Sharp model independent no-arbitrage bounds and optimal hedge ratios for basket options

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Main aims of this contribution

Joint work with David Hobson (U. of Bath and Princeton) and Tai-Ho Wang(National Chung Cheng U., Chia Yi, Taiwan).

- Review recent literature on distribution free bounds for option prices
- Review recent joint work with Wang on **obtaining sharp bounds and optimal hedge ratios in a static no arbitrage one period setting.**
- Discuss more general and recent results obtained in collaboration with Hobson and Wang concerning upper and lower bounds for basket options

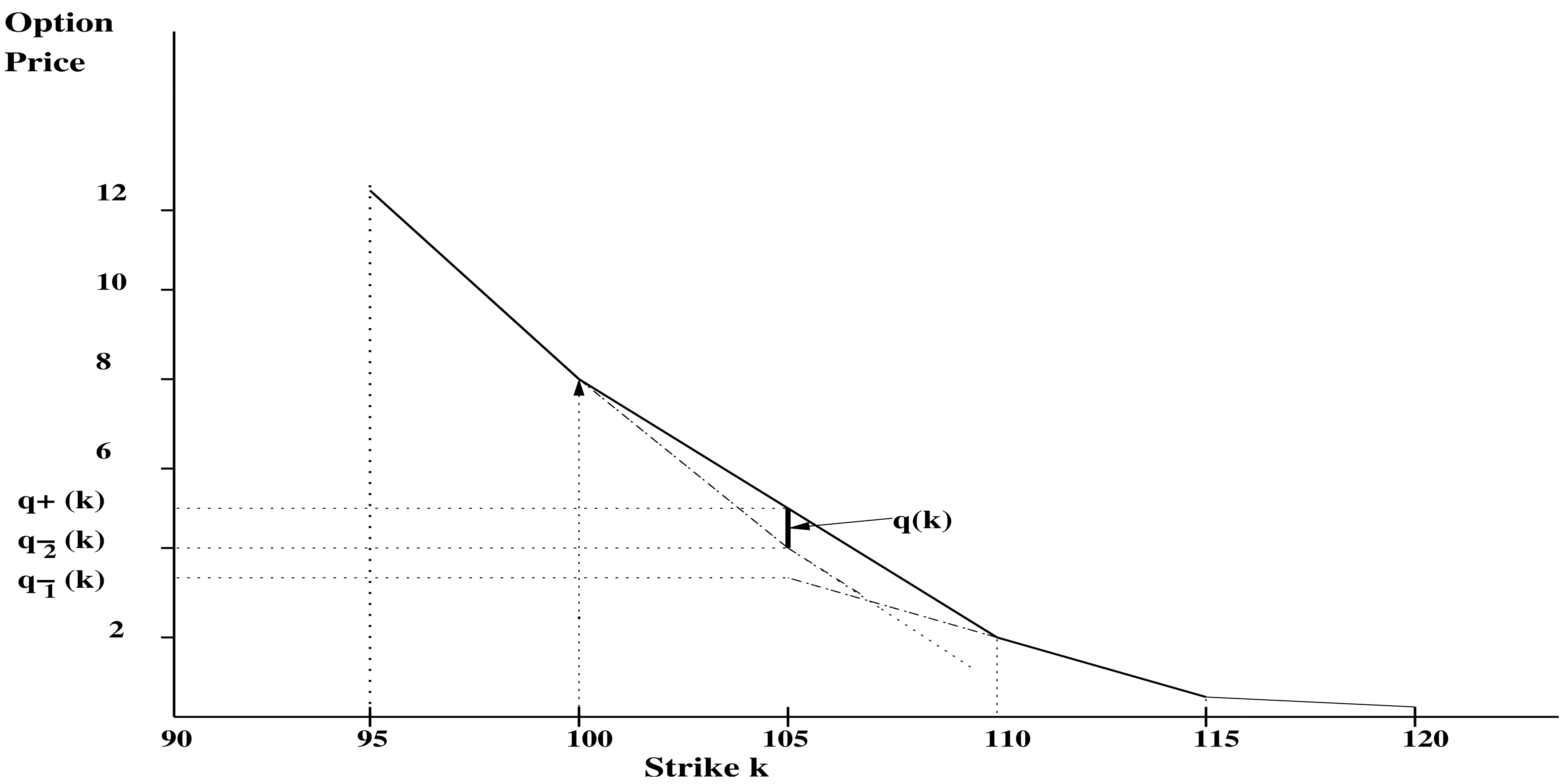
Recent Work on Model Indepedent Option Bounds

Bertsimas and Popescu, 2003, use a LP approach to derive bounds on assets under a variety of constraints. Here is one of their results:

Given prices $C_i(K_i)$ of call options with strikes $0 \leq K_1 \leq .. \leq K_n$ on a stock X, the range of all possible prices for a call option with strike K where $K \in (K_j, K_{j+1})$ for some $j = 0, \cdots, n$ is $[C^-(K), C^+(K)]$ where

$$\begin{aligned} &C^-(K) \\ &= \max\left(C_j \frac{K - K_{j-1}}{K_j - K_{j-1}} + C_{j-1} \frac{K_j - K}{K_k - K_{j-1}},\right. \\ &\qquad \left. C_{j+1} \frac{K_{j+2} - K}{K_{j+2} - K_{j+1}} + C_{j+2} \frac{K - K_{j+1}}{K_{j+2} - K_{j+1}}\right) \quad \text{lower bounds} \\ &C^+(K) = \frac{K_{j+1} - K}{K_{j+1} - K_j} + C_{j+1} \frac{K - K_j}{K_{j+1} - K_j} \quad \text{upper bounds} \end{aligned}$$

Graphic Representation Bertsimas Popescu



Bert-PoP ct'd

- One lesson from Bertsimas and Popescu: already in 1-D lower bounds are more complicated than upper bounds.
- Upper bound corresponds to a linear interpolation and lower bound to a more complicated, albeit still linear interpolation. This theme continues and is accentuated in higher dimension.
- There are many other contributions in this field, notably by Ryan, Perrakis, Ritchken and many others for one dimensional options. Brown, Hobson and Rogers.

♠ Next we introduce Basket options and the work on model independent, no-arbitrage bounds

Basket Options

The Payoff of a basket option:

$$\psi(S_1, \cdots, S_n) = (\sum_i w_i S_i - K)^+$$

- Price weighted –> $w_i = \frac{1}{I(t_0)}$.
- Capitalization Weighted,

$w_i = Cst.\frac{\text{nb. } S_i \text{ shares outstanding}}{\text{Total capitalization}}$.

w_i are readjusted periodically.

- S&P 500, S&P 100, Dow Jones 100, IGBM, IBex 35.

Black-Scholes

The most popular model: **Multidimensional Black-Scholes**

$$dS_t^i = S_t^i(r - d_i)dt + \sigma_i S_t^i dZ_i^t$$
$$S^i(0) = S_0^1$$
$$< dZ_i, dZ_j >= \rho_{ij} dt$$

European Basket Options:

$$u_t + S_i S_j \rho_{ij} \sigma_i \sigma_j u_{S_i S_j} + (r - d_i) S_i u_{S_i} - ru = 0$$
$$u(S_1, \cdots, S_n, T) = (\sum_{i=1}^{n} w_i S_i - K)^+$$

Closed form solution B-S

Solution in closed form

Solution at time 0, for maturity T in closed form

$$e^{-rT}\frac{1}{(2\pi)^{n/2}(detV)^{1/2}}\int\left(\sum w_i e^{(r-d_i-\frac{\sigma_i^2}{2})T-\sqrt{T}X}-K\right)^+ e^{-\frac{1}{2}X^tV^{-1}X}dX$$

where V is the variance-covariance matrix $\{\sigma_i\sigma_i\rho_{ij}\}_{i,j=1}^n$.

