Sharp model independent no-arbitrage bounds and optimal hedge ratios for basket options

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Main aims of this contribution

Joint work with David Hobson (U. of Bath and Princeton) and Tai-Ho Wang(National Chung Cheng U., Chia Yi, Taiwan).

- To briefly review some work done on distribution free bounds for option prices under a variety of different constraints
- To review recent joint work with Wang on **obtaining sharp bounds and optimal hedge ratios in a static no arbitrage one period setting.**
- To discuss even more recent and more general results obtained in collaboration with Hobson and Wang concerning upper and lower bounds for basket options

Basket Options

Lo's result

Following pioneering work by Robert Merton , Andrew Lo in 1987 derived the following distribution free upper bound for the prices C and P of a European call option on one asset give that the variance V^* of the gross return and the short rate r (assumed constant over the period under consideration) are known

$$C \leq \begin{cases} \frac{S(T)-Ke^{-rT}+S_0V^*e^{-2rT}}{1+V^*e^{-2rT}} & \text{if} \quad \frac{S(T)}{K} \geq \frac{2e^{-rT}}{1+V^*e^{-2rT}} \\ \frac{1}{2}\left[S(T)-Ke^{-rT}+\sqrt{(Ke^{-rT}-S(T))^2+S^2(T)V^*e^{-2rT}}\right] & \\ \text{if} \quad \frac{S(T)}{K} < \frac{2e^{-rT}}{1+V^*e^{-2rT}} & \end{cases}$$

Here $V^* = \int \left(\frac{S_T}{S_0} - e^{rT}\right)^2 d\mu^*$, where μ^* is a risk neutral distribution. Lo also showed, using an earlier result of Scarf that the optimizing distribution is a two point distribution.

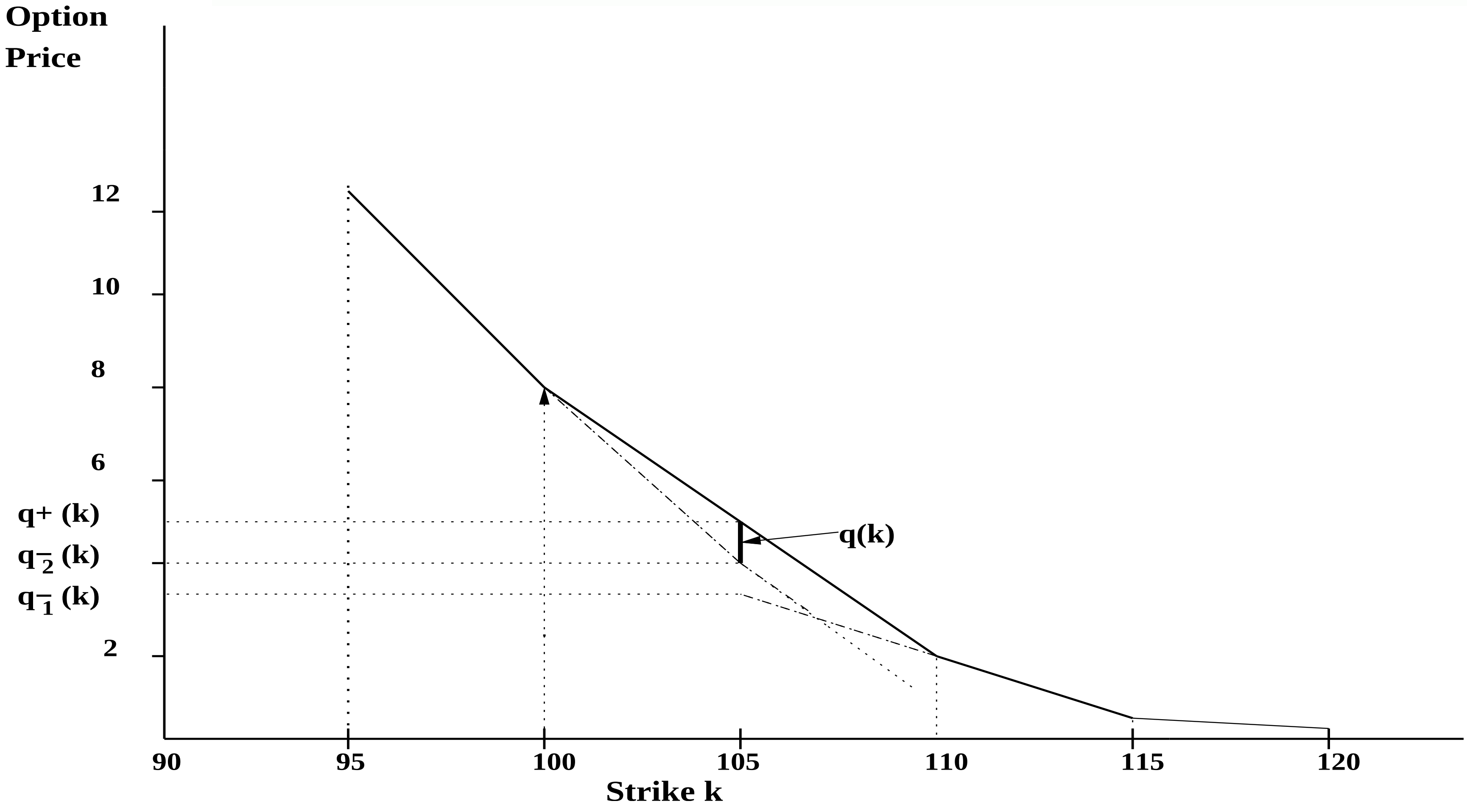
Related Work Ct'd

Bertsimas and Popescu, 2003, use a LP approach to derive bounds on assets under a variety of constraints.

Given prices $C_i(K_i)$ of call options with strikes $0 \leq K_1 \leq .. \leq K_n$ on a stock X, the range of all possible prices for a call option with strike K where $K \in (K_j, K_{j+1})$ for some $j = 0, \cdots, n$ is $[C^-(K), C^+(K)]$ where

$$\begin{aligned} &C^-(K) \\ &= \max\left(C_j \frac{K - K_{j-1}}{K_j - K_{j-1}} + C_{j-1} \frac{K_j - K}{K_k - K_{j-1}}, \right. \\ &\qquad \left. C_{j+1} \frac{K_{j+2} - K}{K_{j+2} - K_{j+1}} + C_{j+2} \frac{K - K_{j+1}}{K_{j+2} - K_{j+1}} \right) \quad \text{lower bounds} \\ &C^+(K) = \frac{K_{j+1} - K}{K_{j+1} - K_j} + C_{j+1} \frac{K - K_j}{K_{j+1} - K_j} \quad \text{upper bounds} \end{aligned}$$

GraphicRepBertsimasPopescu



Bert-PoP ct'd

One lesson from Bertsimas and Popescu: already in 1-D lower bounds are more complicated than upper bounds. Upper bound corresponds to a linear interpolation and lower bound to a more complicated, albeit still linear interpolation. This theme continues and is accentuated in higher dimension.
There are many other contributions in this field, notably by Ryan, Perrakis, Ritchken and many others for one dimensional options. Brown, Hobson and Rogers.

Next we introduce Basket options and our work on model independent, no-arbitrage bounds

Basket Options

Basket Options

The Payoff of a basket option:

$$\psi(S_1, \cdots, S_n) = (\sum_i w_i S_i - K)^+$$

- Price weighted –> $w_i = \frac{1}{I(t_0)}$.
- Capitalization Weighted,

$w_i = Cst.\frac{\text{nb. } S_i \text{ shares outstanding}}{\text{Total capitalization}}$.

w_i are readjusted periodically.

- S&P 500, S&P 100, Dow Jones 100.

Black-Scholes

The most popular model: **Multidimensional Black-Scholes**

$$dS_t^i = S_t^i(r - d_i)dt + \sigma_i S_t^i dZ_i^t$$
$$S^i(0) = S_0^1$$
$$< dZ_i, dZ_j >= \rho_{ij} dt$$

European Basket Options:

$$u_t + S_i S_j \rho_{ij} \sigma_i \sigma_j u_{S_i S_j} + (r - d_i) S_i u_{S_i} - ru = 0$$
$$u(S_1, \cdots, S_n, T) = (\sum_{i=1}^{n} w_i S_i - K)^+$$

Closed form solution B-S

Solution in closed form

Solution at time 0, for maturity T in closed form

$$e^{-rT}\frac{1}{(2\pi)^{n/2}(detV)^{1/2}}\int\left(\sum w_i e^{(r-d_i-\frac{\sigma_i^2}{2})T-\sqrt{T}X}-K\right)^+ e^{\frac{1}{2}X^tV^{-1}X}dX$$

where V is the variance-covariance matrix $\{\sigma_i\sigma_i\rho_{ij}\}_{i,j=1}^n$.

