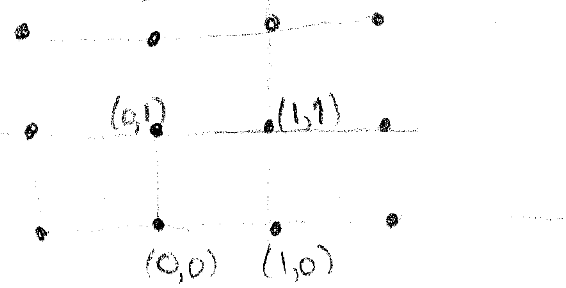
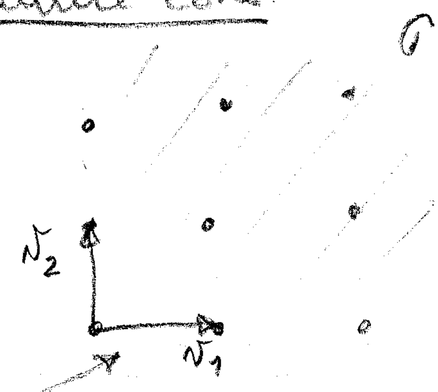
Toric Varieties - Pedro Salomão

Lattice $\mathbb{Z}^2$



Polyhedral lattice cone



strongly convex

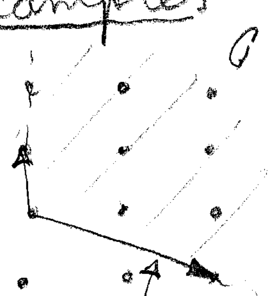
$\sigma \cap -\sigma = \{0\}$

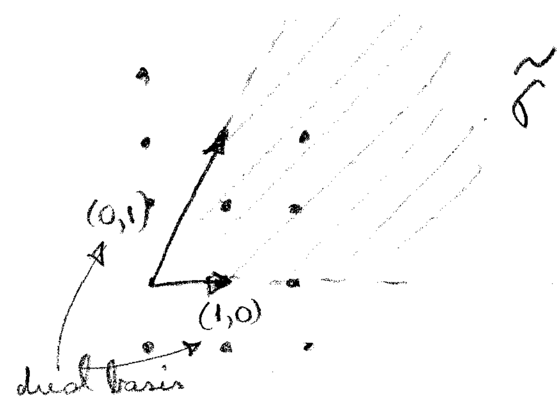
$v_i \in \mathbb{Z}^2$ $\quad \sigma = \{ x \in \mathbb{R}^2 : x = \lambda_i v_i \ , \lambda_i \in \mathbb{R}, \ \lambda_i \geq 0 \}$

Dual cone

$\tilde{\sigma} = \{ x \in (\mathbb{R}^2)^* : \langle u, v \rangle \geq 0 \ \forall v \in \sigma \}$

Examples

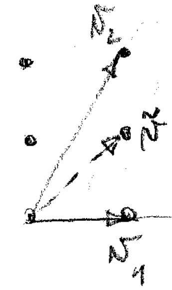




Faces of the cone

②

<u>Gordon's Lemma</u> $\sigma \cap \mathbb{Z}^2$ is finitely generated.



σ is generated by

$v_1 = (1,0)$

$v_2 = (1,2)$

and $\tilde{v} = (1,1)$

<u>Affine Toric Varieties</u> σ - lattice polyhedral strongly convex cone

$\mathbb{C}[z_1, z_2, z_1^{-1}, z_2^{-1}]$ - ring of Laurent polynomials

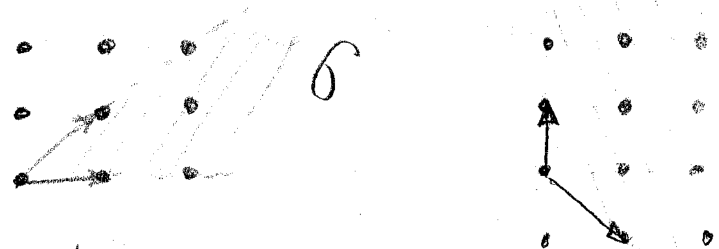
$$R_\sigma := \{ f \in \mathbb{C}[z_1, z_2, z_1^{-1}, z_2^{-1}] : \operatorname{supp}(f) \subset \tilde{\sigma} \cap \mathbb{Z}^2 \}$$

