A New Tool For Correlation Risk Management: The Market Implied Comonotonicity Gap and applications to spread options

Global Derivatives, Paris, 2008

Peter Michael Laurence

Department of Mathematics and Facoltà di Statistica

Universitá di Roma, “ La Sapienza ”

http://www.math.nyu.edu/∼laurence

Main aims of this contribution

- Part I: Optimal arbitrage free static hedging strategies for basket options and new measure of lack of comonotonic or antimonotonic dependence in correlated assets: Market Implied Comonotonicity Gap (Joint work with Tai-Ho Wang, building on earlier work by Hobson, Laurence and Wang).
- Part II: Extension to generalized spread options.

Introducing the GAP

We introduce a quantity called "the Gap", or more precisely "Market Implied Comonotonicity Gap" (for short: MICG), with the property that:

- Gap can be monitored over time and used as a tool in a static (or semi-static) dispersion trading strategy.
- When gap is small ("High correlation") compared to it's historical values: basket (consider case of index option first, later in talk spread) is overpriced.

$\Rightarrow$ Sell basket option, buy options on the components.

- When gap is big compared to it's historical values ("Low correlation"): basket is cheap, undervalued.

$\Rightarrow$ Buy an option on the basket, sell options on the components

Strategy

- This is not (in general) an arbitrage strategy:
- It carries some risk, but downside risk is quite small.
- Downside risk especially small when long the Gap, because of super-replication.
- It is important to find the right time to enter into a "Gap Trade".

Portfolio Implied Correlation

We will describe MICG and contrast with another well known dispersion trading signal, so called "implied correlation.

- Implied correlation is the number ρ such that when ρ_{ij} are replaced by ρ gives same implied variance of index:

  $$\sigma_I^2 = \sum_{i=1}^{n} \sigma_i^2 + \sum_{i \neq j} \sigma_i \sigma_j \rho_{ij} = \sum_{i=1}^{n} \sigma_i^2 + \rho \sum_{i \neq j} \sigma_i \sigma_j$$

  Hence,

  $$\rho = \frac{\sigma_I^2 - \sum\limits_{i=1}^{n} \sigma_i^2}{\sum\limits_{i \neq j} \sigma_i \sigma_j}$$

Instantaneous Implied Correlation for a basket

For an option on an index $I_t = \sum w_i S_{it}$ composed of lognormal assets we have

$$dI_t = \sum w_i dS_{it} = \sum w_i S_i \sigma_i dW_{it}$$

where the W_{it} are correlated Brownian motions $< dW_{it}, dW_{jt} >= \rho_{ij} dt$. The instantaneous variance of the basket is

$$\sum_{i,j} \sigma_i \sigma_j \rho_{ij} S_i S_j dt$$

$$\underbrace{(\sigma_I^{\text{instantaneous}})^2}_{\text{implied}} I_t^2 = \sum_{i,j} \underbrace{\sigma_i}_{\text{implied}} \underbrace{\sigma_j}_{\text{implied}} \rho_{ij} w_i w_j S_{it} S_{jt}$$

Replace ρ_{ij}, for $i \neq j$ by ρ and get

$$\boxed{\underbrace{\rho^{\text{instantaneous}}}_{\text{implied correlation}} = \frac{(\sigma_I^{\text{instantaneous}})^2 I^2 - \sum w_i^2 \sigma_i^2 S_i^2}{\sum_{i,j,i\neq j} \underbrace{\sigma_i}_{\text{implied}} \underbrace{\sigma_j}_{\text{implied}} w_i w_j S_i S_j}}$$

Implied correlation 2

- But

$$\sigma_I = \sigma_I(K^{bask}),$$

so which strikes $K_i, i = 1, \cdots, n$ should we use to select $\sigma_i = \sigma_i(K_i), i = 1, \cdots, n$ in the above formula?
Wide spread practice:

K^{bask} ATM, then choose K_i ATM

But what if K^{bask} is out of or in the money? Or even for ATM in what sense is choice of ATM K_i optimal?

- In contrast MICG gives means of selecting optimal strikes.

A new measure of correlation

- Plan: We will recall the definition of comonotonicity and will illustrate the difference between perfect positive correlation and comonotonicity.
- We introduce as a measure of lack of comonotonicity of components in a basket product:

$$\boxed{\text{Gap} = \mathcal{C} - \mathcal{M}}$$

  - $\mathcal{C}$: the market implied comonotonic price
  - $\mathcal{M}$: true market price

Comonotonicity

Recall the definition of comonotonicity:

A random vector $(X_1, X_2, \cdots X_n)$ is said to be comonotonic if there exists a uniformly distributed random variable U such that

$$U \sim \text{Uniform}(0,1)$$

$$(X_1, X_2, \cdots, X_n) \overset{d}{=} \left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\right),$$

where $F_{X_i}(x)$ is the distribution function of X_i.

Perfect positive correlation $\neq$ comonotonicity

Difference between perfect positive correlation and comononotonicity. Tchen, Dhaene-Denuit's theorem, concerning the relation between linear correlation and comonotonicity:

Theorem 1 *If (X_1, X_2) is a random vector with given margins F_{X_1}, F_{X_2} and let ρ be the Pearson (i.e., linear, standard) correlation coefficient, then we have*

$$\rho(F_{X_1}^{-1}(U), F_{X_2}^{-1}(1-U)) \leq \rho(X_1, Y_1) \leq \rho(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U)),$$

where U is a uniformly distributed random variable.

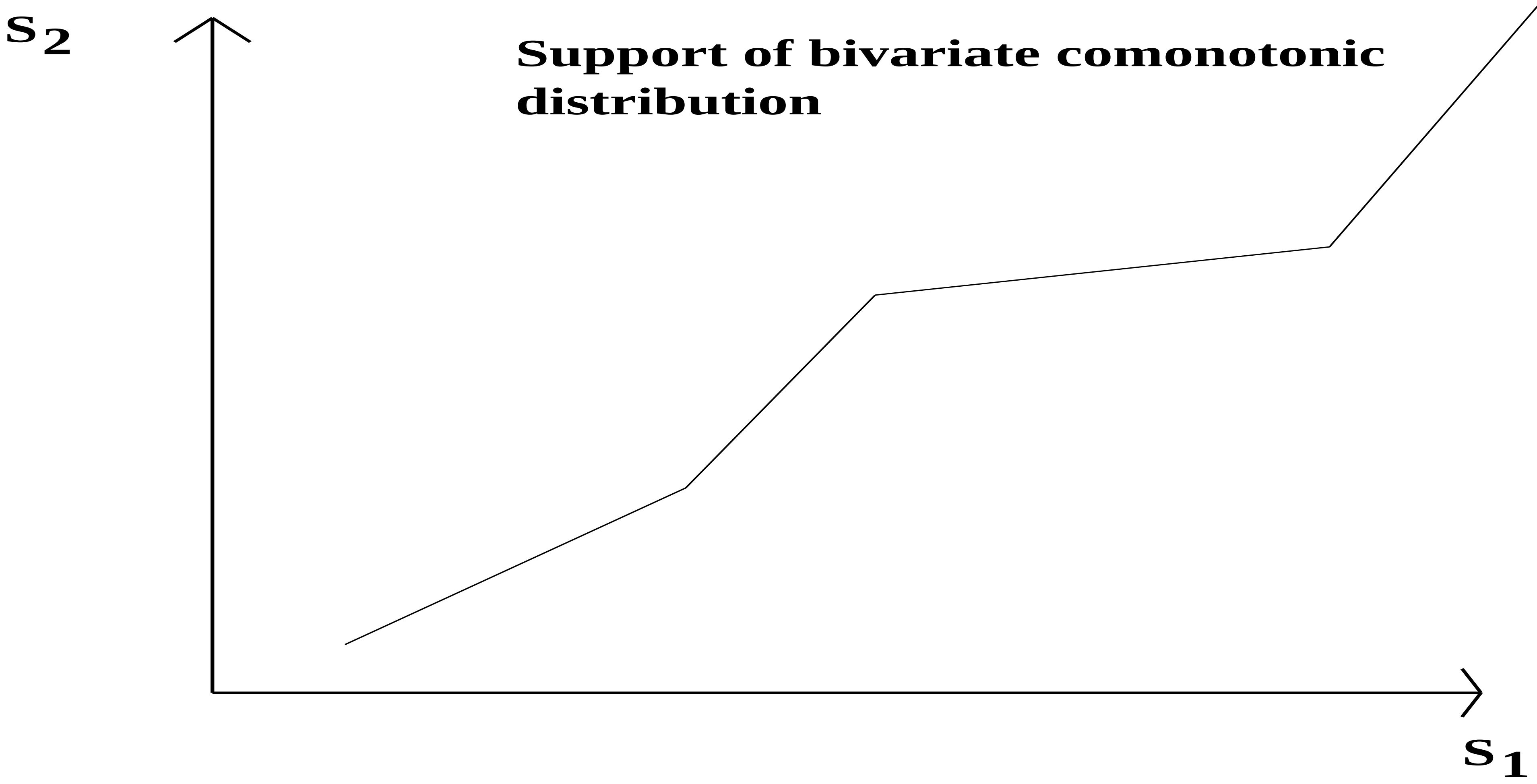
In words:

- Largest value of the correlation for a random vector (X_1, X_2) with given marginals is attained for comonotonic random variables, but is generally not equal to 1 unless they have a linear dependence with positive slope ($X_2 = aX_1 + b, a > 0$).
- Minimal value of the correlation for a random vector $(X_{,}X_2)$ with given marginals is attained for antimonotonic random variables, but is generally not equal to -1.

Does the market offer a comononotonic Index?

- The answer of course is no ! No market traded index is perfectly comonotonic, ie. driven by only *one* market factor.
- But, surprisingly perhaps, we may synthetically create an index option that behaves “ as if” the underlying assets were comonotonic.
- This synthetic comononotonic index option can be created using traded options on the individual components of the index, with judiciously chosen strikes.

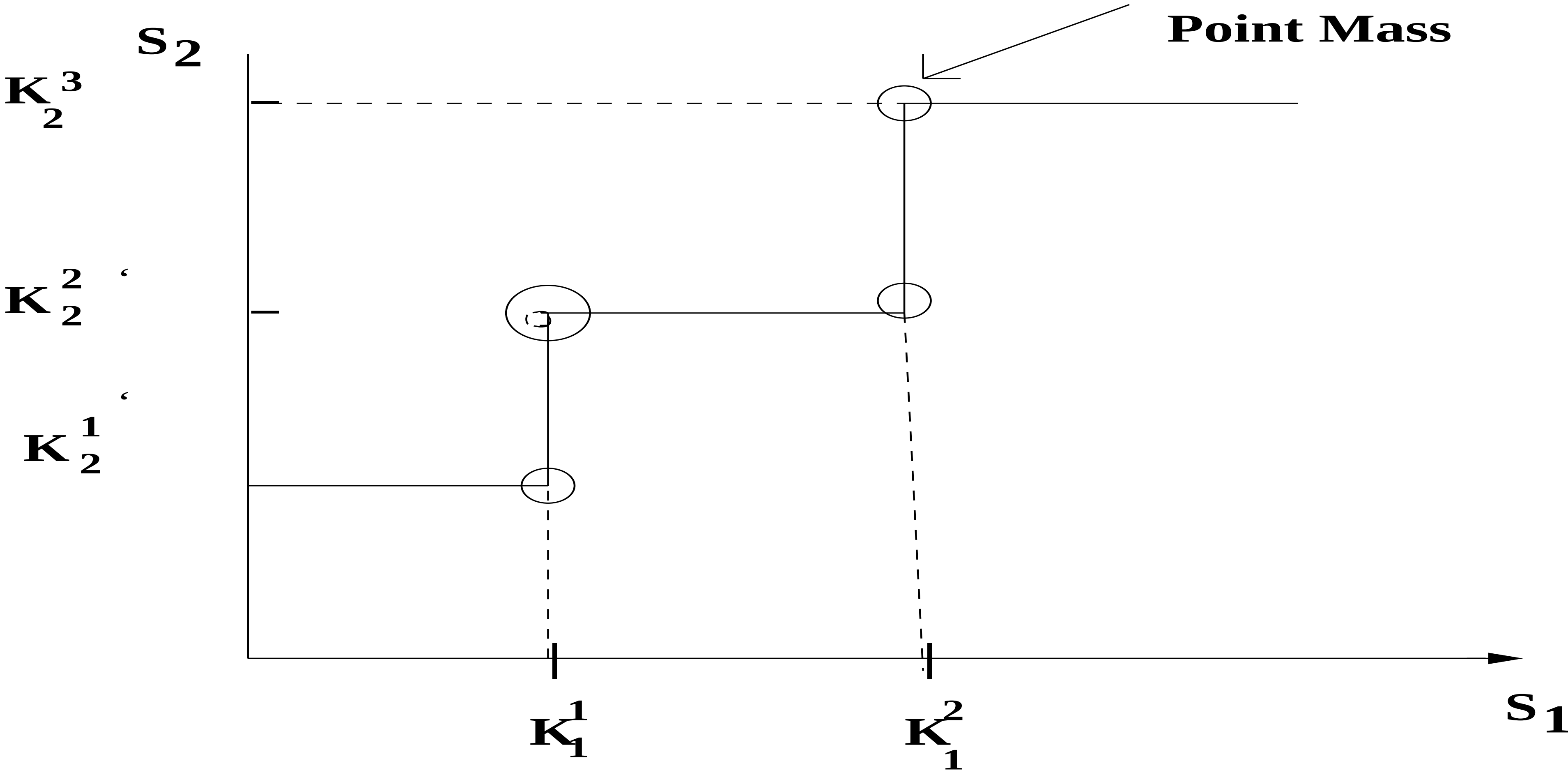
Continuous comonotonic



Support of bivariate comonotonic distribution

S_2 is non−decreasing function of S_1

Comonotonic Distribution: purely atomic, with jumps



A comonotonic distribution with jumps

How to determine $\mathcal{C}$?