First Model independent Results for Basket Options

Robert Merton 1973 (Bell Journal) established the following result. Given n assets $S_1, \cdots, S_n$ and n options C_i with strike K_i. One option per asset. Let $C_{\mathcal{B}}$ denote the price of a basket option with positive constant weights $w_i, i = 2, \cdots, n$. Suppose that in addition the following condition holds:

- $$w_1 K_1 + w_2 K_2 + \cdots + w_n K_n = K$$

Then

$$C_{\mathcal{B}} \leq w_1 C_1 + w_2 C_2 + \cdots + w_n C_n$$

Merton ct'd

Result has a simple proof.

$$
\begin{aligned}
&(w_1 S_1 + w_2 S_2 + \cdots + w_n S_n - K)^+ \\
&= (w_1(S_1 - K_1) + w_2(S_2 - K_2) + \cdots + w_n(S_n - K_n))^+ \\
&\leq w_1(S_1 - K_1)^+ + w_2(S_2 - K_2)^+ + \cdots w_n(S_n - K_n)^+
\end{aligned}
$$

Taking expectations in the above formula yields

$$
\begin{aligned}
C_{\mathcal{B}} &= e^{-rT} E\left[w_1 S_1 + w_2 S_2 + \cdots + w_n S_n - K\right]^+ \\
&\leq e^{-rT} w_1 E[S_1 - K_1)^+] + \cdots + e^{-rT} w_1 E[(S_n - K_n)^+] \\
&= w_1 C_1(K_1) + \cdots w_n C_n(K_n)
\end{aligned}
$$

Merton ct'd

Merton did not however characterize the conditions for equality in this relation. This was done later in P.L.-Wang Risk 2004, AMF 2005.

Note: Letting $D = K - \sum w_i K_i > 0$, it is easy to see that the above proof goes through, but,

• When $D < 0$ the above bound is no longer the the optimal one.

DJX Index

Symbol	Name	Last Price	Weight
AA	ALCOA, INC.	34.270	2.40%
AIG	AMERICAN INTERNATIONAL GROUP INC	61.030	4.28%
AXP	AMERICAN EXPRESS CO	55.630	3.90%
BA	BOEING CO	53.930	3.78%
C	CITIGROUP	47.070	3.30%
CAT	CATERPILLAR INC.	89.970	6.31%
DD	DU PONT EI DE NEMOURS	44.500	3.12%
DIS	WALT DISNEY CO	26.800	1.88%
GE	GENERAL ELECTRIC CO	36.250	2.54%
GM	GENERAL MOTORS CORP	40.210	2.82%
HD	HOME DEPOT INC	43.220	3.03%
HON	HONEYWELL INTERNATIONAL INC.	36.620	2.57%
HPQ	HEWLETT PACKARD CO	19.340	1.36%
IBM	INTERNATIONAL BUSINESS MACHINES	95.320	6.68%
INTC	INTEL CORP	23.690	1.66%
JNJ	JOHNSON AND JOHNSON	61.000	4.28%
JPM	JP MORGAN CHASE AND CO INC	39.170	2.75%
KO	COCA COLA CO	40.790	2.86%
MCD	MCDONALDS CORP	30.500	2.14%
MMM	3M COMPANY	82.680	5.80%
MO	ALTRIA GROUP INC.	54.740	3.84%
MRK	MERCK AND COMPANY INC	26.450	1.85%
MSFT	MICROSOFT CORP	26.970	1.89%
PFE	PFIZER INC.	27.450	1.92%
PG	PROCTER AND GAMBLE CO	54.600	3.83%

A typical Component Option, Procter& Gamble

May, 2004 | July, 2004 | October, 2004 | January, 2005 | January, 2006

PROCTER & GAMBLE CO 105.97 ▼ -0.15 -0.1414% 105.91 106.37 2,727,800

Calls							Strike	Puts						
Symbol	Last	Chg	Bid	Ask	Vol	Int	Price	Symbol	Last	Chg	Bid	Ask	Vol	Int
PG EM	41.50	0.00	40.80	41.10	0	15	65	PG QM	0.00	0.00	0.00	0.05	0	.
PG EN	36.50	0.00	35.80	36.10	0	65	70	PG QN	0.00	0.00	0.00	0.05	0	.
PG EO	31.50	0.00	30.90	31.10	0	15	75	PG QO	0.00	0.00	0.00	0.05	0	.
PG EP	26.00	0.00	25.90	26.10	0	.	80	PG QP	0.05	0.00	0.00	0.05	0	20
PG EQ	21.00	0.00	20.90	21.10	0	40	85	PG QQ	0.00	0.00	0.00	0.05	0	.
PG ER	16.00	0.00	15.90	16.10	0	58	90	PG QR	0.10	0.00	0.00	0.10	0	90
PG ES	11.30	0.00	10.90	11.10	0	204	95	PG QS	0.20	0.00	0.10	0.20	0	173
PG ET	6.00	-0.10	6.00	6.20	132	220	100	PG QT						
PG EA							105	PG QA	1.60	0.00	1.70	1.75	680	2,065
PG EB	0.50	0.00	0.45	0.50	193	2,921	110	PG QB						
PG EC	0.05	0.00	0.05	0.10	15	258	115	PG QC	10.10	0.00	9.50	9.70	0	64
PG ED	0.00	0.00	0.00	0.05	0	.	120	PG QD	14.40	0.00	14.40	14.70	0	75
PG EE	0.00	0.00	0.00	0.05	0	.	125	PG QE	19.70	0.00	19.40	19.70	0	138
PG EF	0.00	0.00	0.00	0.05	0	.	130	PG QF						

- Calibration of basket option price to Prices of Option Components. Three different approaches
  - I. Calibrate to Only one option per asset of a given maturity and forward prices
  - II. Calibrate to all available options with a given maturity
  - III. Assume (à la Dupire) that options are traded with a continuum of strikes $\Leftrightarrow$ Equivalent to assuming Marginals Prescribed.

Calibrate to fixed single name option prices, one per asset, and to fixed forward prices

Let $d\mu(S_T^1, \cdots, S_T^n)$ be the probability density associated to the distribution of the $S_T^i, i = 1, \cdots n$ at time T . Let μ_i be the marginal of μ in the i-th stock. Then we require that

- **Prescribed Option Prices**

$$e^{-rT} \int (S_T^i - K)^+ d\mu_i = C_i, \quad i = 1, \cdots, n$$

- **Prescribed Forward Prices**

$$e^{-rT} \int S_T^i d\mu_i = S_0^i, \quad i = 1, \cdots n$$

Formulation Ct'd

We seek to minimize and maximize the basket option price

Basket Option Price

$$\mathcal{B} = e^{-rT} \int (\sum_{i=1}^{n} w_i S_i - K)^+ d\mu_{\mathcal{B}}(S_1, \cdots, S_n)$$

Thus obtaining *Distribution Free* Bounds

$$\mathcal{B}_L \leq \mathcal{B} \leq \mathcal{B}_U$$

Super and Subreplicating portfolio

Upper bound, resp. Lower Bound (made explicit later on) can be expressed as

$$\sum_{i=1}^{n} \alpha_i^U C_i + \beta_i^U S_0^i + \gamma^U = \mathcal{B}_U$$

$$\sum_{i=1}^{n} \alpha_i^L C_i + \beta_i^L S_0^i + \gamma^L = \mathcal{B}_L$$

ie. $(\alpha_1^U, \cdots, \alpha_n^U, \beta_1^U \gamma^U, \cdots, \beta_n^U, \gamma^U)$

is the least expensive super-replicating portfolio

$\&(\alpha_1^L, \cdots, \alpha_n^L, \beta_1^L, \cdots, \beta_n^L, \gamma^L)$

is the most expensive sub-replicating portfolio

. Note that in general there is a option component, a stock component and a cash component.

Super and Sub

- The former therefore corresponds to a reasonable *ask price* for the seller of the basket option in the absence of other information on the joint distribution of the assets
- The latter is a reasonable *bid price*.

Moment Problems

- Constraints on the joint distribution

From the mathematical point of view, problems of Type I, prescribe (One option/per asset and forward prices) and of Type II(prescribe n options per asset and forward prices)

are MOMENT PROBLEMS, with a long tradition in mathematics.

Krein, Tchebychev, Karlin $\&$ Studden. In last few years Dimitris Bertsimas at MIT and collaborators have extensively studied these problems.

Bertsimas and Popescu: multidimensional moment problems

In 2002 Bersimas and Popescu considered a general class of moment problems of the form

$$E_\pi \left[w \cdot x \right]$$

subject to the general moment constraints

$$E[f_i(x)] = q_i, i = 1, \cdots, n$$
$$\int \pi(x) = 1 \quad \pi(x) \geq 0$$

and concluded that such problems are NP-Hard in general, ie. generically. Thus the outlook for applying such moment problems in fi nance seemed bleak. unless special cases were amenable to special techniques

L-W, d'Aspremont-El Ghaoui

However in 2003 and 2004 Laurence and Wang and D'Aspremont and El-Ghaoui found solutions to some problems of Type I. (One option per asset and/or forward prices).
The problems solved are the following

- Closed form solution for the optimal upper bound problem and optimal super-replicating strategies for n assets. [L-W] $\&$ [DA-EG]
- Closed form solutions for the optimal lower bound problem but without prescribed forward prices and for n assets. [DA-EG]
- Closed form solutions for the optimal lower bound problem with prescribed forward prices and optimal sub-replicating strategies in the case of two assets. [L-W]
- Explicit discrete two dimensional distributions attaining the upper and lower bounds [L-W]. These are 3 and 4-point distributions.

