Hamiltonian group actions

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1 Introduction

In this lecture we will define the Hamiltonian group action on a symplectic manifold. Furthermore, we will give an explanation of the physical background of symplectic manifolds and moment maps coming from the study of classical mechanics. In the last part we will state a definition of moment maps for toric actions and prove it is equivalent to the general definition.

2 Definition

Let (M, ω) be a symplectic manifold on which a Lie group G acts symplectically. We will use the following notation for group actions:

$$\begin{aligned} G \times M &\rightarrow M \\ (g, x) &\rightarrow g \cdot x. \end{aligned}$$

Definition 1. *Let $\mathfrak{g}$ be the Lie algebra of G. Then the* **adjoint action** *of G on $\mathfrak{g}$ is the derivative at e of the function*

$$G \rightarrow G, \quad a \rightarrow gag^{-1}$$

or, using the exponential map, we could say

$$\mathrm{Ad}_g(\xi) := \left.\frac{d}{dt}\right|_{t=0} g \exp(t\xi) g^{-1}$$

so that

$$\mathrm{Ad}_g : \mathfrak{g} \rightarrow \mathfrak{g}$$

is an invertible linear map and defines a left group action on $\mathfrak{g}$.

To define the group action on the dual of the Lie algebra $\mathfrak{g}^*$ consider the dual of the adjoint action

$$\langle \mathrm{Ad}_g^* \ell, \xi \rangle = \langle \ell, \mathrm{Ad}_g \xi \rangle, \qquad \xi \in \mathfrak{g}, \ell \in \mathfrak{g}^*.$$

This induces an action on $\mathfrak{g}^*$:

$$g \cdot \ell = \mathrm{Ad}_{g^{-1}}^* \ell \qquad \ell \in \mathfrak{g}^*$$

which is a left group action and called the **coadjoint action**. Sometimes people write Ad_g^* instead of $\mathrm{Ad}_{g^{-1}}^*$ but still meaning the dual of the inverse adjoint action.

Definition 2. *A group action of a Lie group G on a manifold M induces a vectorfield called the* **infinitesimal action** *defined by*

$$\xi_M(x) := \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi)x.$$

It is the vector field generated by $g_t := \exp(t\xi) \in G$.

If we have an $\mathbb{R}$-action we have a direct duality between the generating vector field and the group action.

Definition 3. *A moment map for a symplectic G-action on a symplectic manifold (M, ω) is a smooth map $\mu : M \to \mathfrak{g}^*$ with the following properties:*

1. *$\langle d\mu(x)v, \xi \rangle = \omega_x(v, \xi_M(x))$for all $x \in M,\ v \in T_xM,\ \xi \in \mathfrak{g}$ and X_ξ is the infinitesimal action, sometimes written as*

   $$d\mu^\xi = -i_{\xi_M}\omega$$

   where $\mu^\xi(x) = \langle \mu(x), \xi \rangle$

2. *μ is equivariant with respect to the coadjoint action on $\mathfrak{g}^*$:*

   $$\mu(gx) = g \cdot \mu(x) = \mathrm{Ad}_{g^{-1}}^* \mu(x).$$

Definition 4. *A symplectic Lie group action on a symplectic manifold is called a* **Hamiltonian group action** *if a moment map μ exists.*

(M, ω, G, μ) is called a Hamiltonian G-space.

3 Remarks on Moment Maps

Definition 5. *Let H be a function on a symplectic manifold (M, ω) and X be a vector field on M.*

1. *We call X_H the hamiltonian vector field of H if $i_{X_H}\omega = dH$. To every function H such a vector field exists.*

2. *We call a function F the Hamiltonian function to X if $i_X\omega = dF$ which of course only exists for special vector fields called Hamiltonian vectorfields.*

For a symplectic manifold (M, ω) the vector space $C^\infty(M)$ is a Lie algebra, where the Lie bracket is given by

$$\{f, g\} := \omega(X_f, X_g)$$

for X_f, X_g are the Hamiltonian vectorfield of f and g respectivly.