So, given a basket options with payoff

$$\left(\sum w_i S_i - K\right)^+$$

how do we determine the comonotonic price?

- ANSWER: If we knew with certainty the marginals F_{S_i} of the individual assets S_i in the basket, the the procedure would be:
- First determine the joint probability distribution for the stocks in the basket via

$$\begin{aligned} &P\left(S_1 \leq x_1, S_2 \leq x_2, \cdots, S_n \leq x_n\right) \\ &= C^U_{\text{Fréchet}}\left(F_{S_1}(x_1), F_{S_2}(x_2), \ldots, F_{S_n}(x_n)\right) \end{aligned}$$

where

$$\boxed{C^U_{\text{Fréchet}}(y_1, y_2, \cdots, y_n) = \min\left(y_1, y_2, \ldots, y_n\right) \quad \text{upper Fréchet bound}}$$

The Gap II

- Second: Determine the density of joint prob. distribution of the basket via

$$
\begin{aligned}
& p(x_1, x_2, \cdots, x_n) \\
& = \frac{\partial^n}{\partial x_1 \partial x_2 \cdots \partial x_n} \left[P\left(S_1 \leq x_1, S_2 \leq x_2, \cdots, S_n \leq x_n \right) \right]
\end{aligned}
$$

- Third:

$$BasketPrice = \int_{\Re_n^+} \left(\sum_{i=1}^{n} S_i - K \right)^+ p(S_1, S_2, \ldots, S_n) dS_1 \ldots dS_n$$

Where do marginals come from?

- Recall Breeden-Litzenberger theorem (Journal of Finance, 1978):

  Theorem 2 *Let $C(S,t,K,T)$ be call prices corresponding at time t and given that the spot price is at S, for a call option struck at K and expiring at T, assuming a continuum of strikes is traded.*
  Then

  $$\frac{\partial^2}{\partial K^2}C(S,t,K,T) = e^{-r(T-t)}p(S,t,K,T) \quad \textit{where } p \textit{ is the transition probability}$$

  $$\Rightarrow \textit{marginal distribution function of } S \textit{ i.e. } F_S(s) \textit{ is therefore known}$$

- In reality, the market provides us only with a finite number of strikes for each expiry and for each stock $S = S_i, i = 1, \cdots, n$. So how do we fill in Call price functions for each asset for all strikes? Answer related (but only very partially explained) by work on distribution free bounds for one asset, of which give a reminder (three slides later).

A typical Component Option, Procter& Gamble

May, 2004 | July, 2004 | October, 2004 | January, 2005 | January, 2006

PROCTER & GAMBLE CO 105.97 ▼ -0.15 -0.1414% 105.91 106.37 2,727,800

Calls							Strike	Puts						
Symbol	Last	Chg	Bid	Ask	Vol	Int	Price	Symbol	Last	Chg	Bid	Ask	Vol	Int
PG EM	41.50	0.00	40.80	41.10	0	15	65	PG QM	0.00	0.00	0.00	0.05	0	.
PG EN	36.50	0.00	35.80	36.10	0	65	70	PG QN	0.00	0.00	0.00	0.05	0	.
PG EO	31.50	0.00	30.90	31.10	0	15	75	PG QO	0.00	0.00	0.00	0.05	0	.
PG EP	26.00	0.00	25.90	26.10	0	.	80	PG QP	0.05	0.00	0.00	0.05	0	20
PG EQ	21.00	0.00	20.90	21.10	0	40	85	PG QQ	0.00	0.00	0.00	0.05	0	.
PG ER	16.00	0.00	15.90	16.10	0	58	90	PG QR	0.10	0.00	0.00	0.10	0	90
PG ES	11.30	0.00	10.90	11.10	0	204	95	PG QS	0.20	0.00	0.10	0.20	0	173
PG ET	6.00	-0.10	6.00	6.20	132	229	100	PG QT	0.45	-0.05	0.45	0.50	158	3,229
PG EA	2.35	-0.35	2.30	2.35	1,287	3,049	105	PG QA	1.60	0.00	1.70	1.75	680	2,065
PG EB	0.50	0.00	0.45	0.50	193	2,921	110	PG QB	4.70	-0.50	4.80	5.00	305	539
PG EC	0.05	0.00	0.05	0.10	15	258	115	PG QC	10.10	0.00	9.50	9.70	0	64
PG ED	0.00	0.00	0.00	0.05	0	.	120	PG QD	14.40	0.00	14.40	14.70	0	75
PG EE	0.00	0.00	0.00	0.05	0	.	125	PG QE	19.70	0.00	19.40	19.70	0	138
PG EF	0.00	0.00	0.00	0.05	0	.	130	PG QF	24.10	-0.60	24.40	24.70	12	163

The "Market Implied" comonotonicity gap

- The market only gives us *partial information* about the marginals through the prices of traded options with various traded strikes $K_1^{(i)}, K_2^{(i)}, \cdots, K_{J(i)}^{(i)}$ for stock S_i at a given maturity t.
- Let UB be the upper bound for basket option, given only this *partial information, then*

$$\boxed{\textit{Market implied comonotonicity Gap} = \textit{UB} - \textit{traded Market Price}}$$

- *Fundamental: Given a basket option on n assets, there is a portfolio $\mathcal{P}$ of $n+1$ options on components, such that*

$$\textit{UB} = \textit{Market Price of } \mathcal{P}$$

Below we will discuss how to determine the upper bound UB.

Recent Work on Model Independent Option

Bertsimas and Popescu, 2003, use a LP approach to derive bounds on assets under a variety of constraints. Here is one of their results:

Given prices $C_i(K_i)$ of call options with strikes $0 \leq K_1 \leq .. \leq K_n$ on a stock X, the range of all possible prices for a call option with strike K where $K \in (K_j, K_{j+1})$ for some $j = 0, \cdots, n$ is $[C^-(K), C^+(K)]$ where

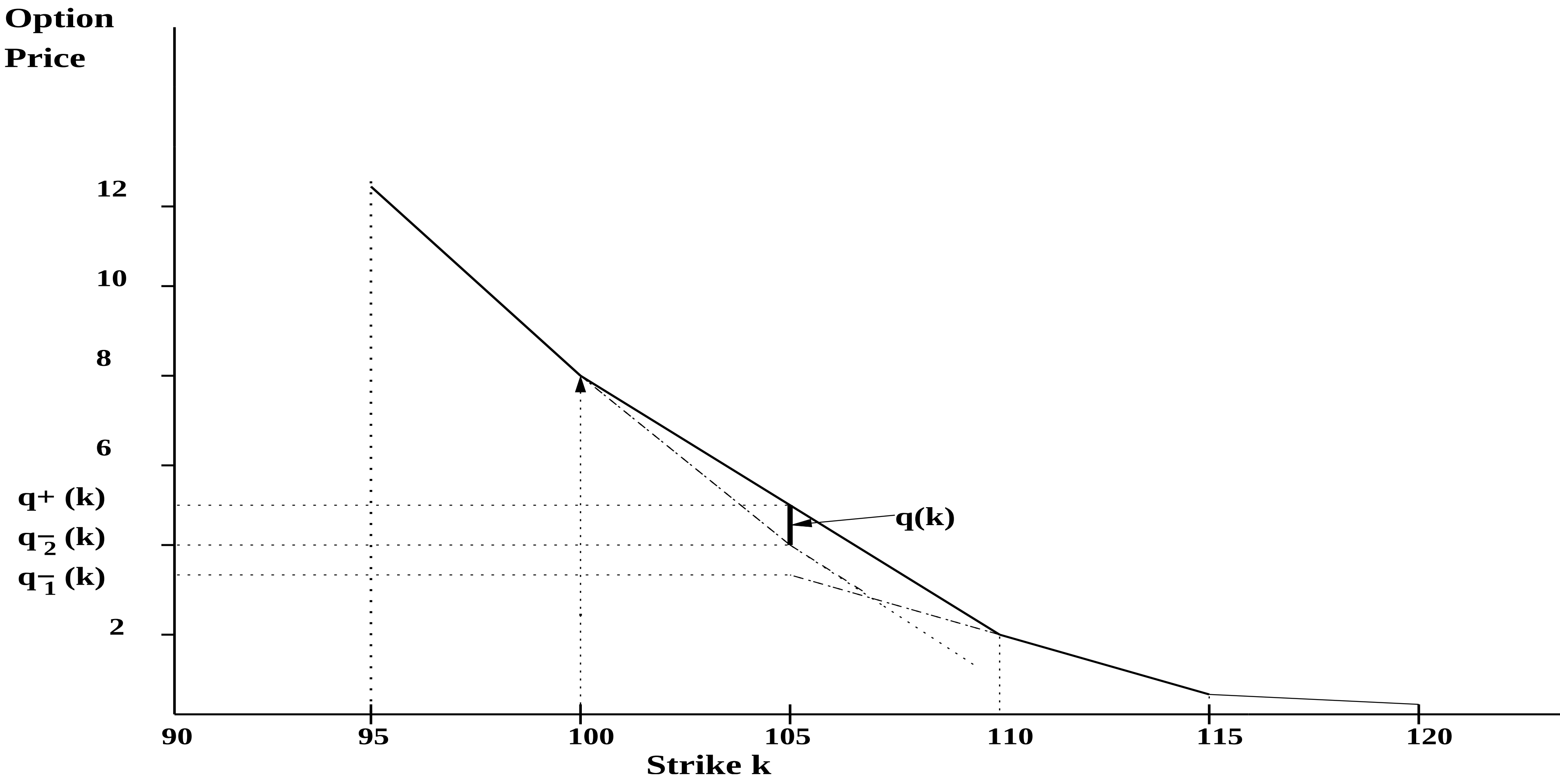
$$\begin{aligned} & C^-(K) \\ & = \max\left(C_j \frac{K - K_{j-1}}{K_j - K_{j-1}} + C_{j-1} \frac{K_j - K}{K_k - K_{j-1}}, \right. \\ & \qquad \left. C_{j+1} \frac{K_{j+2} - K}{K_{j+2} - K_{j+1}} + C_{j+2} \frac{K - K_{j+1}}{K_{j+2} - K_{j+1}} \right) \end{aligned}$$

lower bounds

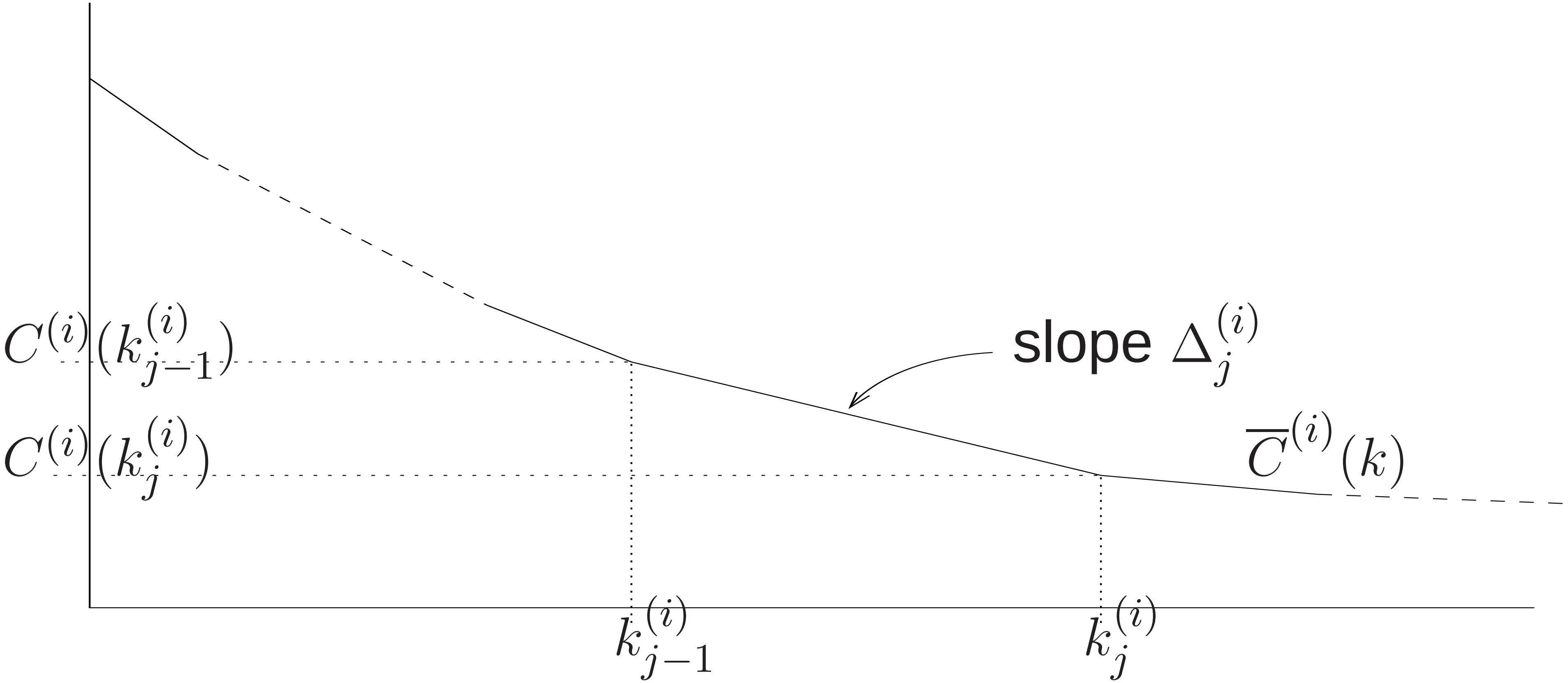
$$C^+(K) = \frac{K_{j+1} - K}{K_{j+1} - K_j} + C_{j+1} \frac{K - K_j}{K_{j+1} - K_j}$$

upper bounds

Bertsimas-Popescu



Linear interpolation



The interpolated call price function. $\Delta^{(i)}_j$ gives the modulus of the slope of $\overline{C}^{(i)}$ over $(k^{(i)}_{j-1}, k^{(i)}_j)$. This graph provides one of many ways of filling in the missing strikes. But it turns out to be the fundamental interpolation, in the case of the upper bound.