Next we briefly review these before passing onto to Type I problems.

Sharp Upper Bound

Recall the important parameter

$$D = K - \sum w_i K_i$$

. The optimal upper bound comes in different forms, according as to whether this parameter $D \geq 0$ or $D < 0$.

Proposition 1 **(Sharp Upper Bound)** *Let* $(m, c) \in \mathring{\mathcal{C}}$. *The upper bound of subject to the 1 option/asset and forward constraints when* $D \geq 0$ *is given by*

$$\sum_{i=1}^{n} w_i c_i \tag{1}$$

Ie. Merton's bound in the case $D = 0$ is sharp both in the case $D > 0$ and in the case $D = 0$.

$D < 0$ *precursor for general Type II problem*

The case $D < 0$ is special not only because it has the most non-trivial form, but also because, as we will now explain, its form holds a clue for the form of the upper bound for bounds of Type II (ie. all options prescribed.)

Upper Bound in case $D < 0$

Upper bound in case $D < 0$. Introduce

$$\mathcal{S}_i = e^{rT}\frac{S_0^i - c_i}{K_i}, i = 1, \cdots n$$

$\mathcal{S}_i$ is an important parameter throughout. Reorder the indices so that

$$\mathcal{S}_1 \leq \mathcal{S}_2 .. \leq \mathcal{S}_n \leq 1$$

and let $\bar{i}$ be the (newly ordered) smallest index such that "tail of series is small enough", ie.

$$K \geq \sum_{i=\hat{i}}^{n} w_i K_i$$

$$K < \sum_{i=\hat{i}+1}^{n} w_i K_i$$

Critical Index $\hat{i}$.

Upper Bound $D < 0$

Upper Bound for $D < 0$

The upper bound in this case is

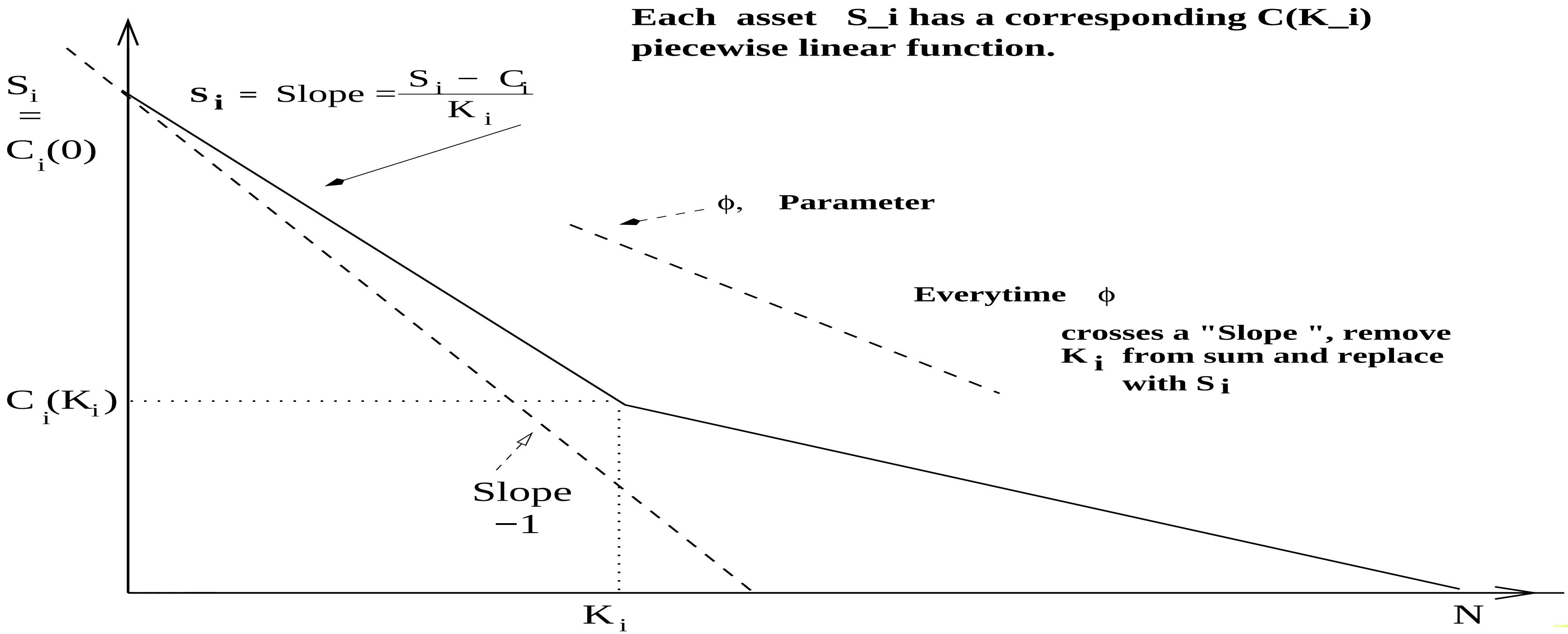
$$\sum_{i=1}^{\hat{i}-1} w_i S_i + \sum_{i=\hat{i}}^{n} w_i K_i - (K - \sum_{i=\hat{i}}^{n} w_i K_i)\mathcal{S}_{\hat{i}},$$

As opposed to the case $D = 0$, the optimal super-replicating portfolio now has options and stock. Still No cash.

Upper bound $D < 0$ ct'd

$$\sum_{i=1}^{\hat{i}-1} w_i S_i + \sum_{i=\hat{i}}^{n} w_i c_i - (K - \sum_{i=\hat{i}}^{n} w_i K_i)\mathcal{S}_{\hat{i}},$$

Illustration in case r=0



Recall $F_i = e^{rT}\frac{S_0^i - C_i}{K_i}$. Introduce the quantities

$$A_i = w_i c_i - \frac{K - w_i K_i}{K_i}(S_0^i - C^i) \quad \text{for} \quad i = 1,2$$
$$F = F_1 + F_2 - 1$$
$$F^+ = \max(F, 0)$$

then

Proposition 2 *Let $(S_0, c) \in \mathcal{C}^o$. The lower bound subject to the constraint $\mu \in \mathcal{M}$ is given by*

$$\max\big(A_1 + e^{-rT} w_2 K_2 F^+,$$
$$A_2 + e^{-rT} w_1 K_1 F^+, A_1 + A_2 + e^{-rT} K F^+, 0\big)$$

for $D < 0$

Lower Bound Ct'd

Proposition Ct'd

$$\max\Big(A_1 + e^{-rT}(K - w_1K_1)F^+,$$
$$A_2 + e^{-rT}(K - w_2K_2)F^+,$$
$$A_1 + A_2 + e^{-rT}KF^+, 0\Big)$$

for $\quad D \geq 0$

Optimal Sub-Replicating Strategies

Moral: Optimal sub-replicating strategies associated to the optimal lower bound sometimes involve all three of cash, stock and options A subreplicating portfolio in the 2-asset case is given, (u_{*1}, u_{*2}) (nb. of calls) and (v_{*1}, v_{*2}) (nb of stocks) and for brevity, we denote the cases where lower bound is equal to one of A_1, A_2, $A_1 + A_2$, $A_1 + A_2 + KF$, $A_1 + (K - w_1 K_1)F$, $A_2 + (K - w_2 K_2)F$, $A_1 + w_2 K_2 F$, $A_2 + w_1 K_1 F$, which, taken in the same order, we refer to as Cases $1 : 8$.

Case	u_{*1}	u_{*2}	$\bar{\nu}_*$	v_{*1}	v_{*1}
1	$\frac{K}{K_1}$	0	0	$-\frac{K-w_1K_1}{K_1}$	0
2	0	$\frac{K}{K_2}$	0	0	$-\frac{K-w_2K_2}{K_2}$
3	$\frac{K}{K_1}$	$\frac{K}{K_2}$	0	$-\frac{K-w_1K_1}{K_1}$	$-\frac{K-w_2K_2}{K_2}$
4	0	0	$-e^{-rT}K$	w_1	w_2
5	w_1	$-\frac{K-w_1K_1}{K_2}$	$-e^{-rT}(K - w_1K_1)$	0	$-\frac{K-w_1K_1}{K_2}$

Constraints of Type III

It is assumed that call prices are known corresponding to strikes extending from 0 to $+\infty$. By the Breeden-Litzenberger theorem

$$\frac{\partial^2 C}{\partial K^2} = \rho$$

where ρ is the density of the distribution of stock $S = P\left(S < K\right)$, the latter assumption is equivalent to full knowledge of the marginals ie. marginals prescribed.
On the one hand restrictive since not realistic to assume full knowledge of the marginals. But on the other hand still allows a rich choice of joint distributions compatible with the given marginals, by using theory of copulas. This is the route Hobson-Laurence-Wang take for optimal sub-replicating strategy when all prices of calls with strikes prescribed . More later, time permitting.