- Numerical evaluation even with a closed form formula can be time consuming if the number of assets is large. Most efficient methods an active field of research.
- Simplest approximation that the index is lognormal not bad for pricing, but not useful for hedging purposes.
- Next 10 years will see a birth of multi-dimensional stochastic volatility and jump models

Motivations and Limitations

- Model independent bounds provide a sanity check for the output of numerical algorithms. In the presence of multiple assets, these run into the curse of dimension.
- Since they are model independent the bounds obtained are necessarily quite wide. So the bounds described today should be seen as only a first step in a research program.
- By the same token, the hedging strategies that will be described today will often be too expensive but being model independent they provide risk free hedges.

Putting into perspective I

- The method we will present today can be used to look for arbitrages involving a static hedging strategy.
- This strategy involves i) selling the index option and ii) buying options on the component assets.
- The key will be a judicious choice of the strikes of the options on the components. This choice will make the basket of options the cheapest super-replicating portfolio available.

Putting into perspective II

- When this super-replicating portfolio happens to be cheaper than the index option, one has found an arbitrage.
- But, unfortunately, such cases are rare. On DJX index unable to detect such an arbitrage but came close.
- We advocate: The tool to detect such arbitrages should be a part of software library because, even when it fails to do so, it carries interesting information about key strikes, which is hard to detect by other means.
- Most likely to detect arbitrages/mispricing on large or illiquid indices.

Mathematical Formulation of the problem

A hierarchy of different mathematical problems Problem: In all cases, Calibrate basket option Prices to market Data.

- Calibration of basket option price to Prices of Option Components.

  Three different approaches:

- I. Calibrate to Only one option per asset of a given maturity and forward prices.
- II. Calibrate to all available options with a given maturity.
- III. Assume (à la Dupire) that options are traded with a continuum of strikes $\Leftrightarrow$ Equivalent to assuming Marginals Prescribed.

Calibrate to fixed single name option prices, one per asset, and to fixed forward prices

Let $d\mu(S_T^1, \cdots, S_T^n)$ be the probability density associated to the distribution of the $S_T^i, i = 1, \cdots n$ at time T . Let μ_i be the marginal of μ in the i-th stock. Then we require that

- **Prescribed Option Prices**

$$e^{-rT} \int (S_T^i - K_{i,j})^+ d\mu_i = C_{ij}, \quad i = 1, \cdots, n, j = 1, \cdots, J(i)$$

- **Prescribed Forward Prices**

$$e^{-rT} \int S_T^i d\mu_i = S_0^i, \quad i = 1, \cdots n \qquad K_{ij} = S_0^i \quad \text{spot price}$$

Let $C_{\mathcal{B}}$ be the price of an index option.
Optimal Model Independent (Distribution Free Bounds) are bounds with the following properties:

- $$C_{\mathcal{B}}^{L} \leq C_{\mathcal{B}} \leq C_{\mathcal{B}}^{U}$$
  LHS and RHS, provide bounds
- The pricing distribution that produces these bounds is consistent with the information about the marginal distributions of the stocks provided by the quoted option prices.

- The bounds are optimal, meaning that there exists a multidimensional stochastic process and associated time T joint distribution for the stock process $(S_{1,t}, S_{2,t}, \cdots, S_{n,t})$ that is consistent with the option prices on single-name and attains the upper and lower bounds.

DJX Index

Symbol	Name	Last Price	Weight
AA	ALCOA, INC.	34.270	2.40%
AIG	AMERICAN INTERNATIONAL GROUP INC	61.030	4.28%
AXP	AMERICAN EXPRESS CO	55.630	3.90%
BA	BOEING CO	53.930	3.78%
C	CITIGROUP	47.070	3.30%
CAT	CATERPILLAR INC.	89.970	6.31%
DD	DU PONT EI DE NEMOURS	44.500	3.12%
DIS	WALT DISNEY CO	26.800	1.88%
GE	GENERAL ELECTRIC CO	36.250	2.54%
GM	GENERAL MOTORS CORP	40.210	2.82%
HD	HOME DEPOT INC	43.220	3.03%
HON	HONEYWELL INTERNATIONAL INC.	36.620	2.57%
HPQ	HEWLETT PACKARD CO	19.340	1.36%
IBM	INTERNATIONAL BUSINESS MACHINES	95.320	6.68%
INTC	INTEL CORP	23.690	1.66%
JNJ	JOHNSON AND JOHNSON	61.000	4.28%
JPM	JP MORGAN CHASE AND CO INC	39.170	2.75%
KO	COCA COLA CO	40.790	2.86%
MCD	MCDONALDS CORP	30.500	2.14%
MMM	3M COMPANY	82.680	5.80%
MO	ALTRIA GROUP INC.	54.740	3.84%
MRK	MERCK AND COMPANY INC	26.450	1.85%
MSFT	MICROSOFT CORP	26.970	1.89%
PFE	PFIZER INC.	27.450	1.92%
PG	PROCTER AND GAMBLE CO	54.600	3.83%

A typical Component Option, Procter& Gamble

May, 2004 | July, 2004 | October, 2004 | January, 2005 | January, 2006

PROCTER & GAMBLE CO 105.97 ▼ -0.15 -0.1414% 105.91 106.37 2,727,800

Calls							Strike	Puts						
Symbol	Last	Chg	Bid	Ask	Vol	Int	Price	Symbol	Last	Chg	Bid	Ask	Vol	Int
PG EM	41.50	0.00	40.80	41.10	0	15	65	PG QM	0.00	0.00	0.00	0.05	0	.
PG EN	36.50	0.00	35.80	36.10	0	65	70	PG QN	0.00	0.00	0.00	0.05	0	.
PG EO	31.50	0.00	30.90	31.10	0	15	75	PG QO	0.00	0.00	0.00	0.05	0	.
PG EP	26.00	0.00	25.90	26.10	0	.	80	PG QP	0.05	0.00	0.00	0.05	0	20
PG EQ	21.00	0.00	20.90	21.10	0	40	85	PG QQ	0.00	0.00	0.00	0.05	0	.
PG ER	16.00	0.00	15.90	16.10	0	58	90	PG QR	0.10	0.00	0.00	0.10	0	90
PG ES	11.30	0.00	10.90	11.10	0	204	95	PG QS	0.20	0.00	0.10	0.20	0	173
PG ET	6.00	-0.10	6.00	6.20	132	229	100	PG QT	0.45	-0.05	0.45	0.50	158	3,229
PG EA	2.35	-0.35	2.30	2.35	1,287	3,049	105	PG QA	1.60	0.00	1.70	1.75	680	2,065
PG EB	0.50	0.00	0.45	0.50	193	2,921	110	PG QB	4.70	-0.60	4.80	5.00	305	5,599
PG EC	0.05	0.00	0.05	0.10	15	258	115	PG QC	10.10	0.00	9.50	9.70	0	64
PG ED	0.00	0.00	0.00	0.05	0	.	120	PG QD	14.40	0.00	14.40	14.70	0	75
PG EE	0.00	0.00	0.00	0.05	0	.	125	PG QE	19.70	0.00	19.40	19.70	0	138
PG EF	0.00	0.00	0.00	0.05	0	.	130	PG QF	24.10	-0.60	24.40	24.70	12	163

Robert Merton 1973 (Bell Journal) established the following result. Given n assets $S_1, \cdots, S_n$ and n options C_i with strike K_i. One option per asset. Let $C_{\mathcal{B}}$ denote the price of a basket option with positive constant weights $w_i, i = 2, \cdots, n$. Suppose that in addition the following condition holds:

- $$w_1 K_1 + w_2 K_2 + \cdots + w_n K_n = K$$

Then

$$C_{\mathcal{B}} \leq w_1 C_1 + w_2 C_2 + \cdots + w_n C_n$$

Merton ct'd

Result has a simple proof:

$$
\begin{aligned}
&(w_1S_1 + w_2S_2 + \cdots + w_nS_n - K)^+ \\
&= (w_1(S_1 - K_1) + w_2(S_2 - K_2) + \cdots + w_n(S_n - K_n))^+ \\
&\leq w_1(S_1 - K_1)^+ + w_2(S_2 - K_2)^+ + \cdots w_n(S_n - K_n)^+
\end{aligned}
$$

Taking expectations in the above formula yields

$$
\begin{aligned}
&C_{\mathcal{B}} = e^{-rT}E\left[w_1S_1 + w_2S_2 + \cdots + w_nS_n - K\right]^+ \\
&\leq e^{-rT}w_1E[S_1 - K_1)^+] + \cdots + e^{-rT}w_1E[(S_n - K_n)^+] \\
&= w_1C_1(K_1) + \cdots w_nC_n(K_n)
\end{aligned}
$$

Merton ct'd

Merton did not however characterize the conditions for equality in this relation. This was done later in P.L.-Wang Risk 2004, AMF 2005.