Co-monotonic copula & Option Prices

- The marginals corresponding to piecewise linear call prices are discontinuous at every strike price and constant between strike prices.
  Because:
-

$$\frac{\partial^2 C^{(i)}}{\partial K^2} = \text{density}$$

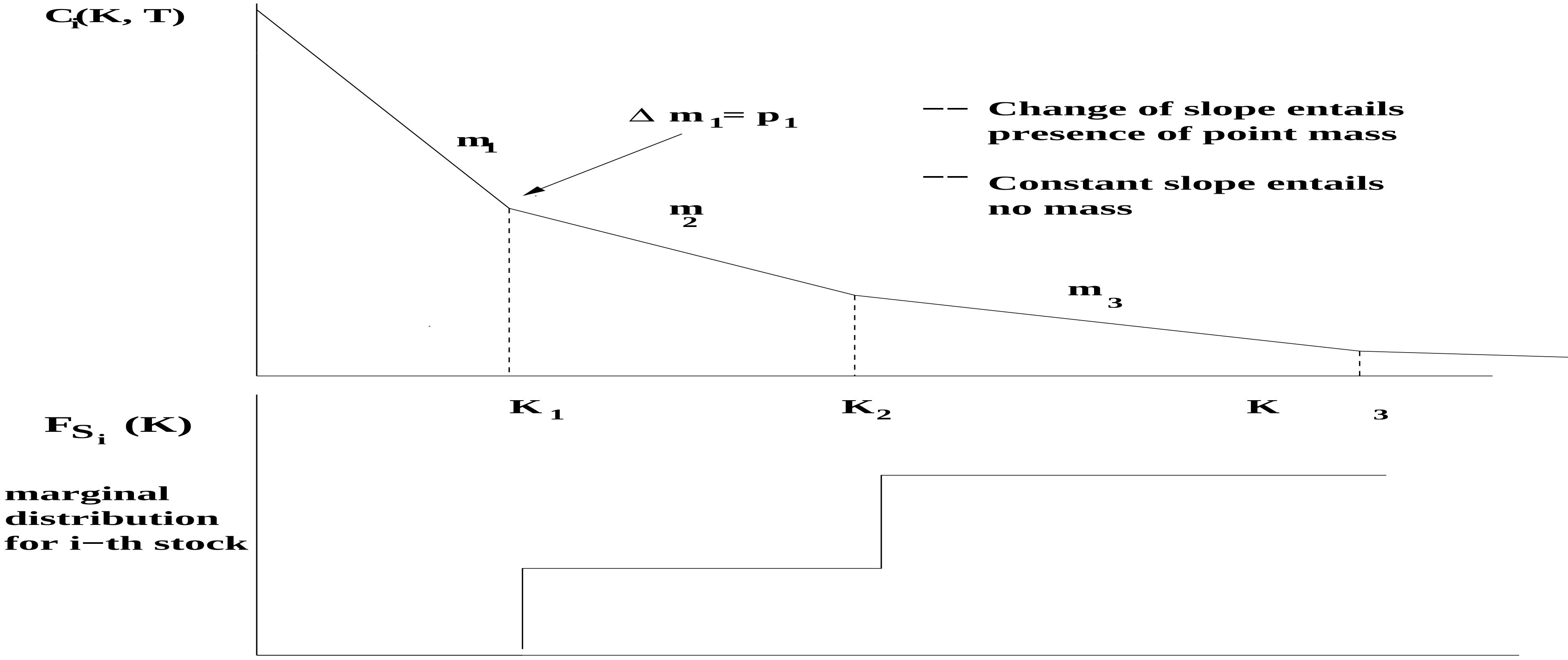
and because our call price functions are piecewise linear *between* two strikes so

$$\frac{\partial^2 C}{\partial K^2} = 0, \quad K_i^j \leq K \leq K_i^{j+1}$$

$$\frac{\partial^2 C}{\partial K^2} = \delta(K_i^j) \times \left(\text{change of slope at } K_i^j\right),$$

This is illustrated in following slide:

Underlying assets have jumps and regions with no mass



The interpolated call price function. $\Delta_j^{(i)}$ gives the modulus of the slope of $\overline{C}^{(i)}$ over $(k_{j-1}^{(i)}, k_j^{(i)})$.

Optimizer

- Now the market implied comonotonic optimizer $(\bar{S_1}, \bar{S_2}, \ldots, \bar{S_n})$ is a random variable which is distributed like the vector random variable

$$\Big((F^M_{S_1})^{-1}(U), (F^M_{S_2})^{-1}(U), \ldots, (F^M_{S_n})^{-1}(U)\Big)$$

  where $F^M_{S_i}, i = 1, \cdots, n$ are the market implied marginals with point masses at the strikes.

- It can be shown (Laurence and Wang (2004, 2005) and Hobson, Laurence and Wang (2005)), (Laurence and Wang, 2008) that the market implied comonotonic optimizer is a solution of optimization problem on next slide:

Optimization - primal

Constrained optimization problem. Determine

$$\sup_{\mu} \int \left(\sum_i w_i S_i - K \right)^+ \mu(dS)$$

subject to

$$\int (S_i - k_j^{(i)})^+ \mu(dS) = C^{(i)}(k_j^{(i)}), \text{ for } i = 1, \dots, n, j = 1, \dots, J^{(i}$$

$$\int \mu(dS) = 1$$

Optimization - dual

Dual problem

$$\inf_{\nu,\psi} \sum_{i=1}^{n} \sum_{j=1}^{J^{(i)}} C^{(i)}(k_j^{(i)})\nu_i^j + \psi$$

subject to

$$\left(\sum_i w_i S_i - K\right)^+ \leq \sum_{i,j} \left(S_i - k_j^{(i)}\right)^+ \nu_i^j + \psi \qquad \text{(*)}$$

$\nu_j^i \in \mathbb{R}$, for $i = 1, \ldots, n, \quad j = 1, \ldots, J^{(i)}$

$\psi \in \mathbb{R}$

(*) is the super-replication condition

Here ψ is cash component and ν_j^i is number of options with strike k_j^i in hedging portfolio.

Finite market - Using all traded options

- **Preliminaries** For simplicity of exposition assume all slopes $\left.\frac{\partial C^{(i)}(u)}{\partial u}\right|_{u=k_j^{(i)}}$ are different as i and j vary. Let $I_n = \{1, 2, \cdots, n\}$ where n is the number of assets.
- There is a privileged index $\hat{i} \in I_n$ such that:

  For any model which is consistent with the observed call prices $C^{(i)}(K_j)$, the price $B(K)$ for the basket option is bounded above by $\overline{\mathcal{B}}_F(K)$, where
- Case I: $\sum_i w_i k^{(i)}_{J(i)} > K$:

$$\overline{\mathcal{B}}_F(K) = \sum_{i \in I_n \setminus \hat{i}} w_i C^{(i)}\left(k^{(i)}_{\overline{j}(i)}\right) + w_{\hat{i}}\left\{(1-\theta^*_{\hat{i}})C^{(\hat{i})}\left(K^{(\hat{i})}_{\overline{j}(\hat{i})-1}\right) + \theta^*_{\hat{i}} C^{(\hat{i})}\left(k^{(\hat{i})}_{\overline{j}(\hat{i})}\right)\right\}$$

- $\theta^*_{\hat{i}}$ is defined as $\theta^*_{\hat{i}} = \frac{\overline{\lambda}^*_{\hat{i}} - \overline{\lambda}^-_{\hat{i}}(\phi^*)}{\overline{\lambda}^+_{\hat{i}}(\phi^*) - \overline{\lambda}^-_{\hat{i}}(\phi^*)} = \frac{(K\overline{\lambda}^*_{\hat{i}}/w_{\hat{i}}) - k^{(i)}_{\overline{j}(\hat{i})-1}}{k^{(i)}_{\overline{j}(\hat{i})} - k^{(i)}_{\overline{j}(\hat{i})-1}}, \overline{\lambda}^*_{\hat{i}} \in [k^{(i)}_{\overline{j}(\hat{i})-1}, k^{(i)}_{\overline{j}(\hat{i})}]$.

Finite market - Result

- Case II: $\sum_i w_i K_{J(i)} \leq K$:

$$\overline{\mathcal{B}}_F(K) = \sum_i w_i C^{(i)} \left(k^{(i)}_{J(i)} \right)$$

****

- Based on experiments with real data, the second case essentially never arises in practice.
- Moreover, the upper bound is optimal in the sense that we can find co-monotonic models which are consistent with the observed call prices and for which the arbitrage-free price for the basket option is arbitrarily close to $\overline{\mathcal{B}}_F(K)$.
- So where's the beef in Case I?
- All the beef in fleshing out the estimate in the first case is in determining the special index $\hat{i}$ and the indices $j(i), i = 1 \cdots, n$.

How to find which options to choose?

- Possible to show that there is No cash component ψ in the optimal portfolio. So can consider super-replicating portfolios consisting entirely of options with various strikes (some of which may have strike zero).
- The upper bound is available in quasi-closed form, meaning there is a simple algorithm to determine the solution, modulo a slope ordering algorithm: Order all slopes of all call price functions and cycle through.
- To get the intuition as to how to proceed, note that if $\sum \lambda_i = 1$ then

$$\left(\sum_i w_i X_M^{(i)} - K \right)^+ \leq \sum_i w_i \left(X_M^{(i)} - \frac{\lambda_i K}{w_i} \right)^+ , \quad \text{due to Merton}$$

So that

$$C_{\mathcal{B}}(K) \leq \sum_i w_i C^{(i)} \left(\lambda_i K / w_i\right).$$

The λ_i are arbitrary and so $C_{\mathcal{B}}(K) \leq \inf_{\lambda_i \geq 0, \sum \lambda_i = 1} \sum_i w_i C^{(i)} \left(\lambda_i K / w_i\right).$

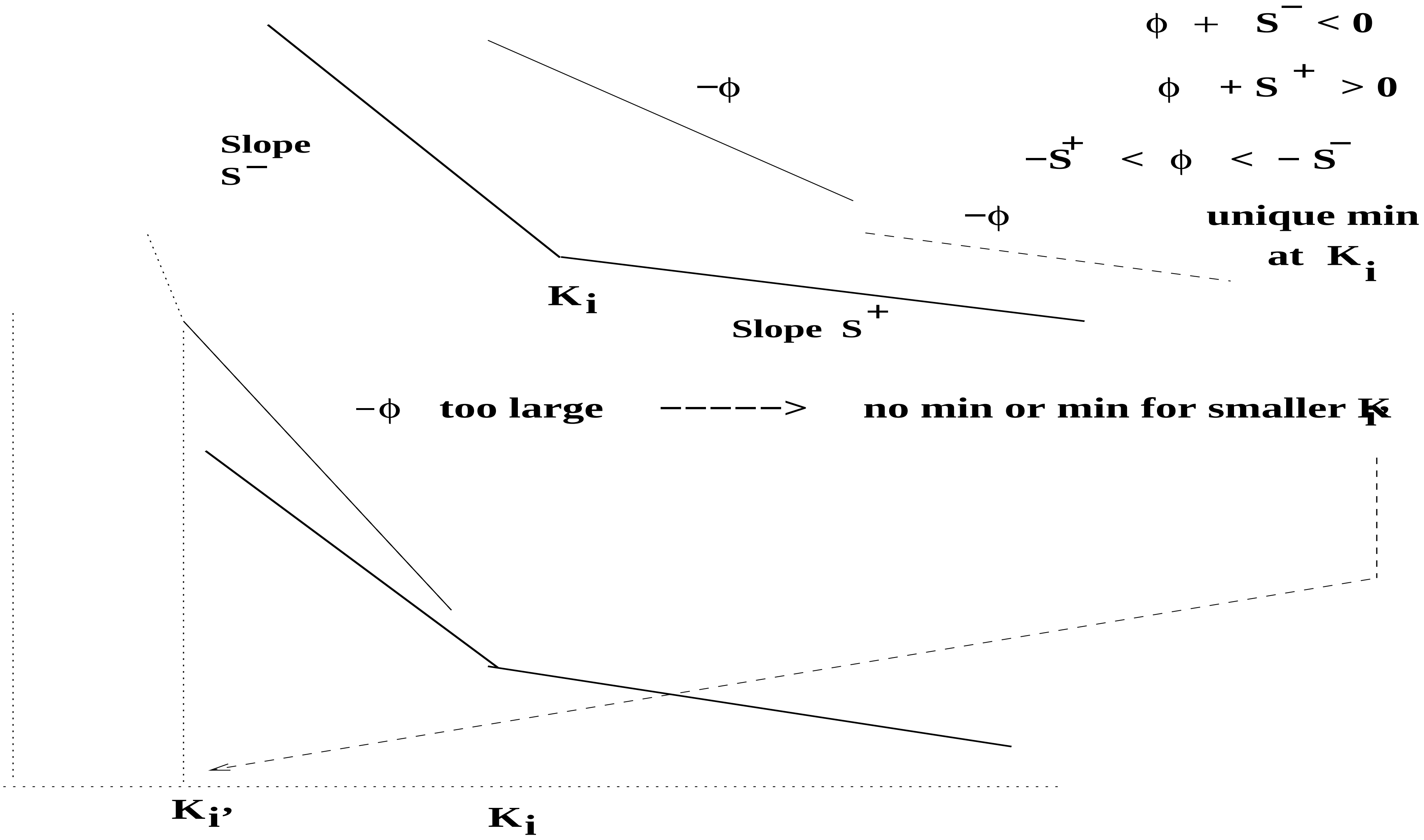
Intuition

- We wish to find the infimum of $\sum_i w_i C^{(i)}\left(\lambda_i K/w_i\right)$ over choices λ_i satisfying $\lambda_i \geq 0, \sum \lambda_i = 1$. Define the Lagrangian

$$L(\lambda, \phi) = \sum_i w_i C^{(i)}\left(\lambda_i K/w_i\right) + \phi\left(\sum_i \lambda_i - 1\right).$$