Finite market

- Observe only options prices on asset i of strikes

$$0 < k_1^{(i)} < \ldots < k_{J^{(i)}}^{(i)}$$

- Stock prices are regarded as options of zero strike.

Optimization - primal

Constrained optimization problem

$$\sup_{\mu} \int \left(\sum_i w_i S_i - K \right)^+ \mu(dS)$$

subject to

$$\int (S_i - k_j^{(i)})^+ \mu(dS) = C^{(i)}(k_j^{(i)}), \text{ for } i = 1, \ldots, n, j = 1, \ldots,$$

$$\int \mu(dS) = 1$$

Optimization - dual

Dual problem

$$\inf_{\nu,\psi} \sum_{i=1}^{n} \sum_{j=1}^{J^{(i)}} C^{(i)}(k_j^{(i)})\nu_i^j + \psi$$

subject to

$$\left(\sum_i w_i S_i - K\right)^+ \leq \sum_{i,j} \left(S_i - k_j^{(i)}\right)^+ \nu_i^j + \psi$$

$\nu_i^j \in \mathbb{R}$, for $i = 1, \ldots, n, \quad j = 1, \ldots, J^{(i)}$

$\psi \in \mathbb{R}$

Optimization - dual

Dual problem

$$\inf_{\nu,\psi} \sum_{i=1}^{n} \sum_{j=1}^{J^{(i)}} C^{(i)}(k_j^{(i)})\nu_i^j + \psi$$

subject to

$$\left(\sum_i w_i S_i - K\right)^+ \leq \sum_{i,j} \left(S_i - k_j^{(i)}\right)^+ \nu_i^j + \psi$$

$\nu_i^j \in \mathbb{R},$ for $i = 1, \ldots, n, \quad j = 1, \ldots, J^{(i)}$

$\psi \in \mathbb{R}$

Huge LP problem !!

Linear interpolation

- For $1 \leq i \leq n$ and $0 \leq j \leq J^{(i)}$ define $\Delta_j^{(i)}$ by $\Delta_0^{(i)} = 1$ and
$$\Delta_j^{(i)} = \frac{C^{(i)}\left(k_{j-1}^{(i)}\right) - C^{(i)}\left(k_j^{(i)}\right)}{k_j^{(i)} - k_{j-1}^{(i)}}$$
- "Fills-in" the missing values of the call price functions by linear interpolation completes the partial information about the marginal to full information.

Linear interpolation

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Linear interpolation

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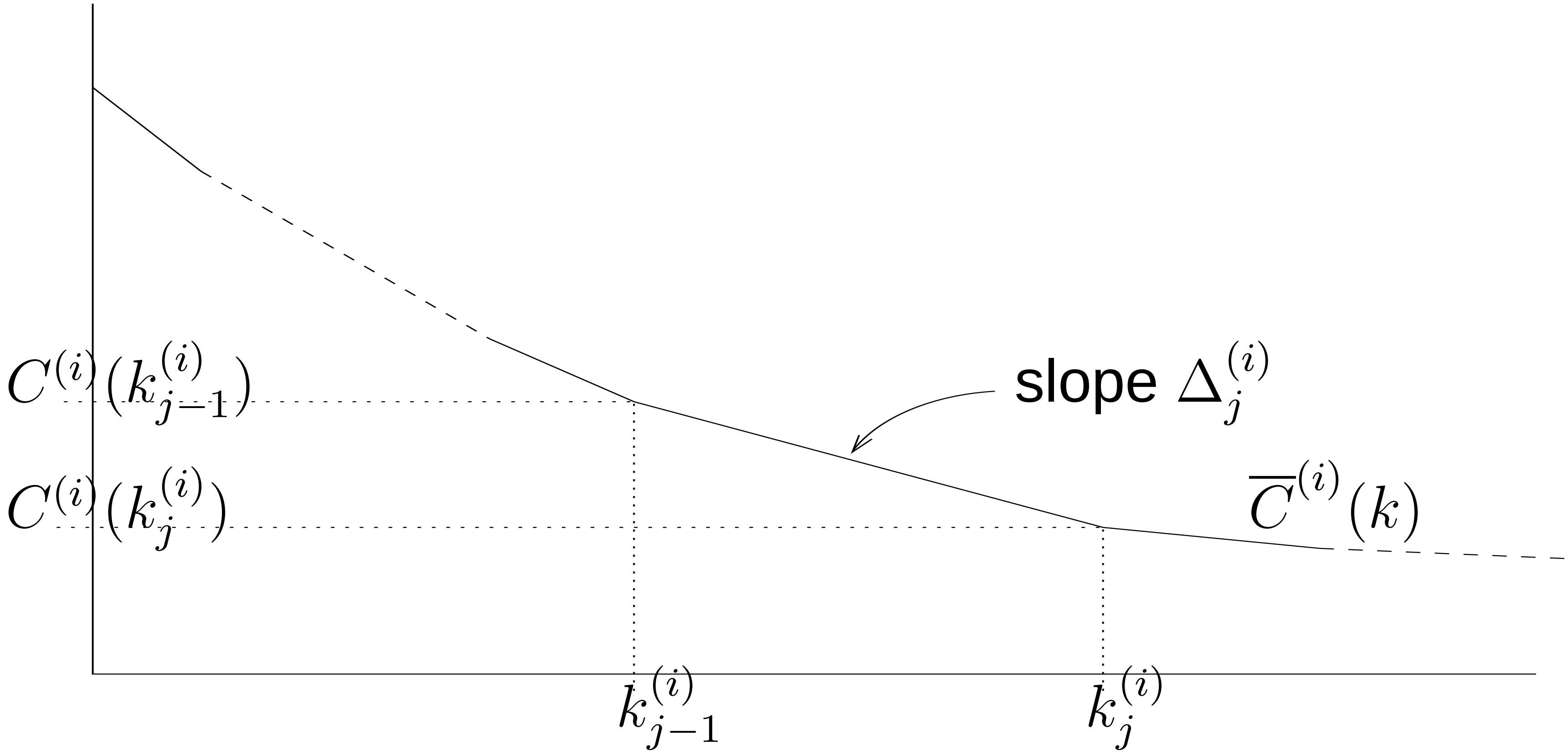
- "Fills-in" the missing values of the call price functions by linear interpolation completes the partial information about the marginal to full information.
- Key observation: the largest convex function passing through given points is the *linearly interpolated* function.
- There exists an optimizing portfolio which consists of options on no more than two strikes per asset, and involves at most $n + 1$ separate options.

Linear interpolation



The interpolated call price function. $\Delta_j^{(i)}$ gives the modulus of the gradient of $\overline{C}^{(i)}$ over $(k_{j-1}^{(i)}, k_j^{(i)})$.

Finite market - Result

- **Preliminaries** For simplicity of exposition assume all slopes $\frac{\partial C^{(i)}(u)}{\partial u}|_{u=k_j^{(i)}}$ are different as i and j vary. Let $I_n = \{1, 2, \cdots, n\}$ where n is the number of assets.
- There is a privileged index $\hat{i} \in I_n$ such that:
- For any model which is consistent with the observed call prices $C^{(i)}(k_j^{(i)})$, the price $B(K)$ for the basket option is bounded above by $\overline{\mathcal{B}}_F(K)$, where
- Case I: $\sum_i w_i k_{J(i)}^{(i)} > K$:

$$\overline{\mathcal{B}}_F(K) = \sum_{i \in I_n \setminus \hat{i}} w_i C^{(i)}\left(k_{\overline{j}(i)}^{(i)}\right) + w_{\hat{i}} \left\{ (1-\theta_{\hat{i}}^*) C^{(\hat{i})}\left(k_{\overline{j}(\hat{i})-1}^{(\hat{i})}\right) + \theta_{\hat{i}}^* C^{(\hat{i})}\left(k_{\overline{j}(\hat{i})}^{(\hat{i})}\right)\right\}$$

- $\theta_{\hat{i}}^*$ is defined as $\theta_{\hat{i}}^* = \frac{\overline{\lambda}_{\hat{i}}^* - \overline{\lambda}_{\hat{i}}^-(\phi^*)}{\overline{\lambda}_{\hat{i}}^+(\phi^*) - \overline{\lambda}_{\hat{i}}^-(\phi^*)} = \frac{(K\overline{\lambda}_{\hat{i}}^*/w_{\hat{i}}) - k_{\overline{j}(\hat{i})-1}^{(i)}}{k_{\overline{j}(\hat{i})}^{(i)} - k_{\overline{j}(\hat{i})-1}^{(i)}}$.