Note: Letting $D = K - \sum w_i K_i > 0$, it is easy to see that the above proof goes through, but,

- When $D < 0$ the above bound is no longer the optimal one.

Optimal Super-Replicating Portfolios: Computational Complexity

• Consider the sub-problem of optimally super-replicating the basket with a portfolio consisting of only options on the components, one of which may be stock (option with zero strike). The optimal super-replicating strategy and optimal UB are closely connected.

• Ie. for an index with 100 components, for each component i, choose one strike per component, say $\bar{K}(i)$ from the say eight strikes trading on that component, with a given maturity. If $D = K - \sum w_i K_i$ we know that

$$C_{\mathcal{B}} \leq \sum_{i}^{n} w_i C_i(\bar{K}_i) \qquad (*)$$

• For real market data, Merton's condition $\sum w_i K_i = K$ almost never holds exactly. Appropriate optimal adjustments were made, when condition doesn't hold, by L. $\&$ Wang (Risk 2004, Applied Mathematical Finance 2005) and by Hobson, L. $\&$ Wang (Quantitative Finance 2005, Insurance Mathematics and Economics, 2005). But, for the sake of illustration, suppose (*) holds.

Optimal Super-Replicating Portfolios: Computational Complexity II

Given an index with 100 components we are faced with the following problem: Determine among all possible ways of selecting, for each asset i one strike from the 8-13 available ones, that strike $\bar{K}_i$ which minimizes

$$C_{\mathcal{B}} \leq \sum w_i C(\bar{K}_i)$$

Complexity in brute force approach : Huge, of order 8^{100} possible combinations.
♣ Good News: There exists a simple and extremely fast algorithm, whose complexity rises linearly with the number of assets, for determining the optimal strikes. (HLW: Quantitative Finance, 2005). Since Merton's inequality does not hold in general it turns out to be necessary for one of the component assets to choose two strikes and for all other components, one is enough.

1. • Is this a method to detect an arbitrage?: Experiments on DJX index (30 stocks) close to an arbitrage (neglecting transaction costs), but no cigar. Experiments with larger indices in progress.
2. • However, even if arbitrages are not present, the method allows one to determine, among the myriad possible strikes, those "where a good part of the action is".
3. • If you are not happy with (1) and (2), you can still use the fast algorithm just to compute the optimal upper bound.

Mathematical Formulation Type II Problems

- Observe only options prices on asset i of strikes

$$0 < k_1^{(i)} < \ldots < k_{J^{(i)}}^{(i)}$$

- Stock prices are regarded as options of zero strike.

Optimization - primal

Constrained optimization problem. Determine

$$\sup_{\mu} \int \left(\sum_i w_i S_i - K \right)^+ \mu(dS)$$

subject to

$$\int (S_i - k_j^{(i)})^+ \mu(dS) = C^{(i)}(k_j^{(i)}), \text{ for } i = 1, \ldots, n, j = 1, \ldots, J^{(i)}$$

$$\int \mu(dS) = 1$$

Optimization - dual

Dual problem

$$\inf_{\nu,\psi} \sum_{i=1}^{n} \sum_{j=1}^{J^{(i)}} C^{(i)}(k_j^{(i)})\nu_i^j + \psi$$

subject to

$$\left(\sum_i w_i S_i - K\right)^+ \leq \sum_{i,j} \left(S_i - k_j^{(i)}\right)^+ \nu_i^j + \psi \qquad \text{(*)}$$

$\nu_i^j \in \mathbb{R}$, for $i = 1, \ldots, n, \quad j = 1, \ldots, J^{(i)}$

$\psi \in \mathbb{R}$

(*) is the super-replication condition

Piecewise Linear Constraints: Can be factorized into collection of Huge LP problems !!

Moment Problems

- Constraints on the joint distribution

From the mathematical point of view, problems of Type I, prescribe (One option/per asset and forward prices) and of Type II(prescribe n options per asset and forward prices) are MOMENT PROBLEMS, with a long tradition in mathematics. Krein, Tchebychev, Karlin $\&$ Studden. In last few years Dimitris Bertsimas at MIT and collaborators have extensively studied these problems.

In 2002 Bertsimas and Popescu considered a general class of moment problems of the form

$$E_\pi\left[w \cdot x\right]$$

subject to the general moment constraints

$$E[f_i(x)] = q_i, i = 1, \cdots, n$$
$$\int \pi(x) = 1 \quad \pi(x) \geq 0$$

and concluded that such problems are NP-Hard in general, ie. generically. Thus the outlook for applying such moment problems in finance seemed bleak. unless special cases were amenable to special techniques

Moment Problem approach: Wrong approach for Upper Bound

- Inspired by the rich history of moment problems and the vast literature on numerical algorithms for semi-definite and semi-infinite programming for Type I problems it was applied by Laurence and Wang and by D'Aspremont and El-Ghaoui to obtain *closed form* expressions for upper bounds and, for lower bounds, but in the latter case, not in full generality.
- For Type II problems, it turns out that the " moment problem approach" is computationally intractible for large indices. In the case of the upper bound a much more effective procedure was found by Hobson L. and Wang wherein the Type II problem is replaced by a much simpler Type III type problem (continuum of strikes) and it is then shown that the solution of this auxiliary problem is also the solution of the original Type II problem.

