_y$$

$$X_3 = u\partial_u$$

Classification

- $b(x) = Ke^{rx}$ and $v(y) = By^{\frac{1}{A}}$, $A \neq 1$.
- dimension = 3
- Basis

$$X_1 = \partial_t$$
$$X_2 = t\partial_t - \frac{1-A}{2r}\partial_x - \frac{Ay}{2}\partial_y$$
$$X_3 = u\partial_u$$

Classification

- $b(x) = (b_1 x + b_0)^2 + r_0$ and $v(y) = 2b_1\sqrt{r_0}\, y$
- dimension = 9
- Basis

$$X_1 = 2t(b_1x+b_0)y\partial_x - \frac{2b_1t[(b_1x+b_0)^2-r_0]y^2}{b(x)}\partial_y + \left[\frac{(b_1x+b_0)^2-r_0}{2b_1r_0b(x)y} + b_1ty\right]u\partial_u,$$

$$X_2 = \frac{t[(b_1x+b_0)^2-r_0]y}{2}\partial_x + \frac{2b_1r_0t(b_1x+b_0)y^2}{b(x)}\partial_y + \left[-\frac{b_1x+b_0}{2b_1b(x)y} + \frac{b_1t(b_1x+b_0)y}{2}\right]$$

$$X_3 = 2(b_1x+b_0)y\partial_x - \frac{2b_1[(b_1x+b_0)^2-r_0]y^2}{b(x)}\partial_y + b_1yu\partial_u,$$

$$X_4 = \frac{[(b_1x+b_0)^2-r_0]y}{2}\partial_x + \frac{2b_1r_0(b_1x+b_0)y^2}{b(x)}\partial_y + \frac{b_1(b_1x+b_0)y}{2}\,u\partial_u,$$

$$X_5 = t^2\partial_t - ty\partial_y + \left(-\frac{3}{2}t + \frac{1}{8b_1^2r_0y^2}\right)u\partial_u,$$

$$X_6 = t\partial_t - \frac{y}{2}\partial_y,$$

$$X_7 = b(x)\partial_x + b_1^2xu\partial_u,$$

Determining equations

Let

$$X = \tau\partial_t + \xi\partial_x + \eta\partial_y + \phi\partial_u$$

be an infinitesimal generator. By a theorem of Bluman, if X is a symmetry generator of the pricing equation, then

- $\tau = \tau(t)$
- $\xi = \xi(x, y, t)$, $\eta = \eta(x, y, t)$
- $\phi = \alpha(x, y, t)u$

We obtain the determining equations by

$$\mathbf{pr}^{(2)}X[Lu] = 0 \quad \text{whenever } Lu = 0$$

as

Determining equations

$$(\tau_t - 2\xi_x)\frac{y^2}{2}b^2 + y^2bb'\xi + yb^2\eta - y^2\rho v b\xi_y = 0$$

$$(\tau_t - 2\eta_y)\frac{y^2}{2}v^2 - y^2\rho v b\eta_x + vv_xy^2\xi + (v^2y + vv_yy^2)\eta = 0$$

$$(\tau_t - \xi_x - \eta_y)y^2\rho v b - y^2b^2\eta_x + \xi(y^2\rho v b' + y^2\rho_x v b + y^2\rho v_x b)$$
$$+\eta(2y\rho v b + y^2\rho_y v b + y^2\rho v_y b) - v^2y^2\xi_y = 0$$

$$-L\xi + y^2b^2\alpha_x + y^2\rho v b\alpha_y = 0$$

$$-L\eta + y^2\rho v b\alpha_x + y^2v^2\alpha_y = 0$$

$$\alpha_t + \frac{y^2}{2}b^2\alpha_{xx} + \frac{y^2}{2}\rho v b\alpha_{xy} + v^2\frac{y^2}{2}\alpha_{yy} = 0$$

Note that hereafter subindices denote partial derivatives.

Special case

We consider the special case that

- $\rho = 0$, i.e., the driving Brownian motions are uncorrelated.
- $v_x = 0$, i.e., volatility of volatility v is independent of the level of the spot.

Hence the pricing equation reduces to

$$\frac{\partial u}{\partial t} + \frac{y^2}{2} b^2(x) \frac{\partial^2 u}{\partial^2 x} + v^2(y) \frac{y^2}{2} \frac{\partial^2 u}{\partial^2 y} = 0$$

Determining equations for the special case

$$(\tau_t - 2\xi_x)\frac{y^2}{2}b^2 + y^2 bb'\xi + yb^2\eta = 0$$

$$(\tau_t - 2\eta_y)\frac{y^2}{2}v^2 + (v^2 y + vv'y^2)\eta = 0$$

$$y^2(b^2\eta_x + v^2\xi_y) = 0$$

$$-L\xi + y^2 b^2 \alpha_x = 0$$

$$-L\eta + y^2 v^2 \alpha_y = 0$$

$$\alpha_t + \frac{y^2}{2}b^2\alpha_{xx} + v^2\frac{y^2}{2}\alpha_{yy} = 0$$

Solving the determining equations

Our strategy for solving the determining equations is

- Solve the first three equations to obtain specific forms for ξ and η.
- Plug the obtained forms for ξ and η into the last three equations to obtain the symmetry generators.