Finite market - Result, Ct'd

- Case II: $\sum_i w_i k^{(i)}_{J(i)} \leq K$:

$$\overline{\mathcal{B}}_F(K) = \sum_i w_i C^{(i)} \left(k^{(i)}_{J(i)} \right)$$

****

- Based on experiments with real data, the second case essentially never arises in practice.
- Moreover, the upper bound is optimal in the sense that we can fi nd co-monotonic models which are consistent with the observed call prices and for which the arbitrage-free price for the basket option is arbitrarily close to $\overline{\mathcal{B}}_F(K)$.
- So where's the beef in Case I?
- All the beef in fleshing out the estimate in the fi rst case is in determining the special index $\hat{i}$ and the indices $j(i), i = 1 \cdots, n$.

How to find which options to choose?

- Possible to show that there is No cash component ψ in the optimal portfolio. So can consider super-replicating portfolios consisting entirely of options with various strikes (some of which may have strike zero).
- The upper bound is available in quasi-closed form, meaning there is a simple algorithm to determine the solution, modulo a slope ordering algorithm: Order all slopes of all call price functions and cycle through.
- To get the intuition as to how to proceed, note that if $\sum \lambda_i = 1$ then

$$\left(\sum_i w_i X_M^{(i)} - K\right)^+ \leq \sum_i w_i \left(X_M^{(i)} - \frac{\lambda_i K}{w_i}\right)^+$$

So that

$$C_{\mathcal{B}}(K) \leq \sum_i w_i C^{(i)}\left(\lambda_i K / w_i\right).$$

The λ_i are arbitrary and so $C_{\mathcal{B}}(K) \leq \inf_{\lambda_i \geq 0, \sum \lambda_i = 1} \sum_i w_i C^{(i)}\left(\lambda_i K / w_i\right).$

Intuition

- We wish to find the infimum of $\sum_i w_i C^{(i)}(\lambda_i K/w_i)$ over choices λ_i satisfying $\lambda_i \geq 0, \sum \lambda_i = 1$. Define the Lagrangian

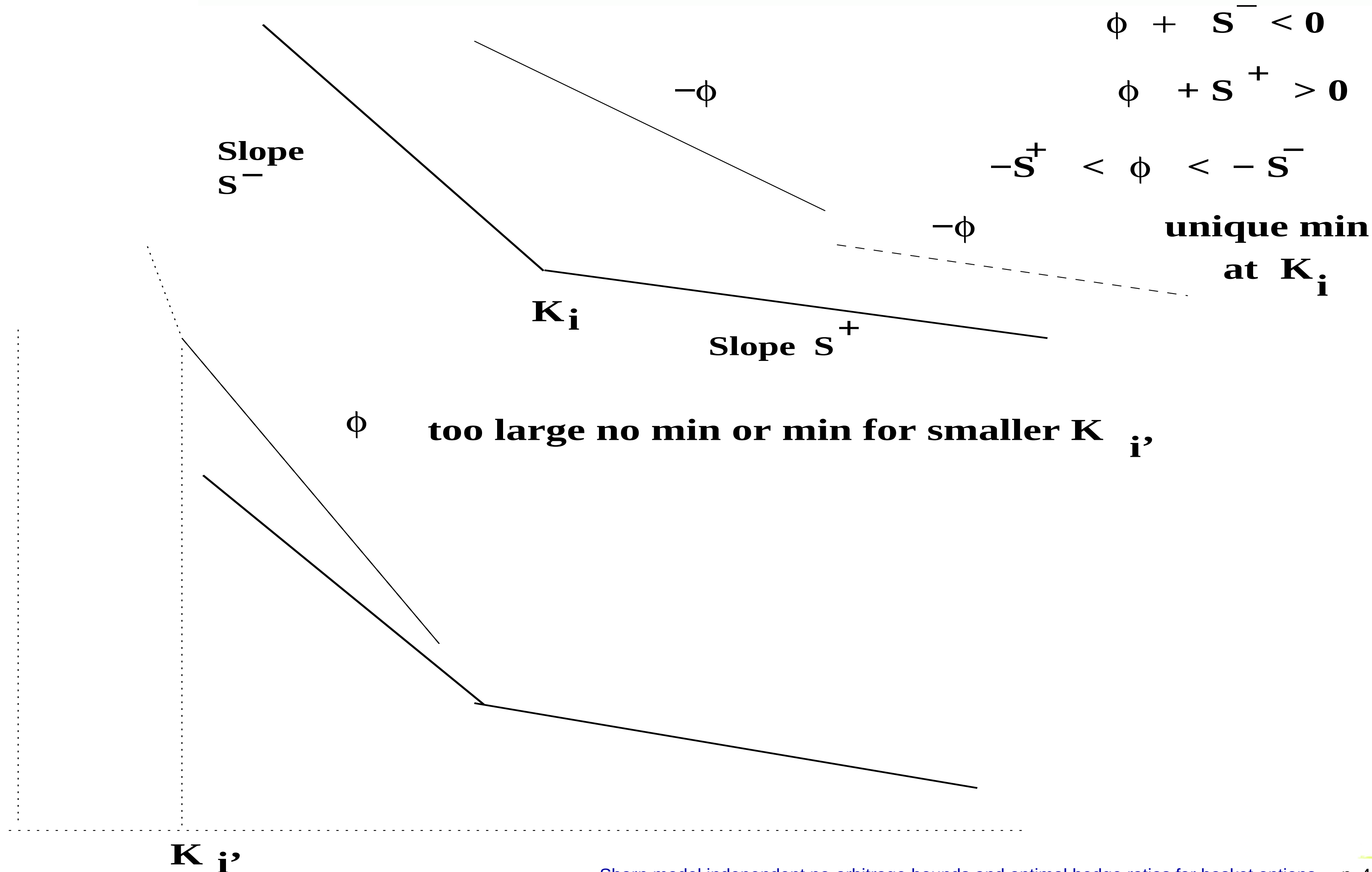
$$L(\lambda, \phi) = \sum_i w_i C^{(i)}(\lambda_i K/w_i) + \phi\left(\sum_i \lambda_i - 1\right).$$

- Objective function is convex but only $C^{0,1}$, because each piecewise linear call price functions $C^{(i)}$, is $C^{0,1}$, ie. $\frac{\partial C^i}{\partial K}$ has a jump at each strike $K_i^j, j = 1, \cdot, n_i$.

- Note that objective functional is separable function of 1-dimensional functions.

- Therefore for each fixed Lagrange Multiplier ϕ, the gradient can point in a cone of different directions. In the terminology of convex analysis we have $\phi/\beta K \in \bar{\partial} C^{(i)}(\lambda_i K/w_i)$, where $\bar{\partial}$ is the *subdifferential* of the function $C^{(i)}$. .

Illustration Min



Algorithm

For each ϕ there is either a unique $\lambda(\phi)$ or an interval $[\lambda^{-}(\phi), \lambda^{+}(\phi)]$.
Essentially:

- $[\lambda(\phi)^{-}, \lambda(\phi)^{+}] \sim [w_i K_i^j / K, w_i K_i^{j+1} / K]$ for some i and j.
- So Algorithm:

  - Order all the slopes of all call price functions. Ie. if 30 assets and 8 non zero strikes , order 240 slopes.

$$\mathcal{S}_1 \leq \mathcal{S}_1 \leq \cdots\cdots \mathcal{S}_{180}$$

  - Now starting with $\phi = \epsilon << 1$ increase ϕ while monitoring the quantity

$$\Lambda(\phi) = \sum \lambda^{+}(\phi)$$

    which starts very large for small ϕ ($\rightleftharpoons$ large K_i^j) and decreases as $\phi \uparrow$.

  - The fi rst time $\Lambda(\phi)$ crosses**1**. STOP! $\mapsto$ Optimal value of $\phi = \phi^*$ has been reached.

Experiment on Real DJX Data

We now illustrate the output on real DJX data.