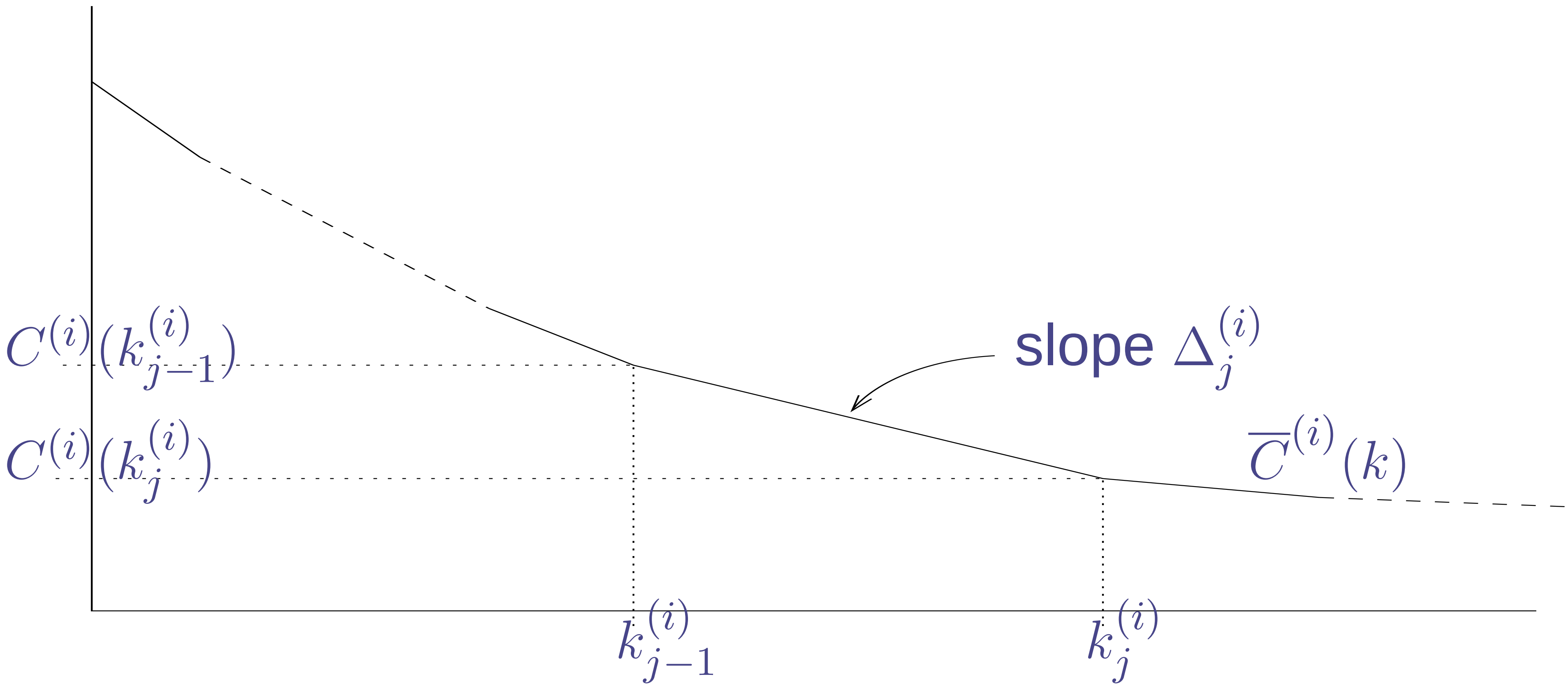
Linear interpolation: Right Approach

- IDEA: "Fill-in" the missing values of the call price functions by a simple linear interpolation that "completes" the partial information about the marginals to (seemingly artificial) full information. Ie. solve the "right" Type 3 problem in order to solve the Type II problem.
- For $1 \leq i \leq n$ and $0 \leq j \leq J^{(i)}$ define $\Delta_j^{(i)}$ by $\Delta_0^{(i)} = 1$ and

$$\Delta_j^{(i)} = \frac{C^{(i)}\left(k_{j-1}^{(i)}\right) - C^{(i)}\left(k_j^{(i)}\right)}{k_j^{(i)} - k_{j-1}^{(i)}}$$

- Key observation: the largest convex function passing through given points is the *linearly interpolated* function.
- There exists an optimizing portfolio which consists of options on no more than two strikes per asset, and involves at most $n+1$ separate options.

Linear interpolation



The interpolated call price function. $\Delta_j^{(i)}$ gives the modulus of the slope of $\overline{C}^{(i)}$ over $(k_{j-1}^{(i)}, k_j^{(i)})$.

Upper Bound in case $D < 0$: One non zero strike

Introduce

$$\mathcal{S}_i = e^{rT}\frac{S_0^i - c_i}{K_i}, i = 1, \cdots n$$

$\mathcal{S}_i$ is an important parameter throughout. Reorder the indices so that

$$\mathcal{S}_1 \leq \mathcal{S}_2 .. \leq \mathcal{S}_n \leq 1$$

and let $\hat{i}$ be the largest index such that "tail of series is large enough", ie.

$$\sum_{i=\hat{i}}^{n} w_i K_i \geq K$$

n

Upper Bound $D < 0$. One non zero strike per asset

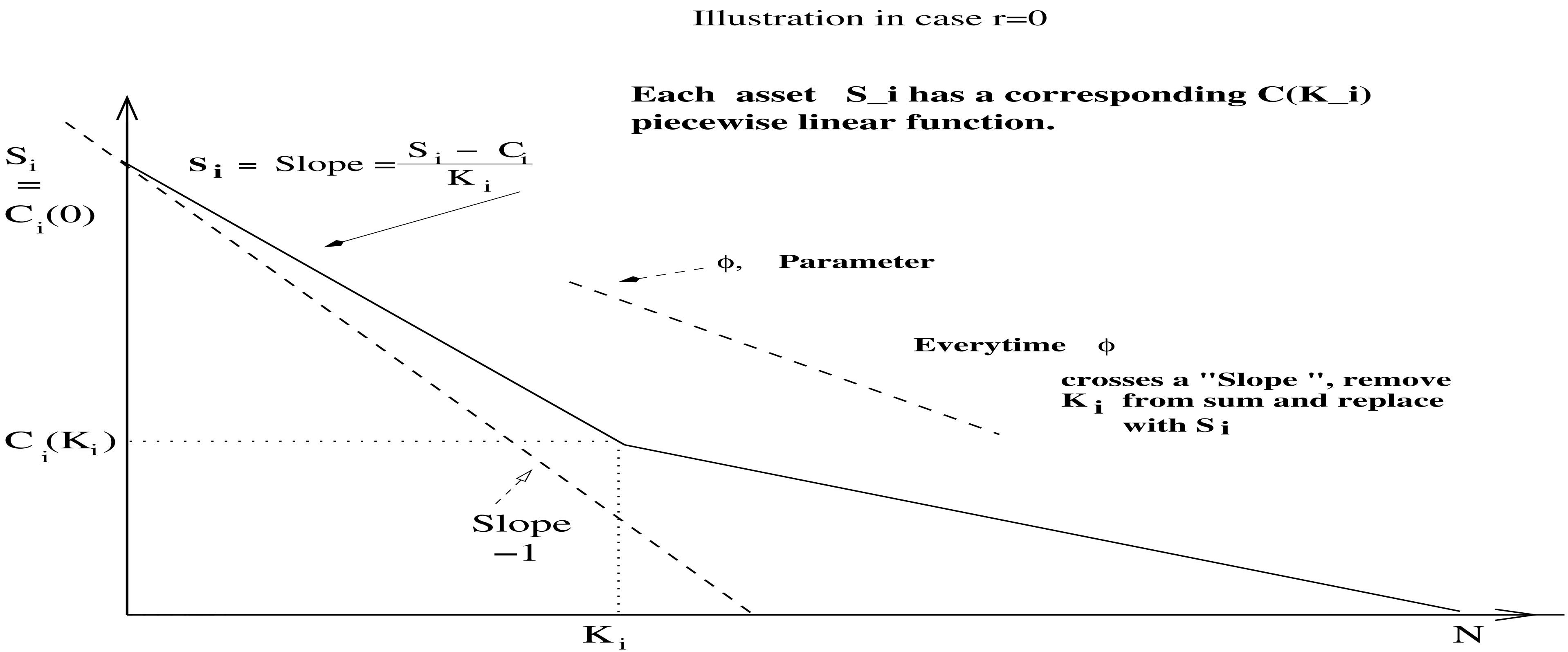
The upper bound in this case is

$$\sum_{i=1}^{\hat{i}-1} w_i S_i + \sum_{i=\hat{i}}^{n} w_i K_i - (K - \sum_{i=\hat{i}}^{n} w_i K_i)\mathcal{S}_{\hat{i}},$$

As opposed to the case $D = 0$, the optimal super-replicating portfolio now has options and stock. Still No cash.

Upper bound $D < 0$ ct'd

$$\sum_{i=1}^{\hat{i}-1} w_i S_i + \sum_{i=\hat{i}}^{n} w_i K_i - (K - \sum_{i=\hat{i}}^{n} w_i K_i) \mathcal{S}_{\hat{i}},$$



Finite market - Result: Using all traded options

- **Preliminaries** For simplicity of exposition assume all slopes $\frac{\partial C^{(i)}(u)}{\partial u}|_{u=k_j^{(i)}}$ are different as i and j vary. Let $I_n = \{1, 2, \cdots, n\}$ where n is the number of assets.
- There is a privileged index $\hat{i} \in I_n$ such that:
- For any model which is consistent with the observed call prices $C^{(i)}(k_j^{(i)})$, the price $B(K)$ for the basket option is bounded above by $\overline{\mathcal{B}}_F(K)$, where
- Case I: $\sum_i w_i k^{(i)}_{J(i)} > K$:

$$\overline{\mathcal{B}}_F(K) = \sum_{i \in I_n \setminus \hat{i}} w_i C^{(i)}\left(k^{(i)}_{\overline{j}(i)}\right) + w_{\hat{i}} \left\{(1 - \theta^*_{\hat{i}}) C^{(\hat{i})}\left(k^{(\hat{i})}_{\overline{j}(\hat{i})-1}\right) + \theta^*_{\hat{i}} C^{(\hat{i})}\left(k^{(\hat{i})}_{\overline{j}(\hat{i})}\right)\right\}$$

- $\theta^*_{\hat{i}}$ is defined as $\theta^*_{\hat{i}} = \frac{\overline{\lambda}^*_{\hat{i}} - \overline{\lambda}^-_{\hat{i}}(\phi^*)}{\overline{\lambda}^+_{\hat{i}}(\phi^*) - \overline{\lambda}^-_{\hat{i}}(\phi^*)} = \frac{(K\overline{\lambda}^*_{\hat{i}}/w_{\hat{i}}) - k^{(i)}_{\overline{j}(\hat{i})-1}}{k^{(i)}_{\overline{j}(\hat{i})} - k^{(i)}_{\overline{j}(\hat{i})-1}}$.

