CONVEXITY THEOREM

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Theorem 1. Convexity theorem

Let (M^{2n},ω) be a compact, connected symplectic manifold, T^m a torus, $\psi : T^m \longrightarrow Sympl(M,\omega)$ a hamiltonian action with moment map $\mu : M \longrightarrow \mathbf{R}^m$. Then:

1.- The level sets of μ are connected.

2.- The image of μ is convex.

3.- The image of μ is the convex hull of the images of the fixed points of ψ.

Proof. **Atiyah, 1982**

We prove the first two claims by induction over m, the dimension of the torus. Consider the statements:

A_m: "The level sets of μ are connected, for any T^m-action".
B_m: "The image of μ is convex, for any T^m-action".

We will prove first A_1 and B_1, and then prove that A_{m-1} implies both A_m and B_m.

- A_1: follows from the following lemma in Morse Theory:

  Lemma 1. *Let $f : M \longrightarrow \mathbf{R}$ be a Morse-Bott function on a compact manifold M so that the critical manifolds have all have index and coindex different than 1. Then $f^{-1}(c)$ is connected for all $c \in \mathbf{R}$.*

  A moment map $\mu : M \longrightarrow \mathbf{R}$ of a hamiltonian T^1-action is a Morse-Bott function, and we have seen that the critical manifolds have even index and dimension, therefore we can apply the lemma to obtain A_1.

- B_1: for a T_1-action, the moment map $\mu : M \longrightarrow \mathbf{R}$ is continuous and M is connected, hence the image of μ is connected in $\mathbf{R}$, which implies it is convex.

- $A_{m-1} \Rightarrow A_m$

First, we can assume that ψ is an effective action. If ψ is effective, then for any $X \in \mathbf{R}^m - \{0\}$, the component μ^X is not constant. Now, we can prove easily that:

$$Crit(\mu^X) = \bigcap_{\theta \in T_X} Fix(\psi_\theta) \tag{1}$$

where T^X is the closure of the subgroup of T generated by X.

We want to prove that $\mu^{-1}(\xi)$ is connected, for all $\xi \in \mathbf{R}^m$. We will first show it for all $\xi = (\xi_1, \ldots, \xi_m)$ so that $(\xi_1, \ldots, \xi_{m-1})$ is a regular value for the reduced moment map $(\mu_1, \ldots, \mu_{m-1})$. The set A of such ξ is dense in the image of μ, by Sard's theorem. By continuity, this proves the result for all values of ξ.

Let ξ be an element of A. By induction hypothesis, the set:

$$Q = \bigcap_{j=1}^{m-1} \mu_j^{-1}(\xi_j) = (\mu_1, \ldots, \mu_{m-1})^{-1}(\xi_1, \ldots, \xi_{m-1}) \tag{2}$$

is connected.

Now consider the function $\mu_m : Q \longrightarrow \mathbf{R}$. A point $x \in Q$ is critical for $\mu_m|_Q$ if and only if there exist real numbers $\theta_1, \ldots, \theta_{m-1}$ such that

$$\sum_{j=1}^{m-1} \theta_j d\mu_j(x) + d\mu_m(x) = 0. \tag{3}$$

This means that x is a critical point of the function μ^X, where $X = (\theta_1, \ldots, \theta_{m-1}, 1)$. Let $C \subset M$ be the critical connected manifold of μ^X which contains x, which we know is even-dimensional and have even index (since all the components of the moment map are Morse-Bott functions). We observe that C intersects Q transversally at x, that is, $T_xM = T_xC + T_xQ$. This is true, since the linear functionals $d\mu_j(x) : T_xM \longrightarrow \mathbf{R}, j = 1, \ldots, m-1$ are linearly independent when restricted to the subspace T_xC.

This transversality implies that the subspace $T_xQ \cap T_xC^\perp$ is a complement of T_xC in T_xM. Hence the Hessian of μ^X is nondegenerate on it, with even index and coindex. In other words, $C \cap Q$ is a critical manifold of $\mu^X|_Q$ of even index and coindex. The same holds for $\mu_m|_Q$, since it only differs from μ^X by a constant. Thus we have prove that the function $\mu_m : Q \longrightarrow \mathbf{R}$ has only critical manifolds of even index and coindex. Hence, by the lemma cited above, the level sets

$$\mu^{-1}(\xi) = Q \cap \mu_m^{-1}(\xi_m) = \mu|_Q^{-1}(\xi) \tag{4}$$

are connected for every $\xi \in A$. This finish the proof of A_m.

- $A_{m-1} \Rightarrow B_m$

Let $A \in \mathbf{Z}^{m\times(m-1)}$ be an injective matrix. Then, for a torus T^{m-1}, define the action:

$$\psi_A : T^{m-1} \longrightarrow Sympl(M,\omega) \tag{5}$$

$$\theta \longrightarrow \psi_{exp(A\xi)} \tag{6}$$

where $\theta = exp(\xi)$ for $\xi \in [0,1)^{m-1}$. We will prove that ψ_A is a hamiltonian action with moment map $\mu_A = A^t\mu : M \longrightarrow \mathbf{R}^{m-1}$.