DJX Strikes	DJX Call Prices	AA	AIG	AXP	BA	C	CAT	DD	DIS	GE
52	47.1	0	0	0	37.5	0	0	0	17.5	25
56	43.1	0	0	0/42.5	37.5	0	0	0	17.5	25
60	39.1	22.5	0	42.5	37.5	0/37.5	0	0	17.5	25
64	35.1	22.5	0	42.5	37.5	37.5	0/60	0	17.5	25
68	31.1	22.5	0	42.5	37.5	37.5	60	0	17.5	25
70	29.1	22.5	0	42.5	37.5	37.5	60	0	17.5	25
72	27.1	22.5	0/60	42.5	37.5	37.5	60	0	17.5	25
76	23.1	22.5	60	42.5	37.5	37.5	60	0	17.5	25
80	19.1	22.5	60	42.5	37.5	37.5	60	0/37.5	17.5	25
84	15.2	22.5	60	42.5	37.5	37.5	60	37.5	20	25
88	11.3	22.5	60	42.5	37.5	40	65	37.5	20	27.5
90	9.4	25	60	45	37.5/40	40	65	37.5	20	27.5
92	7.5	25	65	45	40	42.5	70	37.5/40	20	27.5
94	5.8	25	65	47.5	40	42.5	70	40	22.5	27.5
95	4.95	27.5	65	47.5	40	42.5	70	40	22.5	27.5
96	4.15	27.5	65	47.5	40	42.5	70	40	22.5	30
97	3.35	27.5	70	47.5	42.5	42.5	70	40	22.5	30
98	2.725	27.5	70	47.5	42.5	45	70	40	22.5	30
99	2.125	27.5	70	50	42.5	45	75	40/42.5	22.5	30
100	1.6	30	70	50	42.5	45	75	42.5	22.5	30
102	0.775	30	70	50	45	47.5	75	42.5	22.5	30
103	0.5	30	75	50	45	47.5	75/80	42.5	25	30/32.5
104	0.325	32.5	75	50	45	47.5	80	42.5	25	32.5
105	0.15	32.5	75	50	45	47.5	80	42.5	25	32.5
106	0.15	32.5	75	50	45	47.5	80	45	25	32.5
107	0.15	32.5	75	50	45	47.5	80	45	25	32.5

TABLE 4. The super-replicating portfolio. For each strike on the DJX, and for each component of the basket, we list the relevant strike to hold in the cheapest super-replicacting portfolio. A strike of 0 corresponds to holding the asset. For space reasons we only give the strikes for the first 10 components. In most cases there is a single strike listed. In other cases the optimal portfolio involves a combination of two strikes. Note that the optimal strike to hold on each component asset

Experiment on Real DJX Data: How good is the Upper Bound?

Spot 99.07

DJX Strikes	DJX Prices	UB Unclean Data	UB Clean Data	BS Price $\rho = 0$	BS Price $\rho = .5$	BS Price $\rho = .75$	BS Price $\rho = .9$	BS Price $\rho = .99$
52	47.10	47.09	47.05	47.14	47.14	47.15	47.10	47.18
56	43.10	43.10	43.09	43.16	43.18	43.17	43.15	43.17
60	39.10	39.11	39.10	39.16	39.18	39.13	39.12	39.14
64	35.10	35.11	34.30	35.16	35.16	35.16	35.20	35.17
68	31.10	31.12	30.83	31.17	31.17	31.22	31.17	31.16
70	29.10	29.13	29.12	29.18	29.19	29.18	29.17	29.11
72	27.10	27.14	27.14	27.19	27.22	27.18	27.13	27.18
76	23.10	23.15	22.38	23.18	23.16	23.18	23.15	23.19
80	19.10	19.18	19.18	19.20	19.18	19.15	19.19	19.22
84	15.20	15.24	14.95	15.21	15.24	15.23	15.18	15.23
88	11.30	11.42	11.42	11.20	11.26	11.25	11.25	11.36
90	9.40	9.61	9.61	9.21	9.28	9.35	9.41	9.44
92	7.50	7.90	7.90	7.21	7.34	7.53	7.67	7.73
94	5.80	6.32	6.32	5.22	5.58	5.83	6.01	6.08
95	4.95	5.57	5.57	4.22	4.79	5.06	5.26	5.34
96	4.15	4.85	4.85	3.22	4.01	4.35	4.54	4.66
97	3.35	4.19	4.19	2.24	3.28	3.69	3.92	4.01
98	2.73	3.58	3.58	1.35	2.70	3.12	3.34	3.44
99	2.13	3.02	3.02	0.67	2.16	2.58	2.75	2.96
100	1.60	2.53	2.53	0.25	1.69	2.10	2.33	2.43
102	0.78	1.73	1.73	0.01	0.99	1.37	1.55	1.71
103	0.50	1.42	1.42	0.00	0.71	1.05	1.26	1.36
104	0.33	1.16	1.16	0.00	0.52	0.82	1.02	1.13
105	0.15	0.95	0.95	0.00	0.36	0.63	0.79	0.89
106	0.15	0.75	0.75	0.00	0.25	0.48	0.60	0.70

Lower Bound and Copulas

Prescribed Marginals

- (2002) Rapuch and Roncalli had already determined the optimal lower bound in this setting using copulas and a result of Muller and Scarsini on the super-modular ordering. They do not provide a corresponding optimal sub-replicating strategy
- (2004) Hobson, Laurence and Wang established an optimal sub-replicating strategy for a basket on two assets (with > 0 coeffi cients) subject to the constraint of prescribed marginals, or equivalently, the constraint that prices of call options of all possible strikes are known.
- This new approach yields a new proof of Rapuch and Roncalli's result that does not require the use of copulas and enhances the result by providing such a strategy involving a new class of portfolios called sheeptrack portfolios.

Copulas

One might wish to maximize or minimize the basket option price under the constraint that the marginals are all log-normal. The solution to this problem (or for arbitrary prescribed marginals) is easily expressed in terms of copulas. Reminder: A copula is joint distribution

$$C(x_1, x_2, \cdots, x_n) = P\left(X_1 \leq x_1, \cdots, X_n \leq x_n\right),$$

with uniform marginals and obeying certain natural structure conditions.

Sklar

Sklar's theorem Any joint distribution with continuous marginal distribution functions $F_i, i = 1, \cdots, n$, can be expressed as

$$C(F_1^{-1}, F_2^{-1}, \cdots, F_n^{-1})$$

where $C(x_1, \cdots, x_n)$ is a copula and where F^{-1} is the generalized inverse of F, ie $F^{-1}(t) = \inf\{x \in \Re | F(x) \geq t\}$

Copula2

Minimization problem with fixed marginals can be reduced to problem of finding *optimal copula*.

This problem was solved **in the case** $n = 2$ by Rapuch and Roncalli (Crédit Lyonnais web site) based on earlier results of Muller and Scarsini and Chen.

The Frechet Copulas $C^-(u_1, u_2)$ and $C^+(u_1, u_2)$ given by

$$C^- = \max(u_1 + u_2 - 1, 0)$$
$$C^+ = \min(u_1, u_2)$$

Let $C^-(\mathcal{M}_1, \mathcal{M}_2)$ and $C^+(\mathcal{M}_1, \mathcal{M}_2)$ be the corresponding call option prices. Then for a generic basket option on two assets with the same marginals we have

$$C^-(\mathcal{M}_1, \mathcal{M}_2) \leq C(\mathcal{M}_1, \mathcal{M}_2) \leq C^+(\mathcal{M}_1, \mathcal{M}_2)$$

Frechet ct'd

Their result is deduced from the following more general result of Muller and Scarsini.

Theorem Let F_1 and F_2 be the probability distribution functions of X_1 and X_2. Let $E_C[f(X_1, X_2)]$ denote the expectation of the function $f(X_1, X_2)$ when the copula of the random vector (X_1, X_2) is C. If $C_1 \prec C_2$ (concordance order, same as ptwise in 2-D) then •

$$E_{C_1}[f(X_1, X_2)] \leq E_{C_2}[f(X_1, X_2)]$$

for all supermodular functions f such that the expectations exist.
Supermodular is a natural generalization of non-negative mixed second derivative $\frac{\partial^2 f}{\partial x_1 \partial x_2} \geq 0$ for a C^2 function . Supermodular $\Leftrightarrow$

$$\begin{aligned} \Delta^{(2)} f = f(x_1 + \epsilon_1, x_2 + \epsilon_2) - f(x_1 + \epsilon_1, x_2) \\ - f(x_1, x_2 + \epsilon_2) + f(x_1, x_2) \geq 0 \end{aligned}$$

Rapuch-Roncalli, Ct'd

Basket Option Payoff is supermodular.

Intuitive Proof:

Use the fact that

$$\frac{\partial^2 (w_1 S_1 + w_2 S_2 - K)^+}{\partial S_1 \partial S_2} = w_1 w_2 \delta(\{w_1 S_1 + w_2 S_2 = K\})$$

is a positive distribution. So can apply theorem.

• Therefore optimal upper bound for basket option with prescribed marginals is associated with the Fréchet upper bound.

• Also optimal lower bound for basket option with prescribed marginals is associated with the Fréchet lower bound.

Rapuch-Roncalli's Copula Approach

When the distribution functions F_X and F_Y are continuous, we have

$$\mathcal{C} = \mathcal{C}^- \Leftrightarrow Y = F_Y^{-1}(1 - F_X(X), \quad \text{anti-monotonic}$$
$$\mathcal{C} = \mathcal{C}^+ \Leftrightarrow Y = F_Y^{-1}(F_X(X)). \quad \text{co-monotonic}$$

For the lower bound note this means that

$$C_{\mathcal{B}}^- = \int_{\mathbb{R}^+} [x + F_Y^{-1}(1 - F_X(x)) - K]^+ dF_X(x). \qquad (2)$$

Mathematical Formulation - LB

Constrained optimization problem

$$\inf_{\mu} \int_{\mathbb{R}^2_+} (x+y-K)^+ \mu(dxdy)$$

subject to the constraints on the marginal distributions

$$\int_{\mathbb{R}^+} (x-k_1)^+ \mu_X(dx) = C_X(k_1),$$
$$\int_{\mathbb{R}^+} (y-k_2)^+ d\mu_Y(dy) = C_Y(k_2),$$
$$\int_{\mathbb{R}^2_+} \mu(dx,dy) = 1.$$

μ_X and μ_Y are the marginal distributions.