Finite market - Result, Ct'd

- Case II: $\sum_i w_i k^{(i)}_{J(i)} \leq K$:

$$\overline{\mathcal{B}}_F(K) = \sum_i w_i C^{(i)}\left(k^{(i)}_{J(i)}\right)$$

****

- Based on experiments with real data, the second case essentially never arises in practice.
- Moreover, the upper bound is optimal in the sense that we can find co-monotonic models which are consistent with the observed call prices and for which the arbitrage-free price for the basket option is arbitrarily close to $\overline{\mathcal{B}}_F(K)$.
- So where's the beef in Case I?
- All the beef in fleshing out the estimate in the first case is in determining the special index $\hat{i}$ and the indices $j(i), i = 1 \cdots, n$.

How to find which options to choose?

- Possible to show that there is No cash component ψ in the optimal portfolio. So can consider super-replicating portfolios consisting entirely of options with various strikes (some of which may have strike zero).
- The upper bound is available in quasi-closed form, meaning there is a simple algorithm to determine the solution, modulo a slope ordering algorithm: Order all slopes of all call price functions and cycle through.
- To get the intuition as to how to proceed, note that if $\sum \lambda_i = 1$ then

$$\left(\sum_i w_i X_M^{(i)} - K\right)^+ \leq \sum_i w_i \left(X_M^{(i)} - \frac{\lambda_i K}{w_i}\right)^+$$

So that

$$C_{\mathcal{B}}(K) \leq \sum_i w_i C^{(i)} \left(\lambda_i K / w_i\right).$$

The λ_i are arbitrary and so $C_{\mathcal{B}}(K) \leq \inf_{\lambda_i \geq 0, \sum \lambda_i = 1} \sum_i w_i C^{(i)} \left(\lambda_i K / w_i\right).$

Intuition

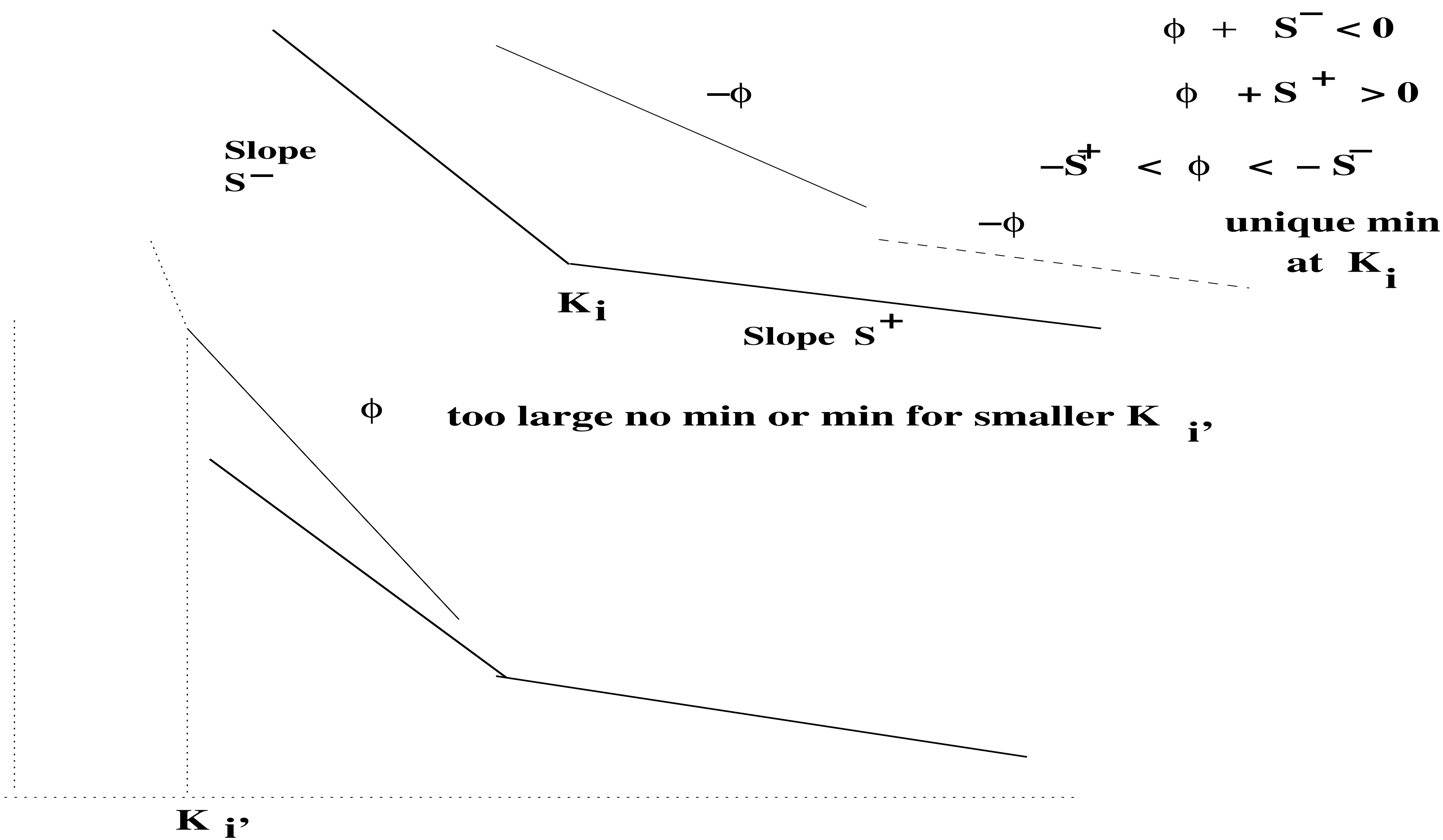
- We wish to find the infimum of $\sum_i w_i C^{(i)}\left(\lambda_i K/w_i\right)$ over choices λ_i satisfying $\lambda_i \geq 0, \sum \lambda_i = 1$. Define the Lagrangian

$$L(\lambda,\phi) = \sum_i w_i C^{(i)}\left(\lambda_i K/w_i\right) + \phi\left(\sum_i \lambda_i - 1\right).$$

- Objective function is convex but only $C^{0,1}$, because each piecewise linear call price functions $C^{(i)}$, is $C^{0,1}$, ie. $\frac{\partial C^i}{\partial K}$ has a jump at each strike $K_i^j, j = 1, \cdot, n_i$.
- Note that objective functional is separable function of 1-dimensional functions.
- Therefore for each fixed Lagrange Multiplier ϕ, the gradient can point in a cone of different directions. In the terminology of convex analysis we have $\phi/\beta K \in \bar{\partial} C^{(i)}(\lambda_i K/w_i)$, where $\bar{\partial}$ is the *subdifferential*

  of the function $C^{(i)}$. .

Illustration Min



Algorithm

For each ϕ there is either a unique $\lambda(\phi)$ or an interval $[\lambda^-(\phi), \lambda^+(\phi)]$.
Essentially:

- $[\lambda(\phi)^-, \lambda(\phi)^+] \sim [w_i K_i^j / K, w_i K_i^{j+1} / K]$ for some i and j.
- So Algorithm:
  - Order all the slopes of all call price functions. Ie. if 30 assets and 8 non zero strikes , order 240 slopes.

$$\mathcal{S}_1 \leq \mathcal{S}_2 \leq \cdots\cdots \mathcal{S}_{240}$$

  - Now starting with $\phi = \epsilon << 1$ increase ϕ while monitoring the quantity

$$\Lambda(\phi) = \sum \lambda^+(\phi)$$

which starts very large for small ϕ ($\Leftarrow$ large K_i^j) and decreases as $\phi \uparrow$.