- Objective function is convex but only $C^{0,1}$, because each piecewise linear call price functions $C^{(i)}$, is $C^{0,1}$, ie. $\frac{\partial C^i}{\partial K}$ has a jump at each strike $K_i^j, j = 1, \cdot, n_i$.
- Note that objective functional is separable function of 1-dimensional functions.
- Therefore for each fixed Lagrange Multiplier ϕ, the gradient can point in a cone of different directions. In the terminology of convex analysis we have $\phi/\beta K \in \bar{\partial} C^{(i)}(\lambda_i K/w_i)$, where $\bar{\partial}$ is the *subdifferential* of the function $C^{(i)}$. .

Illustration Min



$\phi + S^- < 0$

$\phi + S^+ > 0$

$-S^+ < \phi < -S^-$

$-\phi$

$-\phi$ unique min at K_i

Slope S^-

K_i

Slope S^+

$-\phi$ too large -----> no min or min for smaller K_i

$K_{i'}$

K_i

Algorithm

For each ϕ there is either a unique $\lambda(\phi)$ or an interval $[\lambda^-(\phi), \lambda^+(\phi)]$.
Essentially:

- $[\lambda(\phi)^-, \lambda(\phi)^+] \sim [w_i K_i^j/K, w_i K_i^{j+1}/K]$ for some i and j.
- So Algorithm:
  - Order all the slopes of all call price functions. Ie. if 30 assets and 8 non zero strikes , order 240 slopes.

$$\mathcal{S}_1 \leq \mathcal{S}_2 \leq \cdots\cdots \leq \mathcal{S}_{240}$$

  - Now starting with $\phi = \epsilon << 1$ increase ϕ while monitoring the quantity

$$\Lambda(\phi) = \sum \lambda^+(\phi)$$

which starts very large for small ϕ ($\rightleftharpoons$ large K_i^j) and decreases as $\phi \uparrow$.

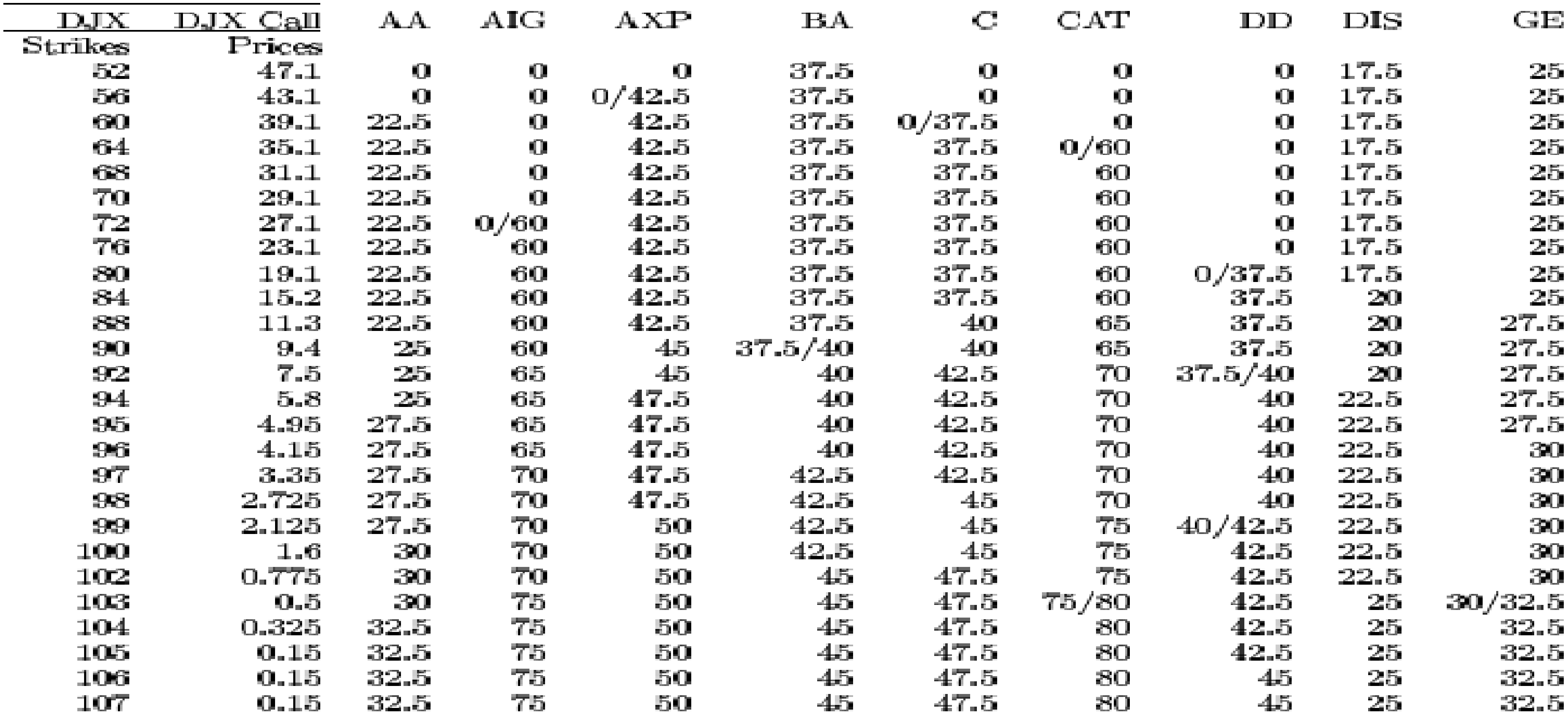
  - The first time $\Lambda(\phi)$ crosses 1. STOP! $\mapsto$ Optimal value of $\phi = \phi^*$ has been reached.

2004 DJX data follows

The following is on old DJX dataset (June 2004).
For a very recent data set, see the end of the talk.

Experiment on Real DJX Data: Spot was 99.

We now illustrate the output on real DJX data.



DJX Strikes	DJX Call Prices	AA	AIG	AXP	BA	C	CAT	DD	DIS	GE
52	47.1	0	0	0	37.5	0	0	0	17.5	25
56	43.1	0	0	0/42.5	37.5	0	0	0	17.5	25
60	39.1	22.5	0	42.5	37.5	0/37.5	0	0	17.5	25
64	35.1	22.5	0	42.5	37.5	37.5	0/60	0	17.5	25
68	31.1	22.5	0	42.5	37.5	37.5	60	0	17.5	25
70	29.1	22.5	0	42.5	37.5	37.5	60	0	17.5	25
72	27.1	22.5	0/60	42.5	37.5	37.5	60	0	17.5	25
76	23.1	22.5	60	42.5	37.5	37.5	60	0	17.5	25
80	19.1	22.5	60	42.5	37.5	37.5	60	0/37.5	17.5	25
84	15.2	22.5	60	42.5	37.5	37.5	60	37.5	20	25
88	11.3	22.5	60	42.5	37.5	40	65	37.5	20	27.5
90	9.4	25	60	45	37.5/40	40	65	37.5	20	27.5
92	7.5	25	65	45	40	42.5	70	37.5/40	20	27.5
94	5.8	25	65	47.5	40	42.5	70	40	22.5	27.5
95	4.95	27.5	65	47.5	40	42.5	70	40	22.5	27.5
96	4.15	27.5	65	47.5	40	42.5	70	40	22.5	30
97	3.35	27.5	70	47.5	42.5	42.5	70	40	22.5	30
98	2.725	27.5	70	47.5	42.5	45	70	40	22.5	30
99	2.125	27.5	70	50	42.5	45	75	40/42.5	22.5	30
100	1.6	30	70	50	42.5	45	75	42.5	22.5	30
102	0.775	30	70	50	45	47.5	75	42.5	22.5	30
103	0.5	30	75	50	45	47.5	75/80	42.5	25	30/32.5
104	0.325	32.5	75	50	45	47.5	80	42.5	25	32.5
105	0.15	32.5	75	50	45	47.5	80	42.5	25	32.5
106	0.15	32.5	75	50	45	47.5	80	45	25	32.5
107	0.15	32.5	75	50	45	47.5	80	45	25	32.5

TABLE 4. The super-replicating portfolio. For each strike on the DJX, and for each component of the basket, we list the relevant strike to hold in the cheapest super-replicacting portfolio. A strike of 0 corresponds to holding the asset. For space reasons we only give the strikes for the first 10 components. In most cases there is a single strike listed. In others the optimal portfolio involves a combination of two strikes. Note that the optimal strike to hold on each component asset increases as the strike on the DJX increases.

How good is the Upper Bound? Spot was 99.

DJX Strikes	DJX Prices	UB Unclean Data	UB Clean Data	BS Price $\rho = 0$	BS Price $\rho = .5$	BS Price $\rho = .75$	BS Price $\rho = .9$	BS Pric $\rho = .99$
52	47.10	47.09	47.05	47.14	47.14	47.15	47.10	47.18
56	43.10	43.10	43.09	43.16	43.18	43.17	43.15	43.17
60	39.10	39.11	39.10	39.16	39.18	39.13	39.12	39.14
64	35.10	35.11	34.30	35.16	35.16	35.16	35.20	35.17
68	31.10	31.12	30.83	31.17	31.17	31.22	31.17	31.16
70	29.10	29.13	29.12	29.18	29.19	29.18	29.17	29.11
72	27.10	27.14	27.14	27.19	27.22	27.18	27.13	27.18
76	23.10	23.15	22.38	23.18	23.16	23.18	23.15	23.19
80	19.10	19.18	19.18	19.20	19.18	19.15	19.19	19.22
84	15.20	15.24	14.95	15.21	15.24	15.23	15.18	15.23
88	11.30	11.42	11.42	11.20	11.26	11.25	11.25	11.36
90	9.40	9.61	9.61	9.21	9.28	9.35	9.41	9.44
92	7.50	7.90	7.90	7.21	7.34	7.53	7.67	7.73
94	5.80	6.32	6.32	5.22	5.58	5.83	6.01	6.08
95	4.95	5.57	5.57	4.22	4.79	5.06	5.26	5.34
96	4.15	4.85	4.85	3.22	4.01	4.35	4.54	4.66
97	3.35	4.19	4.19	2.24	3.28	3.69	3.92	4.01
98	2.73	3.58	3.58	1.35	2.70	3.12	3.34	3.44
99	2.13	3.02	3.02	0.67	2.16	2.58	2.75	2.96
100	1.60	2.53	2.53	0.25	1.69	2.10	2.33	2.43
102	0.78	1.73	1.73	0.01	0.99	1.37	1.55	1.71
103	0.50	1.42	1.42	0.00	0.71	1.05	1.26	1.36
104	0.33	1.16	1.16	0.00	0.52	0.82	1.02	1.13
105	0.15	0.95	0.95	0.00	0.36	0.63	0.79	0.89
106	0.15	0.75	0.75	0.00	0.25	0.48	0.60	0.70
107	0.15	0.59	0.59	0.00	0.16	0.35	0.48	0.53

spread options

PART II

Spread option case

- The methodology for basket options can also be applied to generalized spread options.
- The payoff ψ of the generalized spread options

$$\psi(S_1, \cdots, S_n) = \left(\sum_{i=1}^{n} w_i S_i - K\right)^{+}$$

where the weights w_i are constants of arbitrary sign.

