GROUP ACTIONS ON SYMPLECTIC MANIFOLDS: LECTURE 6

LIAT KESSLER

1. The convexity theorem

Let (M, ω) be a connected compact symplectic manifold, T a torus, $T \times M \to M$ a Hamiltonian action of T on M, and $\Phi \colon M \to \mathfrak{t}^*$ the assoaciated moment map. We claim that the image of Φ is a convex polytope. This is the convexity theorem (Atiyah, Guillemin and Sternberg, 1982).

We saw before that $\Phi(M)$ is a finite union of convex sets. We show now that $\Phi(M)$ looks like a convex polytope near each of its boundary points, and conclude the convexity theorem. (The proof given here is from the course's textbook, by Guillemin and Sternberg. Atiyah's proof will be given in Maria Calle's talk.)

Let p be a point in the boundary of $\Phi(M)$, and $m \in \Phi^{-1}(p)$; let T_1 be the stabilizer group of m. By the generalized local convexity theorem, there exists a neighbourhood U of m and U' of p such that $\Phi(U) = U' \cap (p + S'(\alpha_1, \dots, \alpha_n)$, where $\alpha_i \in \mathfrak{t}^*$ are the weights of the representation of T_1 on $T_m M$. It is enough to show that $\Phi(M) \subseteq p + S'(\alpha_1, \dots, \alpha_n)$.

To see that, Let S_i be a boundary component of $S'(\alpha_1, \dots, \alpha_n)$. Choose $\xi \in \mathfrak{t}$ such that $e_\xi = 0$ on S_i and e_ξ is negative on the interior of $S'(\alpha_1, \dots, \alpha_n)$, where $e_\xi \colon \mathfrak{t}^* \to \mathbb{R}$ is the evaluation $f \mapsto f(\xi)$. (Choosing ξ is choosing 'directions' that are outward normal to S_i; $\mathfrak{t}^{**} = \mathfrak{t}$) Then, if $e_\xi(p) = a$,

$$\Phi^\xi(x) = (e_\xi \circ \Phi)(x) \leq a$$

for $x \in U$, so a is a local maximum of Φ^ξ. Recall that Φ^ξ has a unique local maximum (as a Morse-Bott function with even indices and dimensions of crit. manifolds). Hence, applying the above argument to all faces S_i of S' we get $\Phi(M) \subseteq p + S'(\alpha_1, \dots, \alpha_n)$.

Example. The standard toric action on $\mathbb{CP}^2$ is $(a, b) \cdot [z_0 : z_1 : z_2] = [z_0 : az_1 : bz_2]$. The moment map image is the triangle

$$\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1\}. \tag{1}$$

Whenever we refer to $\mathbb{CP}^2$ as a symplectic manifold, we assume that the symplectic form is ω_{FS}, normalized so that

$$\frac{1}{2\pi}\int_{\mathbb{CP}^1}\omega_{\mathrm{FS}} = 1.$$

The Delzant theorem. If $\dim T = \frac{1}{2}\dim M$, the triple (M,ω,Φ) is a *symplectic toric manifold*, and the T-action is called *toric*. By the Delzant theorem, (M,ω,Φ) is determined by Δ up to an equivariant symplectomorphism, preserving Φ.

The inverse image under Φ of a vertex of Δ is a fixed point for the T-action, and the image of a T-fixed point is a vertex of Δ.

A necessary condition for Δ to occur as the moment map image of a symplectic toric manifold is that it be a *Delzant polytope*, meaning that the edges emanating from each vertex are generated by vectors $v_1,\ldots,v_n$ that span the lattice $\mathbb{Z}^n$.