Instead of considering the moment map μ we could equivalently consider the so called **comoment map** μ^*:

$$\mu^* : \mathfrak{g} \to C^\infty(M) \qquad \mu^*(\xi)(x) = \langle \mu(x), \xi \rangle.$$

The properties then reads:

1. $\mu^*(\xi)$ is a Hamiltonian function for ξ_M

2. μ^* is a Lie algebra anti-homomorphism

$$\mu^*[\xi, \eta] = -\{\mu^*(\xi), \mu^*(\eta)\}.$$

We can easily check that the first properties are equivalent. And it follows that ξ_M (the infinitesimal action of the Lie algebra element ξ) is the Hamiltonian vectorfield to the function $\mu^*(\xi)$:

$$\xi_M = X_{\mu^*(\xi)}.$$

Let's prove that the two second conditions are equivalent. If μ is equivariant we have

$$\langle \mu(gx), \xi \rangle = \langle \mathrm{Ad}^*_{g^{-1}}\mu(x), \xi \rangle = \langle \mu(x), \mathrm{Ad}_{g^{-1}}(\xi) \rangle \qquad \text{for all } \xi, g, x.$$

Then take the path $g_t := \exp(t\eta) \in G$:

$$\begin{aligned} 0 &= \left.\frac{d}{dt}\right|_{t=0} \left[\langle \mu(gx), \xi\rangle - \langle \mu(x), \mathrm{Ad}_{g^{-1}}(\xi)\rangle\right] \\ &= \omega_x(X_\eta(x), X_\xi(x)) - \langle \mu(x), [\xi, \eta]\rangle \\ &= \{\mu^*(\xi), \mu^*(\eta)\}(x) - \mu^*([\xi, \eta])(x). \end{aligned}$$

Left to prove is the other direction. Let μ^* be a Lie algebra homomorphism. We will show that $\langle \mu(gx), \mathrm{Ad}_g \xi\rangle$ is constant. Therefore consider the function

$$T : G \to \mathbb{R}, \qquad T(g) := \langle \mu(gx), \mathrm{Ad}_g \xi\rangle$$

and compute its derivative:

$$dT(g)\eta := \left.\frac{d}{dt}\right|_{t=0} \langle \mu(g_t x), Ad_g \xi\rangle$$
$$= \langle d\mu(gx) X_\eta(gx), \mathrm{Ad}_g \xi\rangle + \langle \mu(gx), [\eta, \mathrm{Ad}_g \xi]\rangle = 0.$$

We sometimes use the following notation $\mu^\xi := \mu^*(\xi) \in C^\infty(M)$.

4 Physical motivation

One model of a classical physical system is given by

$$(T^*N, H)$$

which is a cotangent bundle and a function $H \in C^\infty(T^*N)$ called the Hamiltonian function. The cotangent bundle is canonically symplectic as we saw in the last lecture. If X_H is the Hamiltonian vector field of the function H induced by this symplectic structure, the equations of motion are given by

$$\frac{d}{dt}\rho_t = X_H \circ \rho_t,$$

and called Hamilton's equations. This means that the integral curves of the Hamiltonian vector field describes the motion of the physical system. Out of this statement two major questions arise:

1. Why is this a model of a classical physical system?
2. Are there reasons that this is a good description?

In the following I will try to answer these questions as well as I can.

1. Let N be the configuration space of the physical system and $F = -\nabla V$ the force arising from a potential V. We know that the equations of motions are then given by Newton's law:

$$\ddot{x} = -\nabla V(x).$$

Out of this given physical system we can now construct the previous model and prove that the equations of motions are equivalent. Let H defined on the cotangent space

$$T^*N := \{(p,q) : q \in N, p \in T_q^*N\},$$

the so called phase space, given by:

$$H(q,p) = \frac{\|p\|^2}{2m} + V(q),$$

where the norm comes from a Riemannian metric on the configuration space. The canonical symplectic form is given by $\sum dq_i \wedge dp_i$ so that it is easy to check that the Hamiltonian vector field to H as above is given by:

$$X_H = \sum\left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}\right).$$

Hamilton's equations in coordinates are then given by:

$$\frac{d}{dt}q_i = \frac{\partial H}{\partial p_i} = \frac{p_i}{m} \qquad \frac{d}{dt}q_i = -\frac{\partial H}{\partial q_i} = -\frac{\partial V}{\partial q_i}.$$

These are equivalent to Newton's equations of motion.

2. But this does not explain why we should accept this description. I will try to explain the advantages of this systems in the following.

   - We get a first order differential equation instead of a second order differential equations. This means that to determine the future we do not need initial values of derivatives.
   - Coordinate free description since the symplectic form is canonical.
   - The Hamiltonian description comes out of the lagrangian description by taking the convex conjugate where we go over from the tangent bundle to the contangent bundle.

This general setup of a physical system can be generalized by considering

$$(M, \omega, H)$$

a so called hamiltonian system were (M, ω) is a symplectic manifold and $H \in C^\infty(M)$.

Physicists are interested in symmetries which correspond to group action because they give conserved quantities. In the following we will see that moment maps can be considered to be conserved quantities.