- Examples contain heating oil crack spread ($(42 \times [HO] - [CO] - K)^{+}$), 3:2:1 crack spread ($\left(42 \times \frac{2}{3}[UG] + 42 \times \frac{1}{3}[HO] - [CO] - K\right)$)
Note: 1 barrel = 42 gallons

Antimonotonicity instead

- Let us group the payoff function for the generalized spread option as

$$\psi(S_1,\cdots,S_n)=\left(\sum_{i\in I^+} w_iS_i-\sum_{i\in I^-}|w_i|S_i-K\right)^+$$

  where I^+ denotes the set of indices with positive weights and I^- the negative weights.

- The upper bound is attained when
  - Assets indexed in I^+ are comonotonic to one another.
  - Assets indexed in I^- are also c-monotonic to one another.
  - Any asset in I^+ is antimonotonic to every asset in I^-.
- Special case: $\psi(S_1,S_2)=(S_1-S_2-K)^+$
  Upper bound is attained when S_1 and S_2 are antimonotonic.

$$\underbrace{\text{LB}}_{\text{comonotonic}} \leq \mathcal{M} \leq \underbrace{\text{UB}}_{\text{antimonotonic}}$$

Anti-monotonicity

Recall the definition of anti-monotonicity:

A two dimensional random vector (X_1, X_2) is said to be anti-monotonic if there exists a uniformly distributed random variable U such that

$$U \sim \text{Uniform}(0,1)$$

$$(X_1, X_2) \overset{d}{=} \left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(1-U)\right),$$

where $F_{X_i}(x)$ is the distribution function of X_i.

Spread option

Therefore, for the generalized spread options with payoff

$$\left(\sum_{i\in I^+} w_i S_i - \sum_{i\in I^-} |w_i| S_i - K\right)^+ ,$$

the upper bound is attained if there exists a uniformly distributed random variable $U \sim$ Uniform$(0,1)$ such that

- $S_i \overset{d}{=} F_{S_i}^{-1}(U)$ for $i \in I^+$
- $S_i \overset{d}{=} F_{S_i}^{-1}(1-U)$ for $i \in I^-$

where $F_{S_i}(x)$ is the distribution function of S_i.

Super hedge portfolio

- Observe the inequality

$$\left(\sum_{i\in I^+} w_i S_i - \sum_{i\in I^-} |w_i| S_i - K\right)^+ \le \sum_{i\in I^+} w_i \left(S_i - \frac{\lambda_i K}{w_i}\right)^+ + \sum_{i\in I^-} |w_i| \left(\frac{\lambda_i K}{|w_i|} - S_i\right)^+$$

  where $\lambda_i \ge 0$ and $\sum_{i\in I^+} \lambda_i - \sum_{i\in I^-} \lambda_i = 1$.

- Taking expectation on both sides of the inequality we have

$$\text{Spread option price} \le \sum_{i\in I^+} w_i C_{S_i}\left(\frac{\lambda_i K}{w_i}\right) + \sum_{i\in I^-} |w_i| P_{S_i}\left(\frac{\lambda_i K}{|w_i|}\right)$$

  where $C_{S_i}(k)$ and $P_{S_i}(k)$ are the call and put prices of S_i struck at k respectively.

- The super hedge portfolio is therefore obtained by minimizing the right hand side over the constrained parameters $\lambda_1, \cdots, \lambda_n$.
- The portfolio consists of buying calls for the components with positive weight and puts for components with negative weights.

Optimal solution

- As in the basket case, the constrained minimization problem is solved by the method of Lagrange multipliers.
- Again the slopes $\Delta_j^{(i)}$ are ordered as a (strictly) decreasing sequence $\Delta_1, \cdots, \Delta_N$ with repetitions removed, where

$$\Delta_j^{(i)} = \frac{c_{j-1}^{(i)} - c_j^{(i)}}{k_j^{(i)} - k_{j-1}^{(i)}} \quad \text{for } i \in I^+$$

$$\Delta_j^{(i)} = \frac{p_j^{(i)} - p_{j-1}^{(i)}}{k_j^{(i)} - k_{j-1}^{(i)}} \quad \text{for } i \in I^-$$

Gather together *all slopes*

Puts and calls

- Corresponding to each slope Δ_l, $\lambda_i(l) = \frac{w_i k^{(i)}{}_{j_i(l)}}{K}$ is assigned to asset i, where

$$j_i(l) = \max\{j \in \{1, \cdots, J(i)\} : \Delta_j^{(i)} \geq \Delta_l\} \quad \text{for } i \in I^+$$

$$j_i(l) = \min\{j \in \{1, \cdots, J(i)\} : \Delta_j^{(i)} \geq \Delta_l\} \quad \text{for } i \in I^-$$

Optimal solution

- Starting with $l = N$, let us iteratively decrease l by one, until

$$\sum_{i \in I^+} \lambda_i(l) - \sum_{i \in I^-} \lambda_i(l) = 1.$$

  Denote the critical l by l^*. If the condition $\sum_{i \in I^+} \lambda_i(l) - \sum_{i \in I^-} \lambda_i(l) = 1$ is not exactly satisfied, linearly interpolate the λ_i's for those indices i, which change when l decreases from l^* to $l^* - 1$. Denote the interpolation factor by θ^* and these indices by $I^+_{l^*}$ and $I^-_{l^*}$ for positive and negative weights respectively.

- Case I: $\sum_{i \in I^+} w_i k_i^{(i)} > K$ and $\sum_{i \in I^+} \lambda_i(l^*) - \sum_{i \in I^-} \lambda_i(l^*) = 1$

$$\text{UB} = \sum_{i \in I^+} C^{(i)} \left(\frac{w_i k^{(i)}_{j_i(l^*)}}{K} \right) + \sum_{i \in I^-} P^{(i)} \left(\frac{w_i k^{(i)}_{j_i(l^*)}}{K} \right)$$

Optimal solution

- Case II: $\sum_{i\in I^+} w_i k_i^{(i)} > K$ and $\sum_{i\in I^+} \lambda_i(l^*) - \sum_{i\in I^-} \lambda_i(l^*) > 1$

$$
\begin{aligned}
\text{UB} &= \sum_{i\in I^+\backslash I^+_{l*}} w_i C^{(i)}\left(\frac{w_i k^{(i)}_{j_i(l^*)}}{K}\right) + \sum_{i\in I^-\backslash I^-_{l*}} w_i P^{(i)}\left(\frac{w_i k^{(i)}_{j_i(l^*)}}{K}\right) \\
&+ \sum_{i\in I^+_{l*}} w_i \left[\theta^* C^{(i)}\left(\frac{w_i k^{(i)}_{j_i(l^*)}}{K}\right) + (1-\theta^*)\theta^* C^{(i)}\left(\frac{w_i k^{(i)}_{j_i(l^*)-1}}{K}\right)\right] \\
&+ \sum_{iI^-_{l*}} w_i \left[\theta^* P^{(i)}\left(\frac{w_i k^{(i)}_{j_i(l^*)}}{K}\right) + (1-\theta^*) P^{(i)}\left(\frac{w_i k^{(i)}_{j_i(l^*)} + 1}{K}\right)\right]
\end{aligned}
$$

- Case III: $\sum_{i\in I^+} w_i k_i^{(i)} \leq K$,

$$\text{UB} = \sum_{i\in I^+} w_i C^{(i)}(k_i^{(i)})$$

Simulation illustration

K	Hedging Price	MC Price	MC accuracy	S_1 strike	C	S_2 strike	P
2	10.03	10.12	0.07	1.46	0.16	59/59.5	3.43/3
2.5	9.77	9.71	0.07	1.46	0.16	58.5	3.17/2
3.5	9.29	9.29	0.07	1.48	0.15	58	2.68
4.5	8.83	8.83	0.06	1.48	0.15	57.5/58	2.68/2
13	5.60	5.64	0.05	1.65/1.60	0.09/0.1	54.5	1.35

S_1 and S_2 are distributed like two antimonotonic geometric Brownian motions (equivalently the instantaneous correlation ρ equals -1) with parameters $\sigma_1 = .355$, $\sigma_2 = .2$, $T = .5$, $r = 0$, $d_1 = d_2 = 0$. The Monte Carlo prices are computed using $n = 50,000$ paths. The spot prices are $S_1 = 1.48$, $S_2 = 59.33$, and the weights are $w_1 = 42$, $w_2 = 1$. The strikes that were actually trading are given by the NYMEX data for the December 2006 contract.

Empirical analysis

The results of monitoring the crack spread option, difference between heating and crude oil for the contract that expired December 2006 are shown in the following table. The table shows the true price in the third column and the lower and upper bounds in column 2 and 4. The comononotonicity and antimonotonicity gaps are shown next, as well as their relative counterparts.

Empirical analysis

Day	LB	TP	UB	TP - LB	UB - TP	UB - LB	$\frac{TP-LB}{UB-TP}$	$\frac{UB-TP}{UB-LB}$
6-Oct	1.39	2.65	7.52	1.25	4.88	6.13	0.20	0.80
13-Oct	1.53	3.06	7.53	1.52	4.47	6.00	0.25	0.75
20-Oct	1.26	2.55	6.72	1.30	4.17	5.46	0.24	0.76
23-Oct	0.95	2.40	5.22	1.45	2.82	4.27	0.34	0.66
26-Oct	1.29	2.24	6.15	0.95	3.91	4.86	0.20	0.80
30-Oct	0.57	1.39	5.17	0.81	3.78	4.60	0.18	0.82
31-Oct	0.57	1.36	5.10	0.79	3.73	4.52	0.17	0.83
1-Nov	0.49	1.09	4.75	0.60	3.65	4.26	0.14	0.86
2-Nov	0.47	2.26	4.69	1.79	2.43	4.22	0.42	0.58
3-Nov	0.60	2.50	4.92	1.90	2.42	4.32	0.44	0.56
6-Nov	0.85	2.96	5.17	2.11	2.21	4.32	0.49	0.51
7-Nov	1.00	1.45	5.04	0.45	3.59	4.04	0.11	0.89
8-Nov	0.83	1.25	4.87	0.42	3.62	4.04	0.10	0.90
9-Nov	1.13	1.10	5.19	-0.03	4.09	4.05	-0.01	1.01

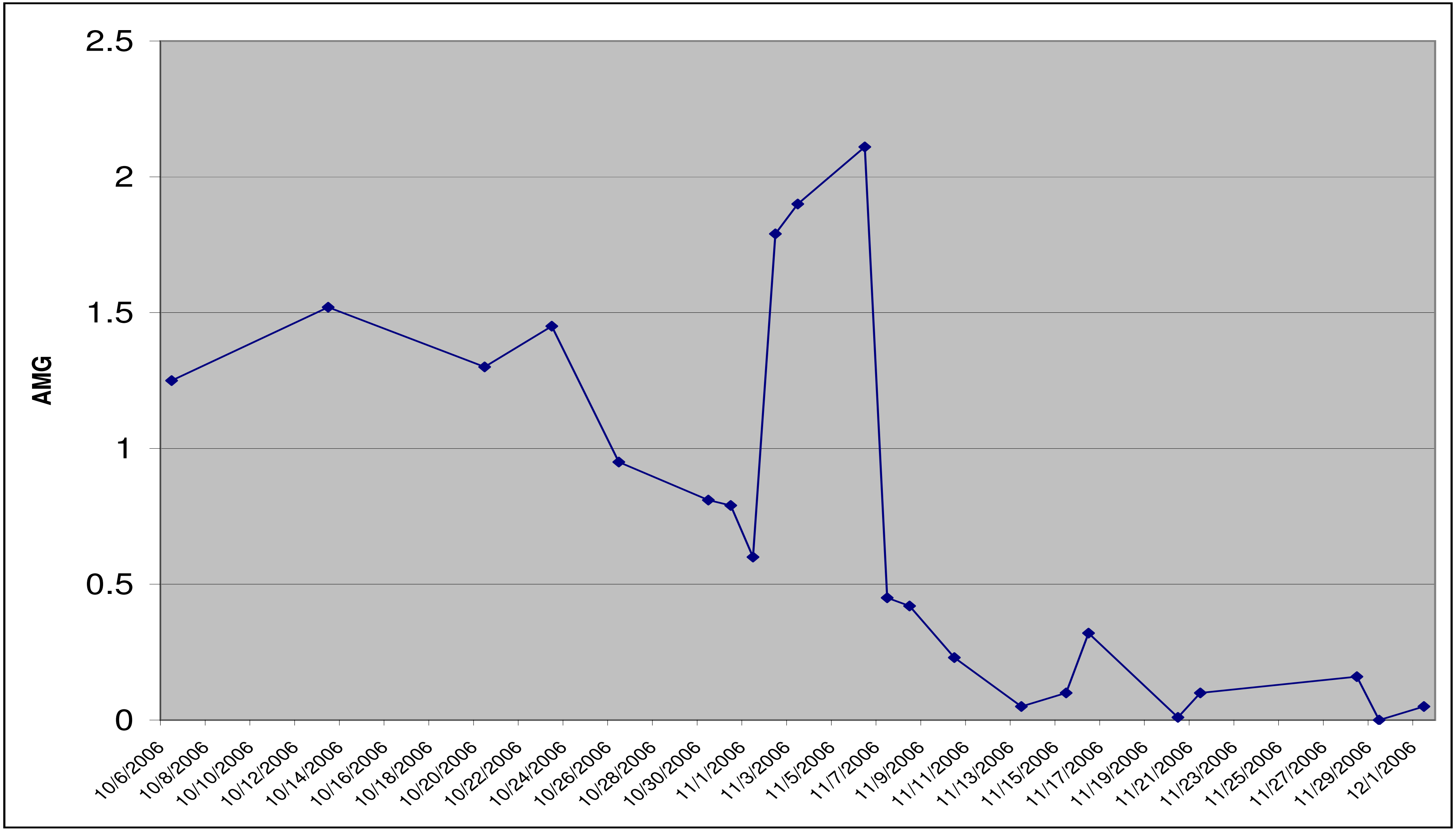
Empirical analysis

Day	LB	TP	UB	TP - LB	UB - TP	UB - LB	$\frac{TP-LB}{UB-TP}$	$\frac{UB-TP}{UB-LB}$
10-Nov	0.87	1.10	4.87	0.23	3.77	4.00	0.06	0.94
13-Nov	0.60	0.65	4.36	0.05	3.71	3.76	0.01	0.99
14-Nov	0.93	0.80	4.69	-0.13	3.89	3.76	-0.04	1.04
15-Nov	1.05	1.15	4.87	0.10	3.72	3.83	0.03	0.97
16-Nov	1.21	1.53	4.92	0.32	3.39	3.71	0.09	0.91
20-Nov	1.36	1.37	4.82	0.01	3.45	3.46	0.00	1.00
21-Nov	2.13	2.23	5.47	0.10	3.24	3.35	0.03	0.97
28-Nov	1.35	1.51	4.28	0.16	2.77	2.93	0.05	0.95
29-Nov	2.10	2.10	4.83	0.00	2.73	2.73	0.00	1.00
1-Dec	1.70	1.75	4.25	0.05	2.50	2.55	0.02	0.98
4-Dec	1.30	1.20	3.69	-0.10	2.49	2.39	-0.04	1.04
5-Dec	1.09	0.82	3.35	-0.27	2.53	2.26	-0.12	1.12
6-Dec	1.03	0.97	3.14	-0.06	2.17	2.11	-0.03	1.03
7-Dec	0.72	0.56	2.64	-0.16	2.08	1.93	-0.08	1.08