Mathematical Formulation - dual

$$\sup_{\nu_1,\nu_2,\lambda} \int_{\mathbb{R}^+} C_X(k_1)\nu_1(k_1) + \int_{\mathbb{R}^+} C_Y(k_2)\nu_2(k_2) + \lambda$$

subject to the constraints on the measures $\nu_i, i = 1, 2$

$$\left[(x+y-K)^+ - \int (x-k_1)^+\nu_1(dk_1) - \int (y-k_2)^+\nu_1(dk_1) - \lambda\right] \geq 0$$

for all $\mu \in \mathbb{M}_+$

where ν_i's range over all finite signed measure and $\lambda \in \mathbb{R}$

Weak Duality

For any feasible primal variable μ and any dual variables ν_1, ν_2 and λ the corresponding primal value is always greater than or equal to the corresponding dual value. Namely,

$$\int_{\mathbb{R}^2_+} (x+y-K)^+ \mu(dxdy)$$

$$\geq \int_{\mathbb{R}^+} C_X(k_1)\nu_1(k_1) + \int_{\mathbb{R}^+} C_Y(k_2)\nu_2(k_2) + \lambda$$

Hence, dual variables ν_1, ν_2 and λ represent a sub-replicating portfolio for basket option consists of (buying or selling) individual options and cash.

Optimality

Find feasible primal variable $\bar{\mu}$ and dual variables $\bar{\nu}_1$, $\bar{\nu}_2$ and $\bar{\lambda}$ with the same value, i.e.,

$$\int_{\mathbb{R}^2_+} (x+y-K)^+ \bar{\mu}(dxdy)$$
$$= \int_{\mathbb{R}^+} C_X(k_1)\bar{\nu}_1(k_1) + \int_{\mathbb{R}^+} C_Y(k_2)\bar{\nu}_2(k_2) + \bar{\lambda}$$

Optimal Primal Variable

Let F_X and F_Y be the marginal cdfs of X and Y respectively. The optimal primal variable $\bar{\mu}$ is characterized as

- $(X, Y) \sim (F_X^{-1}(U), F_X^{-1}(1 - U))$ where $U \sim$ Uniform$(0, 1)$
- Equivalently, the joint cdf F of X and Y is

$$F(x, y) = (F_X(x) + F_Y(y) - 1)^+$$

- That is, X and Y are countermonotonic.
- Feasibility of $\bar{\mu}$ is immediate.

Auxiliary Function

- C'^{+}: the right derivative of call price function
  C'^{-} the left derivative.
- $\phi(x) := {C'_X}^{+}(x) + {C'_Y}^{-}(K - x) + 1$
  ϕ is only defined on $[0, K]$ and right-continuous.

Auxiliary Function

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  C'^{-} the left derivative.
- $\phi(x) := C_X'^{+}(x) + C_Y'^{-}(K-x) + 1$
  ϕ is only defined on $[0, K]$ and right-continuous.
- ϕ is of finite variation and hence has at most countably many "crossings" over zero.
- $A := \{x : \phi(x) > 0\}$ is a countable union of disjoint intervals, $A = \cup_j A_j$.

Assumption - Finite Crossings

- **Assumption**: A is a union of a finite number n of intervals.
- Let the intervals $(A_j)_{1\leq j\leq n}$ be placed in their natural order, and that the boundary of A_j is given by the points $\{K_1^{2j-1}, K_1^{2j}\}$. Define $K_2^j = K - K_1^{2n-j+1}$

A Remark on the set A

- In the case that ϕ is C^0 in $[0, K]$, the determination of the strikes K_1^i's reduces to find the zeros $\{x : \phi(x) = 0\}$ of ϕ.

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- A lemma
  Let $I \subset \mathbb{R}$ be a bounded open interval, $h \in C^1(\bar{I})$ and p be a regular value of h, i.e., $h'(x) \neq 0$ for every $x \in h^{-1}(p)$. Then $h^{-1}(p)$ is finite.

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- A lemma
  Let $I \subset \mathbb{R}$ be a bounded open interval, $h \in C^1(\bar{I})$ and p be a regular value of h, i.e., $h'(x) \neq 0$ for every $x \in h^{-1}(p)$. Then $h^{-1}(p)$ is finite.
- In our case, x is such that $C'_X(x) = -C'_Y(K - x) - 1$, then $C''_X(x) \neq C''_Y(K - x)$,
  i.e., the densities of X and Y are different at such points.

Optimal dual: Sheep-Track Portfolio

$$\bar{\nu}_1(dk_1) = \delta_0(k_1)dk_1 + \sum_{i=1}^{2n}(-1)^i \delta_{K_1^i}(k_1)dk_1,$$

$$\bar{\nu}_2(dk_2) = \delta_0(k_2)dk_2 + \sum_{i=1}^{2n}(-1)^i \delta_{K_2^i}(k_2)dk_2,$$

$$\bar{\lambda} = \sum_{i=1}^{n}(K_1^{2i} - K_1^{2i-1}) - K = \sum_{i=1}^{n}(K_2^{2i} - K_2^{2i-1}) - K.$$

We call such portfolio "STP", short for "sheep-track portfolios", since the graph of such a portfolio is reminiscent of such tracks on British hillsides.

Figure for STP

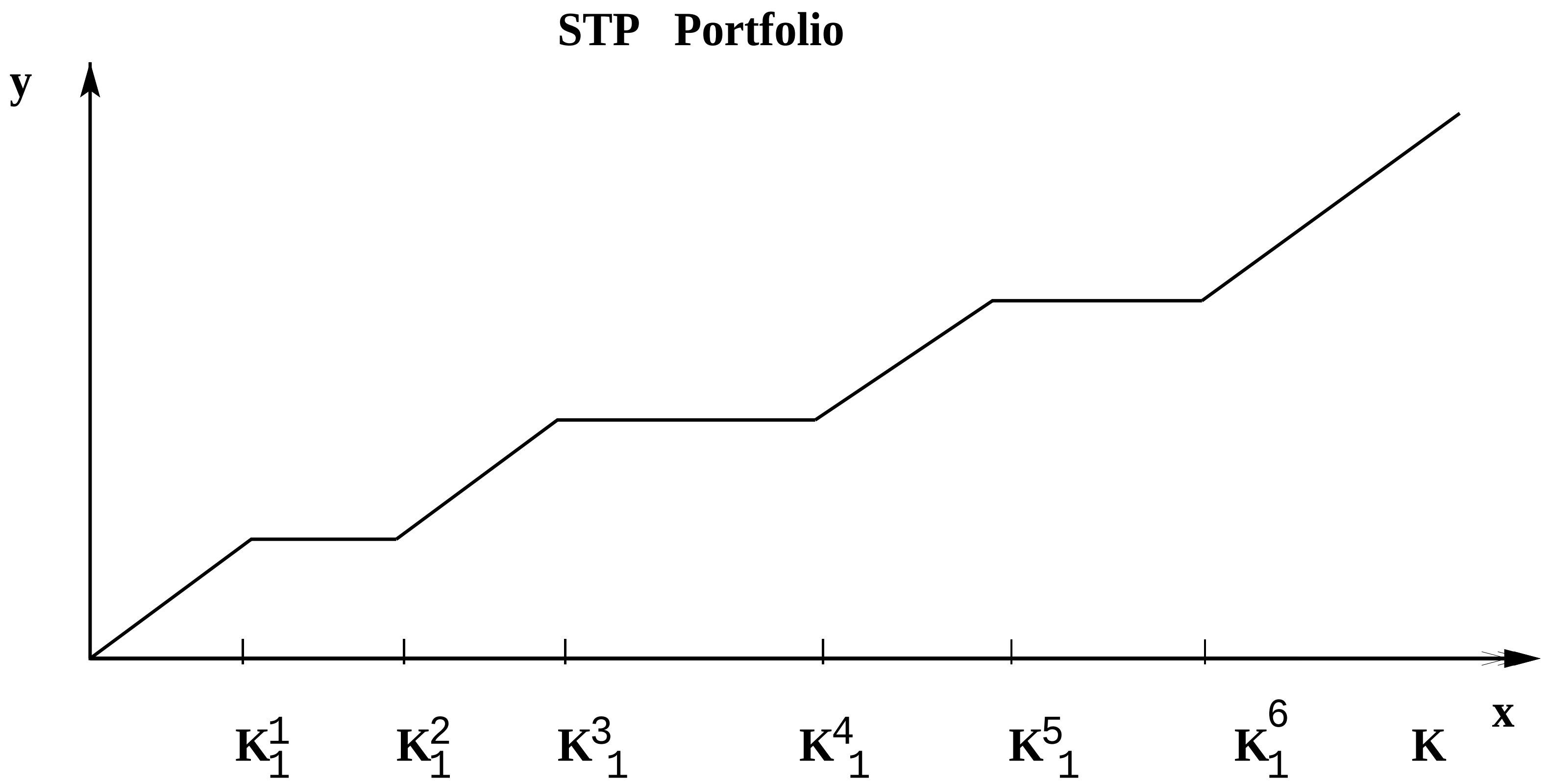


Figure 1: The figure illustrates an STP portfolio which, starting at zero, is piecewise linear with alternating slopes one and zero. The points of transition define K_1^i for odd and even i.