finitely generated monomial algebra

$(a_1, a_2) \in \mathbb{Z}^2 \longrightarrow z_1^{a_1} z_2^{a_2}$

$+ \qquad\qquad \cdot \qquad$ isomorphism

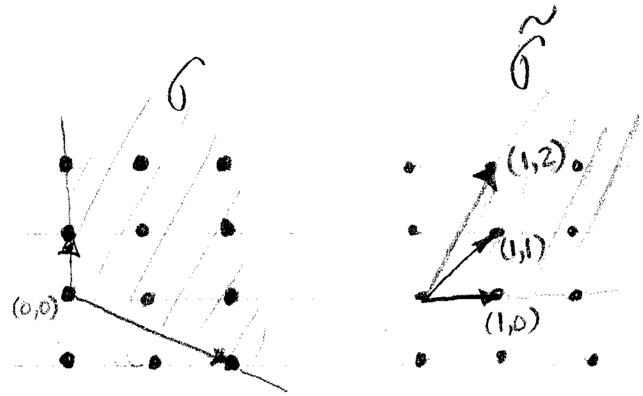
<u>Ex 1:</u>



$\tilde{\sigma}$ generated by $(1,1)$ and $(0,1)$

R_σ is generated by $z_1 z_2$ and z_2

Ex 2



R_σ is generated by

$$\underset{\mu_1}{\overset{\|}{z_1}}, \quad \underset{\mu_2}{\overset{\|}{z_1 z_2}} \text{ and } \underset{\mu_3}{\overset{\|}{z_1 z_2^2}} \qquad \mu_1 \mu_3 = \mu_2^2$$

Let

$$X_\sigma = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 z_3 = z_2^2\} \leftarrow \text{singular cone with } \dim_{\mathbb{C}} X_\sigma = 2$$

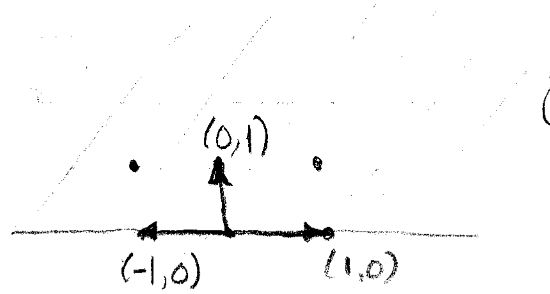
In Ex 1, $X_\sigma = \{(z_1, z_2) \in \mathbb{C}^2 \}$

X_σ is the affine toric variety corresponding to σ

Proposition X_σ contains the torus $(\mathbb{C}^*)^2$ as a Zariski open dense subset

Cor $\dim_{\mathbb{C}} X_\sigma = 2$

Ex 3



R_σ is generated by $\underset{u_1}{\overset{\shortparallel}{z_1}}$, $\underset{u_2}{\overset{\shortparallel}{z_2}}$ and $\underset{u_3}{\overset{\shortparallel}{z_1^{-1}}}$

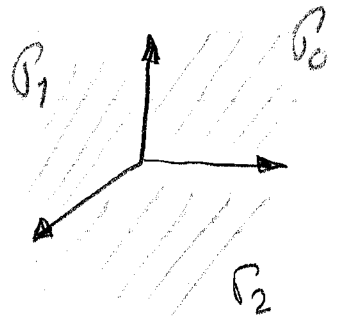
$$u_1^{-1} = u_3$$

$$X_\sigma = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^{-1} = z_3\} \simeq \mathbb{C}^* \times \mathbb{C}$$

Fans

A **fan** Δ in $\mathbb{R}^2$ is a finite union of cones s.t.