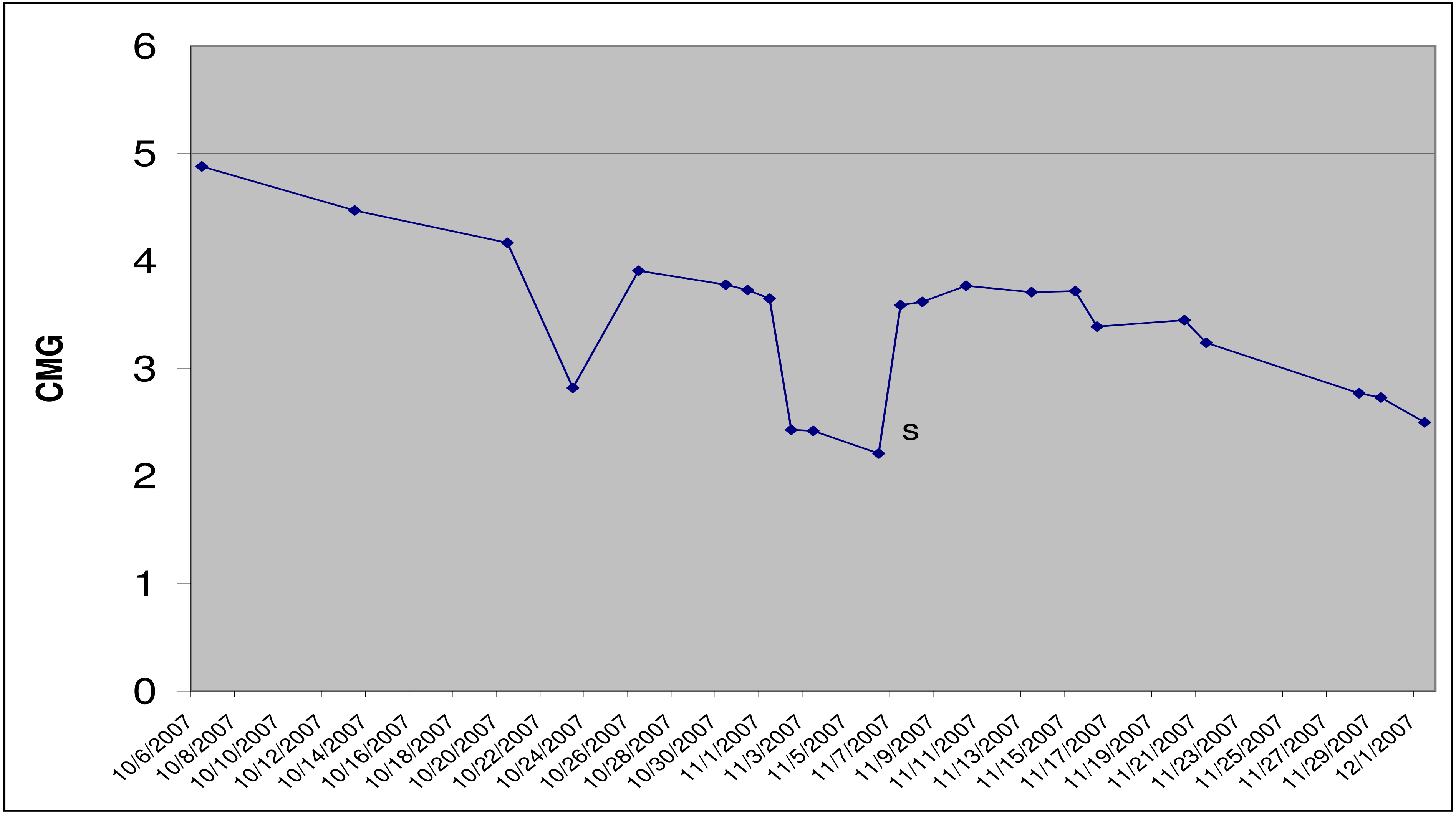
Empirical analysis

Day	LB	TP	UB	TP - LB	UB - TP	UB - LB	$\frac{TP-LB}{UB-TP}$	$\frac{UB-TP}{UB-LB}$
8-Dec	0.64	0.38	2.36	-0.26	1.98	1.72	-0.15	1.15
11-Dec	0.50	0.15	1.84	-0.35	1.69	1.34	-0.26	1.26
12-Dec	0.53	0.14	1.74	-0.39	1.60	1.21	-0.33	1.33
13-Dec	0.62	0.16	1.56	-0.46	1.40	0.94	-0.49	1.49

day by day, CMG



day by day AMG



day by day

To see how the gaps can generate a profit, suppose for instance tha on October 13th we sell the comonotonicity gap $\mathcal{G}^c$ for 1.52 (sell sprea option and buy optimal *subreplicating* portfolio). Then on November 21st we buy back the gap for 0.1. If the annualized interest rate is 0.05, w have made a profit of 1.51. Also, in our data set, $\mathcal{G}^a$ is monotonicall decreasing, so we can sell the antimonotonicity gap on October 6th an buy it back for a profit at almost any later date. The data set also appear to indicate some arbitrage opportunities, but this may be offset by bid as spreads or lack of liquidity.

Constraints of Type III

It is assumed that call prices are known corresponding to strikes extending from 0 to $+\infty$. By the Breeden-Litzenberger theorem

$$\frac{\partial^2 C}{\partial K^2} = e^{-rT} \rho$$

where ρ is the density of the distribution of stock $S = P\left(S < K\right)$, the latter assumption is equivalent to full knowledge of the marginals ie. marginals prescribed.

On the one hand restrictive since not realistic to assume full knowledge of the marginals. But o

the other hand still allows a rich choice of joint distributions compatible with the given marginals, by

using theory of copulas. This is the route Hobson-Laurence-Wang take for optimal sub-replicatin

strategy when all prices of calls with strikes prescribed .

Sklar

Sklar's theorem Any joint distribution with continuous marginal distribution functions $F_i, i = 1, \cdots, n$, can be expressed as

$$C(F_1^{-1}, F_2^{-1}, \cdots, F_n^{-1})$$

where $C(x_1, \cdots, x_n)$ is a copula and where F^{-1} is the gener-alized inverse of F, ie $F^{-1}(t) = \inf\{x \in \Re | F(x) \geq t\}$

Copula2

Minimization problem with fixed marginals can be reduced to problem of finding *optimal copula*. This problem was solved **in the case** $n = 2$ by Rapuch and Roncalli (Crédit Lyonnais web site) based on earlier results of Muller and Scarsini and Chen.
The Frechet Copulas $C^-(u_1, u_2)$ and $C^+(u_1, u_2)$ given by

$$\begin{aligned} C^- &= \max(u_1 + u_2 - 1, 0) \\ C^+ &= \min(u_1, u_2) \end{aligned}$$

Let $C^-(\mathcal{M}_1, \mathcal{M}_2)$ and $C^+(\mathcal{M}_1, \mathcal{M}_2)$ be the corresponding call option prices. Then for a generic basket option on two assets with the same marginals we have

$$C^-(\mathcal{M}_1, \mathcal{M}_2) \leq C(\mathcal{M}_1, \mathcal{M}_2) \leq C^+(\mathcal{M}_1, \mathcal{M}_2)$$

Dhaene and Gooverts and Rapuch-Roncalli's Copula Approach

When the distribution functions F_X and F_Y are continuous, we have, in the case of a basket on $X+Y$, that

$$\mathcal{C} = \mathcal{C}^- \Leftrightarrow Y = F_Y^{-1}(1 - F_X(X)), \quad \text{anti-monotonic}$$

$$\mathcal{C} = \mathcal{C}^+ \Leftrightarrow Y = F_Y^{-1}(F_X(X)). \quad \text{co-monotonic}$$

For the lower bound note this means that

$$C_{\mathcal{B}}^- = \int_{\mathbb{R}+} [x + F_Y^{-1}(1 - F_X(x)) - K]^+ dF_X(x). \quad (1$$

Spread Options

- Then Lower bound in the case of spread options is attained when the assets
  are Comonotonic.
- This means that: The spread options's price is:

$$E[(X - Y - K)^+] = \int_0^1 (F_X^{-1}(u) - F_Y^{-1}(u) - K)^+ du$$

ie., both, X and Y are driven by the same random factor.

The Primal Problem: Unified basket and spread

$$\inf_{\mu} \int_{\mathbb{R}^2_+} (x \pm y - K)^+ \mu(dx, dy)$$ take “+” for basket, “-”, for spread

where μ ranges over the space of all risk neutral distributions on $\mathbb{R}^2_+$, subject to the constraints on the marginal distributions

$$(\int_{\mathbb{R}^+} (x - k_1)^+ \mu_X(dx) = C_X(k_1),$$

$$\int_{\mathbb{R}^+} (y - k_2)^+ d\mu_Y(dy) = C_Y(k_2),$$

$$\int_{\mathbb{R}^2_+} \mu(dx, dy) = 1.$$

The Dual Problem

$$\sup_{\nu_1,\nu_2,\lambda} \langle C_X, \nu_1\rangle + \langle C_Y, \nu_2\rangle + \lambda \qquad (2)$$

subject to the constraints

$$\begin{aligned}
&(x \pm y - K)^+ - \int (x - k_1)^+ d\nu_1(k_1) - \int (y - k_2)^+ d\nu_2(k_2) - \lambda \\
&\geq 0, \\
&\forall x \geq 0 \quad y \geq 0.
\end{aligned}$$

where $\nu_i, i = 1, 2$ are signed measures associated to the constraints on the marginals. The optimal ν_i's will turn out to be alternating sums of delta functions.