Introdution of STP

- Recall

$$\bar{\nu}_1(dk_1) = \delta_0(k_1)dk_1 + \sum_{i=1}^{2n}(-1)^i \delta_{K_1^i}(k_1)dk_1$$

- Integrate this against $x - k_1$, ie $\int (x-k_1)^+ dk_1$ and get

$$f_1(z) = z + \sum_{a=1}^{n}\{(z-K_1^{2a})^+ - (z-K_1^{2a-1})^+\}$$

- So defi ne STP payoff functions:

$$f_i(z) = z + \sum_{a=1}^{n}\{(z-K_i^{2a})^+ - (z-K_i^{2a-1})^+\}.$$

- In this notation: Hedging strategy by STP: $f_1(x) + f_2(y) + \bar{\lambda} = f_1(x) - f_1(K-y)$

Feasibility of STP

- Feasibility of STP is then equivalent to establish

$$f_1(x) - f_1(K - y) \leq (x + y - K)^+$$

which can be seen by using the inequality

$$f_1(z + u) - f_1(z) \leq u^+$$

since f_1 is piecewise linear with alternately slopes 0 and 1

Construction of optimal bivariate processes

Notice that equality is achieved in the feasibility inequality

$$f_1(x) - f_1(K - y) \leq (x + y - K)^+$$

provided that one of the following 3 cases holds:

- $x = K - y$
- $x > K - y$ and $f_1(x) = f_1(K - y) + x + y - K$
  i.e., $(K - y, x) \subset \quad (K_1^{2j}, K_1^{2j+1})$
- $x < K - y$ and $f_1(x) = f_1(K - y)$
  i.e., $(x, K - y) \subset \quad (K_1^{2j-1}, K_1^{2j})$

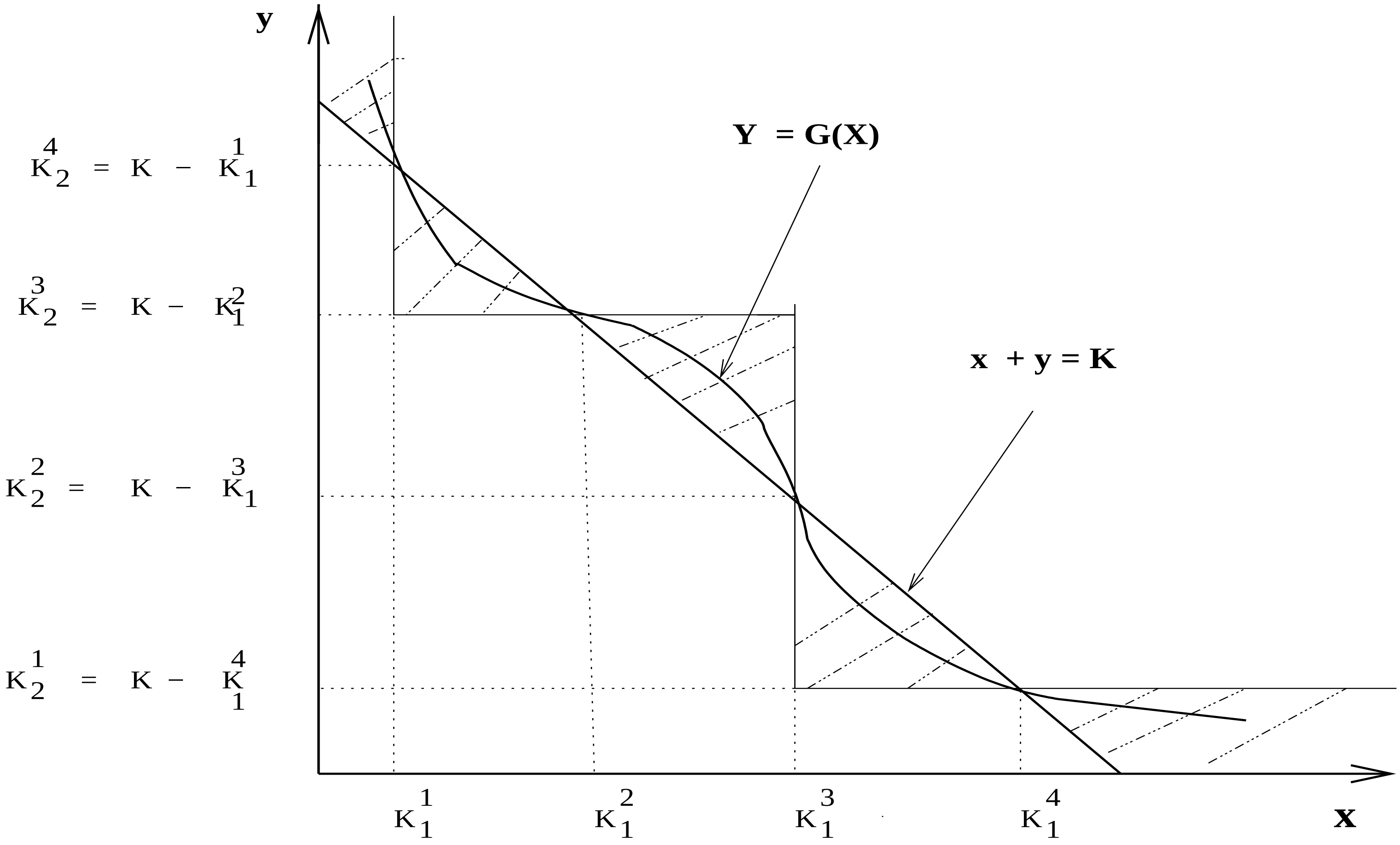
Illustrative Case

- X and Y are continuous random variables with strictly positive densities on $[0, K]$.
- $Y = G(X) := F_Y^{-1}(1 - F_X(X))$
  (X and Y are countermonotonic.)
- Note that

$$y = G(x) < K - x \iff F_X(x) + F_Y(K - x) - 1 > 0$$
$$\iff \phi(x) > 0$$

  Hence the points where G crosses the line $x + y = K$ are exactly the boundary points of the set A
- For general case, take $X = F_X^{-1}(U)$ and $Y = F_Y^{-1}(1 - U)$ where $U \sim$ Uniform$[0, 1]$

Countermonotonicity Distribution



Optimality

Primal Value = Dual Value and

$$
\begin{aligned}
& \langle C_X, \bar{\nu}_1 \rangle + \langle C_Y, \bar{\nu}_2 \rangle + \bar{\lambda} \\
\leq \quad & \sup_{\nu_1, \nu_2, \lambda} \langle C_X, \nu_1 \rangle + \langle C_Y, \nu_2 \rangle + \lambda \\
\leq \quad & \inf_{\mu} \quad \int_{\mathbb{R}^2_+} (x + y - K)^+ \mu(dx, dy) \\
\leq \quad & \int_{\mathbb{R}^2_+} (x + y - K)^+ \bar{\mu}(dx, dy)
\end{aligned}
$$

Last and first are equal, so equality throughout!

Finiteness Assumption on A

Two key lemmas

- Let E be a measurable subset of an bounded open interval I. Then its characteristic function χ_E is of bounded variation if and only if E consists of finitely many intervals.
- Let f be a function of bounded variation defined on an open interval I. Denote by E_t the level set $\{x \in I : f(x) > t\}$ for f. Then, for almost every $t \in \mathbb{R}$, the characteristic function χ_{E_t} of E_t is of bounded variation.

Beyond Finite Crossings

- $\Phi_t := \{x \in (0, K) : \phi(x) > t\}$
- χ_{Φ_t} is of bounded variation for almost every t
- Suppose that χ_A (recall that $\Phi_0 = A$) is not of bounded variation
- Given any $\epsilon > 0$, there exists a positive $t_\epsilon < \epsilon/K$ with $\chi_{\Phi_{t_\epsilon}}$ of bounded variation (hence Φ_{t_ϵ} is a finite union of intervals)
- Form a portfolio $(\nu_1^\epsilon, \nu_2^\epsilon, \lambda^\epsilon)$ by using $\Phi_\epsilon := \Phi_{t_\epsilon}$

Conclusions

- The optimal lower bound and optimal sub-replicating strategys is unknown in the discrete strike case.
- Optimal lower bound is open even in the full marginals prescribed (cts strike case) when $n \geq 3$.
- We have treated only a 1 period model. Multi-period models are open.
- Corresponding problems in the American Basket option case are open
- Upper bound could be made closer if add additional constraints. Which ones? Correlation prescribed? Entropy constraints?