LINEAR ACTION OF A TORUS: GUIDED EXERCISE

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Let $T \cong (S^1)^r$ and suppose that we are given a linear symplectic T-action on some symplectic vector space (V, ω) of dimension $2n$.

Step I: Show that there is a T-invariant positive inner product g on V. (Hint: start from any positive inner product on V, and use the fact that T is compact to average along T.)

Step II: Since ω and g are nondegenerate the equation

$$\omega(u, v) = g(Au, v) \tag{0.1}$$

is satisfied for some linear map $A\colon V \to V$. After some manipulation on A, you get a complex structure J (i.e., $J^2 = -\operatorname{Id}$) that is compatible with ω. (This is proven in A. C. da Silva's book in section 12.2 "Complex Structures on Vector Spaces".) Show that the positive inner product $\omega(u, Jv)$ is again T-invariant.

Step III: Prove the general claim: Let (V, ω) be a symplectic vector space and J a complex structure on V. Show that J is compatible with ω iff the form $H\colon V \times V \to \mathbb{C}$ defined by

$$H(u, v) = \omega(u, Jv) + i\omega(u, v)$$

is complex linear in v, complex anti-linear in u, satisfies $H(v, u) = \overline{H(u, v)}$, and has a positive-definite real part. Such a form is called a *Hermitian inner product* on (V, J).

Deduce from steps II and III that there is a T-invariant Hermitian form on (V, J). In other words, T is a subtorus of a maximal torus in $U(n)$, (where $U(n)$ consists of the linear transformations of V that preserve the Hermitian inner product). Notice that (V, J) is understood as a complex vector space.

Step IV: Prove the following (linear algebra)-claim: Let $A_1, \ldots, A_r$ be commuting linear operators on a complex vector space V.

(1) If each A_i is diagonalizable, then $A_1, \ldots, A_r$ are simultaneously diagonalizable, i.e., there is a basis of simultaneous eigen-vectors for the A_i-s.

(2) If $H\colon V \times V \to \mathbb{C}$ is a Hermitian inner product and each A_i is unitary with respect to H, then there is an orthonormal basis consisting of simultaneous eigen-vectors for the A_i-s. For

this part, assume that $A_1, \ldots, A_r$ and the inverses $A_1^{-1}, \ldots, A_r^{-1}$ commute. (First, explain why, in this part, $A_1, \ldots, A_r$ are invertible.)

(Hint for part I: show this by induction on r; in the induction step: let A_r have an eigenvalue λ, and let $E_\lambda = \{v \colon A_r v = \lambda v\}$ be the λ-eigenspace for A_r; show that $A_i(E_\lambda) \subset E_\lambda$ for all i; notice that by hypothesis, the whole space is the direct sum of the eigenspaces for A_r.)

(Hint for part II: consider the orthogonal complement $E_\lambda^\perp$ of E_λ; use induction on $\dim V$.)

Deduce that there is an orthonormal basis consisting of simultaneous eigen-vectors corresponding to the simultaneous eigen-values (to be called weights) $\alpha_1, \ldots, \alpha_n$ of the elements of $\mathfrak{t}$, (where $\mathfrak{t}$ is the Lie algebra of T). (Here the α_i are linear functions on $\mathfrak{t}$.) (Recall: to move from T to the Lie algebra $\mathfrak{t}$ we use the exp map.)

This means, (explain how), that there is a symplectic linear isomorphism of V with $\mathbb{R}^{2n}$, such that the linear symplectic action of $\mathfrak{t}$ on (V, ω) is given by the homomorphism $\rho \colon \mathfrak{t} \to \mathbb{R}^n$,

$$\rho(\xi) = (\alpha_1(\xi), \ldots, \alpha_n(\xi)).$$

Step V: show that the moment map of the T-action on (V, ω) is the composition of the standard moment map of $T^n = (S^1)^n$ on $\mathbb{R}^{2n}$,

$$\Psi_{T^n}(p_1, \ldots, p_n, q_1, \ldots, q_n) = (\frac{1}{2}({p_1}^2 + {q_1}^2), \ldots, \frac{1}{2}({p_n}^2 + {q_n}^2)). \tag{0.2}$$

with $\rho^* \colon \mathbb{R}^n \to \mathfrak{t}^*$, and so is given by

$$\Phi(q_1, \ldots, q_n, p_1, \ldots, p_n) = \frac{1}{2}({p_1}^2 + {q_1}^2)\alpha_1 + \ldots + \frac{1}{2}({p_n}^2 + {q_n}^2)\alpha_n.$$

Deduce that the image of Φ is the convex region in $\mathfrak{t}^*$

$$S(\alpha_1, \ldots, \alpha_n) = \{\sum_{i=1}^{n} s_i \alpha_i \ \mid \ s_1, \ldots, s_n \geq 0\}, \tag{0.3}$$

the α_i-s being the weights of the representation of T on V.