Indeed, let $\xi_j = (0,\dots,1,\dots,0) \in \mathbf{R}^{m-1}$ be the standard basis of $\mathbf{R}^{m-1}$ and $X_j(x) = \frac{d}{dt}|_{t=0}\psi_A(exp(t\xi_j)) \cdot x$ the vector field generated by ξ_j and ψ_A. Let $\eta_j \in \mathbf{R}^m$ and $Y_j(x)$ the analogous basis of $\mathbf{R}^m$ and vector fields generated by ψ. Then:

$$X_j(x) = \frac{d}{dt}|_{t=0}\psi_A(exp(t\xi_j)) \cdot x = \frac{d}{dt}|_{t=0}\psi(exp(tA\xi_j)) \cdot x \tag{7}$$

$$\frac{d}{dt}|_{t=0}\psi(exp(tA_j)) \cdot x = \frac{d}{dt}|_{t=0}\psi(exp(t\sum_{k=1}^{m} a_{kj}\eta_k)) \cdot x \tag{8}$$

$$\frac{d}{dt}|_{t=0}\prod_{k=1}^{m}\psi(exp(ta_{kj}\eta_k)) \cdot x = \sum_{k=1}^{m} a_{kj}Y_k(x), \tag{9}$$

where $A_j = (a_{1j},\dots,a_{mj})$ are the columns of A, the second equality comes from the definition of ψ_A, the fifth equality comes from the fact that the exponential map is a homomorphism, and the last equality comes from the product rule and the definition of Y_k. Then, since μ is the moment map of ψ, we have:

$$-i_{X_j}\omega = -i_{\sum_{k=1}^{m} a_{kj}Y_k}\omega = -\sum_{k=1}^{m} a_{kj}i_{Y_k}\omega = \sum_{k=1}^{m} a_{kj}\mu_k = (A^t\mu)_j. \tag{10}$$

Take now any $p_0 \in \mu_A^{-1}(\xi)$, for $\xi \in \mathbf{R}^{m-1}$. Then:

$$p \in \mu_A^{-1}(\xi) \Leftrightarrow A^t\mu(p) = \xi = A^t\mu(p_0). \tag{11}$$

Therefore $\mu_A^{-1}(\xi) = \{p \in M | \mu(p) - \mu(p_0) \in ker(A^t)\}$. By hypothesis (A_{m-1}), $\mu_A^{-1}(\xi)$ is connected. Take p_t a path connecting p_0 to p_1. Then $\gamma(t) = \mu(p_t) - \mu(p_0)$ is a path in $ker(A^t)$ connecting 0 to $\mu(p_1) - \mu(p_0)$. But $ker(A^t)$ is 1-dimensional, hence $\gamma(t)$ goes through any convex combination of 0 and $\mu(p_1) - \mu(p_0)$, that is, $\mu(p_t)$ goes through any convex combination of $\mu(p_0)$ and $\mu(p_1)$. But this implies:

$$t\mu(p_0) + (1-t)\mu(p_1) \in \mu(M)\, \forall t \in [0,1]. \tag{12}$$

This is true for any p_0 and p_1 such that there is an injective matrix A with $A^t(\mu(p_0)) = A^t(\mu(p_1))$. For two arbitrary points p_0 and p_1 in M, we can choose a matrix A and points p_0' and p_1' in M arbitrarily close to p_0 and p_1 respectively with the condition above. Taking limits, we get that:

$$t\mu(p_0) + (1-t)\mu(p_1) \in \mu(M)\, \forall t \in [0,1], \tag{13}$$

for any p_0 and p_1 in M, which proves that $\mu(M)$ is convex.

Finally, we want to prove that $\mu(M)$ is the convex hull of the images of the fixed points of the action. Consider the set $C = \{\text{fixed points of } \psi\} = C_1 \cup \cdots \cup C_N$. Here the C_i's are the connected components of C, which we know are symplectic manifolds. We know also that μ is constant on each C_i, we call then $a_i = \mu(C_i)$. We have just proved that $\mu(M)$ is convex, therefore we know:

$$co\{a_1, \ldots, a_N\} \subset \mu(M), \tag{14}$$

where *co* denotes convex hull. For the converse, take a point $\xi \in \mathbf{R}^m - co\{a_1, \ldots, a_N\}$. Then we can separate ξ from the convex hull by an hyperplane, that is, there is a vector $\eta \in \mathbf{R}^m$ such that:

$$\langle \eta, \xi \rangle > \langle \eta, a_i \rangle \tag{15}$$

for $i = 1, \ldots, N$. Without loss of generality, we can assume that η has rationally independent components.

By the irrationality of the components of η, the set $T_\eta = \{exp(t\eta) : t \in \mathbf{R}\} \subset T$ is dense in T. Therefore, $d\mu^\eta(x) = 0$ if and only if $x \in C$. But that means that μ^η attains its maximum at one of the submanifolds C_i. Thus it holds:

$$\langle \eta, \xi \rangle > \sup_{1 \le i \le N} \langle \eta, a_i \rangle = \sup_{p \in M} \langle \eta, \mu(p) \rangle. \tag{16}$$

This implies that ξ is not in $\mu(M)$, which proves the result.

$\square$