  - The first time $\Lambda(\phi)$ crosses 1. STOP! $\mapsto$ Optimal value of $\phi = \phi^*$ has been reached.

Experiment on Real DJX Data: Spot was 99.07

We now illustrate the output on real DJX data.

DJX Strikes	DJX Call Prices	AA	AIG	AXP	BA	C	CAT	DD	DIS	GE
52	47.1	0	0	0	37.5	0	0	0	17.5	25
56	43.1	0	0	0/42.5	37.5	0	0	0	17.5	25
60	39.1	22.5	0	42.5	37.5	0/37.5	0	0	17.5	25
64	35.1	22.5	0	42.5	37.5	37.5	0/60	0	17.5	25
68	31.1	22.5	0	42.5	37.5	37.5	60	0	17.5	25
70	29.1	22.5	0	42.5	37.5	37.5	60	0	17.5	25
72	27.1	22.5	0/60	42.5	37.5	37.5	60	0	17.5	25
76	23.1	22.5	60	42.5	37.5	37.5	60	0	17.5	25
80	19.1	22.5	60	42.5	37.5	37.5	60	0/37.5	17.5	25
84	15.2	22.5	60	42.5	37.5	37.5	60	37.5	20	25
88	11.3	22.5	60	42.5	37.5	40	65	37.5	20	27.5
90	9.4	25	60	45	37.5/40	40	65	37.5	20	27.5
92	7.5	25	65	45	40	42.5	70	37.5/40	20	27.5
94	5.8	25	65	47.5	40	42.5	70	40	22.5	27.5
95	4.95	27.5	65	47.5	40	42.5	70	40	22.5	27.5
96	4.15	27.5	65	47.5	40	42.5	70	40	22.5	30
97	3.35	27.5	70	47.5	42.5	42.5	70	40	22.5	30
98	2.725	27.5	70	47.5	42.5	45	70	40	22.5	30
99	2.125	27.5	70	50	42.5	45	75	40/42.5	22.5	30
100	1.6	30	70	50	42.5	45	75	42.5	22.5	30
102	0.775	30	70	50	45	47.5	75	42.5	22.5	30
103	0.5	30	75	50	45	47.5	75/80	42.5	25	30/32.5
104	0.325	32.5	75	50	45	47.5	80	42.5	25	32.5
105	0.15	32.5	75	50	45	47.5	80	42.5	25	32.5
106	0.15	32.5	75	50	45	47.5	80	45	25	32.5
107	0.15	32.5	75	50	45	47.5	80	45	25	32.5

TABLE 4. The super-replicating portfolio. For each strike on the DJX, and for each component of the basket, we list the relevant strike to hold in the cheapest super-replicacting portfolio. A strike of 0 corresponds to holding the asset. For space reasons we only give the strikes for the first 10 components. In most cases there is a single strike listed. In others the optimal portfolio involves a combination of two strikes. Note that the optimal strike to hold on each component asset increases as the strike on the DJX increases.

Experiment on Real DJX Data: How good is the Upper Bound? Spot was 99.07

DJX Strikes	DJX Prices	UB Unclean Data	UB Clean Data	BS Price $\rho = 0$	BS Price $\rho = .5$	BS Price $\rho = .75$	BS Price $\rho = .9$	BS Price $\rho = .99$
52	47.10	47.09	47.05	47.14	47.14	47.15	47.10	47.18
56	43.10	43.10	43.09	43.16	43.18	43.17	43.15	43.17
60	39.10	39.11	39.10	39.16	39.18	39.13	39.12	39.14
64	35.10	35.11	34.30	35.16	35.16	35.16	35.20	35.17
68	31.10	31.12	30.83	31.17	31.17	31.22	31.17	31.16
70	29.10	29.13	29.12	29.18	29.19	29.18	29.17	29.11
72	27.10	27.14	27.14	27.19	27.22	27.18	27.13	27.18
76	23.10	23.15	22.38	23.18	23.16	23.18	23.15	23.19
80	19.10	19.18	19.18	19.20	19.18	19.15	19.19	19.22
84	15.20	15.24	14.95	15.21	15.24	15.23	15.18	15.23
88	11.30	11.42	11.42	11.20	11.26	11.25	11.25	11.36
90	9.40	9.61	9.61	9.21	9.28	9.35	9.41	9.44
92	7.50	7.90	7.90	7.21	7.34	7.53	7.67	7.73
94	5.80	6.32	6.32	5.22	5.58	5.83	6.01	6.08
95	4.95	5.57	5.57	4.22	4.79	5.06	5.26	5.34
96	4.15	4.85	4.85	3.22	4.01	4.35	4.54	4.66
97	3.35	4.19	4.19	2.24	3.28	3.69	3.92	4.01
98	2.73	3.58	3.58	1.35	2.70	3.12	3.34	3.44
99	2.13	3.02	3.02	0.67	2.16	2.58	2.75	2.96
100	1.60	2.53	2.53	0.25	1.69	2.10	2.33	2.43
102	0.78	1.73	1.73	0.01	0.99	1.37	1.55	1.71
103	0.50	1.42	1.42	0.00	0.71	1.05	1.26	1.36
104	0.33	1.16	1.16	0.00	0.52	0.82	1.02	1.13
105	0.15	0.95	0.95	0.00	0.36	0.63	0.79	0.89
106	0.15	0.75	0.75	0.00	0.25	0.48	0.60	0.70
107	0.15	0.59	0.59	0.00	0.16	0.35	0.48	0.53

Sharp Lower Bound for $n = 2$

- Type I Problem: prescribed options and forward prices

Recall $F_i = e^{rT} \frac{S_0^i - C_i}{K_i}$. Introduce the quantities

$$
\begin{aligned}
A_i &= w_i c_i - \frac{K - w_i K_i}{K_i}(S_0^i - C^i) \quad \text{for} \quad i = 1, 2 \\
F &= F_1 + F_2 - 1 \\
F^+ &= \max(F, 0)
\end{aligned}
$$

then

Proposition 1 *Let $(S_0, c) \in \mathcal{C}^o$. The lower bound subject to the constraint $\mu \in \mathcal{M}$ is given by*

$$
\max\Big(A_1 + e^{-rT} w_2 K_2 F^+,
$$

$$
A_2 + e^{-rT} w_1 K_1 F^+, A_1 + A_2 + e^{-rT} K F^+, 0\Big)
$$

$$
\textit{for} \quad D \leq 0
$$

.

Lower Bound Ct'd

Proposition Ct'd

$$\max\Big(A_1 + e^{-rT}(K - w_1K_1)F^+,$$
$$A_2 + e^{-rT}(K - w_2K_2)F^+,$$
$$A_1 + A_2 + e^{-rT}KF^+, 0\Big)$$

for $D \geq 0$

Optimal Sub-Replicating Strategies

Moral: Optimal sub-replicating strategies associated to the optimal lower bound sometimes involve all three of cash, stock and options A subreplicating portfolio in the 2-asset case is given, (u_{*1}, u_{*2}) (nb. of calls) and (v_{*1}, v_{*2}) (nb of stocks) and for brevity, we denote the cases where lower bound is equal to one of A_1, A_2, $A_1 + A_2$, $A_1 + A_2 + KF$, $A_1 + (K - w_1 K_1)F$, $A_2 + (K - w_2 K_2)F$, $A_1 + w_2 K_2 F$, $A_2 + w_1 K_1 F$, which, taken in the same order, we refer to as

Cases $1 : 8$.