Solving the determining equations

Note that the first two equations have general solution given as

$$\xi = \frac{\tau_t}{2} M b + W b + b l(t, y)$$

$$\eta = y v \frac{\tau_t}{2} U + y v h(t, x)$$

where

$$U(y) = \int \frac{dy}{yv}, \quad M(x) = \int \frac{dx}{b(x)},$$

$$W(x, y) = \frac{\tau_t}{2} v U M + v \int \frac{h(t, x)}{b} dx$$

and h, l are arbitrary functions.

Classifying equation

Plugging the expressions for ξ and η into the third equation we obtain

$$bh_x = -\frac{v}{y}\left(\frac{\tau_t}{2}v'UM + \frac{\tau_t}{2}\frac{M}{y} + v'\int \frac{h}{b}dx + l_y\right)$$

Differentiating the above equation with respect to x and multiplying the resulting equation by b then differentiating with respect to x again we obtain the following classifying equation

$$(bb'h_x + b^2 h_{xx})_x + \frac{vv'}{y}h_x = 0$$

Two big categories

Now we separate it into two cases.

- $h_x = 0$ (hence $h = h(t)$)
- $h_x \neq 0$. The classifying equation becomes

$$\frac{(b(bh_x)_x)_x}{h_x} = -\frac{vv'}{y}$$

For either case we end up with specific form of v in order to have nontrivial symmetry generators.

Obtaining generators

- Plugging the obtained specific forms for v into the last three equations in determining equations yields a differential-integral identity relation for b and its derivatives and integrals.
- Assuming linear independence between b and its derivatives and integrals we obtain specific forms for b and the corresponding symmetry generators.

Conclusion

- We have tried to ilustrate the role of heat kernels in finding good approximate solutions for a family of stochastic volatility models
- We have outlined the basic principles of Lie symmetry analysis and how it applies to one and two dimensional parabolic problems
- Open problems:
  – Symmetry analysis for $\rho \neq 0$ open.
  – Once we know the symmetry operators, we still need to determine the cases for which a fundamental solution can be expressed in closed form. Ongoing, computationally intensive, even using Mathematica.
  – Further interaction between heat kernels and Lie symmetry analysis through the investigation of integrable geodesic flows.

Appendices: CEV

CEV diffusion

$$dS_t - \mu S_t dt + \sigma S_t^{\beta} dW_t$$
$$S_0 = S > 0$$

- For $\beta < 1$, infinity is a natural boundary (unreachable)
- For $\frac{1}{2} \leq \beta < 1$ the origin is an exit (absorbing boundary). Ie. hit zero and never leave.
- For $\beta < \frac{1}{2}$ the origin is a regular boundary point and can be specified as a killing boundary by adjoining a killing boundary condition,
- For $\beta > 1$ the origin is a natural boundary at infinity and is an entrance boundary
- For $\beta < 1$ the probability of absorption at zero given $S_0 = S$ is

$$Q(S_T = 0) = G(\frac{1}{2(|\beta - 1|}, \zeta)$$

$G(\nu, x)$ is the complementary Gamma distribution and $\zeta = \frac{2\mu S^{-2(\beta-1)}}{\sigma^2(\beta-1)(e^{2\mu(\beta-1)T}-1)}$.

Other approaches: WKB

- Note on relation to WKB method: Lesniewski in an unpublished paper that came before his work with Hagan, considers the case of

$$K_t - \frac{1}{2} y^2 (b^2(x) K_{xx} + 2\rho v u_{xy} + v^2 K_{yy}) = 0$$

In small time limit, after appropriate scaling $\tau = \frac{t}{T}$, $\epsilon = v^2 T = o(1)$

$$K_t - \frac{1}{2} \epsilon y^2 (b^2(x) K_{xx} + 2\rho v K_{xy} + v^2 K_{yy})$$

where $K = K(z, Z, \tau)$ ($z = (x, y)$). Now use WKB ansatz:

$$K = \frac{1}{2\pi\epsilon} R(z, Z, \tau, \epsilon) e^{-\frac{1}{\epsilon} S(z, Z, \tau)}$$

where $S(z, Z, \tau)$ is independent of ϵ, S satisfies Hamilton-Jacobi equation:

$$S_\tau + \frac{1}{2} \partial^\mu S \partial_\mu S = 0, \quad (R^2)_\tau + \partial^\mu (R^2 \partial_\mu S) = \epsilon R \partial^\mu \partial_\mu R$$

Other approaches 2

Now key is that there is always a solution of time dependent HJB equation of the form

$$S = \frac{1}{\tau}\Gamma$$

where $\Gamma = d^2$ solves stationary equation. I.e.

$$\begin{aligned} S_\tau &= -\frac{1}{4\tau^2}\Gamma \\ \partial^\mu S \partial_\mu S &= g^{ij} S_i S_j \\ &= \frac{1}{16\tau^2} \underbrace{g^{ij}\Gamma_i\Gamma_j}_{=4\Gamma} \\ &= \frac{1}{4\tau^2}\Gamma \end{aligned}$$

Therefore we see that

$$\frac{S}{\epsilon} = \frac{d^2(x,y)}{4\underbrace{\tau}_{t/T}\underbrace{\epsilon}_{v^2 T}} = \frac{d^2(x,y)}{4v^2 t}$$