Theorem 1. *Let (M, ω, H) be a Hamiltonian system and $\mu : M \to \mathfrak{g}^*$ a moment map to the G action on M under which H is invariant. Then μ is constant along the integral curves of the Hamiltonian flow generated by H.*

Proof. Let ρ_t be the integral curves of the Hamiltonian vector field to H. Then:

$$\begin{aligned} \frac{d}{dt}\mu(\rho_t)[\xi] &= d\mu(\rho_t)\frac{d}{dt}\rho_t[\xi] \\ &= \omega_{\rho_t}(X_H \circ \rho_t, \xi_M \circ \rho_t) = (\iota_{X_H}\omega)_{\rho_t}(\xi_M \circ \rho_t) \\ &= dH(\rho_t)(\xi_M \circ \rho_t) = \left.\frac{d}{ds}\right|_{s=0} H(\exp(s\xi)\rho_t) = 0 \end{aligned}$$

□

Example

Let $M = \mathbb{R}^6 \cong T^*\mathbb{R}^3$ with the standard symplectic form on $\mathbb{R}^6$. This is the model for a particle moving in $\mathbb{R}^3$. Use coordinates for $\mathbb{R}^6$ such that

$$T^*\mathbb{R}^3 \cong \mathbb{R}^6 := \{(x, p) \in \mathbb{R}^3 \times \mathbb{R}^3\}.$$

For a potential invariant under rotations we get a Hamiltonian function invariant under the action of $SO(3)$ given by:

$$R(x, p) = (Rx, Rp), \quad R \in SO(3).$$

The Lie algebra $\mathfrak{so}(3)$, of the special orthogonal group, are the antisymmetric matrices and the infinitesimal action is given by

$$X_A(x, p) = (Ax, Ap), \quad A \in \mathfrak{so}(3).$$

The moment map for that action is given by

$$\mu((x, p))(A) = \langle p, Ax \rangle.$$

By identifying the Lie algebra $\mathfrak{so}(3)$ with $\mathbb{R}^3$ via the isomorphism

$$\begin{array}{ccc} \Phi : \mathbb{R}^3 & \to & \mathfrak{g} \\ \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} & \to & \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix} \end{array}$$

we get that $\Phi(\xi)x = \xi \times x$ and therefore:

$$\mu((x,p))(\xi) = \langle p, \xi \times x \rangle = \langle x \times p, \xi \rangle.$$

It follows that

$$\mu(x,p) = x \times p,$$

which is the angular momentum.

5 Equivalence to the other definition

In this section we see that in the case of a toric action we can use a simpler definition of the moment map which is equivalent.

Definition 6. *Let $T = (S^1)^r$ act on (M, ω) symplectically. Then the action is called Hamiltonian if there exists a map*

$$\mu : M \to \mathbb{R}^r$$

such that

$$d\mu_j = -i_{X_j}\omega,$$

where the X_j are the generating vector fields of the action of each component:

$$X_j(x) := \left.\frac{d}{dt}\right|_{t=0} g_t^j \cdot x$$

where $g_t^j = (1, ...1, e^{it}, 1, ...1)$.

Proposition 1. *Definition 4 is equivalent to Definiton 6 in the case of a torus action.*

Proof. There is a basis of the Lie algebra of the torus

$$\{\xi^1, ..., \xi^r\}$$

such that $\exp(t\xi^j) = g_t^j$. It follows that

$$\left.\frac{d}{dt}\right|_{t=0} g_t^j = \xi^j \text{ and } \xi_M^j = X_j.$$

"⇒" If we have been given a moment map $\mu : M \to \mathfrak{t}^*$ to the dual of the Lie algebra of T, define

$$\mu_j(x) := \langle \mu(x), \xi_j \rangle$$

and the property is fullfilled.

"⇒" Let $\xi \in \mathfrak{t}$. Then

$$\xi := \sum \alpha_i \xi_i$$

for some $\alpha_i \in \mathbb{R}$. Define the moment map by:

$$\langle \mu(x), \xi \rangle := \sum \alpha_i \mu_i(x).$$

The moment map property is again easy to check. So it is left only to prove that it is equivariant. Considering the equivalent property for the comoment map, we have to prove that

$$\{\mu^*(\xi), \mu^*(\eta)\} = \omega(\xi_M, \eta_M) = 0.$$

First we see that this is constant by taking the derivative:

$$d(\omega(\xi_M, \eta_M)) = i_{[\xi_M, \eta_M]}\omega = 0.$$

Now we conclude this must be zero since

$$\omega(\xi_M, \eta_M) = d\mu^\xi \eta_M = \eta_M \mu^\xi$$

which is the derivative of function along a $\exp(i\eta)$-orbit on M. Since this orbit is compact there must be a maximum and therefore it must be zero somewhere. Since it is constant it is zero everywhere.

□