Case	u_{*1}	u_{*2}	$\bar{\nu}_*$	v_{*1}	v_{*1}
1	$\frac{K}{K_1}$	0	0	$-\frac{K-w_1K_1}{K_1}$	0
2	0	$\frac{K}{K_2}$	0	0	$-\frac{K-w_2K_2}{K_2}$
3	$\frac{K}{K_1}$	$\frac{K}{K_2}$	0	$-\frac{K-w_1K_1}{K_1}$	$-\frac{K-w_2K_2}{K_2}$
4	0	0	$-e^{-rT}K$	w_1	w_2
5	w_1	$-\frac{K-w_1K_1}{K_2}$	$-e^{-rT}(K - w_1K_1)$	0	$-\frac{K-w_1K_1}{K_2}$

Constraints of Type III

It is assumed that call prices are known corresponding to strikes extending from 0 to $+\infty$. By the Breeden-Litzenberger theorem

$$\frac{\partial^2 C}{\partial K^2} = e^{-rT} \rho$$

where ρ is the density of the distribution of stock $S = P(S < K)$, the latter assumption is equivalent to full knowledge of the marginals ie. marginals prescribed.
On the one hand restrictive since not realistic to assume full knowledge of the marginals. But on the other hand still allows a rich choice of joint distributions compatible with the given marginals, by using theory of copulas. This is the route Hobson-Laurence-Wang take for optimal sub-replicating strategy when all prices of calls with strikes prescribed .

Sklar

Sklar's theorem Any joint distribution with continuous marginal distribution functions $F_i, i = 1, \cdots, n$, can be expressed as

$$C(F_1^{-1}, F_2^{-1}, \cdots, F_n^{-1})$$

where $C(x_1, \cdots, x_n)$ is a copula and where F^{-1} is the generalized inverse of F, ie $F^{-1}(t) = \inf\{x \in \Re | F(x) \geq t\}$

Copula2

Minimization problem with fixed marginals can be reduced to problem of finding *optimal copula*.
This problem was solved **in the case** $n = 2$ by Rapuch and Roncalli (Crédit Lyonnais web site) based on earlier results of Muller and Scarsini and Chen.
The Frechet Copulas $C^-(u_1, u_2)$ and $C^+(u_1, u_2)$ given by

$$C^- = \max(u_1 + u_2 - 1, 0)$$
$$C^+ = \min(u_1, u_2)$$

Let $C^-(\mathcal{M}_1, \mathcal{M}_2)$ and $C^+(\mathcal{M}_1, \mathcal{M}_2)$ be the corresponding call option prices. Then for a generic basket option on two assets with the same marginals we have

$$C^-(\mathcal{M}_1, \mathcal{M}_2) \le C(\mathcal{M}_1, \mathcal{M}_2) \le C^+(\mathcal{M}_1, \mathcal{M}_2)$$

Dhaene and Gooverts and Rapuch-Roncalli's Copula Approach

When the distribution functions F_X and F_Y are continuous, we have

$$\mathcal{C} = \mathcal{C}^- \Leftrightarrow Y = F_Y^{-1}(1 - F_X(X)), \quad \text{anti-monotonic}$$
$$\mathcal{C} = \mathcal{C}^+ \Leftrightarrow Y = F_Y^{-1}(F_X(X)). \quad \text{co-monotonic}$$

For the lower bound note this means that

$$C_{\mathcal{B}}^- = \int_{\mathbb{R}+} [x + F_Y^{-1}(1 - F_X(x)) - K]^+ dF_X(x). \tag{1}$$

A Remarkable set of hedging portfolios: The “sheep track portfolios”

What is a sheeptrack portfolio? Let X and Y denote the two stocks. Let denote the price of a call option on X, resp. Y, by $C_X(\cdot)$, resp. $C_Y(\cdot)$ and similarly for puts by P_X, P_Y.
A sheeptrack portfolio has the form

$$X + \sum_{i=1}^{n} C_X(K_X^{2i}) - C_Y(K_Y^{2i-1})$$

$$-$$

$$\left(P_Y(K_{bask}) + \sum_{i=1}^{n} P_X(K_Y^{2i}) - P_Y(K_Y^{2i-1}) \right),$$

where the strikes $K_Y^i = K_{bask} - K_X^{2n-i+1}$ and where $K_X^i, i = 1, \cdots$ are determined by finding the zeroes of the following function

$$\phi(x) = C_X(x) + C_Y(K_{bask} - x) + 1$$

Using Sheep-Tracks in Black-Scholes setting

Assume geometric brownian motions for both stocks. We took $T = .5$, $S_0^1 = S_0^2 = 100$ and $\sigma_1 = .355, \sigma_2 = .2$ e $w_1 = w_2 = .5$. In the first column the basket strike is shown. In the second column is the Monte Carlo price of the basket in the case where stock S_1 and S_2 are anti-monotonic with the prescribed marginals.

The STP Portfolio - Black-Scholes

Kbask	MC Value	Hedging Portfolio's Value	Strike K_1^1	$Strike K_1^2$
81.5	18.52	18.50	absent	absent
84	16.03	16.00	absent	absent
86.5	13.55	13.50	absent	absent
89	11.02	11.00	absent	absent
91.5	8.5	8.50	absent	absent
94	5.97	6.00	absent	absent
96.5	3.98	3.99	51.24	89.40
99	2.73	2.69	44.47	101.61
100	2.28	2.29	42.50	105.76
102.5	1.54	1.54	38.52	115.19
105	1.02	1.03	35.41	123.73
107.5	0.69	0.69	32.83	131.73
110	0.45	0.46	30.65	139.30
112.5	0.32	0.31	28.78	146.59
115	0.21	0.21	27.12	153.64
117.5	0.14	0.14	25.64	160.48

Sheep-Track Portfolios

The term sheep track portfolio arises because such portfolios correspond to integrating a payoff function of the form

$$f_i(z) = z + \sum_{a=1}^{n} \{(z - K_i^{2a})^+ - (z - K_i^{2a-1})^+\}.$$

As a function of z these look like this:

