Lemma 1 The Lie algebra of the stabilizer G_m is $\mathfrak{g}_m = \{X \in \mathfrak{g} \mid X_M|_m = 0\}$.

pf Since G_m is a Lie subgroup of G,

$\mathfrak{g}_m = \{X \in \mathfrak{g} \mid \exp(tX) \in G_m \text{ for all } t \in \mathbb{R}\}$.

Suppose $X \in \mathfrak{g}_m$, then $\exp(tX) \in G_m$ $\forall t$ and so

$\exp(tX) \cdot m \equiv m$ and hence it is a constant curve.

Thus its tangent is zero: $X_M|_m = 0$.

Suppose that $X_M|_m = 0$. I must show that $\exp(tX) \in G_m$ $\forall t$.

Hence it is sufficient to prove that $\exp(tX) \cdot m$ is a constant curve: $\frac{d}{dt}(\exp(tX) \cdot m)\Big|_{t=s} = \frac{d}{dt}((\exp(sX))(\exp(tX)) \cdot m)\Big|_{t=0} =$

$= (\exp(sX))_* (X_M|_m) = 0$ by the hypothesis and the definition of the push-forward.

Lemma 2 The map $G/G_m \to M$, $aG_m \mapsto a\cdot m$, is a one to one immersion. If the action is proper, the orbit $G\cdot m$ is an embedded submanifold in M, and the map $aG_m \mapsto a\cdot m$ is a diffeomorphism between G/G_m and the orbit $G\cdot m$.

Pf The map is clearly 1-1. The diff. at e of the map $a \mapsto a\cdot m$ is $X \to X_M|_m$. By lemma 1, its kernel is $\mathfrak{g}_m$. ~~Hence~~ The tangent space of G/G_m is $\mathfrak{g}/\mathfrak{g}_m$. Hence the map is an immersion at the base point eG_m. By equivariance, it is an immersion everywhere. If the action is proper, the map $aG_m \mapsto a\cdot m$ is proper as a map to $G\cdot m$. Being a proper 1-1 ~~map~~ immersion, the map is an embedding.

__Lemma 3__ The tangent space to the orbit $G \cdot m$ is $T_m(G \cdot m) = \{X_M \mid X \in \mathfrak{g}\}$.

If this space is all of $T_m(M)$, the orbit of m contains a nbhd of m.

__pf__ The first assertion follows from Lemma 2. The second follows from the inverse function thm.

__prop 1__ Let a compact Lie group G act on a manifold M and let $m \in M^G$ be a fixed point. Then $\exists$ a G-equivariant diff from a nbhd of the origin in $T_m M$ onto a nbhd of m in M.

__pf__ This was shown in ~~Ban~~ a previous talk. (All you really need is a G-equivariant exponential map).

Defn 1 A principal G-bundle over a manifold M is a manifold M is a manifold P, a free right ~~action~~ G-action on P, and a map $\pi : P \to M$ whose level sets are exactly the orbits of G, such that every point in M has a nbhd U and a diff. φ, $\varphi : \pi^{-1}(U) \to U \times G$ that sends the fiber $\pi^{-1}(m)$ to the fiber $\{m\} \times G$ and that is equivariant with respect to the G-action which, on $U \times G$, is given by multiplication on the right in the second component.

Defn 2 A rank k vector bundle over a manifold M is a manifold E, a map $\pi : E \to M$, and the structure of a k-dim. vector space on each fiber $\pi^{-1}(m)$, such that every point in M has a nbhd U and a vectorspace V and a diff $\varphi : \pi^{-1}(U) \to U \times V$ that sends the fiber $\pi^{-1}(m)$ to the fiber $\{m\} \times V$ and that is a linear isomorphism on each fiber.

Defn 3
Let $P \to M$ be a principal H-bundle, and let H act linearly on a vector space W. The associated bundle is $P \times_H W := (P \times W)/H$, where the action of H on $P \times W$ by which one quotients is

$$a : (p, w) \mapsto (pa^{-1}, a \cdot w).$$

Denote by $[p, w]$ the equivalence class of $(p, w) \in P \times W$ in $P \times_H W$, so that $[pa, w] = [p, aw]$ for all $a \in H$.
The projection $P \to M$ induces the map $P \times_H W \to M$ which sends $[p, w] \mapsto \pi(p)$.

Lemma 4 The above construction yields a vector bundle over M.

Pf Let $U \subset M$ be an open set whose preimage in P is $U \times H$. Then its preimage in $P \times_H W$ is $(U \times H) \times_H W = U \times W$.

Let a Lie group G act smoothly and properly on a manifold M. Let G_m denote the stabilizer of a point $m \in M$. For all $a \in G_m$, the differential of $a: M \to M$ sends $T_m M \to T_m M$. These maps form a linear representation of G_m on $T_m M$. The tangent space to the orbit, $T_m(G \cdot m)$, is a G_m invariant subspace of $T_m M$. Since the action is proper, the stabilizer G_m is compact. Hence, there exists a G_m invariant decomposition, $T_m(M) = T_m(G \cdot m) \oplus W$, where W is the normal to the orbit with respect to any G_m invariant inner product. Since G_m acts on W, we can form the associated bundle $G \times_{G_m} W$. Let D be a small disc in W around the origin w.r.t. some G_m invariant metric.

<u>Thm 1</u> (Slice Thm). There exists a G-equivariant diffeomorphism from the disc bundle $G \times_{G_m} D$ onto a neighborhood of the orbit $G \cdot m$ in M, whose restriction to the zero section $G \times_{G_m} \{0\} = G/G_m$ is the map $aG_m \mapsto a \cdot m$.

<u>Pf</u> Since the action is proper, the stabilizer G_m is compact. By prop 1 , there exists a G_m-equivariant diffeomorphism, E: (a nbhd of 0 in $T_m M$) $\to$ (nbhd of m in M). Let $D \subset W$ be a ball small enough to be contained in the domain of E. The map $\Psi : G \times_{G_m} D \to M$, $[a,v] \mapsto a \cdot E(v)$, is well defined and is a local diffeomorphism at the point $[e,0]$. Note that Ψ is G-equivariant: $\Psi(g[a,v]) = \Psi([ga,v]) = ga \cdot E(v) = g\Psi([a,v])$. By G-equivariance, Ψ is a local diffeomorphism at all points of the form $[a,0]$. It remains to show that Ψ is one-to-one if D is sufficiently small.

Assume the contrary, then $\exists u_n, v_n \to 0$ in W and $a_n, b_n \in G$ s.t. $[a_n, u_n] \neq [b_n, v_n]$ and such that $a_n \cdot E(u_n) = b_n \cdot E(v_n)$. WLOG, assume $b_n = e$ (else act by b_n^{-1}). Then $a_n \cdot E(u_n) = E(v_n) \to m$. This does not imply that $a_n \to e$. But since the action is proper note that under the map $G \times M \to M \times M$ the sequence $(a_n, E(u_n))$ goes to the converging sequence $(E(v_n), E(u_n)) \to (m, m)$. ~~Since~~ Hence properness implies the ~~exiting~~ existence of a converging subsequence $a_{n_j} \to a_\infty$, and so this leads to a contradiction with the fact that Ψ is a local diffeomorphism at $[a_\infty, 0]$.