Warm up Problem: Bounding basket or spread below with portfolio of two

Using the elementary inequality:

$$(a-b)^+ \geq a^+ - b^+$$

We have, by setting $K = K_1 + K_2$

$$\begin{aligned}
&(X+Y-K)^+ \\
&= (X - K_1 - (K_2 - Y))^+ \\
&\geq (X - K_1)^+ - (K_2 - Y)^+
\end{aligned}$$

so, integrating: $\int (X+Y-K)^+ \mu(dx,dy) \geq C_X(K_1) - P_Y(K_2)$.
Similarly, by taking $K_1 - K_2 = K$, we get:

$$\begin{aligned}
&(X-Y-K)^+ \\
&= (X - K_1 - (Y - K_2))^+ \geq (X - K_1)^+ - (Y - K_2)^+
\end{aligned}$$

So $\int (X-Y-K)^+ \mu(dx,dy) \geq C_X(K_1) - P_Y(K_2)$

A Remarkable set of hedging portfolios: The “sheep track portfolios, Case 1, positive weights”

The optimal potfolios associated to the dual problems turn out to be intimately connected to the the sheep track portfolios.
There are two kinds (almost the same):

$$\phi_B(z) = \sum_{i=1}^{n} z^+ + (z - K_i^+)^- - (z - K_i^-)+ \qquad \text{for basket}$$

$$\phi_S(z) = \sum_{i=1}^{n} (z - K_i^-) - (z - K_i^+)^+ \qquad \text{for spread}$$

Sheep-Track Portfolios

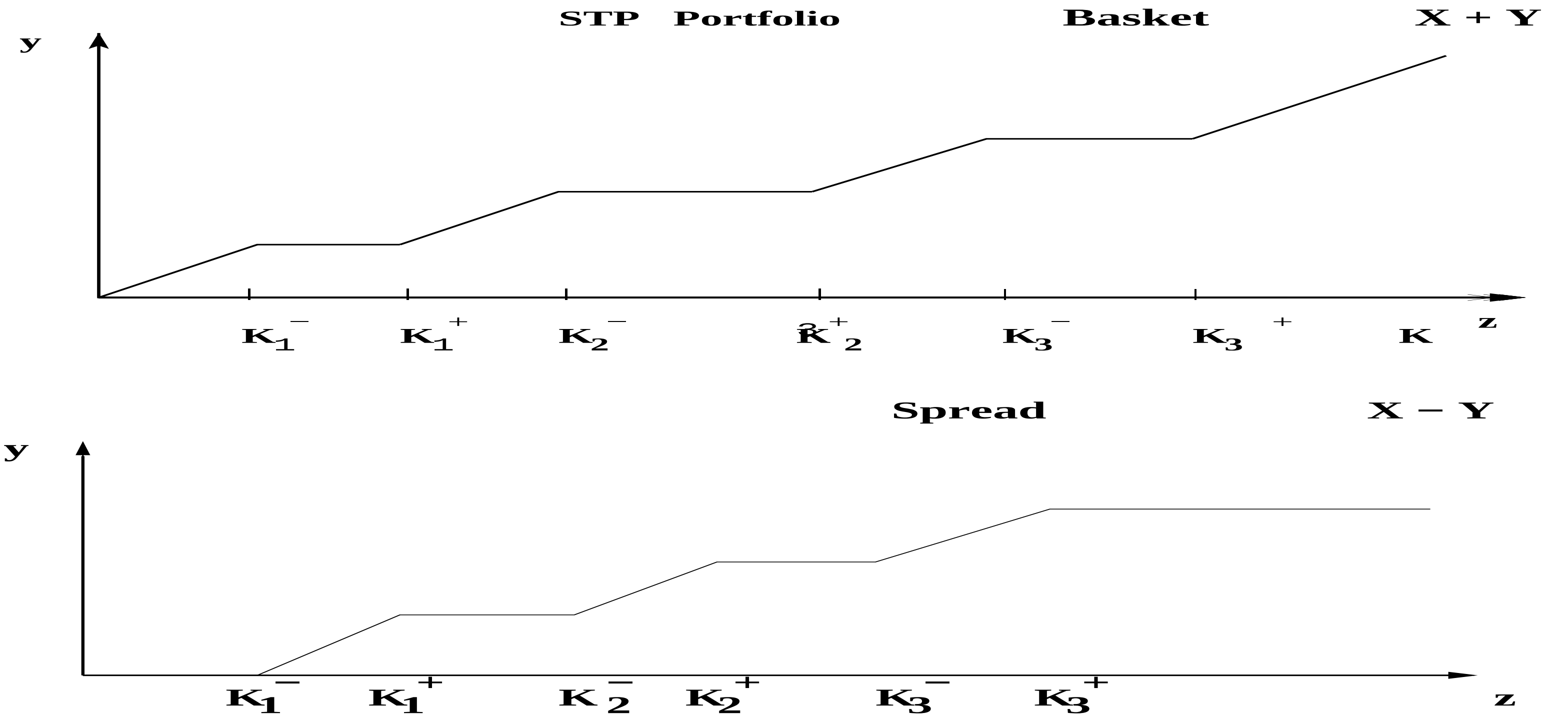


Figure 1: The figure illustrates an STP portfolio which, starting at zero, is piecewise linea with alternating slopes one and zero. The points of transition define $K_i^{\pm}$.

The Optimal subreplicating payoff

Using the Sheeptrack portfolios we have just defined, the key is to observe the following analogues of the elementary inequalities mentioned earlier:

- Basket:

$$\phi_B(x) - \phi_B(K - y) \leq (x + y - K)^+$$

- Spread:

$$\phi_S(x - K) - \phi_S(y) \leq (x - y - K)^+$$

Both of these follow immediately from the mean value theorem, since both ϕ_B and ϕ_S have slope bounded by 1.

What is a sheeptrack portfolio?

Let X and Y denote the two stocks. Let denote the price of a call option on X, resp. Y, by $C_X(\cdot)$, resp. $C_Y(\cdot)$ and similarly for puts by P_X, P_Y. The sheep track portfolios are obtained by taking the expectations of the above sheeptrack payoffs. So we get:

- For Basket:

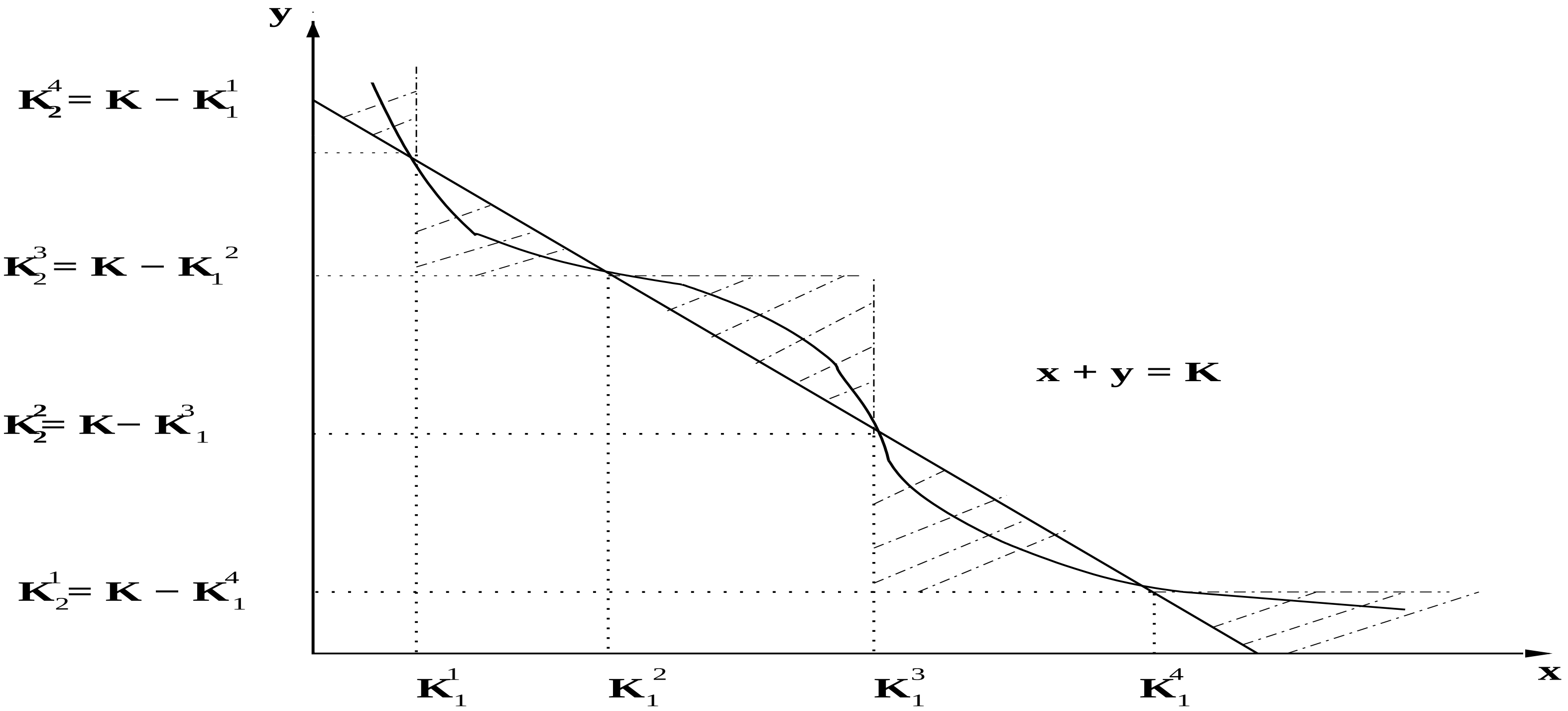
$$X + K - Y + \sum_{i=1}^{n} C_X(K_i^+) - C_X(K_i^-) - \left(\sum_{i=1}^{n} P_Y(K - K_i^+) - \sum_{i=1}^{n} P_Y(K - K_i^-) \right)$$

- For Spread:

$$\sum_{i=1}^{n} C_X(K_i^- + K) - C_X(K_i^+ + K) - \sum_{i=1}^{n} \left(C_Y(K_i^-) - C_Y(K_i^+) \right)$$

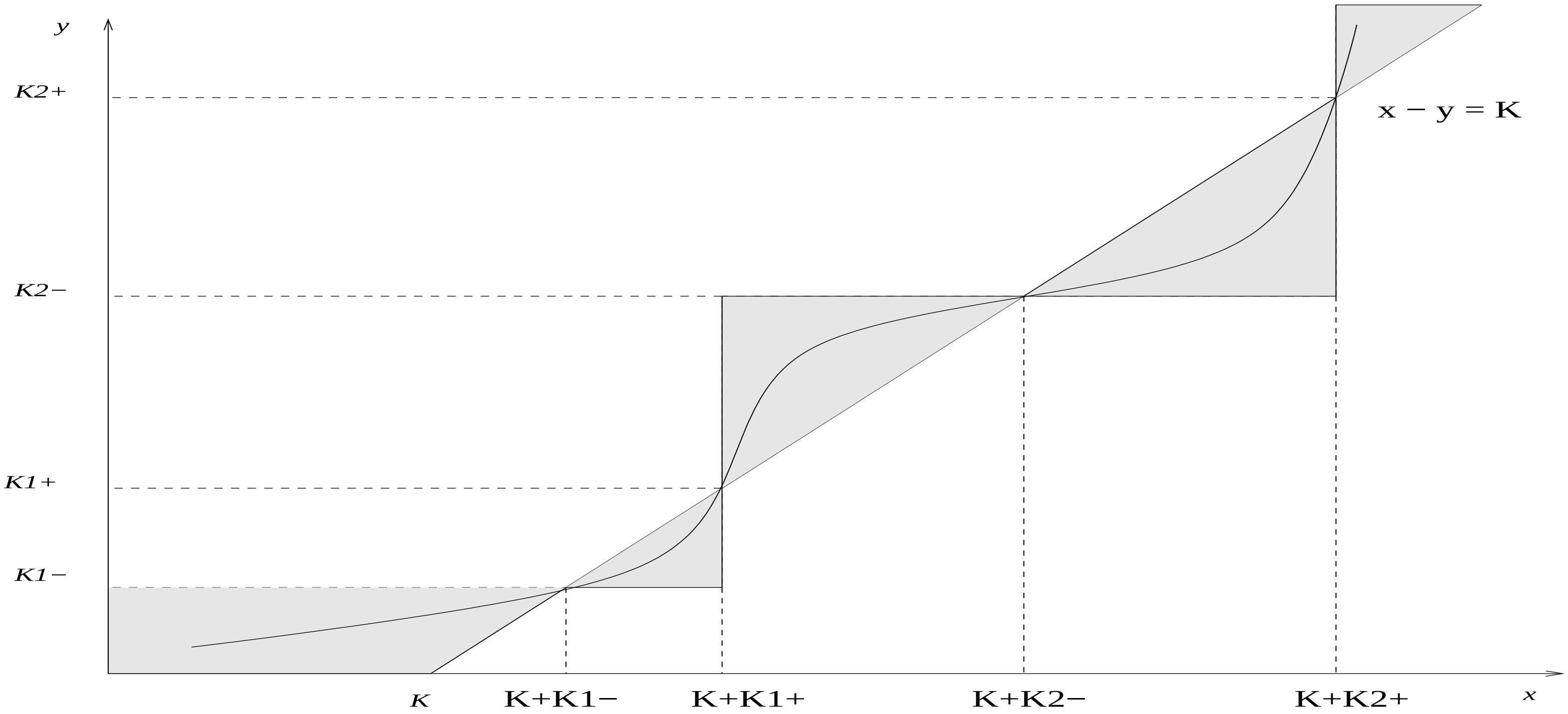
How to determine the $K_i^{\pm}$: Basket

Below, we show the support of a joint antimonotonic distribution with the given marginals in the continuous marginal case, where $Y = F_Y^{-1}(1 - F_X(X))$



How to determine the $K_i^{\pm}$: Spread

Below, we show the support of a joint antimonotonic distribution with the given marginals in the continuous marginal case, where $Y = F_Y^{-1}(F_X(X))$



Using Sheep-Tracks in Black-Scholes setting

Assume geometric brownian motions for both stocks. We took $T = .5$, $S_0^1 = S_0^2 = 100$ anc $\sigma_1 = .355, \sigma_2 = .2$ e $w_1 = w_2 = .5$. In the first column the basket strike is shown. In the seconc column is the Monte Carlo price of the basket in the case where stock S_1 and S_2 are anti-monotonic with the prescribed marginals.