1. Every cone of Δ is a strongly convex, polyhedral and lattice cone
2. Every face of a cone of Δ is a cone of Δ
3. If σ and σ' are cones of Δ, then $\sigma \cap \sigma'$ is a common face of σ and σ'



Toric Variety (two-dimensional)

$\sigma \in \Delta$, consider X_σ

Consider the disjoint union $\bigcup_{\sigma \in \Delta} X_\sigma$ with a special* identification of "some points of X_σ and $X_{\sigma'}$ "along" $X_{\sigma \cap \sigma'}$

The resulting space is called a <u>toric variety</u>.

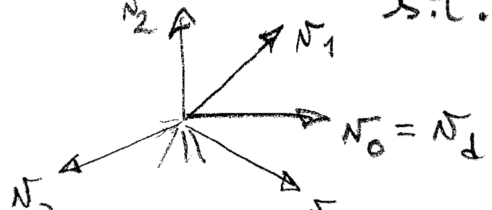
Definition

A fan is complete if it covers $\mathbb{R}^2$ (Prop $\Leftrightarrow X_\Delta$ is compact)
A fan is regular if the vectors generating each of its cones can be completed in a bases of $\mathbb{Z}^2$
(Prop $\Leftrightarrow X_\Delta$ is regular)

Two-dimensional regular complete Toric Varieties

Sequence of lattice points

$v_0, v_1, \ldots\ldots, v_{d-1}, v_d = v_0$ (counterclockwise order)
s.t. successive pairs generate the lattice

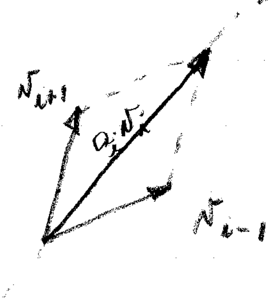


OBS: $\det \begin{pmatrix} v_{ix} & v_{iy} \\ v_{i+1\,x} & v_{i+1\,y} \end{pmatrix} = 1$

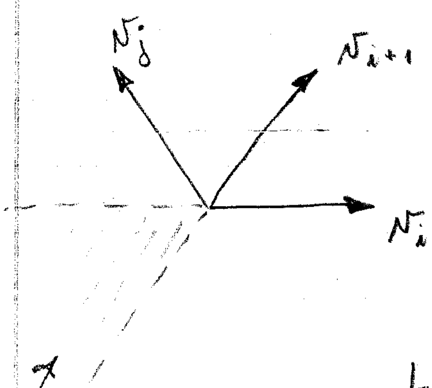
(6)

Some Topological Contrains of these configuration

(1) $\exists\, a_i \in \mathbb{Z}$ s.t. $a_i v_i = v_{i-1} + v_{i+1}$ $\forall\ 1 \leq i \leq d$



(2)



v_{j+1} cannot be there

(3) If $d \geq 4$, there exist $v_j = -v_i$ for some i, j

(4) If $v_i = -v_0$ and $i \geq 3$ then $v_j = v_{j-1} + v_{j+1}$ for some $0 < j < i$

(3) and (4) imply that if $d \geq 5$, then
$v_j = v_{j-1} + v_{j+1}$ for some j and

v_{j-1} and v_{j+1} generate a strongly convex cone

5 The integers $a_1, \ldots, a_d$ satisfy

$$\begin{pmatrix} 0 & -1 \\ 1 & a_1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & a_2 \end{pmatrix} \cdots\cdots \begin{pmatrix} 0 & -1 \\ 1 & a_d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

6 Inserting $v' = v_k + v_{k+1}$ between v_k and v_{k+1} changes the sequence

$$\ldots, a_k, a_{k+1}, \ldots$$
$$\downarrow$$
$$\ldots, a_k + 1, 1, a_{k+1} + 1, \ldots$$

It follows that $\sum a_i = 3d - 12$

 $\left.\begin{array}{l} d=3 \\ d=4 \end{array}\right\}$ Well-know toric varieties $\Rightarrow \mathbb{C}P^2$ and Hirzebruch surface


Ex 3 and 4 $d \geq 5 \Rightarrow$ the toric variety can be constructed indutively