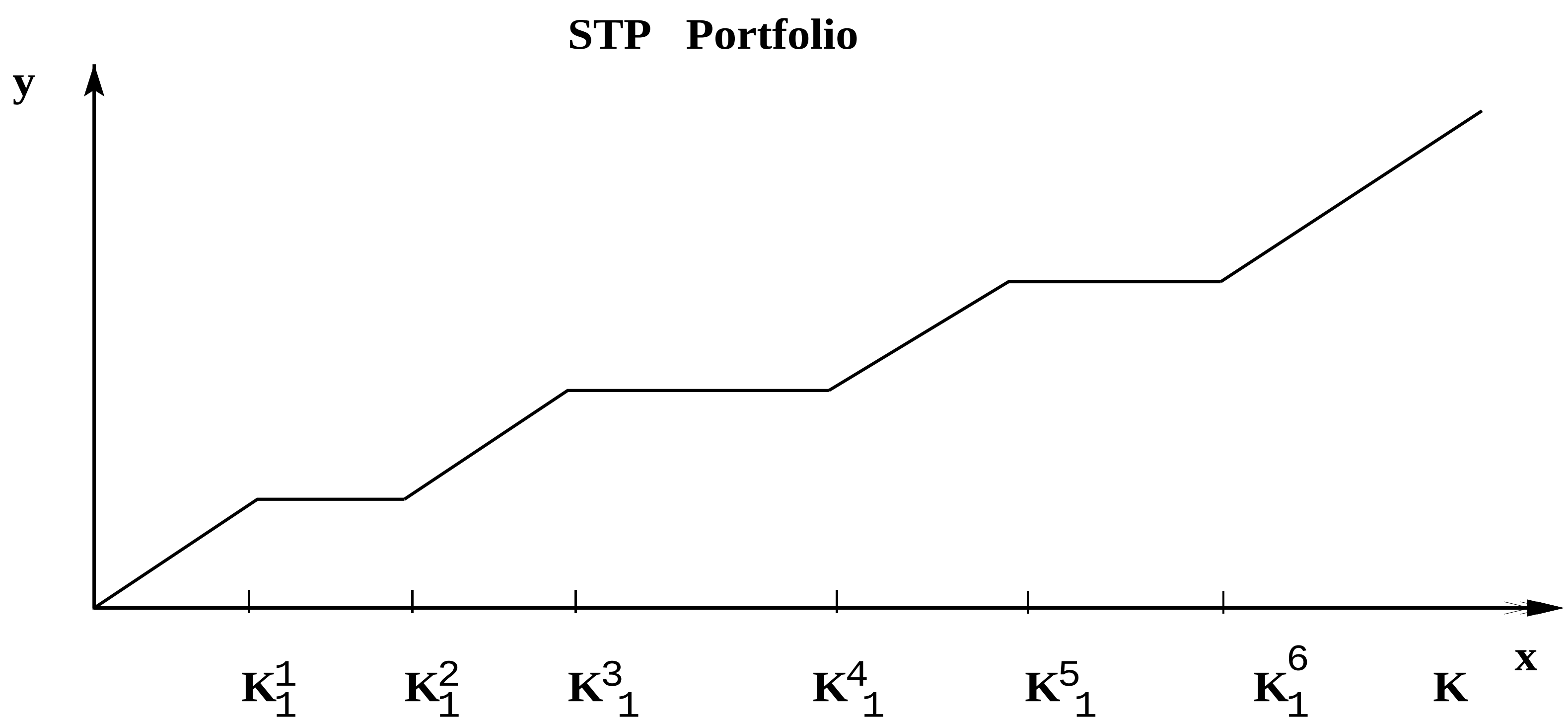


Figure 1:

Mathematical Formulation - LB

Constrained optimization problem

$$\inf_{\mu} \int_{\mathbb{R}_+^2} (x+y-K)^+ \mu(dxdy)$$

subject to the constraints on the marginal distributions

$$\int_{\mathbb{R}+} (x-k_1)^+ \mu_X(dx) = C_X(k_1),$$
$$\int_{\mathbb{R}+} (y-k_2)^+ d\mu_Y(dy) = C_Y(k_2),$$
$$\int_{\mathbb{R}_+^2} \mu(dx,dy) = 1.$$

μ_X and μ_Y are the marginal distributions.

Mathematical Formulation - dual

$$\sup_{\nu_1,\nu_2,\lambda} \int_{\mathbb{R}^+} C_X(k_1)\nu_1(k_1) + \int_{\mathbb{R}^+} C_Y(k_2)\nu_2(k_2) + \lambda$$

subject to the constraints on the measures $\nu_i, i = 1, 2$

$$\left[(x+y-K)^+ - \int (x-k_1)^+\nu_1(dk_1) - \int (y-k_2)^+\nu_1(dk_1) - \lambda\right] \geq 0$$

for all $\mu \in \mathbb{M}_+$

where ν_i's range over all finite signed measures and $\lambda \in \mathbb{R}$

Optimal dual: Sheep-Track Portfolio

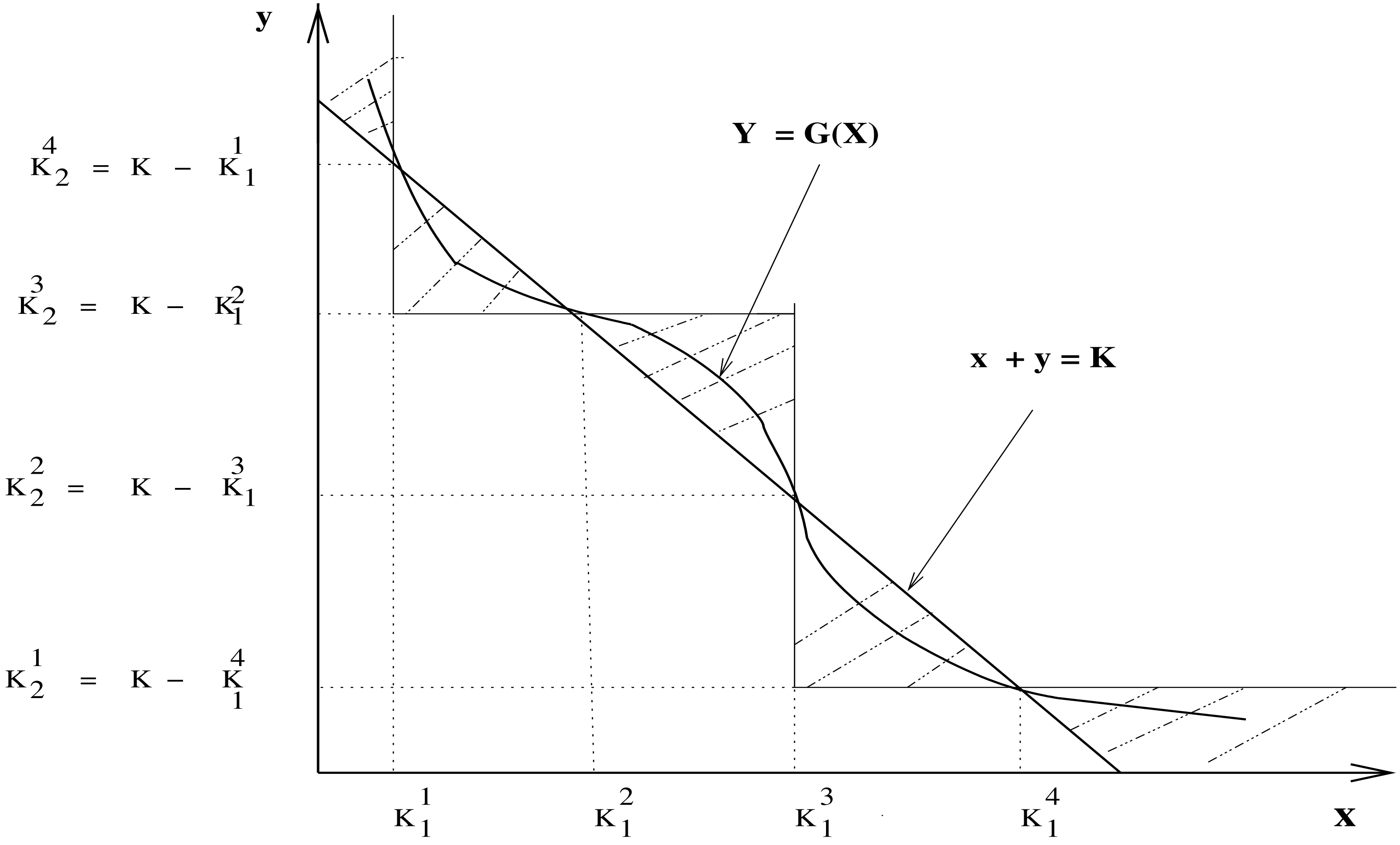
$$\bar{\nu}_1(dk_1) = \delta_0(k_1)dk_1 + \sum_{i=1}^{2n}(-1)^i \delta_{K_1^i}(k_1)dk_1,$$

$$\bar{\nu}_2(dk_2) = \delta_0(k_2)dk_2 + \sum_{i=1}^{2n}(-1)^i \delta_{K_2^i}(k_2)dk_2,$$

$$\bar{\lambda} = \sum_{i=1}^{n}(K_1^{2i} - K_1^{2i-1}) - K = \sum_{i=1}^{n}(K_2^{2i} - K_2^{2i-1}) - K.$$

We call such portfolio "STP", short for "sheep-track portfolios", since the graph of such a portfolio is reminiscent of such tracks on British hillsides.

Countermonotonicity Distribution



Conclusions

- (OPEN PROBLEM) The optimal lower bound and optimal sub-replicating strategy is unknown in the discrete strike case.
- Is there a "half-way house" between solving the incredible "moment problems" and using tricks like copulas which use the full marginals?
- Optimal lower bound is open even in the full marginals prescribed (cts strike case) when $n \geq 3$.
- We have treated only a 1 period model. Multi-period models are open.
- Corresponding problems in the American Basket option case are open, ie. S&P 100.
- Upper bound could be made closer if add additional constraints. Which ones? Correlation prescribed? Entropy constraints? . Ie. make **bet** on the correlations being in a certain range. Then static hedge no longer risk-free, but works with high likelihood.
- Quien desea ver los slides puede navigar al seguente sitio:

  http://www.math.nyu.edu/ laurence