The STP Portfolio - Black-Scholes

Kbask	MC Value	Hedging Portfolio's Value	Strike K_1^1	$StrikeK_1^2$
81.5	18.52	18.50	absent	absent
84	16.03	16.00	absent	absent
86.5	13.55	13.50	absent	absent
89	11.02	11.00	absent	absent
91.5	8.5	8.50	absent	absent
94	5.97	6.00	absent	absent
96.5	3.98	3.99	51.24	89.40
99	2.73	2.69	44.47	101.61
100	2.28	2.29	42.50	105.76
102.5	1.54	1.54	38.52	115.19
105	1.02	1.03	35.41	123.73
107.5	0.69	0.69	32.83	131.73
110	0.45	0.46	30.65	139.30
112.5	0.32	0.31	28.78	146.59
115	0.21	0.21	27.12	153.64
117.5	0.14	0.14	25.64	160.48

Numerical accuracy of hedge: Spread

	$S_1 = 7$		$S_1 = 7.5$		$S_1 = 8$		$S_1 = 8.5$	
K	MC	Hedge	MC	Hedge	MC	Hedge	MC	Hedge
2	1.2698	1.2484	1.6999	1.6607	2.0965	2.1054	2.5728	2.5664
3	0.7226	0.7198	1.0181	1.0228	1.3597	1.3642	1.7837	1.7465
4	0.39702	0.37984	0.58286	0.58533	0.84274	0.8476	1.1252	1.148
5	0.2024	0.20514	0.34267	0.3295	0.50977	0.48878	0.7199	0.7109
6	0.10327	0.005244	0.18111	0.1058	0.29421	0.23783	0.42807	0.40338

Table 1: In this figure we compare the optimal lower bound, achieved by the comonotonic distribution, with the discrete lower bound for various values of the spread option strike K and for various values of the first asset's spot value S_1. The second asset $S_2 = 4$ is taken the same throughout.

Optimal dual: Sheep-Track Portfolio

$$\bar{\nu}_1(dk_1) = \delta_0(k_1)dk_1 + \sum_{i=1}^{2n}(-1)^i \delta_{K_1^i}(k_1)dk_1,$$

$$\bar{\nu}_2(dk_2) = \delta_0(k_2)dk_2 + \sum_{i=1}^{2n}(-1)^i \delta_{K_1^i}(k_2)dk_2,$$

Empirical Work Gap Trading

The next slides report on applying the Methodology to the DJX index.

- The period was February 22, 2008 to April 18'th, 2008 for the first data set and April 2 to April 30 for the second.
- The basket option and the options on the individual asset expired the 21'st of April.
- A few days we were unable to download data. The days with data are indicated in tables tha follow.
- The study considered several different measures:
- 1 - RLGap and Gap
- Implied (lognormal) correlation
- Historical correlations: average and lognormal.

Dates	DJX ATM	Im.Cor	1 - ATM Gap	GapATM	B. ATM 9	% P&L 9	S. ATM 3	
22/02/08	122.00	0.44	0.75	1.38				
25/02/08	124.00	0.38	0.73	1.54				
26/02/08	126.00	0.33	0.72	1.59			1.40	0.00
28/02/08	126.00	0.40	0.72	1.43			1.61	0.02
29/02/08	123.00	0.50	0.78	1.20			1.69	0.08
04/03/08	121.00	0.42	0.75	1.29			1.55	-0.08
05/03/08	122.00	0.40	0.75	1.33			1.31	-0.21
06/03/08	122.00	0.36	0.71	1.47			1.36	-0.16
07/03/08	119.00	0.52	0.80	1.05	1.05	0.00	1.44	-0.10
10/03/08	117.00	0.48	0.80	1.12	1.22	0.16	1.53	-0.14
11/03/08	122.00	0.47	0.76	1.18	0.79	-0.25	1.14	-0.23
12/03/08	121.00	0.45	0.76	1.24	1.00	-0.04	1.27	-0.29
13/03/08	122.00	0.39	0.72	1.36	1.14	0.09	1.29	-0.34
14/03/08	119.00	0.53	0.81	1.00	1.18	0.12	1.29	-0.14
17/03/08	120.00	0.44	0.76	1.26	1.38	0.32	1.39	-0.26
19/03/08	121.00	0.44	0.75	1.20	1.30	0.23	1.12	-0.32
02/04/08	126.00	0.28	0.77	0.63	1.01	-0.04	1.38	-1.13
03/04/08	126.00	0.18	0.71	0.79	1.28	0.22	0.64	-1.25
07/04/08	127.00	0.23	0.70	0.67	1.52	0.45	0.37	-1.39
09/04/08	125.00	0.38	0.83	0.39	0.83	-0.21	0.38	-0.26
10/04/08	125.00	0.23	0.70	0.69	1.24	0.18	0.31	-0.30
11/04/08	123.00	0.23	0.75	0.52	2.02	0.92	0.13	0.26
14/04/08	123.00	0.17	0.66	0.66	0.99	-0.05	0.13	0.27
15/04/08	124.00	0.04	0.50	0.76	0.94	-0.11	0.06	0.16
16/04/08	126.00	0.17	0.77	0.33	2.74	1.61	0.05	-0.96
17/04/08	126.00	0.00	0.74	0.28	1.59	0.51	0.04	-1.07
18/03/08	128.00	0.26	0.00	0.76			0.04	-2.64

Table 1: This table illustrates the results of trading the comonotoncity gap. We use the ATM gap in column 5 or one minus the relative ATMgap (ie. Index/UB) as a *signal*. Buying the gap when it is relatively cheap, here on March 3'd, we then sell the index option struck at 119, and buy the options with optimally chosen strikes. These strikes are not necessarily ATM. Our P&L is then determined by the evolution in time of the prices of that basket option and of the portfolio of options on single names. When we sell, we are guaranteed to have a riskless situation at expiry, due to superreplicating nature of portfolio. The percentage P&L of the strategy is illustrated in column 6. It is quite successful. A sell strategy on 26 of Feb, when gap is relatively large is less successful and riskier since no downside protection at expiration

	Im. Cor	Gap ATM	DJX	GapOFM (ATM +3)	1-RLGap	OTM, 126 ATM+3	% P&L OTM(126)	Av. His Cor
22/02/08	0.44	1.38	121.70	1.40	0.65			0.61
25/02/08	0.38	1.54	124.30	1.61	0.62			0.86
26/02/08	0.33	1.59	126.45	1.69	0.59			0.55
28/02/08	0.40	1.43	125.72	1.55	0.59			0.79
29/02/08	0.50	1.20	122.66	1.31	0.67	1.31	0.00	0.53
04/03/08	0.42	1.29	120.77	1.36	0.64	1.50	0.14	0.52
05/03/08	0.40	1.33	122.21	1.44	0.63	1.70	0.30	0.15
06/03/08	0.36	1.47	121.57	1.53	0.59	1.64	0.25	0.82
07/03/08	0.52	1.05	119.07	1.14	0.71	1.55	0.18	0.77
10/03/08	0.48	1.12	117.40	1.27	0.69	1.60	0.22	0.77
11/03/08	0.47	1.18	121.57	1.29	0.62	1.70	0.30	0.57
12/03/08	0.45	1.24	121.10	1.29	0.64	1.77	0.35	0.61
13/03/08	0.39	1.36	121.69	1.39	0.60	1.86	0.42	0.61
14/03/08	0.53	1.00	119.03	1.12	0.71	1.59	0.21	0.61
17/03/08	0.44	1.26	119.72	1.38	0.64	1.80	0.37	0.61
19/03/08	0.44	1.20	121.00	0.64	1.00	1.88	0.44	0.38
02/04/08	0.28	0.63	126.00	0.37	0.09	3.11	1.37	0.40
03/04/08	0.18	0.79	125.84	0.38	0.06	3.34	1.55	0.42
07/03/08	0.23	0.67	126.53	0.31	0.06	3.60	1.75	0.35
09/03/08	0.38	0.39	125.27	0.13	0.06	1.76	0.34	0.31
10/03/08	0.23	0.69	125.30	0.13	0.13	1.83	0.40	0.41
11/03/08	0.23	0.52	123.25	0.06	0.13	0.92	-0.29	0.44
14/03/08	0.17	0.66	123.02	0.05	0.24	0.91	-0.31	0.39
15/03/08	0.04	0.76	123.62	0.04	0.29	1.10	-0.16	
16/03/08	0.17	0.33	126.19	0.04	0.35	2.71	1.07	
17/03/08	0.00	0.28	126.20	0	0.33	2.90	1.22	
18/03/08	0.26	0.76	128.40			5.42	3.13	

Table 2: This table illustrates a gap trading strategy on a basket option that is out of the money by three dollars. The signal for the buy strategy is a relatively low gap on February 29-th. The strike of the OTM basket option is 126 (ATM is 123). This strategy proves to be extremely successful as illustrated by its P&L in column 7. The last colum illustrates the historical correlation caculated with four day periods starting February 22'd. This and the implied correlation in column two provide additional information which can be potentially used in conjunction with the gap trading. This needs to be explored further.

Ticker		Closest ATM						OTM(+3)		
	Spot	Str. date 9					Date5	Str.		
MMM	77.08	75	75	75		78.4	80	80	80	
AA	36.54	37.5	35	35		37.14	37.5	37.5	37.5	
MO	74.23	75	75	75		73.14	75	75	75	
AXP	41.59	42.5	42.5	40	Interp.	42.3	42.5	42.5	45	Interp.
AIG	43.08	45	45	45		46.86	45	50	50	
T	35.15	35	35	35		34.83	35	35	35	
BA	77.33	75	75	75		82.79	85	85	85	
CAT	69.49	70	70	70		72.33	72.5	75	75	
C	20.92	20	20	20		23.71	22.5	25	25	
KO	59.29	60	60	57.5		58.46	57.5	60	60	
DD	45.23	45	45	45		46.42	47.5	47.5	47.5	
XOM	83.01	85	85	85		87.01	85	90	90	
GE	32.39	32	32	32		33.14	33	34	34	
GM	21.92	22.5	22.5	22.5		23.28	22.5	25	25	
HPQ	47.24	47.5	47.5	47.5		47.77	47.5	50	50	
HD	25.84	25	25	25		26.55	27.5	27.5	27.5	
HON	56.73	57.5	57.5	57.5		57.54	57.5	60	60	
INTC	20.05	20	20	20		19.97	20	21	21	
IBM	113.42	115	115	115		113.86	115	115	115	
JNJ	61.85	60	60	60		61.96	60	65	65	
JPM	37.75	37.5	37.5	37.5		40.65	40	42.5	42.5	
MCD	52.55	52.5	52.5	52.5		54.11	55	55	55	
MRK	42.25	42.5	42.5	42.5		44.3	45	45	45	
MSFT	27.62	27.5	27.5	27.5		27.2	27.5	27.5	27.5	
PFE	21.39	22.5	22.5	22.5		22.28	22.5	22.5	22.5	
PG	66.08	65	65	65		66.18	65	67.5	67.5	
UTX	68.19	70	70	70		70.51	70	70	70	
VZ	35.22	35	35	35		36.32	37.5	37.5	37.5	
WMT	50.43	50	50	50		49.59	50	50	50	
DIS	30.93	30	30	30		32.41	32.5	32.5	32.5	

Table 3: This table illustrates the right choice of strikes to invest in for a buy strategy at date 9 (columns 2-5) and for a buy strategy at date 5. For this dataset the right strikes for the ATM strategy at date 9 are the same as ATM strikes on that date except for AXP that needs to be interpolated between two strikes. For the more successful OTM (+3) strategy at date 5, some strikes are still ATM and others are not. We also need to interpolate AXP, ie. choose a linear combination of two adjacent strikes.

	DJX	ATM K	I. corr	1- RL	Gap	Freeze	P&L
4/2/2008	126	126	0.25	0.74	1.22	0	
4/3/2008	125.84	126	0.23	0.73	1.31	0	
4/7/2008	126.53	127	0.26	0.72	1.21	1.22	0
4/8/2008	125.43	125	0.28	0.76	1.17	1.26	0.04
4/9/2008	125.27	125	0.34	0.79	0.99	1.23	0.02
4/10/2008	125.3	125	0.27	0.73	1.25	1.46	0.20
4/11/2008	123.25	123	0.27	0.75	1.14	1.65	0.36
4/14/2008	123.02	123	0.27	0.73	1.17	1.79	0.48
4/15/2008	123.62	124	0.20	0.67	1.34	1.82	0.50
4/16/2008	126.19	126	0.24	0.72	1.13	1.59	0.32
4/17/2008	126.2	126	0.26	0.73	1.04	1.59	0.32
4/18/2008	128.4	128	0.19	0.68	1.22	1.76	0.45
4/21/2008	128.25	128	0.19	0.67	1.13	1.79	0.48
4/22/2008	126.86	127	0.25	0.69	1.00	1.63	0.35
4/29/2008	128.8	129	0.20	0.67	0.77	1.52	0.26
4/30/2008	128.2	128	0.34	0.81	0.45	1.27	0.05

Table 1: Trading the gap on **May 2008** maturity ATM basket option. We buy the Gap on April 7'th, since 1 - RLG is low (though not particularly low) by selling the index option struck at 127 and by buying the portfolio with the optimally selected strikes. The P&L of this trade is positive on all successive dates.

Results ATM Gap

See Tables in next slides

Conclusions

- We have discussed the market implied comonotonicity gap as a tool for dispersion trading. Here it has been illustrated empirically in the case of spread options.
- Many open problems: – Closed Form Lower bound for basket options for more than two assets, forward prices and one or more strikes (One strike, no forward, OK d’Aspremont-El Ghaoui, 2006)
  –Closed form lower bound for two assets spread options and more than one strike constraint. (One strike OK + forward prices OK).
- Add constraints on the correlation(s).

- Statistical testing needed to determine optimal time to enter into a Gap strategy. Studies of profit and loss over periods of a year or more needed.
- To monitor and/or guidelines to invest in gap: http://www.indexandspreadoptions.com

Slogan: MIND THE GAP !